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BLOCH CONSTANTS OF BOUNDED SYMMETRIC DOMAINS

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ABSTRACT. Let D_1 and D_2 be two irreducible bounded symmetric domains in the complex spaces V_1 and V_2 respectively. Let E be the Euclidean metric on V_2 and h the Bergman metric on V_1. The Bloch constant $b(D_1, D_2)$ is defined to be the supremum of $E(f'(z)x, f'(z)x)^{\frac{1}{2}}/h_z(x,x)^{1/2}$, taken over all the holomorphic functions $f : D_1 \to D_2$ and $z \in D_1$, and nonzero vectors $x \in V_1$. We find the constants for all the irreducible bounded symmetric domains D_1 and D_2. As a special case we answer an open question of Cohen and Colonna.

0. INTRODUCTION

The well-known Schwarz lemma states that a holomorphic mapping f from the unit disk Δ into itself is contractive in the Bergman metric, namely if z_1, z_2 are two points in Δ and $d(z_1, z_2)$ is their Bergman distance then $d(f(z_1), f(z_2)) \leq d(z_1, z_2)$. Moreover if f is a holomorphic mapping and is unitary at $z = 0$ in the Bergman metric, then f is a rotation. There has been considerable interest in studying various generalizations of the Schwarz lemma and certain extremal mappings. See [R], [CC] and [Y]. Now if f is a holomorphic mapping from one complex domain D_1 in a complex space V_1 into another domain D_2 in a space V_2, one can study the mapping properties of f with respect to various metrics on the domains. The case we will consider here is when D_1 and D_2 are bounded symmetric domains in V_1 and V_2 respectively with the Bergman metric on D_1 (or rather on its tangent space V_1) and the Euclidean metric on D_2. Following Cohen and Colonna [CC] we define the Bloch constant $b(D_1, D_2)$, and we will find the constants for all irreducible domains (or Cartan domains) D_1 and D_2. When D_2 is the unit disk in the complex plane we answer an open question in [CC]. We proceed to explain in more detail our main result.

Let D_1, D_2 be two bounded domains in the complex spaces V_1 and V_2, respectively. The Bergman metric of D induces a metric h_z on V_1 for each $z \in D_1$. We equip V_2 with the Euclidean metric E obtained from the Bergman metric at 0 of D_2.

Denote by $H(D_1, D_2)$ the space of all holomorphic mappings from D_1 to D_2. For $f \in H(D_1, D_2)$ we define the Bloch constant $b_f(D_1, D_2)(z)$ of f at $z \in D_1$ and

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the Bloch constant $b_f(D_1, D_2)$ of f by

$$b_f(D_1, D_2)(z) = \sup_{x \in V_1} \frac{E(f'(z)x, f'(z)x)^{\frac{1}{2}}}{h_z(x,x)^{\frac{1}{2}}},$$

and

$$b_f(D_1, D_2) = \sup_{z \in D_1} b_f(D_1, D_2)(z),$$

respectively. The Bloch constant $b(D_1, D_2)$ of (D_1, D_2) is defined by

$$b(D_1, D_2) = \sup_{f \in H(D_1, D_2)} b_f(D_1, D_2).$$

Let D_1 and D_2 be two *irreducible bounded symmetric domains*. Every irreducible bounded symmetric domain is uniquely determined by a triple of integers (r, a, b); see §1 below. Here r is the rank. The integer $p = (r-1)a + 2 + b$ is called the genus of D. Let r_j and p_j be the corresponding rank and genus of D_j, $j = 1, 2$. The main result of this paper is

Theorem A. *The Bloch constant* $b(D_1, D_2)$ *is given by*

$$b(D_1, D_2) = \frac{r_2^{\frac{1}{2}} p_2^{\frac{1}{2}}}{p_1^{\frac{1}{2}}}.$$

When D_1 is a classical domain (see below) and D_2 is the unit disk in the complex plane (with $p_2 = 2$) the above result is proved by Cohen and Colonna [CC] through a case by case calculation. We give here a unified solution and express the constants in term of the rank and genus. We will use the Jordan-triple characterization of bounded symmetric domains and the Jordan-triple theoretic description of the topological boundary of the domain ([L1] and [W]). Nevertheless the basic idea is similar to that in [CC].

Finally we remark that the Bloch constant b_f can be defined for any Riemannian manifolds as the Lipschitz constant. Let $f : M \to N$ be a mapping of Riemannian manifolds. The Lipschitz constant of f is defined by

$$\lambda_f = \sup_{x \neq y} \frac{d_N(f(x), f(y))}{d_M(x,y)},$$

which depends only on the metric space structure of the two spaces. When $M = D_1$ and $N = D_2$ are bounded symmetric domains equipped with the Bergman metric and Euclidean metric respectively, and f is holomorphic, one can easily prove that $\lambda_f = b_f$.

The paper is organized as follows. In §1 we give some preliminaries on bounded symmetric domains. In §2 we prove that the radius of an inscribed Hilbert ball in an irreducible bounded symmetric domain D is less than $p^{\frac{1}{2}}$. Using this fact and the Schwarz lemma, we prove the main theorem in §3.

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1. Preliminaries on bounded symmetric domains

We begin by recalling the Jordan-triple characterization of bounded symmetric domains. Our general references here are [L1], [Up] and [Sa].

Let D be an irreducible bounded symmetric domain in a complex n-dimensional space V and let $h_z(\cdot,\cdot)$ be the Bergman metric of D at z. We identify the tangent space at z with V, so a vector $x \in V$ has the Bergman norm $h_z(x,x)^{\frac{1}{2}}$.

Let $Aut(D)_0$ be the identity component of the group of all biholomorphic automorphisms of D, and K the isotropy subgroup of 0. The Lie algebra $aut(D)$ of $Aut(D)_0$ is identified with the Lie algebra of all completely integrable holomorphic vector fields on D, equipped with the Lie product $[X,Y](z) := X'(z)Y(z) - Y'(z)X(z),\ \ X, Y \in aut(D),\ z \in D$.

Let $aut(D) = \mathfrak{k} + \mathfrak{p}$ be the *Cartan decomposition* of $aut(D)$ with respect to the involution $\theta(X)(z) := -X(-z)$. There exists a quadratic form $Q : V \to End(\overline{V}, V)$ (where $\overline{V}$ is the complex conjugate of V), such that $\mathfrak{p} = \{\xi_v; v \in V\}$, where $\xi_v(z) := v - Q(z)v$.

Let $\{z, v, w\}$ be the polarization of $Q(z)v$, i.e.,

$$\{z, v, w\} = Q(z + w)v - Q(z)v - Q(w)v.$$

This defines a *triple product* $V \times \overline{V} \times V \to V$, with respect to which V is a *$JB^\star$-triple*, see [Up].

We define $D(z, v) \in \mathrm{End}(V, V)$ by $D(z, v)w = \{z, v, w\}$. Let $B(z, w)$ be the Bergman operator on V,

$$B(z, w) = 1 - D(z, w) + Q(z)Q(w).$$

The following is proved in [L1, Theorem 2.10].

Lemma 1. *The Bergman metric on V at $z = 0$ is given by*

$$h_0(u, v) := \langle u, v\rangle = \mathrm{tr}\, D(u, v) \tag{1.1}$$

and it is K-invariant, where "tr" *is the trace functional on $End(V)$; at $z \in D$ the Bergman metric is*

$$h_z(u, v) = h_0(B^{-1}(z, z)u, v) = \left\langle B^{-1}(z, z)u, v\right\rangle.$$

We take the Euclidean norm on V given by $h_0(u,u)^{\frac{1}{2}}$. Besides the Euclidean norm, V carries also the *spectral norm*

$$\|z\| := \|\frac{1}{2}D(z, z)\|^{1/2}, \tag{1.2}$$

where the norm of an operator in $End(V)$ is taken with respect to the Hilbert norm $\langle\cdot,\cdot\rangle^{\frac{1}{2}}$ on V. The domain D can now be realized as the open unit ball of V with respect to the spectral norm, i.e.

$$D = \{z \in V : \|z\| < 1\}. \tag{1.3}$$

An element $v \in V$ is a *tripotent* if $\{v, v, v\} = v$. In the matrix Cartan domains (of types I, II, and III, see below) the tripotents are exactly the partial isometries. Each tripotent $v \in V$ gives rise to a Peirce decomposition of V,

$$V = V_0(v) \oplus V_1(v) \oplus V_2(v) \tag{1.4}$$

where

$$V_j(v) = \{u \in V : D(v, v)u = ju\}.$$

Two tripotents v and u are *orthogonal* if $D(v,u)=0$. Orthogonality is a symmetric relation. A tripotent v is *minimal* if it can not be written as a sum of two non-zero orthogonal tripotents. A *frame* is a maximal family of pairwise orthogonal, minimal tripotents. It is known that the group K acts transitively on frames. In particular, the cardinality of all frames is the same, and is equal to the rank r of D. Every $z\in V$ admits a *spectral decomposition* $z=\sum_{j=1}^r s_j v_j$, where $\{v_j\}_{j=1}^r$ is a frame and $s_1\ge s_2\ge\cdots\ge s_r\ge 0$ are the *singular numbers* of z. The spectral norm of z is equal to the largest singular value s_1.

Let us choose and fix a frame $\{e_j\}_{j=1}^r$ in V. Then, by the transitivity of K on the frames, each element $z\in V$ admits a *polar decomposition* $z=k\sum_{j=1}^r s_j e_j$, where $k\in K$ and $s_j=s_j(z)$ are the singular numbers of z.

Let $e:=e_1+\cdots+e_r$; then e is a *maximal tripotent*, namely the only tripotent which is orthogonal to e is 0. Let

$$V=\sum_{0\le j\le k\le r}\oplus V_{j,k}$$

be the *joint Peirce decomposition* of V associated with $\{e_j\}_{j=1}^r$, where

$$V_{j,k}=\{v\in V; D(e_l,e_l)v=(\delta_{l,j}+\delta_{l,k})v,\ 1\le l\le r\},$$

for $(j,k)\ne(0,0)$ and $V_{0,0}=\{0\}$. By the minimality of $\{e_j\}_{j=1}^r$, $V_{j,j}=\mathbb{C}\,e_j$, $1\le j\le r$. The transitivity of K on the frames implies that the integers

$$a:=\dim V_{j,k}\quad(1\le j<k\le r),\qquad b:=\dim V_{0,j}\quad(1\le j\le r)$$

are independent of the choice of the frame and of $1\le j<k\le r$. The triple of integers (r,a,b) uniquely determines D.

The *genus* $p=p(D)$ is defined by

$$p:=\langle e_1,e_1\rangle=\frac{1}{r}\mathrm{tr}D(e,e)=(r-1)a+b+2. \tag{1.5}$$

Finally we give a list of all the irreducible Jordan triples; see [L1] and [L2].

Type I(n, m) ($n\le m$): $V=M_{n,m}(\mathbb{C})$;

Type II(n): $V=\{z\in M_{n,n}(\mathbb{C}); z^T=-z\}$;

Type III(n): $V=\{z\in M_{n,n}(\mathbb{C}); z^T=z\}$;

Type IV(n): $V=\mathbb{C}^n$;

Type V: $V=M_{1,2}(\mathbb{O}_\mathbb{C})$;

Type VI: $V=\{z\in M_{3,3}(\mathbb{O}_\mathbb{C}); \tilde z^t=z\}$.

Here $\mathbb{O}_\mathbb{C}$ is the 8-dimensional Cayley algebra. The Q-operator for Type I-III domains is

$$Q(z)v=zv^*z,$$

where v^* is the adjoint of the matrix v. For Type IV it is

$$Q(z)v=q(z,\bar v)z-q(z)\bar v,$$

where $q(z)$ is the standard quadratic form on $\mathbb{C}^n$ and $q(x,y)=q(x+y)-q(x)-q(y)$ is its polarization. Type I-IV domains are also called classical domains. For Type V $Q(z)v=z\cdot(\tilde v^t\cdot z)$, where $v\mapsto\tilde v$ is the canonical involution of $\mathbb{O}_\mathbb{C}$. The Q operator for Type VI is

$$Q(z)v=\frac{1}{2}(z\circ(z\circ v^*)-z^2\circ v^*),$$

where $x\circ y=x\cdot y+y\cdot x$.

We list also the corresponding triple (r, a, b),

$$(r,a,b) = \begin{cases} (n,2,m-n), & \text{Type I}(n,m), \\ (\frac{n}{2},4,0), & \text{Type II}(n) \text{ and } n \text{ even}, \\ (\frac{n-1}{2},4,2), & \text{Type II}(n) \text{ and } n \text{ odd}, \\ (n,1,0), & \text{Type III}(n), \\ (2,n-2,0), & \text{Type IV}, \\ (2,6,4), & \text{Type V}, \\ (3,8,0), & \text{Type VI}. \end{cases}.$$

2. The radius of an inscribed Hilbert ball in D

In this section we will find an upper bound for the radius of an inscribed Hilbert ball in D. The result will be used to calculate the Bloch constants in §3.

We need the following description of the boundary of the bounded symmetric domain D; see §6 in [L1] (especially §6.9), and [W]. Recall the notation in (1.4).

Theorem 1. *The boundary of D is given by*

$$\partial D = \bigcup_{i=1}^{r} X_i,$$

where

$$X_i = \bigcup_{t_i \in M_i} \mathcal{T}_{t_i}$$

and M_i is the set of tripotents of rank i and $\mathcal{T}_{t_i} = t_i + D \cap V_0(t_i)$.

Roughly speaking, ∂D is a convex curvilinear polyhedron whose faces are X_i.

Following [CC], we define

$$\gamma_D = \inf_{x \in \partial D} \langle x, x \rangle^{\frac{1}{2}}.$$

Proposition 1. *We have*

$$\gamma_D = p^{\frac{1}{2}}.$$

Proof. Let $x = t_i + z \in X_i$ with $t_i \in M_i$ and $z \in D \cap V_0(t_i)$. Thus

$$\langle x, x \rangle = \operatorname{tr} D(t_i + z, t_i + z) = \operatorname{tr} D(t_i, t_i) + \operatorname{tr} D(t_i, z) + \operatorname{tr} D(z, t_i) + \operatorname{tr} D(z, z).$$

However it follows from [L1, Theorem 3.13] that $D(t_i, z) = D(z, t_i) = 0$, and thus

$$\langle x, x \rangle = \operatorname{tr} D(t_i, t_i) + \operatorname{tr} D(z, z) \geq D(t_i, t_i).$$

Since the group K acts on M_i transitively and unitarily with respect to $\langle \cdot, \cdot \rangle$, and $e_1 + \cdots + e_i \in M_i$ it follows that

$$\langle x, x \rangle \geq \operatorname{tr} D(t_i, t_i) = \operatorname{tr} D(e_1 + \cdots + e_i, e_1 + \cdots + e_i) = i \operatorname{tr} D(e_1, e_1) = ip \geq p.$$

Moreover by taking $x = e_1 \in M_1 \subset \partial D$ the above inequality becomes an equality. This finishes the proof. □

Geometrically the above proposition says that if B is a ball in the Hilbert space $(V, \langle \cdot, \cdot \rangle)$ inscribed in D with center 0 and radius ρ, then $\rho < p^{\frac{1}{2}}$.

3. The Bloch constants

Let D_1 and D_2 be two irreducible bounded symmetric domains in V_1 and V_2 with ranks r_1 and r_2 respectively. We denote by $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ the corresponding inner products on V_1 and V_2 given by Lemma 1 in §1, and by $\|\cdot\|_1$ and $\|\cdot\|_2$ the corresponding spectral norm given in (1.2). Let h_z be the Bergman metric of D_1. At $z = 0 \in D_1$ we have $h_0(u, v) = \langle u, v\rangle_1$.

Denote by $H(D_1, D_2)$ the space of holomorphic mappings from D_1 to D_2. Recall that $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ are Banach spaces with D_1 and D_2 being their unit balls. The next lemma is well-known, see e.g. [R] and [Y].

Lemma 2 (Schwarz lemma). *Suppose $f \in H(D_1, D_2)$. Then $f'(0)$, the derivative of f at 0, is a contractive mapping from $(V_1, \|\cdot\|_1)$ into $(V_2, \|\cdot\|_2)$.*

Recall the Bloch constant defined in §0. The Bloch constant of $f \in H(D_1, D_2)$ at z is now

$$b_f(z) = b_f(D_1, D_2)(z) = \sup_{x\in\mathbb{C}^N} \frac{\langle f'(z)x, f'(z)x\rangle_2^{\frac{1}{2}}}{h_z(x, x)^{\frac{1}{2}}}$$

Notice that $b_f(\phi(z)) = b_{f\circ\phi}(z)$ for any biholomorphic mapping ϕ of D_1. However D_1 is a symmetric domain and for every $z \in D_1$ there is a biholomorphic mapping ϕ mapping z to 0. Thus the Bloch constant of (D_1, D_2) is

$$b(D_1, D_2) = \sup_{f\in H(D_1,D_2)} b_f(0).$$

Our main result is the following (stated as Theorem A in §0).

Theorem 2. *Let D_1 and D_2 be two irreducible bounded symmetric domains in their realizations as the unit balls in the Jordan triples. Let r_j and p_j be the rank and the genus of D_j, $j = 1, 2$. Then the Bloch constant $b(D_1, D_2)$ is given by*

$$b(D_1, D_2) = \frac{r_2^{\frac{1}{2}} p_2^{\frac{1}{2}}}{p_1^{\frac{1}{2}}}.$$

Proof. Let $f \in H(D_1, D_2)$. For any $x \in V_1$,

$$\frac{\langle f'(0)x, f'(0)x\rangle_2}{h_0(x, x)} = \frac{\langle f'(0)x, f'(0)x\rangle_2}{\langle x, x\rangle_1} = \frac{\langle f'(0)x, f'(0)x\rangle_2}{\|f'(0)x\|_2^2} \frac{\|f'(0)x\|_2^2}{\|x\|_1^2} \frac{\|x\|_1^2}{\langle x, x\rangle_1}.$$

Write $y = f'(0)x$ and let $y = s_1E_1 + s_2E_2 + \cdots + s_{r_2}E_{r_2}$ be the spectral decomposition of y with the E_j being minimal tripotents (and some s_j may be zero). We find the first term in the above product:

$$\frac{\langle y, y\rangle_2}{\|y\|_2^2} = \frac{(s_1^2 + \cdots + s_{r_2}^2)\,\langle E_1, E_1\rangle_2}{\max_j s_j^2} = p_2 \frac{(s_1^2 + \cdots + s_{r_2}^2)}{\max_j s_j^2} \le r_2p_2,$$

since $\langle E_j, E_j\rangle_2 = \langle E_1, E_1\rangle_2 = p_2$ by (1.5). For the second term we use Lemma 2 (the Schwarz lemma) to get

$$\frac{\|f'(0)x\|_2^2}{\|x\|_1^2} \le 1.$$

The third term is

$$\begin{aligned}\frac{\|x\|_1^2}{\langle x,x\rangle_1} &\leq \left(\inf_{u\in V_1}\frac{\langle u,u\rangle_1}{\|u\|_1^2}\right)^{-1}\\ &= \left(\inf_{u\in\partial D}\frac{\langle u,u\rangle_1}{\|u\|_1^2}\right)^{-1}\\ &= \left(\inf_{u\in\partial D}\langle u,u\rangle_1\right)^{-1}\\ &= \frac{1}{\gamma_{D_1}^2}\\ &= \frac{1}{p_1},\end{aligned}$$

where the first equality is obtained from the fact that D is open and convex, the second is because of (1.3) and the last by Proposition 1.

Now, putting the three inequalities together, we have

$$b_f(0)^2 = \sup_{x\in V_1}\frac{\langle f'(0)x, f'(0)x\rangle_2}{h_0(x,x)} \leq r_2p_2\frac{1}{p_1},$$

and

$$b(D_1,D_2)^2 = \sup_{f\in H(D_1,D_2)} b_f(0)^2 \leq r_2p_2\frac{1}{p_1}. \tag{3.1}$$

We now prove the reverse inequality. We fix a minimal tripotent e_1 of V_1 and a maximal tripotent E of V_2. Take $f(z) = \frac{1}{p_1}\langle z,e_1\rangle_1 E$. We claim that f maps D_1 into D_2. It's sufficient to prove that $\frac{1}{p_1}|\langle z,e_1\rangle_1| < 1$ if $z\in D_1$. Otherwise, suppose for some $z\in D_1$, $\frac{1}{p_1}|\langle z,e_1\rangle_1| \geq 1$. Write $z = \lambda e_1 + y$ according to the Peirce decomposition of V with respect to e_1, $V = \mathbb{C}e_1 \oplus V_1(e_1)\oplus V_0(e_1)$, with $y\in V_1(e_1)\oplus V_0(e_1)$ and $\lambda\in\mathbb{C}$. Then

$$\langle z,e_1\rangle_1 = \lambda\langle e_1,e_1\rangle_1 = \lambda p_1$$

and

$$|\lambda| = \frac{1}{p_1}|\langle z,e_1\rangle_1| \geq 1. \tag{3.2}$$

Furthermore

$$\begin{aligned}\langle D(z,z)e_1,e_1\rangle_1 &= \langle D(\lambda e_1+y,\lambda e_1+y)e_1,e_1\rangle_1\\ &= |\lambda|^2\langle D(e_1,e_1)e_1,e_1\rangle_1 + \lambda\langle D(e_1,y)e_1,e_1\rangle_1\\ &\qquad + \bar{\lambda}\langle D(y,e_1)e_1,e_1\rangle_1 + \langle D(y,y)e_1,e_1\rangle_1.\end{aligned}$$

By the Peirce rule, $D(e_1,y)e_1 = 0$ and $D(y,e_1)e_1\in V_1(e_1)\oplus V_0(e_1)$; see [L1]. So the second and third term, $\langle D(e_1,y)e_1,e_1\rangle_1$ and $\langle D(y,e_1)e_1,e_1\rangle_1$, are 0. Thus

$$\begin{aligned}\langle D(z,z)e_1,e_1\rangle_1 &= |\lambda|^2\langle D(e_1,e_1)e_1,e_1\rangle_1 + \langle D(y,y)e_1,e_1\rangle_1\\ &= 2|\lambda|^2p_1 + \langle D(y,y)e_1,e_1\rangle_1\\ &\geq 2p_1 \quad (\text{by (3.2)}),\end{aligned} \tag{3.3}$$

since $D(y,y)$ is a positive operator (see [L1]).

However since $z \in D_1$ we have $||D(z,z)||_1 = 2||z||_1^2 < 2$ by (1.3), and thus

$$|\langle D(z,z)e_1, e_1\rangle_1| < 2\langle e_1, e_1\rangle_1 = 2p_1.$$

This contradicts (3.3). Thus f maps D_1 to D_2.

Now $f'(0)u = \frac{1}{p_1}\langle u, e_1\rangle_1 E$, $f'(0)e_1 = E$, $\langle E, E\rangle = r_2 p_2$ and

$$h_0(e_1, e_1) = \langle e_1, e_1\rangle_1 = p_1.$$

Thus

$$b(D_1, D_2) \geq b_f(D_1, D_2)(0) = \frac{r_2^{\frac{1}{2}} p_2^{\frac{1}{2}}}{p_1^{\frac{1}{2}}}.$$

□

Let $D_2 = \Delta$ be the unit disk in $\mathbb{C}$. We have thus proved

Theorem 3. *Let D be an irreducible bounded symmetric domain in its realization as the unit ball in the Jordan triple, with rank r and genus p. Then the Bloch constant $b(D, \Delta)$ is given by*

$$b(D,\Delta) = \frac{\sqrt{2}}{\sqrt{p}} = \begin{cases} \frac{\sqrt{2}}{\sqrt{n+m}}, & \text{Type I}(n,m), \\ \frac{1}{\sqrt{n-1}}, & \text{Type II}(n), \\ \frac{\sqrt{2}}{\sqrt{n+1}}, & \text{Type III}(n), \\ \frac{\sqrt{2}}{\sqrt{n}}, & \text{Type IV}(n), \\ \frac{1}{\sqrt{6}}, & \text{Type V}, \\ \frac{1}{3}, & \text{Type VI}. \end{cases}.$$

When D is a classical domain the above result is proved by Cohen and Colonna [CC]. (Note that our Type II domains (antisymmetric matrices) are their Type III and our Type III domains (symmetric matrices) are their Type II; we have followed [L1] and [L2] for the choice of the type while their choice is as in [Hu].)

Let $H^\infty(D_1, (V_2, \|\cdot\|))$ be the space of bounded analytic mappings from D_1 to the Banach space $(V_2, \|\cdot\|)$. For $f \in H^\infty(D_1, (V_2, \|\cdot\|))$ we use the corresponding norm

$$\|f\|_{H^\infty} = \sup_{z\in D_1} \|f(z)\|_2.$$

The following result then follows easily from Theorem 2, which when D_2 is the unit disk is related to the study of Bloch functions [CC].

Corollary. *Let $f \in H^\infty(D_1, (V_2, \|\cdot\|))$. Then*

$$\frac{\langle f'(z)x, f'(z)x\rangle_2}{h_z(x,x)} \leq b(D_1 D_2)^2 \|f\|_{H^\infty}^2.$$

Another immediate consequence of Theorem 2 is a generalization of Theorem 6 of [CC] to all bounded symmetric domains; the proof of it can be done similarly. It characterizes those extremal functions f so that the Bloch constant $b(D, \Delta)$ is achieved by $b_f(D, \Delta)(w)$ for some $w \in D$.

Theorem 4. *Let $D = D_1 \times D_2 \times \cdots \times D_k$ be a bounded symmetric domain with D_j irreducible, and assume that $f : D \mapsto \Delta$ is holomorphic and such that $b_f(D, \Delta)$*

$= b(D,\Delta)(\omega)$ *for some* $w \in D$. *Then* $b(D,\Delta) = b(D_m,\Delta)$ *for some* $m \in \{1,\dots,k\}$, *and there exist* $x_m \in \partial_1 D_m$ *and an automorphism* T *of* D_m *such that*

$$f(z_1,\dots,z_{m-1},T(zx_m),z_{m+1},\dots,z_k) = z$$

for all $z \in \Delta$ *and* $z_j \in D_j$ *for all* $j \neq m$.

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