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TARSKI'S FINITE BASIS PROBLEM VIA $\mathbf{A}(\mathcal{T})$

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Abstract. R. McKenzie has recently associated to each Turing machine $\mathcal{T}$ a finite algebra $\mathbf{A}(\mathcal{T})$ having some remarkable properties. We add to the list of properties, by proving that the equational theory of $\mathbf{A}(\mathcal{T})$ is finitely axiomatizable if $\mathcal{T}$ halts on the empty input. This completes an alternate (and simpler) proof of McKenzie's negative answer to A. Tarski's finite basis problem. It also removes the possibility, raised by McKenzie, of using $\mathbf{A}(\mathcal{T})$ to answer an old question of B. Jónsson.

An algebra $\mathbf{A} = \langle A, (F_i)_{i\in I}\rangle$ is *finite* if its universe A is a finite set, and is of *finite type* if its fundamental operations F_i are finite in number. All algebras to be considered in this paper are assumed to be of finite type.

A *variety* is any nonempty class of algebras (all having the same type of fundamental operations) which is closed under the formation of arbitrary direct products, subalgebras, and homomorphic images. We say that a variety $\mathcal{V}$ is *residually very finite* and write $\kappa(\mathcal{V}) < \omega$ if there exists a finite cardinal n such that every member of $\mathcal{V}$ can be embedded in a product of algebras all having fewer than n elements. $\mathcal{V}$ is *residually small* if the same condition holds but with n not assumed to be finite. If $\mathbf{A}$ is a finite algebra then we write $\kappa(\mathbf{A}) < \omega$ to mean $\kappa(\mathsf{V}(\mathbf{A})) < \omega$, where $\mathsf{V}(\mathbf{A})$ is the smallest variety containing $\mathbf{A}$.

A finite algebra $\mathbf{A}$ is *finitely based* if the equations (or 'identities') which can be built from [names for] the fundamental operations and which are identically true in $\mathbf{A}$ can all be logically derived from some finite number of them. This is equivalent to $\mathsf{V}(\mathbf{A})$ being a strictly elementary or finitely axiomatizable class in the sense of first-order logic.

Two problems well-known to universal algebraists are:

Problem 1. ([9, 10], A. Tarski, 1960s): Is there an algorithm which, given an arbitrary finite algebra as input, determines whether it is finitely based?

Problem 2. ([5, 11], attributed to B. Jónsson, 1970s): If $\mathbf{A}$ is a finite algebra such that $\kappa(\mathbf{A}) < \omega$, must $\mathbf{A}$ be finitely based?

R. McKenzie recently answered Problem 1 negatively [8]. His solution uses an ingenious construction which effectively assigns to each Turing machine $\mathcal{T}$ a finite algebra $\mathbf{F}(\mathcal{T})$ which is finitely based if and only if $\mathcal{T}$ halts. This construction evolved from an earlier one, denoted by $\mathbf{A}(\mathcal{T})$, which McKenzie described in [7] and which he showed satisfies

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1. $\mathsf{V}(\mathbf{A}(\mathcal{T}))$ is residually small, for all $\mathcal{T}$.
2. $\kappa(\mathbf{A}(\mathcal{T})) < \omega$ if and only if $\mathcal{T}$ halts.
3. If $\mathcal{T}$ does not halt, then $\mathbf{A}(\mathcal{T})$ is not finitely based.

This earlier construction was particularly intriguing because, as McKenzie noted, the algebras $\mathbf{A}(\mathcal{T})$ must answer either Problem 1 or Problem 2 negatively, but it was not clear which.

In this paper we show that the algebras $\mathbf{A}(\mathcal{T})$, like $\mathbf{F}(\mathcal{T})$, answer Problem 1. Whereas the techniques McKenzie employed to analyse $\mathbf{F}(\mathcal{T})$ are syntactic, our methods are algebraic and have wider application. Our result completes a proof of the negative solution to Problem 1 which is significantly shorter than McKenzie's original proof. It also yields a strong negative answer to Problem 1, namely, that the finite basis problem is unsolvable even when restricted to finite algebras which generate a residually small variety.

The proof is spread equally across two papers, this one and McKenzie's [7]. We have organized our paper so that much of it can be read independently of [7], yet so that its intersection with [7] is minimized. A summary of the relevant parts of [7] may be found in Section 4. Readers of earlier drafts of this paper will notice a simplification, due to K. Kearnes, of the argument in Section 1.

1. Tools

Throughout this paper $\mathcal{F}$ will denote a finite language containing the symbols $\wedge$ (binary) and 0 (nullary). Let $\mathcal{W}_\mathcal{F}$ be the class of all algebras of type $\mathcal{F}$ whose reduct to $\{\wedge, 0\}$ is a meet semilattice with least element 0, and whose remaining fundamental operations are monotone with respect to the semilattice ordering. Observe that $\mathcal{W}_\mathcal{F}$ is a finitely based variety.

Definition 1.1. Let $\mathbf{A} \in \mathcal{W}_\mathcal{F}$. We say that $\mathbf{A}$ is 0-*absorbing* if $F(a_1, \dots, a_n) = 0$ whenever F is an n-ary fundamental operation ($n > 0$) and $0 \in \{a_1, \dots, a_n\} \subseteq A$. We say that $\mathbf{A}$ *commutes with* $\wedge$ if

$$F(a_1, \dots, a_n) \wedge F(b_1, \dots, b_n) = F(a_1 \wedge b_1, \dots, a_n \wedge b_n)$$

for each n-ary fundamental operation F ($n > 0$) and all $a_i, b_i \in A$.

Lemma 1.2. *Suppose that $\mathbf{A} \in \mathcal{W}_\mathcal{F}$ has height 1 and is 0-absorbing, and that $\mathsf{V}(\mathbf{A})$ is residually small. Then $\mathbf{A}$ commutes with $\wedge$.*

Proof. Suppose not. Pick a failure $F, \bar{a}, \bar{b}$ of the commuting condition. As F is order-preserving, we must have $F(\bar{a}) \wedge F(\bar{b}) > F(a_1 \wedge b_1, \dots, a_n \wedge b_n)$. Since $\mathbf{A}$ has height 1, this implies $F(\bar{a}) = F(\bar{b}) =: c \neq 0$ (and $\bar{a} \neq \bar{b}$). Moreover, no a_i or b_i is 0 since $\mathbf{A}$ is 0-absorbing.

Let κ be any infinite cardinal and for $i < \kappa$ define $f_1^{(i)}, \dots, f_n^{(i)} \in A^\kappa$ by

$$\langle f_1^{(i)}(j), \dots, f_n^{(i)}(j) \rangle = \begin{cases} \bar{a} & \text{if } j < \kappa \text{ and } j \neq i, \\ \bar{b} & \text{if } j = i. \end{cases}$$

Let $X = \{f_t^{(i)} : i < \kappa,\ 1 \le t \le n\}$ and let $\hat{c} \in A^\kappa$ be the constant map with value c. Observe that $F(f_1^{(i)}, \dots, f_n^{(i)}) = \hat{c}$ for each $i < \kappa$, and that $|X| = \kappa$.

Now let $\delta = 0_{A^\kappa} \cup \{f \in A^\kappa : 0 \in \operatorname{range}(f)\}^2$. Clearly $(\hat{0}, \hat{c}) \notin \delta$, and $\delta \in \operatorname{Con} \mathbf{A}^\kappa$ because $\mathbf{A}$ is 0-absorbing. We claim that if $f, g \in X$ with $f \neq g$, then $(\hat{0}, \hat{c}) \in \operatorname{Cg}^{\mathbf{A}^\kappa}(f, g) \vee \delta$. Indeed, suppose $f = f_t^{(i)}$ and let $\phi = \operatorname{Cg}^{\mathbf{A}^\kappa}(f, g)$. Then

$f = f \wedge f \overset{\phi}{\equiv} f \wedge g \overset{\delta}{\equiv} \hat{0}$, proving $(f, \hat{0}) \in \phi \vee \delta$. Let $\lambda(x)$ be the polynomial $F(f_1^{(i)}, \dots, f_{t-1}^{(i)}, x, f_{t+1}^{(i)}, \dots, f_n^{(i)})$; then $\lambda(f) = \hat{c}$ while $\lambda(\hat{0}) = \hat{0}$, proving $(\hat{0}, \hat{c}) \in \phi \vee \delta$.

Thus if θ is a congruence of $\mathbf{A}^\kappa$ extending δ and maximal with respect to omitting $(\hat{0}, \hat{c})$, then $\mathbf{A}^\kappa/\theta$ is subdirectly irreducible and has cardinality at least κ. As κ was arbitrary, $\mathsf{V}(\mathbf{A})$ must be residually large. $\square$

Remark. Lemma 1.2 can also be deduced from [4, Lemma 2.5].

Lemma 1.3. *Suppose that $\mathcal{K}$ is a finite set of finite algebras in $\mathcal{W}_{\mathcal{F}}$ and that $\mathcal{V} = \mathsf{V}(\mathcal{K})$. Suppose furthermore that each member of $\mathcal{K}$ has height 1, is 0-absorbing, and commutes with $\wedge$. Then there exists $n > 0$ with the following property: for every term $t(x, \bar{y})$ there exist terms $s(x, z_1, \dots, z_n)$ and $p_1(\bar{y}), \dots, p_n(\bar{y})$ such that*

$$\mathcal{V} \models t(x, \bar{y}) \approx s(x, p_1(\bar{y}), \dots, p_n(\bar{y})).$$

Proof. Let $t(x, y_1, \dots, y_m)$ be given, where we assume that each y_i occurs explicitly in t. We may also assume that x occurs explicitly in t, since otherwise the claim is trivial. List $\mathcal{K} = \{\mathbf{A}_1, \dots, \mathbf{A}_q\}$. For each $k = 1, \dots, q$ define $R_k^t = \operatorname{range}(t^{\mathbf{A}_k}) \setminus \{0\}$. We claim that if $e \in R_k^t$ then there is a unique solution $(a, \bar{c}) \in A_k^{m+1}$ to $t(a, \bar{c}) = e$. For if $t(a, \bar{c}) = t(b, \bar{d}) = e$ then $t(a \wedge b, c_1 \wedge d_1, \dots, c_m \wedge d_m) = e$, since $\mathbf{A}_k$ commutes with $\wedge$. Since $e \neq 0$ and $\mathbf{A}_k$ is 0-absorbing, we have $0 \notin \{a \wedge b, c_1 \wedge d_1, \dots, c_m \wedge d_m\}$. This forces $(a, \bar{c}) = (b, \bar{d})$, since $\mathbf{A}_k$ has height 1.

Now for each $k = 1, \dots, q$ and $e \in R_k^t$ define an equivalence relation $E_{k,e}^t$ on $\{1, \dots, m\}$ by

$$(i, j) \in E_{k,e}^t \quad \text{iff} \quad (\mathbf{A}_k \models t(x, \bar{y}) = e \ \rightarrow \ y_i = y_j).$$

By the earlier claim, $E_{k,e}^t$ has at most $|A_k|$ equivalence classes. It follows that if

$$\begin{aligned} E^t &= \bigcap \{E_{k,e}^t \, : \, 1 \leq k \leq q, \, e \in R_k^t\}, \\ M &= \max\{|A_k| \ : \ 1 \leq k \leq q\}, \\ n &= M^{Mq}, \end{aligned}$$

then E^t has at most n equivalence classes. Note that n does not depend on t, and that

$$(*) \qquad \mathcal{K} \models \forall x \bar{y} [t(x, \bar{y}) \neq 0 \rightarrow \bigwedge_{(i,j) \in E^t} y_i = y_j].$$

Now enumerate the equivalence classes of E^t as $C_1, \dots, C_r$. For each $k = 1, \dots, r$ and $i = 1, \dots, m$, define

$$\begin{aligned} p_k(\bar{y}) &= \bigwedge \{y_j \, : \, j \in C_k \}, \\ f_i &= p_k, \quad \text{where } i \in C_k. \end{aligned}$$

Condition $(*)$ implies that the identity $t(x, \bar{y}) \approx t(x, f_1, \dots, f_m)$ is true in $\mathcal{K}$ and hence in $\mathcal{V}$. Clearly $t(x, f_1, \dots, f_m)$ can be rewritten as $s(x, p_1, \dots, p_r)$ for some term $s(x, z_1, \dots, z_r)$. Since $r \leq n$, this proves the lemma. $\square$

Recall that a variety $\mathcal{V}$ has *definable principal congruences* if there is a first-order formula $\Phi(x, y, u, v)$ in the language of $\mathcal{V}$ such that for all $\mathbf{A} \in \mathcal{V}$ and $a, b, c, d \in A$, $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$ if and only if $\mathbf{A} \models \Phi(a, b, c, d)$.

Theorem 1.4. *Let $\mathcal{V}$ be a subvariety of $\mathcal{W}_{\mathcal{F}}$ satisfying $\kappa(\mathcal{V}) < \omega$ and such that every subdirectly irreducible member of $\mathcal{V}$ has height 1 and is 0-absorbing. Then $\mathcal{V}$ has definable principal congruences and hence is finitely based.*

Remark. The fact that $\mathcal{V}$ has definable principal congruences was first observed by K. Kearnes, who also supplied the following proof. In fact, Theorem 1.4 is a special case of a much more general result which will appear in [3].

Proof. Let $\mathcal{K}$ be a finite set containing an isomorphic copy of each subdirectly irreducible member of $\mathcal{V}$. By Lemma 1.2, each member of $\mathcal{K}$ commutes with $\wedge$. Thus there exists an $n > 0$ witnessing the conclusion of Lemma 1.3. $\mathcal{V}$ is locally finite by hypothesis; thus there exists a finite set S of terms in the variables $\{x, z_1, \dots, z_n\}$ such that every term t in these variables satisfies $\mathcal{V} \models t \approx s$ for some $s \in S$. Hence:

Claim 1. For every $\mathbf{A} \in \mathcal{V}$, $\mathrm{Pol}_1 \mathbf{A} = \{s(x, \bar{b}) : s \in S, \bar{b} \in A^n\}$.

Recall that $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$ if and only if there exist elements $a_0, a_1, \dots, a_m$ with $a_0 = a$ and $a_m = b$, such that for each $i < m$ there exists $\lambda_i \in \mathrm{Pol}_1 \mathbf{A}$ satisfying $\{\lambda_i(c), \lambda_i(d)\} = \{a_i, a_{i+1}\}$. The sequence $\langle a_i \rangle$ is called a *Mal'cev sequence witnessing* $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$.

Claim 2. If $\mathbf{A} \in \mathcal{V}$ and $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$, then there exists such a sequence $a = a_0, a_1, \dots, a_m = b$ with $m \leq 2 \cdot |S|$.

To prove it, begin with an arbitrary Mal'cev sequence $a = a_0, a_1, \dots, a_m = b$ witnessing $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$. Define a sequence $a = b_0, b_1, \dots, b_{2m} = b$ by

$$b_i = \begin{cases} a_0 \wedge a_1 \wedge \cdots \wedge a_i & \text{if } i \leq m, \\ a_{i-m} \wedge a_{i-m+1} \wedge \cdots \wedge a_m & \text{if } m \leq i \leq 2m. \end{cases}$$

Observe that $b_0, b_1, \dots, b_{2m}$ is also a Mal'cev sequence witnessing $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$, and that $b_i \geq b_{i+1}$ for $i < m$ while $b_i \leq b_{i+1}$ for $m \leq i < 2m$. Let $J = \{i < 2m : b_i \neq b_{i+1}\}$. We shall show that $|J| \leq 2 \cdot |S|$, which will prove the claim.

First suppose $i, j \in J$ with $i < j < m$. Thus $b_i > b_{i+1} \geq b_j > b_{j+1}$. Pick $\lambda, \mu \in \mathrm{Pol}_1 \mathbf{A}$ such $\{\lambda(c), \lambda(d)\} = \{b_i, b_{i+1}\}$ and $\{\mu(c), \mu(d)\} = \{b_j, b_{j+1}\}$. By Claim 1 there exist $s, t \in S$ and $\bar{u}, \bar{v} \in A^n$ such that $\lambda(x) = s(x, \bar{u})$ and $\mu(x) = t(x, \bar{v})$. We claim that $s \neq t$. For if $s = t$ then we would have

$$\begin{aligned} s(c, \bar{v}) &= s(c, \bar{v}) \wedge s(d, \bar{u}) \\ &= s(c \wedge d, v_1 \wedge u_1, \dots, v_n \wedge u_n) \\ &= s(d, \bar{v}) \wedge s(c, \bar{u}) \\ &= s(d, \bar{v}), \end{aligned}$$

which contradicts $b_j \neq b_{j+1}$. This proves that $|J \cap \{i : i < m\}| \leq |S|$. A similar proof shows that $|J \cap \{i : m \leq i < 2m\}| \leq |S|$, proving Claim 2.

It follows immediately from Claims 1 and 2 that $\mathcal{V}$ has definable principal congruences. Since $\kappa(\mathcal{V}) < \omega$, it follows by [5, Theorem 3] that $\mathcal{V}$ is finitely based. ☐

If $F \in \mathcal{F}$ is a k-ary fundamental operation ($k > 0$) and $1 \leq i \leq k$, then $F_{(i)}(x, \bar{y})$ denotes the term $F(y_1, \dots, y_{i-1}, x, y_{i+1}, \dots, y_k)$. If $\mathbf{A} \in \mathcal{W}_{\mathcal{F}}$ and $\bar{a} \in A^{k-1}$, then we call $F_{(i)}(x, \bar{a})$ a *basic translation* of $\mathbf{A}$. Let $\mathcal{G}$ be the set of all basic translations of $\mathbf{A}$; then we let $\mathrm{Pol}_1^* \mathbf{A}$ denote the closure of $\mathcal{G} \cup \{\mathrm{id}\}$ under composition, while $\mathbf{A}^*$ denotes the nonindexed algebra $\langle A, \mathcal{G} \rangle$. The following are either easily verified or well known: (i) $\mathrm{Con}\, \mathbf{A}^* = \mathrm{Con}\, \mathbf{A}$; (ii) $\mathrm{Pol}_1 \mathbf{A}^* = \mathrm{Pol}_1^* \mathbf{A} \cup \{\text{constant maps}\}$.

Lemma 1.5. *Suppose $\mathbf{A} \in \mathcal{W}_{\mathcal{F}}$ and $\mu \in \operatorname{Con} \mathbf{A}$ and $a, b \in A$ with $b < a$.*

1. *If $\mu \neq 0_A$, then there exists $(p,q) \in \mu$ with $q < p$.*
2. *If $0 \prec p$ and $(p,0) \in \operatorname{Cg}^{\mathbf{A}}(a,b)$, then there exists $\lambda \in \operatorname{Pol}_1^* \mathbf{A}$ such that $\lambda(b) = 0$ and $\lambda(a) = p$.*
3. *If $\operatorname{Cg}^{\mathbf{A}}(a,b) \cap \mu \neq 0_A$, then there exist $(p,q) \in \mu$ and $\lambda \in \operatorname{Pol}_1^* \mathbf{A}$ such that $\lambda(b) \leq q < p$ and $\lambda(a) = p$.*

Proof. The first item follows from the presence of $\wedge$. To prove the second item, observe that $(p,0) \in \operatorname{Cg}^{\mathbf{A}^*}(a,b)$ and that $\{p,0\}$ is the range of the idempotent unary polynomial $e(x) := x \wedge p$ of $\mathbf{A}^*$. Thus by Mal'cev's description of principal congruences there must exist $\tau \in \operatorname{Pol}_1 \mathbf{A}^*$ such that $\{e\tau(a), e\tau(b)\} = \{p, 0\}$. In fact, $e\tau(a) = p$ and $e\tau(b) = 0$ as $e\tau$ is order-preserving. Thus $\lambda = e\tau$ witnesses item 2.

To prove item 3, suppose $\operatorname{Cg}^{\mathbf{A}}(a,b) \cap \mu \neq 0_A$. By item 1 there exists $(p_1, q_1) \in \operatorname{Cg}^{\mathbf{A}}(a,b) \cap \mu$ with $q_1 < p_1$. Using the fact that $(p_1,q_1) \in \operatorname{Cg}^{\mathbf{A}^*}(a,b)$ and that every $\lambda \in \operatorname{Pol}_1^* \mathbf{A}$ is order-preserving, and using the basic translation $x \mapsto x \wedge p_1$, one sees that there must exist $\lambda_1 \in \operatorname{Pol}_1^* \mathbf{A}$ such that $\lambda_1(b) \leq q_1$, $\lambda_1(a) \not\leq q_1$, and $\operatorname{range}(\lambda_1) \subseteq [0, p_1]$; hence in particular $\lambda_1(a) \leq p_1$. Let $p = \lambda_1(a)$, $q = \lambda_1(a) \wedge q_1$, and $\lambda(x) = \lambda_1(x) \wedge p$. Then $q < p$, $(p,q) \in \operatorname{Cg}^{\mathbf{A}}(p_1,q_1) \subseteq \mu$, $\lambda \in \operatorname{Pol}_1^* \mathbf{A}$, $\lambda(a) = p$, and $\lambda(b) \leq q$, as required. ☐

2. McKenzie's method

In Section 3 we shall prove some properties of the equational theory of $\mathbf{A}(\mathcal{T})$ which hold regardless of whether $\mathcal{T}$ halts. In fact, the results of that section are valid for a large class of algebras constructed according to a method invented by McKenzie in [6] and explained in [12]. Therefore, in this section we shall describe this class.

The results in Section 3 will apply to the algebra $\mathbf{A}$ provided the universe and fundamental operations of $\mathbf{A}$ can be described in the following way. The universe of $\mathbf{A}$ is the disjoint union of the finite sets $\{0\}$, X and Y. The set X is itself the disjoint union of two sets X_0, X_1; moreover, there is a given bijection from X_0 to X_1 denoted by $x \mapsto \tilde{x}$. We define functions $\partial : X \cup \{0\} \to X \cup \{0\}$ and $\nu : A \to A$ as follows:

$$\partial(x) = \begin{cases} \tilde{x} & \text{if } x \in X_0, \\ u & \text{if } x = \tilde{u} \in X_1, \\ 0 & \text{if } x = 0, \end{cases}$$

$$\nu(x) = \begin{cases} u & \text{if } x = \tilde{u} \in X_1, \\ x & \text{otherwise.} \end{cases}$$

Also define $A_0 = A \setminus X_1$. In the terminology of [6, 7], A_0 consists of the "unbarred" elements of A.

The language $\mathcal{F}$ of $\mathbf{A}$ is of the form

$$\mathcal{F} = \{0, \wedge\} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \{J, J'\},$$

where the indicated sets are finite and pairwise disjoint, and whose members are interpreted in $\mathbf{A}$ according to the following restrictions.

1. 0 is interpreted by itself. $\wedge$ is the binary operation which makes A a height-1 meet semilattice with 0 being the least element.

2. The members of $\mathcal{A}$ satisfy the following conditions. If $F \in \mathcal{A}$ and F is k-ary, then (i) $k > 0$; (ii) range$(F) \subseteq X \cup \{0\}$; (iii) A_0 is a subuniverse of $\langle A, F\rangle$; (iv) for every $i = 1, \dots, k$ and all $a \in X$, all $b, c, \bar{d}$ in A_0, and all $\bar{e}$ in A,

$$F_{(i)}(0, \bar{e}) = 0,$$
$$F_{(i)}(\partial(a), \bar{e}) = \partial(F_{(i)}(a, \bar{e})),$$
$$F_{(i)}(b, \bar{d}) = F_{(i)}(c, \bar{d}) \neq 0 \to b = c \qquad \text{('injectivity in } A_0\text{')}.$$

3. Each member S of $\mathcal{B}$ is $d{+}3$-ary for some $d > 0$ (depending on S), and satisfies the following conditions: (i) for every $\bar{a} \in A^d$, either $\mathbf{A} \models S(\bar{a}, x, y, z) \approx (x \wedge y) \vee (x \wedge z)$ or $\mathbf{A} \models S(\bar{a}, x, y, z) \approx 0$; (ii) if some $a_i = 0$ then $\mathbf{A} \models S(\bar{a}, x, y, z) \approx 0$. Included in $\mathcal{B}$ is S_2, whose definition is

$$S_2(u, v, x, y, z) = \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } u = \partial(v) \in X, \\ 0 & \text{otherwise.} \end{cases}$$

4. For each n-ary member T of $\mathcal{C}$ there are an $\mathcal{A} \cup \{\wedge\}$-term t in the variables $\{x_1, \dots, x_n\}$, a conjunction $\Sigma(\bar{x})$ of equations between pairs of these variables, and an n-ary predicate $\Phi(\bar{x})$ on A, all of which satisfy: (i) some member of $\mathcal{A}$ occurs in t (hence range$(t) \subseteq X \cup \{0\}$); (ii) if $I = \{i : x_i \text{ occurs in } t\}$, then for every $j \in \{1, \dots, n\}$ there exists $i \in I$ such that $\vdash \Sigma(\bar{x}) \to x_i \approx x_j$; (iii) $\mathbf{A} \models (t(\bar{x}) \not\approx 0 \ \&\ \Sigma(\bar{x})) \to \Phi(\bar{x})$; (iv) $\mathbf{A} \models (t(\bar{x}) \not\approx 0 \ \&\ \Phi(\bar{x})) \to \&_{i=1}^{n} x_i \not\approx 0$; and (v)

$$T(\bar{x}) = \begin{cases} t(\bar{x}) & \text{if } \Phi(\bar{x}) \text{ and } \Sigma(\bar{x}), \\ \partial(t(\bar{x})) & \text{if } \Phi(\bar{x}) \text{ but not } \Sigma(\bar{x}), \\ 0 & \text{otherwise.} \end{cases}$$

5. $\mathcal{D}$ consists of nullary operations only, while $\mathcal{E} \subseteq \{\nu\}$.
6. J and J' are defined by

$$J(x, y, z) = \begin{cases} x & \text{if } x = \partial(y) = z \in X, \\ x \wedge y & \text{otherwise,} \end{cases}$$
$$J'(x, y, z) = \begin{cases} x & \text{if } x = \partial(y) \in X, \\ x \wedge y \wedge z & \text{otherwise.} \end{cases}$$

This completes the list of requirements for the results in Section 3 to be valid for $\mathbf{A}$. Now fix an algebra $\mathbf{A}$ satisfying these requirements, with language $\mathcal{F}$ as described above. Here are some easy consequences of these requirements.

7. If $F \in \mathcal{F}$ is of arity $k > 0$ and $1 \leq i \leq k$, then we say that $F_{(i)}$ is *0-absorbing* if $\mathbf{A} \models F_{(i)}(0, \bar{z}) \approx 0$. Every $F_{(i)}$ is 0-absorbing *except* for $J_{(3)}$, $J'_{(3)}$, and possibly $S_{(d+2)}$ and $S_{(d+3)}$ ($S \in \mathcal{B}$, arity$(S) = d + 3$). In the exceptional cases it is still true that $F_{(i)}(x, \bar{z}) \leq F_{(i)}(y, \bar{z})$ whenever $x \leq y$. Hence every fundamental operation of $\mathbf{A}$ is monotone, i.e., $\mathbf{A} \in \mathcal{W}_{\mathcal{F}}$.
8. Suppose $F \in \mathcal{A}$ and $F_{(i)}(b, \bar{d}) = F_{(i)}(c, \bar{d}) \neq 0$. Let $e_j = \nu(d_j) \in A_0$ for each j, and let m be the number of j's for which $e_j \neq d_j$. Then $F_{(i)}(x, \bar{e}) = \partial^m(F_{(i)}(x, \bar{d}))$ for all x, so $F_{(i)}(b, \bar{e}) = F_{(i)}(c, \bar{e}) \neq 0$. Furthermore, $F(b, \bar{e}) \notin A_0$ if and only if $b \in X_1$, and similarly with b replaced by c. Thus either $b, c \in A_0$, in which case $b = c$ by the 'injectivity' requirement, or else $b, c \in X_1$, in which case $F_{(i)}(\partial(b), \bar{e}) = F_{(i)}(\partial(c), \bar{e}) \neq 0$, which implies $\partial(b) = \partial(c)$ by 'injectivity' and hence $b = c$. In other words, the 'injectivity' requirement

holds for all $b, c, \bar{d}$ in A (not just in A_0). As a consequence, the basic translation $\lambda(x) = F_{(i)}(x, \bar{d})$ satisfies $\lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$ for all $a, b \in A$.

9. Using similar reasoning, one can show that if $F \in \mathcal{A}$ is of arity $k > 0$ and $1 \leq i \leq k$, then for all $b, c, \bar{d}$ in A,

$$F_{(i)}(b, \bar{d}) = \partial(F_{(i)}(c, \bar{d})) \neq 0 \rightarrow b = \partial(c) \in X.$$

10. Suppose $T \in \mathcal{C}$ is of arity n and $t, \Sigma(\bar{x})$ are associated with T as described in item 4. For simplicity assume that the variables occurring in t are precisely $x_1, \ldots, x_k$. For each $i \in \{1, \ldots, k\}$ let $V_i = \{j : \vdash \Sigma(\bar{x}) \rightarrow x_i \approx x_j\}$ and let $\hat{x}_i = \bigwedge_{j \in V_i} x_j$. Finally, let $\hat{t}(\bar{x}) = t(\hat{x}_1, \ldots, \hat{x}_k)$. Clearly if $\bar{a} \in A^n$ then $\Sigma(\bar{a})$ implies $\hat{t}(\bar{a}) = t(\bar{a})$, while the failure of $\Sigma(\bar{a})$ implies $\hat{t}(\bar{a}) = 0$. Now suppose $\hat{t}(\bar{a}) \neq 0$. Then $a_i \neq 0$ for each $i = 1, \ldots, n$ (since x_i occurs in $\hat{t}$) and $\Sigma(\bar{a})$ holds, hence $t(\bar{a}) = \hat{t}(\bar{a}) \neq 0$, and thus $\Phi(\bar{a})$ holds by requirement 4(iii). Then $T(\bar{a}) = t(\bar{a})$ by definition, which proves that $\mathbf{A} \models \hat{t}(\bar{x}) \leq T(\bar{x})$.

We now list some identities which will turn out to be true in $\mathbf{A}$. The first are the '$\mathcal{A}$-identities'. In the following three axiom schemes, F ranges over $\mathcal{A}$ and $F^{\bar{u}}_{(i)}(x)$ is an abbreviation for $F_{(i)}(x, \bar{u})$.

$$F^{\bar{u}}_{(i)}(x \wedge y) \approx F^{\bar{u}}_{(i)}(x) \wedge F^{\bar{u}}_{(i)}(y), \tag{1}$$
$$F^{\bar{u}}_{(i)}(J(x,y,z)) \approx J(F^{\bar{u}}_{(i)}(x), F^{\bar{u}}_{(i)}(y), F^{\bar{u}}_{(i)}(z)), \tag{2}$$
$$F^{\bar{u}}_{(i)}(J'(x,y,z)) \approx J'(F^{\bar{u}}_{(i)}(x), F^{\bar{u}}_{(i)}(y), F^{\bar{u}}_{(i)}(z)). \tag{3}$$

Next come the '$\mathcal{B}$-identities.' In the following, S ranges over $\mathcal{B}$ and $e^{\bar{u}}_S(x)$ is an abbreviation for $S(\bar{u}, x, x, x)$.

$$S(\bar{u}, x, y, z) \leq x, \tag{4}$$
$$e^{\bar{u}}_S(S(\bar{u}, x, y, z)) \approx S(\bar{u}, x, y, z), \tag{5}$$
$$e^{\bar{u}}_S(F(x_1, \ldots, x_n)) \approx F(e^{\bar{u}}_S(x_1), \ldots, e^{\bar{u}}_S(x_n)) \quad (F \in \mathcal{F}, \text{ arity} > 0), \tag{6}$$
$$e^{\bar{u}}_S(F_{(i)}(x, \bar{y})) \approx F_{(i)}(e^{\bar{u}}_S(x), \bar{y}) \qquad (F_{(i)} \text{ 0-absorbing}). \tag{7}$$

The next group are the '$\mathcal{C}$-identities':

$$\hat{t}(\bar{x}) \leq T(\bar{x}), \tag{8}$$
$$J'(T(\bar{x}), t(\bar{x}), \hat{t}(\bar{x})) \approx T(\bar{x}), \tag{9}$$

where in these schemes T ranges over $\mathcal{C}$ and $t, \hat{t}$ are the $\mathcal{A} \cup \{\wedge\}$-terms associated with T as discussed in item 10 above.

Next are the 'J-identities.'

$$x \wedge y \leq J(x, y, z), \tag{10}$$
$$x \wedge y \wedge z \leq J'(x, y, z), \tag{11}$$
$$J(x, y, 0) \leq x \wedge y, \tag{12}$$
$$J(x, y, z) \leq J'(x, y, x), \tag{13}$$
$$J'(x, y, z) \wedge w \leq J(x, y, S_2(x, y, w, w, w)), \tag{14}$$
$$J'(x, y, z) \wedge w \leq J'(x, y, x \wedge y \wedge z \wedge w), \tag{15}$$
$$J(x, y, z) \wedge J'(x, y, z) \leq z. \tag{16}$$

Finally, we mention the 'ν-identities.' They are to be considered only if $\nu \in \mathcal{F}$.

$$J'(\nu(x), x, \nu(x)) \quad \approx \quad \nu(x), \tag{17}$$

$$x \wedge y \wedge \nu(y) \quad \leq \quad \nu(x \wedge y). \tag{18}$$

Lemma 2.1. *The identities listed in* (1)–(16) *are true in* $\mathbf{A}$*. If* $\nu \in \mathcal{F}$ *then* (17)–(18) *are also true in* $\mathbf{A}$*.*

Proof. (1) was proved above (in item 8). To prove (2), suppose $F \in \mathcal{A}$ is k-ary, $\bar{d} \in A^{k-1}$, $1 \leq i \leq k$, $\lambda(x) = F_{(i)}(x, \bar{d})$, and $a, b, c \in A$; we shall first show that $\lambda(J(a,b,c)) \geq J(\lambda(a), \lambda(b), \lambda(c))$. This is obvious if the right-hand side is 0, so suppose $J(\lambda(a), \lambda(b), \lambda(c)) > 0$. Then by definition of J, either $\lambda(a) = \lambda(b) \neq 0$ or $\lambda(a) = \partial(\lambda(b)) = \lambda(c) \in X$, and in either case $J(\lambda(a), \lambda(b), \lambda(c)) = \lambda(a)$. In the first case we get $a = b \neq 0$ by 'injectivity' (see item 8), while in the second case we get $a = \partial(b) = c \in X$ (see item 9). Thus, in either case, $\lambda(J(a,b,c)) = \lambda(a)$, which proves $\lambda(J(a,b,c)) \geq J(\lambda(a), \lambda(b), \lambda(c))$. The opposite inequality has an easier proof. (3) can be established similarly.

(4) is evident. (5)–(7) can be verified separately in each of the two possible cases: (i) $\mathbf{A} \models S(\bar{a}, x, y, z) \approx (x \wedge y) \vee (x \wedge z)$ and $\mathbf{A} \models e_S^{\bar{a}}(x) \approx x$, or (ii) $\mathbf{A} \models S(\bar{a}, x, y, z) \approx e_S^{\bar{a}}(x) \approx 0$.

(8) was proved above in item 10. To prove (9), first note that $J'(a,b,c) \leq a$ for all $a, b, c \in A$, so it suffices to prove $\mathbf{A} \models T(\bar{x}) \leq J'(T(\bar{x}), t(\bar{x}), \hat{t}(\bar{x}))$. Suppose $T(\bar{a}) \neq 0$. Then either $T(\bar{a}) = t(\bar{a}) \neq 0$ or $T(\bar{a}) = \partial(t(\bar{a})) \neq 0$. In the former case $\Sigma(\bar{a})$ must hold, and hence $\hat{t}(\bar{a}) = t(\bar{a})$ by the discussion in item 10. Thus, in either case, $J'(T(\bar{a}), t(\bar{a}), \hat{t}(\bar{a})) = T(\bar{a})$ by the definition of J', which proves the desired inequality.

Each of (10)–(16) can be easily proved by assuming that the left-hand side of the inequality is not 0 and deducing that both sides must be equal. These and the ν-identities are left as an exercise for the reader. □

In the next lemma and throughout the rest of this paper, $\mathcal{F}_0$ shall denote $\mathcal{F} \setminus \mathcal{D}$. If $\mathbf{B}$ is an $\mathcal{F}$-algebra then $\mathbf{B}|_{\mathcal{A}}$ and $\mathbf{B}|_{\mathcal{F}_0}$ denote the reducts of $\mathbf{B}$ to $\mathcal{A}$ and $\mathcal{F}_0$ respectively.

Lemma 2.2. *Suppose* $\mathbf{B} \in \mathcal{W}_{\mathcal{F}}$*,* $\mathbf{B}$ *is a model of the identities listed in* (1)–(7)*,* $S \in \mathcal{B}$ *is* $d+3$*-ary, and* $\bar{a} \in B^d$*.*

1. *For all* $\lambda \in \mathrm{Pol}_1^* \mathbf{B}|_{\mathcal{A}}$ *and* $x, y, z \in B$*,*

$$\lambda(x \wedge y) = \lambda(x) \wedge \lambda(y),$$
$$\lambda(J(x,y,z)) = J(\lambda(x), \lambda(y), \lambda(z)),$$
$$\lambda(J'(x,y,z)) = J'(\lambda(x), \lambda(y), \lambda(z)),$$
$$\lambda(e_S^{\bar{a}}(x)) = e_S^{\bar{a}}(\lambda(x)).$$

2. *For all* $n > 0$*,* $t \in \mathrm{Clo}_n \mathbf{B}|_{\mathcal{F}_0}$ *and* $\bar{x} \in B^n$*,*

$$e_S^{\bar{a}}(t(x_1, \ldots, x_n)) = t(e_S^{\bar{a}}(x_1), \ldots, e_S^{\bar{a}}(x_n)).$$

3. $e_S^{\bar{a}}$ *is idempotent,* $e^{\bar{a}}(x) \leq x$*, and* $S(\bar{a}, x, y, z) \in \mathrm{range}(e_S^{\bar{a}})$ *for all* $x, y, z \in B$*.*
4. *If* $x \leq y$*, then* $e_S^{\bar{a}}(y) \wedge x = e_S^{\bar{a}}(x)$*.*
5. *If* $y \in \mathrm{range}(e_S^{\bar{a}})$ *and* $x \leq y$*, then* $x \in \mathrm{range}(e_S^{\bar{a}})$*.*

Proof. The equations in item 1 follow from the identities in (1)–(3) and (7) respectively, by arguing inductively on the complexity of λ. Similarly, item 2 follows from the identities in (6) and the fact that $e_S^{\bar a}(0) = 0$ (by identity (4)). Item 3 is a direct consequence of identities (4) and (5). To prove item 4, observe that $e_S^{\bar a}(x) = e_S^{\bar a}(y \wedge x) = e_S^{\bar a}(y) \wedge x$ by identity (7). Item 5 follows from items 3 and 4. □

Lemma 2.3. *Suppose* $\mathbf{B} \in \mathcal{W}_{\mathcal{F}}$*,* $\mathbf{B}$ *is a model of identities* (10)–(16)*, and* $\mathbf{B} \models S_2(u, v, x, y, z) \approx 0$*. Then* $\mathbf{B} \models J(x, y, z) \approx x \wedge y$ *and* $\mathbf{B} \models J'(x, y, z) \approx x \wedge y \wedge z$*.*

Proof. Clearly $x \wedge y \le J(x, y, z)$ and $x \wedge y \wedge z \le J'(x, y, z)$ by identities (10) and (11), so it suffices to prove the opposite inequalities. First,

$$\begin{array}{rcll} J(x,y,z) & \le & J'(x,y,x) & \text{by identity (13)} \\ & \approx & J'(x,y,x) \wedge w & \text{where } w = J'(x,y,x) \\ & \le & J(x,y,S_2(x,y,w,w,w)) & \text{by identity (14)} \\ & \approx & J(x,y,0) & \text{as } S_2 \approx 0 \\ & \le & x \wedge y & \text{by identity (12).} \end{array}$$

Similarly,

$$\begin{array}{rcll} J'(x,y,z) & \approx & J'(x,y,z) \wedge w & \text{where } w = J'(x,y,z) \\ & \le & J(x,y,S_2(x,y,w,w,w)) & \text{by identity (14)} \\ & \approx & J(x,y,0) & \text{as } S_2 \approx 0, \end{array}$$

and so

$$\begin{array}{rcll} J'(x,y,z) & \approx & J(x,y,0) \wedge J'(x,y,z) & \\ & \approx & J(x,y,0) \wedge J(x,y,z) \wedge J'(x,y,z) & \\ & \le & x \wedge y \wedge z & \text{by (12), (16).} \end{array}$$

□

3. Axiomatizing the dichotomy

Throughout this section $\mathbf{A}$ shall denote a fixed algebra in the language

$$\mathcal{F} = \{0, \wedge\} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \{J, J'\}$$

which satisfies the requirements of the previous section.

Recall that $\mathcal{W}_{\mathcal{F}}$ was defined in Section 1, and let $\mathcal{W}_{\mathcal{F}}^{(0)}$ be the subvariety defined relative to $\mathcal{W}_{\mathcal{F}}$ by the identities $S(\bar u, x, y, z) \approx 0$ ($S \in \mathcal{B}$). Two facts of crucial importance for McKenzie's method are the following: (i) every subdirectly irreducible member of $\mathsf{V}(\mathbf{A})$ has height 1; (ii) if $\mathbf{B}$ is a subdirectly irreducible member of $\mathsf{V}(\mathbf{A})$, then either $\mathbf{B} \in \mathsf{HS}(\mathbf{A})$ or $\mathbf{B} \in \mathcal{W}_{\mathcal{F}}^{(0)}$. In this section we shall prove that the dichotomy in (ii) can be deduced from finitely many identities of $\mathbf{A}$.

Until further notice, let S be a fixed member of $\mathcal{B}$ of arity $d + 3$. We assume with no loss of generality that S is not identically 0. Let $\mathsf{c}_1, \dots, \mathsf{c}_d$ be new constant symbols and define

$$\begin{aligned} \mathcal{F}_0 &= \mathcal{F} \setminus \mathcal{D}, \\ \mathcal{F}^+ &= \mathcal{F} \cup \{\mathsf{c}_1, \dots, \mathsf{c}_d\}, \\ \mathcal{F}_0^+ &= \mathcal{F}_0 \cup \{\mathsf{c}_1, \dots, \mathsf{c}_d\}, \\ P &= \{\bar a \in A^d : \mathbf{A} \models S(\bar a, x, y, z) \approx (x \wedge y) \vee (x \wedge z)\}, \\ \mathcal{K} &= \{\langle \mathbf{A}, \bar a \rangle : \bar a \in P\}, \end{aligned}$$

where in the last definition the members of $\mathcal{K}$ are construed as $\mathcal{F}^+$-algebras in the obvious way. Finally, let $\mathcal{K}_0$ be the set of reducts of the members of $\mathcal{K}$ to the language $\mathcal{F}_0^+$.

The observations in this paragraph are due to McKenzie [8, §6]. Both $\mathsf{V}(\mathcal{K})$ and $\mathsf{V}(\mathcal{K}_0)$ are congruence distributive, since in either variety the terms $d_1(x,y,z) = S(\bar{\mathsf{c}},x,y,z)$, $d_2(x,y,z) = x \wedge z$, $d_3 = S(\bar{\mathsf{c}},z,y,x)$ satisfy the Jónsson identities. Therefore, both $\mathsf{V}(\mathcal{K})$ and $\mathsf{V}(\mathcal{K}_0)$ are finitely based by K. Baker's theorem [1]. Let

$$\Delta_S = \{s_i(\bar{x},\bar{\mathsf{c}}) \approx t_i(\bar{x},\bar{\mathsf{c}}) \,:\, i < m\}$$

be a finite basis for $\mathsf{V}(\mathcal{K})$ which contains a finite basis for $\mathsf{V}(\mathcal{K}_0)$, written so that each s_i and t_i is an $\mathcal{F}$-term. Finally, let

$$\Gamma_S = \{e_S^{\bar{u}}(s_i(\bar{x},\bar{u})) \approx e_S^{\bar{u}}(t_i(\bar{x},\bar{u})) \,:\, i < m\}.$$

The reader can verify that Γ_S is a finite set of $\mathcal{F}$-identities true in $\mathbf{A}$.

Lemma 3.1. *Suppose $\mathbf{B}$ is a subdirectly irreducible model of Γ_S. Suppose further that there exists $\bar{a} \in B^d$ such that $\mathbf{B} \models S(\bar{a},x,x,x) \approx x$. Then $\mathbf{B} \in \mathsf{HS}(\mathbf{A})$.*

Proof. Let $\mathbf{B}^+ = \langle \mathbf{B}, \bar{a}\rangle$, an $\mathcal{F}^+$-algebra. Then $\mathbf{B}^+ \models \Delta_S$, so $\mathbf{B}^+ \in \mathsf{V}(\mathcal{K})$. $\mathbf{B}^+$ is itself subdirectly irreducible and $\mathsf{V}(\mathcal{K})$ is congruence distributive; hence $\mathbf{B}^+ \in \mathsf{HSP}_u(\mathcal{K}) = \mathsf{HS}(\mathcal{K})$ by a well-known theorem of B. Jónsson. So $\mathbf{B} \in \mathsf{HS}(\mathbf{A})$. □

We now assume that the set Γ_S has been chosen for each $S \in \mathcal{B}$ as described above, and let $\Gamma = \bigcup_{S\in\mathcal{B}} \Gamma_S$. Let Σ_0 be a finite basis for $\mathcal{W}_{\mathcal{F}}$. Let Σ_1 be the set consisting of the identities listed in (1)–(7) in Section 2, and Σ_2 the set consisting of (8)–(16) if $\nu \notin \mathcal{F}$, or (8)–(18) if $\nu \in \mathcal{F}$.

Lemma 3.2. *Suppose $\mathbf{B}$ is a subdirectly irreducible model of $\Sigma_0 \cup \Sigma_1 \cup \Gamma$ with monolith μ. If there exist $S \in \mathcal{B}$ (of arity $d+3$), $\bar{a} \in B^d$, and $(p,q) \in \mu$ such that $q < p$ and $S(\bar{a},p,p,p) = p$, then $\mathbf{B} \in \mathsf{HS}(\mathbf{A})$.*

Proof. Choose $S \in \mathcal{B}$, $\bar{a} \in B^d$ and $(p,q) \in \mu$ such that $q < p$ and $S(\bar{a},p,p,p) = p$. Throughout this proof we shall let $e(x)$ denote $e_S^{\bar{a}}(x)$, which is idempotent by Lemma 2.2. Let $C = \mathrm{range}(e)$. Our aim (in light of Lemma 3.1) is to prove that $C = B$.

Claim 1. If $x \in B$, $y \in C$, and $x \leq y$, then $x \in C$.

Indeed, this is just Lemma 2.2(5). Note that it follows that $\{0,q,p\} \subseteq C$.

Next, observe that if $F \in \mathcal{F}$ has arity $k > 0$, then for any $x_1,\ldots,x_k \in C$,

$$\begin{aligned} F(x_1,\ldots,x_k) &= F(e(x_1),\ldots,e(x_k)) \\ &= e(F(x_1,\ldots,x_k)) \quad \text{by identity (6)} \\ &\in C. \end{aligned}$$

Hence C is a subuniverse of $\mathbf{B}_0 := \mathbf{B}|_{\mathcal{F}_0}$. Let $\mathbf{C}_0$ be the corresponding subalgebra.

Suppose $t(\bar{x},\bar{y})$ is an $n+m$-ary $\mathcal{F}_0$-term, $\bar{b} \in B^m$, and $c_i = e(b_i) \in C$ for $i = 1,\ldots,m$. Then for any $x_1,\ldots,x_n \in C$,

$$\begin{aligned} e(t(\bar{x},\bar{b})) &= t(e(x_1),\ldots,e(x_n),e(b_1),\ldots,e(b_m)) \quad \text{(Lemma 2.2(2))} \\ &= t(\bar{x},\bar{c}). \end{aligned}$$

This proves $\mathrm{Pol}\,\mathbf{C}_0 = \mathrm{Pol}\,\mathbf{B}_0|_C = \mathrm{Pol}\,\mathbf{B}|_C$, where $\mathbf{B}|_C$ is the algebra induced by $\mathbf{B}$ on C as defined in [2]. Thus the map $|_C : \mathrm{Con}\,\mathbf{B} \to \mathrm{Con}\,\mathbf{C}_0$ is a surjective lattice homomorphism by [2, Lemma 2.3]. Since $(p,q) \in \mu|_C$ we have $\mu|_C \neq 0_C$, and therefore $\mathbf{C}_0$ is itself subdirectly irreducible.

Now $\mathbf{C}_0$ is a subalgebra of $\mathbf{B}_0$ and therefore satisfies the $\mathcal{F}_0$-identities in Γ_S. Since Δ_S contains a basis for $\mathsf{V}(\mathcal{K}_0)$, the proof of Lemma 3.1 (suitably modified) yields $\mathbf{C}_0 \in \mathsf{HS}(\mathbf{A}|_{\mathcal{F}_0})$. In particular, it follows that $\mathbf{C}_0$ has height 1, and therefore $q = 0$ and (by Claim 1) $0 \prec p$ in $\mathbf{B}$.

Claim 2. Suppose $b \in B$, $c \in C$, and $0 < c \leq b$. Then $b = c$.

Assume instead that $c < b$. Then $(p,0) \in \mathrm{Cg}^{\mathbf{B}}(b,c)$. By Lemma 1.5(2) there exists $\lambda \in \mathrm{Pol}_1 \mathbf{B} = \mathrm{Pol}_1 \mathbf{B}|_{\mathcal{F}_0}$ such that $\lambda(c) = 0$ and $\lambda(b) = p$. Write $\lambda(x) = t(x,\bar{d})$, where t is an $\mathcal{F}_0$-term and $\bar{d} \in B^m$. Then

$$\begin{aligned} p \ &= \ e(p) = e(t(b,\bar{d})) \\ &= t(e(b), e(d_1), \dots, e(d_m)) \quad \text{(Lemma 2.2(2))} \end{aligned}$$

and, by similar reasoning,

$$0 = t(e(c), e(d_1), \dots, e(d_m)).$$

But $e(c) = e(b)$, since $0 < e(c) \leq e(b)$ and $\mathbf{C}_0$ has height 1. This would imply $p = 0$, which is false; so $c = b$.

Note that it follows from Claim 2 that for all $b \in B$, if $b \notin C$ then $e(b) = 0$, since otherwise $0 < e(b) < b$ by Lemma 2.2(3).

Claim 3. For every $F \in \mathcal{F}_0$ of arity $k > 0$, and for every $i = 1, \dots, k$ and $\bar{d} \in B^{k-1}$ and $b \in B$, if $F_{(i)}(b,\bar{d}) \in C \setminus \{0\}$ and $F_{(i)}(0,\bar{d}) = 0$, then $b \in C$.

Suppose instead that $b \notin C$; then $e(b) = 0$ by the above observation. But then

$$\begin{aligned} F_{(i)}(b,\bar{d}) &= e(F_{(i)}(b,\bar{d})) \\ &= F_{(i)}(e(b), e(d_1), \dots, e(d_{k-1})) \quad \text{identity (6)} \\ &\leq F_{(i)}(e(b), d_1, \dots, d_{k-1}) \quad \text{Lemma 2.2(3)} \\ &= F_{(i)}(0,\bar{d}) \\ &= 0, \end{aligned}$$

which contradicts the assumption that $F_{(i)}(b,\bar{d}) \neq 0$.

Claim 4. For every $\lambda \in \mathrm{Pol}_1^* \mathbf{B}$ and $b \in B$, if $\lambda(b) \in C \setminus \{0\}$ and $\lambda(0) = 0$, then $b \in C$.

This is proved by writing $\lambda = \lambda_1 \lambda_2 \cdots \lambda_n$, where each λ_i is a basic translation of $\mathbf{B}$, and arguing by induction on n. In the inductive step, let $\lambda' = \lambda_2 \cdots \lambda_n$, $b' = \lambda'(b)$ and $0' = \lambda'(0)$. Then $\lambda_1(b') = \lambda(b) \in C \setminus \{0\}$, while $\lambda_1(0) \leq \lambda_1(0') = \lambda(0) = 0$. Hence $b' \in C$ by Claim 3 and $0' \in C$ by Claim 1, and thus $0' = 0$, since $0' < b'$ and $\mathbf{C}_0$ has height 1. This proves $\lambda'(b) \in C \setminus \{0\}$ and $\lambda'(0) = 0$, so $b \in C$ by induction.

Now we can complete the proof of the lemma. Let $b \in B \setminus \{0\}$. Then $(p,0) \in \mathrm{Cg}^{\mathbf{B}}(b,0)$. By Lemma 1.5(2) there exists $\lambda \in \mathrm{Pol}_1^* \mathbf{B}$ such that $\lambda(b) = p$ and $\lambda(0) = 0$. Hence $b \in C$ by Claim 4, which proves $C = B$ and therefore $\mathbf{B} \in \mathsf{HS}(\mathbf{A})$ by Lemma 3.1. $\square$

Lemma 3.3. *Suppose* $\mathbf{B}$ *is a subdirectly irreducible model of* $\Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ *with monolith* μ. *Suppose moreover that for every* $S \in \mathcal{B}$ *(of arity* $d+3$*) and all* $\bar{a} \in B^d$, *if* $(p,q) \in \mu$ *with* $q < p$ *then* $S(\bar{a},p,p,p) \neq p$. *Then* $\mathbf{B} \in \mathcal{W}_{\mathcal{F}}^{(0)}$.

Proof. The proof is loosely based on Lemmas 6.6 and 6.8 from [6].

Claim 1. If $S \in \mathcal{B}$ (of arity $d+3$), $\bar{a} \in B^d$, $(p,q) \in \mu$ and $q < p$, then $S(\bar{a},p,p,p) \leq q$.

To prove it, let $p' = e_S^{\bar{a}}(p)$ and $q' = e_S^{\bar{a}}(q)$. Clearly $(p',q') \in \mathrm{Cg}^{\mathbf{B}}(p,q) = \mu$, and $q' \leq p'$ by monotonicity. Also, $S(\bar{a},p',p',p') = p'$ by idempotence of $e_S^{\bar{a}}$, so the hypothesis of the lemma forces $q' = p'$. Since $q' \leq q$ by Lemma 2.2(3), we get $p' \leq q$, as required.

Define

$$\begin{aligned}\mathcal{U} = \{(b,a) \in B^2 \,:\, & b < a \text{ and there exist } \lambda \in \mathrm{Pol}_1^* \mathbf{B}|_{\mathcal{A}} \text{ and } (p,q) \in \mu \\ & \text{such that } q < p \text{ and } \lambda(a) \geq p \text{ and } \lambda(b) \wedge p \leq q\}.\end{aligned}$$

Claim 2. Suppose $b \leq a$ and there exist $\lambda \in \mathrm{Pol}_1^* \mathbf{B}|_{\mathcal{A}}$ and $(p,q) \in \mu$ such that $q < p$ and $\lambda(a) \wedge p \not\leq q$ and $\lambda(b) \wedge p \leq q$. Then $(b,a) \in \mathcal{U}$.

The hypotheses imply $b < a$. Let $p' = \lambda(a) \wedge p$ and $q' = \lambda(a) \wedge q$. Then $(p',q') \in \mu$ and $q' < p'$. Clearly $\lambda(a) \geq p'$, while $\lambda(b) \wedge p' \leq \lambda(b) \wedge p \leq q$, and hence $\lambda(b) \wedge p' \leq q'$, which proves $(b,a) \in \mathcal{U}$.

Claim 3. Suppose $b < a$.

1. If $F \in \mathcal{A}$ (of arity k), $\bar{c} \in B^{k-1}$ and $(F_{(i)}(b,\bar{c}), F_{(i)}(a,\bar{c})) \in \mathcal{U}$, then $(b,a) \in \mathcal{U}$.
2. If $c \in B$ and $(b \wedge c, a \wedge c) \in \mathcal{U}$, then $(b,a) \in U$.
3. If $\lambda \in \mathrm{Pol}_1 \mathbf{B}|_{\mathcal{A} \cup \{\wedge\}}$ and $(\lambda(b), \lambda(a)) \in \mathcal{U}$, then $(b,a) \in \mathcal{U}$.

Item 1 is obvious. To prove item 2, pick $\lambda \in \mathrm{Pol}_1^* \mathbf{B}|_{\mathcal{A}}$ and $(p,q) \in \mu$ such that $q < p$ and $\lambda(a \wedge c) \geq p$ and $\lambda(b \wedge c) \wedge p \leq q$. Then $\lambda(a) \geq p$ and also $\lambda(c) \geq p$ by monotonicity, while

$$\begin{aligned}\lambda(b) \wedge p &= \lambda(b) \wedge (\lambda(c) \wedge p) \\ &= \lambda(b \wedge c) \wedge p \quad \text{Lemma 2.2(1)} \\ &\leq q,\end{aligned}$$

which proves $(b,a) \in \mathcal{U}$.

Item 3 is proved by induction on the complexity of λ. Inductively, assume that $\lambda(x) = F(\lambda_1(x), \ldots, \lambda_k(x))$, where $F \in \mathcal{A} \cup \{\wedge\}$, and the claim has already been verified for $\lambda_1, \ldots, \lambda_k$. Choose $\eta \in \mathrm{Pol}_1^* \mathbf{B}|_{\mathcal{A}}$ and $(p,q) \in \mu$ such that $q < p$ and $\eta(\lambda(a)) \geq p$ and $\eta(\lambda(b)) \wedge p \leq q$. Let $a_0 = \lambda(b)$, $a_k = \lambda(a)$, and for each i satisfying $1 \leq i < k$ let

$$a_i = F(\lambda_1(a), \ldots, \lambda_i(a), \lambda_{i+1}(b), \ldots, \lambda_k(b)).$$

Then $a_0 \leq a_1 \leq \cdots \leq a_k$ by monotonicity, so there exists a unique $i < k$ such that $\eta(a_i) \wedge p \leq q$, while $\eta(a_{i+1}) \wedge p \not\leq q$. Then $(a_i, a_{i+1}) \in \mathcal{U}$ by Claim 2. But $a_i = F_{(i+1)}(\lambda_{i+1}(b), \bar{c})$ while $a_{i+1} = F_{(i+1)}(\lambda_{i+1}(a), \bar{c})$ for appropriate $\bar{c}$, and therefore $(b,a) \in \mathcal{U}$ by item 1 or 2 and the inductive hypothesis.

Claim 4. Suppose $(b,a) \in \mathcal{U}$ and $S \in \mathcal{B}$. Then $a \notin \mathrm{range}(S)$.

Suppose instead that $S(\bar{c},r,s,t) = a$. Then $e_S^{\bar{c}}(a) = a$ by Lemma 2.2(3). Choose $\lambda \in \mathrm{Pol}_1^* \mathbf{B}|_{\mathcal{A}}$ and $(p,q) \in \mu$ such that $q < p$ and $\lambda(a) \geq p$ and $\lambda(b) \wedge p \leq q$. Then $\lambda(a) \in \mathrm{range}(e_S^{\bar{c}})$, and hence $p \in \mathrm{range}(e_S^{\bar{c}})$ by Lemma 2.2, implying $S(\bar{c},p,p,p) = p$, which contradicts the hypothesis of this lemma.

Claim 5. Suppose $a', b', a^*, b^*, r_1, r_2, r_3 \in B$ satisfy $b' \le a'$ and $b^* \le a^*$, and

$$\begin{aligned} a' \wedge b^* &\le b', \\ a' &\le J'(r_1, r_2, r_3), \\ r_1 \wedge r_2 \wedge r_3 &\le a^*. \end{aligned}$$

If $(b', a') \in \mathcal{U}$, then $(b^*, a^*) \in \mathcal{U}$.

For the proof, choose $\lambda \in \mathrm{Pol}_1^* \mathbf{B}|_{\mathcal{A}}$ and $(p, q) \in \mu$ such that $q < p$ and $\lambda(a') \ge p$ and $\lambda(b') \wedge p \le q$. Let

$$\begin{aligned} A' &= \lambda(a'), & A^* &= \lambda(a^*), \\ B' &= \lambda(b'), & B^* &= \lambda(b^*), \\ R_i &= \lambda(r_i), & i &= 1, 2, 3. \end{aligned}$$

Then

$$A' \ge p, \tag{19}$$

$$B' \wedge p \le q, \tag{20}$$

by the choice of λ and (p, q), and

$$B^* \le A^*, \tag{21}$$

$$A' \wedge B^* \le B', \tag{22}$$

$$A' \le J'(R_1, R_2, R_3), \tag{23}$$

$$R_1 \wedge R_2 \wedge R_3 \le A^*, \tag{24}$$

by applying Lemma 2.2(1) to the hypotheses of the claim. Observe that

$$\begin{aligned} B^* \wedge p &\le B^* \wedge A' && \text{by (19)} \\ &\le B' && \text{by (22)}, \end{aligned}$$

and hence

$$\begin{aligned} B^* \wedge p &\le B' \wedge p \\ &\le q && \text{by (20)}. \end{aligned}$$

Thus it will suffice to prove $A^* \wedge p \not\le q$, since then $(B^*, A^*) \in \mathcal{U}$ by Claim 2 and hence $(b^*, a^*) \in \mathcal{U}$ by Claim 3(3). So assume to the contrary that $A^* \wedge p \le q$. Then

$$\begin{aligned} J'(R_1, R_2, R_3) \wedge p &\le J(R_1, R_2, S_2(R_1, R_2, p, p, p)) && \text{by identity (14)} \\ &\le J(R_1, R_2, q) && \text{by Claim 1}, \end{aligned}$$

while

$$\begin{aligned} J'(R_1, R_2, R_3) \wedge p &\le J'(R_1, R_2, R_1 \wedge R_2 \wedge R_3 \wedge p) && \text{by identity (15)} \\ &\le J'(R_1, R_2, A^* \wedge p) && \text{by (24)} \\ &\le J'(R_1, R_2, q) && \text{by assumption}, \end{aligned}$$

and so

$$\begin{aligned} p &= A' \wedge p && \text{by (19)} \\ &\le J'(R_1, R_2, R_3) \wedge p && \text{by (23)} \\ &\le J(R_1, R_2, q) \wedge J'(R_1, R_2, q) && \text{by the above} \\ &\le q && \text{by identity (16)}, \end{aligned}$$

which contradicts $q < p$. This proves the claim.

Claim 6. Suppose $b_i \leq a_i$ $(1 \leq i \leq 3)$.

1. If $(J(b_1, b_2, b_3), J(a_1, a_2, a_3)) \in \mathcal{U}$, then $(b_1 \wedge b_2, a_1 \wedge a_2) \in \mathcal{U}$.
2. If $(J'(b_1, b_2, b_3), J'(a_1, a_2, a_3)) \in \mathcal{U}$, then $(b_1 \wedge b_2 \wedge b_3, a_1 \wedge a_2 \wedge a_3) \in \mathcal{U}$.

To prove the first item, let

$$\begin{aligned} a' &= J(a_1, a_2, a_3), \qquad a^* = a_1 \wedge a_2, \\ b' &= J(b_1, b_2, b_3), \qquad b^* = b_1 \wedge b_2, \\ & r_1 = r_3 = a_1, \quad r_2 = a_2. \end{aligned}$$

Then $a' \wedge b^* \leq b^* \leq b'$ by identity (10), $a' \leq J'(r_1, r_2, r_3)$ by identity (13), and $r_1 \wedge r_2 \wedge r_3 \leq a^*$ is obviously true. Thus by Claim 5, $(b^*, a^*) \in \mathcal{U}$, as desired. The second item is proved similarly, this time using

$$\begin{aligned} a' &= J'(a_1, a_2, a_3), \qquad a^* = a_1 \wedge a_2 \wedge a_3, \\ b' &= J'(b_1, b_2, b_3), \qquad b^* = b_1 \wedge b_2 \wedge b_3, \\ & r_i = a_i, \quad i = 1, 2, 3, \end{aligned}$$

and identity (11).

Claim 7. Suppose $T \in \mathcal{C}$ (of arity n), and let $t(\bar{x})$ and $\hat{t}(\bar{x})$ be the accompanying $\mathcal{A} \cup \{\wedge\}$-terms as explained in Section 2, items 4 and 10. Suppose also $b_i \leq a_i$ $(1 \leq i \leq n)$. If $(T(\bar{b}), T(\bar{a})) \in \mathcal{U}$, then $(\hat{t}(\bar{b}), \hat{t}(\bar{a})) \in \mathcal{U}$.

The proof is similar to that of Claim 6, using

$$\begin{aligned} a' &= T(\bar{a}), \qquad a^* = \hat{t}(\bar{a}), \\ b' &= T(\bar{b}), \qquad b^* = \hat{t}(\bar{b}), \end{aligned}$$

$$r_1 = T(\bar{a}), \quad r_2 = t(\bar{a}), \quad r_3 = \hat{t}(\bar{a}),$$

and identities (8) and (9).

Claim 8. (if $\nu \in \mathcal{F}$). If $b < a$ and $(\nu(b), \nu(a)) \in \mathcal{U}$, then $(b, a) \in \mathcal{U}$.

Use

$$\begin{aligned} a &= \nu(a), \qquad a^* = a, \\ b' &= \nu(b), \qquad b^* = b, \end{aligned}$$

$$r_1 = r_3 = \nu(a), \quad r_2 = a,$$

and identities (17) and (18).

Main Claim. For all $\lambda \in \mathrm{Pol}_1\, \mathbf{B}$, if $b < a$ and $(\lambda(b), \lambda(a)) \in \mathcal{U}$, then $(b, a) \in \mathcal{U}$.

First, for each term s containing no constants, define the *height* of s in the usual fashion, namely, $ht(x) = 0$ for each variable x, and

$$ht(F(s_1, \ldots, s_n)) = 1 + \max(ht(s_i) \ : \ 1 \leq i \leq n)$$

for each $F \in \mathcal{F}$ of arity $n > 0$. Also define

$$\begin{aligned} ht^*(s) = \max(ht(s^*) \ : \ & s^* \text{ is a subterm of } s \text{ of the form} \\ & F(s_1, \ldots, s_n), \text{ where } F \notin \mathcal{A} \cup \{\wedge\}). \end{aligned}$$

Now choose a term $s(x, \bar{y})$ containing no constants such that $\lambda(x) = s(x, \bar{c})$ for some parameters $\bar{c}$ from B. The proof of the claim is by induction on $\langle ht^*(s), ht(s)\rangle$. Inductively, assume that $s(x, \bar{y}) = F(s_1(x, \bar{y}), \dots, s_n(x, \bar{y}))$ and that the Claim has already been proved for all terms s' for which $\langle ht^*(s'), ht(s')\rangle$ is lexicographically less than $\langle ht^*(s), ht(s)\rangle$. If $F \in \mathcal{A} \cup \{\wedge\}$ then the argument proving Claim 3(3) can be used. If $F \in \mathcal{B}$ then the implication is vacuously true by Claim 4. If $F \in \mathcal{C}$ with accompanying $\mathcal{A} \cup \{\wedge\}$-terms t and $\hat{t}$, let $s'(x, \bar{y}) = \hat{t}(s_1(x, \bar{y}), \dots, s_n(x, \bar{y}))$ and $\lambda'(x) = s'(x, \bar{c})$. Then $(\lambda(b), \lambda(a)) \in \mathcal{U}$ implies $(\lambda'(b), \lambda'(a)) \in \mathcal{U}$ by Claim 7, and hence $(b, a) \in \mathcal{U}$ by induction. A similar proof works if $F \in \mathcal{E} \cup \{J, J'\}$, using Claim 8 or 6 respectively. (As a very special case, if $\lambda(x) = J(c, d, \lambda'(x))$ then the implication is vacuously true, by Claim 6.)

Now we can finish the proof of the lemma. Suppose $r := S(\bar{a}, b, c, d) \neq 0$ for some $S \in \mathcal{B}$ and $\bar{a}, b, c, d$ in B. Then $\mathrm{Cg}^{\mathbf{B}}(r, 0) \cap \mu \neq 0_B$. Thus by Lemma 1.5(3) there exist $\lambda \in \mathrm{Pol}_1 \mathbf{B}$ and $(p, q) \in \mu$ such that $q < p$ and $\lambda(r) = p$ and $\lambda(0) \leq q$. Clearly $(\lambda(0), \lambda(r)) \in \mathcal{U}$ by definition, so $(0, r) \in \mathcal{U}$ by the main claim. But $r \in \mathrm{range}(S)$, contradicting Claim 4. □

Theorem 3.4. *Suppose* $\mathbf{A}$ *satisfies the requirements of Section 2. Recall that* $\mathcal{W}_{\mathcal{F}}^{(0)}$ *denotes the variety defined relative to* $\mathcal{W}_{\mathcal{F}}$ *by the identities* $S(\bar{u}, x, y, z) \approx 0$ *(*$S \in \mathcal{B}$*). There exists a finitely based variety* $\mathcal{V}_1$ *which contains* $\mathbf{A}$ *and is such that every subdirectly irreducible member of* $\mathcal{V}_1$ *is either in* $\mathsf{HS}(\mathbf{A})$ *or in* $\mathcal{W}_{\mathcal{F}}^{(0)}$*.*

Proof. Let $\mathcal{V}_1$ be the variety axiomatized by $\Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Gamma$. □

Corollary 3.5. *Suppose that* $\mathbf{A}$ *satisfies the requirements of Section 2. Then* $\mathbf{A}$ *is finitely based if and only if* $\mathsf{V}(\mathbf{A}) \cap \mathcal{W}_{\mathcal{F}}^{(0)}$ *is finitely based.*

Proof. Suppose $\mathsf{V}(\mathbf{A}) \cap \mathcal{W}_{\mathcal{F}}^{(0)}$ is finitely based. Let Φ be a first-order sentence axiomatizing $\mathcal{W}_{\mathcal{F}}^{(0)}$, let Ψ be a first-order sentence axiomatizing $\mathsf{V}(\mathbf{A}) \cap \mathcal{W}_{\mathcal{F}}^{(0)}$, and let Θ be a first-order sentence axiomatizing the variety $\mathcal{V}_1$ mentioned in Theorem 3.4. Then $\mathsf{V}(\mathbf{A}) \models \Theta \ \& \ (\Phi \to \Psi)$. Pick a finite set Σ of identities true in $\mathbf{A}$ such that $\Sigma \vdash \Theta \ \& \ (\Phi \to \Psi)$. Then the subdirectly irreducible models of Σ are precisely the subdirectly irreducible members of $\mathsf{V}(\mathbf{A})$, so $\mathsf{V}(\mathbf{A})$ is finitely based. The implication in the other direction is trivial. □

Corollary 3.6. *Suppose that* $\mathbf{A}$ *satisfies the requirements of Section 2 that every subdirectly irreducible member of* $\mathsf{V}(\mathbf{A})$ *has height* 1*, and that* $\kappa(\mathbf{A}) < \omega$*. Then* $\mathbf{A}$ *is finitely based.*

Proof. Let $\mathcal{F}$ be the type of $\mathbf{A}$, let $\mathcal{F}_1$ denote $\mathcal{F} \setminus (\mathcal{B} \cup \{J, J'\})$, and let $\mathcal{V}_0$ denote the reduct of $\mathsf{V}(\mathbf{A}) \cap \mathcal{W}_{\mathcal{F}}^{(0)}$ to $\mathcal{F}_1$. $\mathcal{V}_0$ is term equivalent to $\mathsf{V}(\mathbf{A}) \cap \mathcal{W}_{\mathcal{F}}^{(0)}$ by Lemma 2.3. On the other hand, $\mathbf{A}|_{\mathcal{F}_1}$ is 0-absorbing by the discussion in Section 2, item 7, and this is inherited by all members of $\mathcal{V}_0$. Therefore $\mathcal{V}_0$ and hence $\mathsf{V}(\mathbf{A}) \cap \mathcal{W}_{\mathcal{F}}^{(0)}$ are finitely based by Theorem 1.4. Hence $\mathbf{A}$ is finitely based by Corollary 3.5. □

Remark. It can be shown (though we won't) that the hypothesis that each subdirectly irreducible member of $\mathsf{V}(\mathbf{A})$ have height 1 is redundant.

4. McKenzie's $\mathbf{A}(\mathcal{T})$

Paraphrasing [7] slightly, we define a *Turing machine* to be any (total) function $\mathcal{T} : \{q_1, \dots, q_k\} \times \{0, 1\} \to \{0, 1\} \times \{L, R\} \times \{q_0, q_1, \dots, q_k\}$ with $k > 0$.

Here $\{q_0, q_1, \ldots, q_k\}$ is to be thought of as the set of *internal states* (with $q_0 =$ *halting state* and $q_1 =$ *initial state*), $\{0,1\}$ is the set of *tape symbols* (with $0 =$ *blank*), and L, R represent the *moves* 'Left' and 'Right' respectively.

$\mathcal{T}$ is conceived to act on two-way infinite tapes whose squares are indexed by the integers. Thus an instantaneous description for $\mathcal{T}$ consists of a triple (t, n, q_i), where t is a function $\mathbb{Z} \to \{0,1\}$ (representing the current contents of the tape), n is an integer (representing the current position of the read/write head), and q_i is the current state. $\mathcal{T}$ *halts* if its action starting from the initial instantaneous description $(\hat{0}, 0, q_1)$ eventually leads to an instantaneous description of the form (t, n, q_0).

McKenzie's algebra $\mathbf{A}(\mathcal{T})$ is a finite algebra in a language

$$\mathcal{F}(\mathcal{T}) = \{0, \wedge\} \cup \mathcal{A}(\mathcal{T}) \cup \mathcal{B} \cup \mathcal{C}(\mathcal{T}) \cup \mathcal{D} \cup \{J, J'\}$$

which satisfies the requirements of Section 2. We do not need to know the precise definition of $\mathbf{A}(\mathcal{T})$, but do need to know the language. $\mathcal{A}(\mathcal{T})$ consists of the binary operation symbols $\cdot$ and I and the 3-ary symbols F_{irt}, where (i, r, t) ranges over $\{1, \ldots, k\} \times \{0,1\} \times \{0,1\}$. $\mathcal{B}$ consists of S_0 and S_1 (4-ary) and S_2 (5-ary). $\mathcal{C}(\mathcal{T})$ consists of the 4-ary symbols T and U^j_{irt}, where (i, r, t) ranges over $\{1, \ldots, k\} \times \{0,1\} \times \{0,1\}$ and j ranges over $\{1,2\}$. Finally, $\mathcal{D}$ consists of the nullary constant Δ.

Define the algebra $\mathbf{S}_{\mathbb{Z}}(\mathcal{T})$ of type $\mathcal{F}(\mathcal{T})$ as follows. The universe is

$$S_{\mathbb{Z}}(\mathcal{T}) = \{0\} \cup \{a_n,\ b_n\ :\ n \in \mathbb{Z}\},$$

where the indicated elements are distinct. $\mathbf{S}_{\mathbb{Z}}(\mathcal{T})$ is defined by interpreting 0 as itself, $\wedge$ so that $\langle S_{\mathbb{Z}}(\mathcal{T}), \wedge\rangle$ is a height-1 meet semilattice with least element 0,

$$\begin{aligned} &\begin{cases} a_n \cdot b_{n+1} = b_n & \text{for } n \in \mathbb{Z}, \\ \quad x \cdot y = 0 & \text{otherwise}, \end{cases} \\ &\quad J(x,y,z) = x \wedge y, \\ &\quad J'(x,y,z) = x \wedge y \wedge z, \\ &T(x,y,z,u) = (x \cdot y) \wedge (z \cdot u), \end{aligned}$$

and letting the remaining operations in $\mathcal{F}(\mathcal{T})$ be identically zero. The remarkable results in [7] are summarized in the following theorem.

Theorem 4.1 (McKenzie)**.**

1. $\mathbf{A}(\mathcal{T})$ *satisfies the requirements of Section 2.*
2. *Every subdirectly irreducible member of* $\mathsf{V}(\mathbf{A}(\mathcal{T}))$ *has height* 1*.*
3. $\mathsf{V}(\mathbf{A}(\mathcal{T}))$ *is residually small.*
4. *If* $\mathcal{T}$ *does not halt, then* $\mathbf{S}_{\mathbb{Z}}(\mathcal{T}) \in \mathsf{V}(\mathbf{A}(\mathcal{T}))$*. Hence* $\kappa(\mathbf{A}(\mathcal{T})) \not< \omega$ *and* $\mathbf{A}(\mathcal{T})$ *is inherently nonfinitely based.*
5. *If* $\mathcal{T}$ *halts, then* $\kappa(\mathbf{A}(\mathcal{T})) < \omega$*.*

Proof. We assume that the reader is thoroughly familiar with [6, Section 6] and [7]. Item 1 can then be verified easily. The only condition in Section 2 requiring comment is condition 4. Corresponding to each operation in $\mathcal{C}(\mathcal{T})$ one uses the term t, conjunction of equations Σ, and predicate Φ given by the following table:

member of $\mathcal{C}(\mathcal{T})$	$t(x,y,z,u)$	$\Sigma(x,y,z,u)$	$\Phi(x,y,z,u)$
$T(x,y,z,u)$	$x \cdot y$	$x = z \mathbin{\&} y = u$	$x \cdot y = z \cdot u$
$U^1_{irt}(x,y,z,u)$	$F_{irt}(x,y,u)$	$y = z$	$x \prec z$
$U^2_{irt}(x,y,z,u)$	$F_{irt}(y,z,u)$	$x = y$	$x \prec z$

Every nontrivial member of $\mathsf{HS}(\mathbf{A}(\mathcal{T}))$ has height 1. A subdirectly irreducible member of $\mathsf{V}(\mathbf{A}(\mathcal{T}))$ which is not in $\mathsf{HS}(\mathbf{A}(\mathcal{T}))$ is said by McKenzie to be *large.* To prove item 2, it suffices by an ultraproduct argument to show that every finite large subdirectly irreducible member of $\mathsf{V}(\mathbf{A}(\mathcal{T}))$ has height 1. McKenzie did this in [7, Lemmas 5.2(i), 5.5(3) and 5.7(4)]. As McKenzie notes, this much of his argument does not depend on $\mathcal{T}$ halting.

The first assertion and the first half of the second assertion in item 4 were proved in [7, Lemma 4.1]. An explanation of the second half of the second assertion can be found in [8, proof of Lemma 4.1]. Item 5 follows from [7, Theorem 5.1]. Item 3 follows from item 5 and [7, §6]. □

Corollary 4.2. $\mathbf{A}(\mathcal{T})$ *is finitely based if and only if* $\mathcal{T}$ *halts.*

Proof. By Theorem 4.1 and Corollary 3.6. □

5. Further results

A particularly intriguing special case of Jónsson's Problem 2 (stated at the beginning of this article) is:

Problem 3. Suppose $\mathcal{V}$ is a subvariety of $\mathcal{W}_{\mathcal{F}}$ satisfying $\kappa(\mathcal{V}) < \omega$ and such that every subdirectly irreducible member of $\mathcal{V}$ has height 1 (with respect to the underlying semilattice ordering). Must $\mathcal{V}$ be finitely based?

McKenzie's $\mathsf{V}(\mathbf{A}(\mathcal{T}))$ is a variety of the above form when $\mathcal{T}$ halts. Thus a positive answer to Problem 3 would prove the main result of this paper as a special case. In this section we prove two results which may be useful in the study of Problem 3.

Definition 5.1. Suppose $\mathbf{A} \in \mathcal{W}_{\mathcal{F}}$ has height 1 and $p \in A \setminus \{0\}$. Define the sets $U_n(p)$ and $U(p)$ by

$$\begin{aligned} U_0(p) &= \{p\}, \\ a \in U_{n+1}(p) &\text{ iff } a \in U_n(p) \text{ or there exists a basic translation } F_{(i)}(x,\bar{c}) \\ &\qquad \text{of } \mathbf{A} \text{ such that } F_{(i)}(a,\bar{c}) \in U_n(p) \text{ and } F_{(i)}(0,\bar{c}) = 0, \\ U(p) &= \bigcup_n U_n(p). \end{aligned}$$

Observe that each set $U_n(p)$ is definable in $\mathbf{A}$ by a first-order formula which depends only on n and $\mathcal{F}$, and which includes the parameter p.

Lemma 5.2. *Suppose* $\mathbf{A} \in \mathcal{W}_{\mathcal{F}}$ *has height* 1 *and* $p \in A \setminus \{0\}$.

1. $U_0(p) \subseteq U_1(p) \subseteq U_2(p) \subseteq \cdots$.
2. *If* $n < \omega$ *and* $U_n(p) = U_{n+1}(p)$, *then* $U_n(p) = U(p)$.
3. $a \in U(p)$ *if and only if there exists* $\lambda \in \mathrm{Pol}_1^* \mathbf{A}$ *such that* $\lambda(a) = p$ *and* $\lambda(0) = 0$.
4. $0 \notin U(p)$.

Define $\theta = 0_A \cup (A \setminus U(p))^2$. *Then*

5. $\theta \in \mathrm{Con}\, \mathbf{A}$.

6. $\mathbf{A}/\theta$ *is subdirectly irreducible.*
7. $|A/\theta| = |U(p)| + 1$.
8. *If* $\mathbf{A}$ *is itself subdirectly irreducible with monolith* μ*, and if* $(0,p) \in \mu$*, then* $U(p) = A \setminus \{0\}$ *and* $\theta = 0_A$.

Proof. (Item 3, $\Leftarrow$): Suppose $\lambda \in \operatorname{Pol}_1^* \mathbf{A}$ and $\lambda(a) = p$ and $\lambda(0) = 0$. Write $\lambda = \lambda_n \cdots \lambda_2 \lambda_1$, where each λ_i is a basic translation. Define $a_0 = a$, $b_0 = 0$, $a_{i+1} = \lambda_{i+1}(a_i)$, and $b_{i+1} = \lambda_{i+1}(b_i)$ $(i < n)$. Then $b_i \le a_i$ for $i \le n$, since each λ_i is order-preserving. Since $a_n = p \ne 0 = b_n$ we have $b_i \ne a_i$, and therefore $b_i = 0$ for all $i \le n$ (here we use the fact that $\mathbf{A}$ has height 1). Then inductively $a_i \in U(p)$ for $i = n, \dots, 1, 0$.

(Item 5): It suffices to show that θ is preserved by each basic translation λ of $\mathbf{A}$. So suppose $(a,b) \in \theta$ but $\{\lambda(a), \lambda(b)\} \cap U(p) \ne \emptyset$; say $\lambda(a) \in U(p)$. If $\lambda(0) = 0$ then $a \in U(p)$, and consequently $a = b$, as $(a,b) \in \theta$. If conversely $\lambda(0) \ne 0$, then λ is constant (since λ is order-preserving and $\mathbf{A}$ has height 1). In either case, $\lambda(a) = \lambda(b)$ as required.

(Item 6): $(p,0) \notin \theta$ by item 4. Therefore it suffices to show that $(a,b) \notin \theta$ implies $(p,0) \in \operatorname{Cg}^{\mathbf{A}}(a,b)$. Now $(a,b) \notin \theta$ implies $a \ne b$ and $\{a,b\} \cap U(p) \ne \emptyset$, say $a \in U(p)$. Choose $\lambda \in \operatorname{Pol}_1^* \mathbf{A}$ satisfying $\lambda(a) = p$ and $\lambda(0) = 0$ (by item 3) and let $\tau(x) = \lambda(x \wedge a)$. Then $\tau(a) = p$ while $\tau(b) = 0$, which suffices.

(Item 8): Use item 3 and Lemma 1.5(2) with $b = 0$. $\square$

This leads to the following reduction of Problem 3.

Proposition 5.3. *Let* $\mathcal{V}$ *be a subvariety of* $\mathcal{W}_{\mathcal{F}}$ *satisfying* $\kappa(\mathcal{V}) < \omega$ *and such that every subdirectly irreducible member of* $\mathcal{V}$ *has height* 1*. Then* $\mathcal{V}$ *is finitely based if and only if there exists a finitely based variety* $\mathcal{V}'$ *satisfying* $\mathcal{V} \subseteq \mathcal{V}' \subseteq \mathcal{W}_{\mathcal{F}}$ *and such that every subdirectly irreducible member of* $\mathcal{V}'$ *has height* 1.

Proof. If $\mathcal{V}$ itself is finitely based then we may choose $\mathcal{V}' = \mathcal{V}$. Conversely, suppose that some variety $\mathcal{V}'$ satisfying the above conditions exists. To prove that $\mathcal{V}$ is finitely based, it suffices to find a finitely based variety $\mathcal{V}_1 \supseteq \mathcal{V}$ which satisfies $\kappa(\mathcal{V}_1) < \omega$. Let Σ_1 be a finite equational basis for $\mathcal{V}'$, let N be an upper bound to the cardinalities of the subdirectly irreducible members of $\mathcal{V}$, let ϕ be a first-order sentence defining the height-1 members of $\mathcal{W}_{\mathcal{F}}$, and let ψ be a first-order sentence asserting

$$\phi \;\to\; \forall x(x \ne 0 \;\to\; |U_N(x)| < N).$$

$\mathcal{V} \models \psi$ by Lemma 5.2(5–7) and the choice of N. By the compactness theorem of first-order logic, there exists a finite set Σ_2 of identities true in $\mathcal{V}$ such that $\Sigma_2 \vdash \psi$. Let $\mathcal{V}_1$ be the variety defined by $\Sigma_1 \cup \Sigma_2$.

Clearly $\mathcal{V} \subseteq \mathcal{V}_1$ and $\mathcal{V}_1$ is finitely based. Let $\mathbf{A}$ be a subdirectly irreducible member of $\mathcal{V}_1$, with monolith μ. $\mathbf{A}$ has height 1 since $\mathcal{V}_1 \subseteq \mathcal{V}'$. By Lemmas 1.5(1) and 5.2(8) there exists $p \ne 0$ such that $(0,p) \in \mu$ and $U(p) = A \setminus \{0\}$. Since $\mathbf{A} \models \psi$ we have $|U_N(p)| < N$, which also implies $U_N(p) = U(p)$ by Lemma 5.2(1–3). Hence $|A| \le N$, proving $\kappa(\mathcal{V}_1) < \omega$. $\square$

The next definition and lemma give an infinite axiomatization of the property that all subdirectly irreducibles have height 1. The important feature of this axiomatization is that it is stated in terms of unary polynomials.

Definition 5.4. Let $\Delta_{\mathcal{F}}$ be an infinite set of quasi-identities which assert, in an arbitrary member $\mathbf{A}$ of $\mathcal{W}_{\mathcal{F}}$, that

$$\forall x \forall y[(\tau(0) \leq \lambda(x) \ \&\ x \leq y) \ \rightarrow\ \lambda(y) \wedge \tau(x) \leq \lambda(x)]$$

for all $\lambda, \tau \in \operatorname{Pol}_1 \mathbf{A}$. Also let $\Delta^*_{\mathcal{F}}$ be the subset that one obtains by requiring λ, τ to range over $\operatorname{Pol}^*_1 \mathbf{A}$ only.

Lemma 5.5. *Suppose $\mathcal{V}$ is a subvariety of $\mathcal{W}_{\mathcal{F}}$. Then the following are equivalent:*

1. $\mathcal{V} \models \Delta_{\mathcal{F}}$.
2. *All subdirectly irreducible members of $\mathcal{V}$ satisfy $\Delta^*_{\mathcal{F}}$.*
3. *Every subdirectly irreducible member of $\mathcal{V}$ has height* 1.

Proof. $(3 \Rightarrow 1)$. Since $\Delta_{\mathcal{F}}$ consists of quasi-identities, it suffices to show that each subdirectly irreducible $\mathbf{A} \in \mathcal{V}$ satisfies $\Delta_{\mathcal{F}}$. Suppose $\lambda, \tau \in \operatorname{Pol}_1 \mathbf{A}$ and $a, b \in A$ with $b \leq a$. As $\mathbf{A}$ has height 1, it follows that either $b = 0$ or $b = a$. In either case, if $\tau(0) \leq \lambda(b)$ then $\lambda(a) \wedge \tau(b) \leq \lambda(b)$.

$(2 \Rightarrow 3)$. Suppose that $\mathbf{A}$ is a subdirectly irreducible member of $\mathcal{W}_{\mathcal{F}}$ and $\mathbf{A} \models \Delta^*_{\mathcal{F}}$. Let μ be the monolith of $\mathbf{A}$ and suppose $p, q, r \in A$ with $q < p < r$ and $(p, q) \in \mu$. Then $(p, q) \in \operatorname{Cg}^{\mathbf{A}}(p, r) = \operatorname{Cg}^{\mathbf{A}^*}(p, r)$. As every $\lambda \in \operatorname{Pol}^*_1 \mathbf{A}$ is order-preserving, there must exist some $\lambda \in \operatorname{Pol}^*_1 \mathbf{A}$ such that $p \leq \lambda(r)$ and $p \not\leq \lambda(p)$. Let $\tau(x) = x$. Then

$$(\tau(0) \leq \lambda(x) \ \&\ x \leq y) \ \rightarrow\ \lambda(y) \wedge \tau(x) \leq \lambda(x)$$

is false in $\mathbf{A}$ when $x = p$ and $y = r$, which contradicts the assumption that $\mathbf{A} \models \Delta^*_{\mathcal{F}}$.

Thus if $q < p$ and $(p, q) \in \mu$, then p is maximal (with respect to $\leq$) in $\mathbf{A}$. It also follows that $q \prec p$, since if $q < p' \leq p$ then the map $x \mapsto x \wedge p'$ forces $(p', q) \in \operatorname{Cg}^{\mathbf{A}}(p, q) = \mu$ and hence $p' = p$ by the previous sentence.

Suppose next that $0 < q < p$ and $(p, q) \in \mu$. Then $(p, q) \in \operatorname{Cg}^{\mathbf{A}^*}(q, 0)$. Observe that the map $e : x \mapsto x \wedge p$ is a fundamental operation of $\mathbf{A}^*$, is idempotent, and has range containing $\{p, q\}$. Hence there exists some $\tau \in \operatorname{Pol}^*_1 \mathbf{A}$ such that $\tau(0) \leq q$ and $\tau(q) \not\leq q$ and $\operatorname{range}(\tau) \subseteq \operatorname{range}(e) = [0, p]$. Let $\lambda(x) = x$. Then again we get a failure of the corresponding quasi-identity in $\Delta^*_{\mathcal{F}}$ (with $x = q$ and $y = p$), a contradiction. This proves that if $q < p$ and $(p, q) \in \mu$, then $q = 0$.

Using Lemma 1.5(1) and the above discussion, we deduce the existence of $p \in A$ such that $0 \prec p$ and $(p, 0) \in \mu$. Suppose $b < a$ in A. By Lemma 1.5(2) there exists $\lambda \in \operatorname{Pol}^*_1 \mathbf{A}$ such that $\lambda(b) = 0$ and $\lambda(a) = p$. Next suppose that $0 < b$. Then by the same argument there is another $\tau \in \operatorname{Pol}^*_1 \mathbf{A}$ such that $\tau(0) = 0$ and $\tau(b) = p$. But this leads to a failure of the appropriate member of $\Delta^*_{\mathcal{F}}$ upon setting $x = b$ and $y = a$. Hence if $b < a$ then $b = 0$, which proves that $\mathbf{A}$ has height 1. $\square$

Definition 5.6. In any $\mathbf{A} \in \mathcal{W}_{\mathcal{F}}$ define the 4-ary relations $R_{\mathbf{A}}$ and $R^*_{\mathbf{A}}$ as follows:

$$(x, y, z, w) \in R_{\mathbf{A}} \leftrightarrow \exists \lambda \in \operatorname{Pol}_1 \mathbf{A}[\lambda(x) = z \ \&\ \lambda(y) = w],$$
$$(x, y, z, w) \in R^*_{\mathbf{A}} \leftrightarrow \exists \lambda \in \operatorname{Pol}^*_1 \mathbf{A}[\lambda(x) = z \ \&\ \lambda(y) = w].$$

Proposition 5.7. *Suppose $\mathcal{V}$ is a subvariety of $\mathcal{W}_{\mathcal{F}}$ satisfying $\kappa(\mathcal{V}) < \omega$ and such that every subdirectly irreducible member of $\mathcal{V}$ has height 1. Suppose further that there exist a finitely based variety $\mathcal{V}_1$ and a first-order formula $\rho(x, y, z, w)$ such that $\mathcal{V} \subseteq \mathcal{V}_1 \subseteq \mathcal{W}_{\mathcal{F}}$ and for every $\mathbf{A} \in \mathcal{V}_1$, $R^*_{\mathbf{A}} \subseteq \rho^{\mathbf{A}} \subseteq R_{\mathbf{A}}$. Then $\mathcal{V}$ is finitely based.*

Proof. Let ψ be the following first-order sentence:

$$\forall xyzwuv[(\rho(x,y,z,w) \ \& \ \rho(0,x,u,v) \ \& \ u \leq z \ \& \ x \leq y) \ \rightarrow \ w \wedge v \leq z].$$

Then for every $\mathbf{A} \in \mathcal{V}_1$ the following implications hold: $\mathbf{A} \models \Delta_{\mathcal{F}} \Rightarrow \mathbf{A} \models \psi \Rightarrow \mathbf{A} \models \Delta^*_{\mathcal{F}}$. Thus $\mathcal{V} \models \psi$ by Lemma 5.5. By the compactness theorem there exists a finite set Σ_2 of identities true in $\mathcal{V}$ such that $\Sigma_2 \vdash \psi$. Let Σ_1 be a finite basis for $\mathcal{V}_1$ and let $\mathcal{V}'$ be the variety defined by $\Sigma_1 \cup \Sigma_2$. $\mathcal{V}'$ is finitely based, contains $\mathcal{V}$, and satisfies $\Delta^*_{\mathcal{F}}$. Hence $\mathcal{V}$ is finitely based by Lemma 5.5 and Proposition 5.3. $\square$

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References

[1] K. Baker, Finite equational bases for finite algebras in a congruence-distributive equational class, *Adv. in Math.* **24** (1977), 207–243. MR **56:**5389

[2] D. Hobby and R. McKenzie, *The Structure of Finite Algebras*, Amer. Math. Soc. (Providence, R.I.), 1988. MR **89m:**08001

[3] K. Kearnes and Á. Szendrei, Self-rectangulating varieties of type 5, manuscript, 1995.

[4] E. Kiss and P. Prőhle, Problems and results in tame congruence theory, *Algebra Universalis* **29** (1992), 151–171. MR **93g:**08004

[5] R. McKenzie, Paraprimal varieties: a study of finite axiomatizability and definable principal congruences in locally finite varieties, *Algebra Universalis* **8** (1978), 336–348. MR **57:**9634

[6] ———, The residual bounds of finite algebras, *Int. J. Algebra and Computation* **6** (1996), 1–28. CMP 96:07

[7] ———, The residual bound of a finite algebra is not computable, *Int. J. Algebra and Computation* **6** (1996), 29–48. CMP 96:07

[8] ———, Tarski's finite basis problem is undecidable, *Int. J. Algebra and Computation* **6** (1996), 49–104. CMP 96:07

[9] P. Perkins, Unsolvable problems for equational theories, *Notre Dame J. Formal Logic* **8** (1967), 175–185. MR **38:**4310

[10] A. Tarski, Equational logic and equational theories of algebras, pp. 275–288 in H. A. Schmidt et al., eds., *Contributions to Mathematical Logic*, North-Holland (Amsterdam), 1968. MR **38:**5692

[11] W. Taylor, *Equational Logic*, Survey 1979, Houston J. Math., 1979. MR **80j:**03042

[12] R. Willard, On McKenzie's method, *Periodica Math. Hungarica* **32** (1996), 149–165. CMP 97:01

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