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GENERALIZED WEIL'S RECIPROCITY LAW AND MULTIPLICATIVITY THEOREMS

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Abstract. Fix a one-dimensional group variety G with Euler–characteristic $\chi(G) = 0$, and a quasi–projective variety Y, both defined over $\mathbf{C}$. For any $f \in Hom(Y, G)$ and constructible sheaf $\mathcal{F}$ on Y, we construct an invariant $c_{\mathcal{F}}(f) \in G$, which provides substantial information about the topology of the fiber–structure of f and the structure of $\mathcal{F}$ along the fibers of f. Moreover, $c_{\mathcal{F}} : Hom(Y, G) \to G$ is a group homomorphism.

1. Introduction and the main result

We fix a one–dimensional (abelian) group variety G and a quasi–projective variety Y, both defined over the complex numbers. We will use the multiplicative notation for the group structure $\mu : G \times G \to G$ of G. We would like to relate the topological fiber–structure of morphisms $f : Y \to G$ and the group structure of $Hom(Y, G)$.

If $f : Y \to G$ is a morphism, and $\mathcal{F}$ is a constructible sheaf on Y, then the Euler–characteristic $\chi(t)$ of the cohomology groups $H^*(f^{-1}(t), \mathcal{F})$ is constant on a non–empty Zariski open set of G. This generic value is denoted by $\chi(t_{gen})$. (If $\dim f(Y) = 0$ then, by definition, $\chi(t_{gen}) = 0$.) An element $t \in G$ is called a χ–*critical value* if $\Delta\chi(t) := \chi(t_{gen}) - \chi(t) \neq 0$. The set of χ–critical values is denoted by $C(f, \mathcal{F})$. Our first invariant associated with f and $\mathcal{F}$ is

$$a(f, \mathcal{F}) = \prod_{t \in C(f,\mathcal{F})} t^{\Delta\chi(t)} \in G.$$

If $C(f, \mathcal{F}) = \emptyset$ then, by definition, $a(f, \mathcal{F}) = e_G$, the identity element of G.

If we have two morphisms $f_i : Y \to G$ $(i = 1, 2)$, then we define $f_1 \cdot f_2 : Y \to G$ by $(f_1 \cdot f_2)(y) = \mu(f_1(y), f_2(y))$. The proposed "*multiplicativity problem*" is the following: fix G, and study the existence of the multiplicativity property

$$\text{(M.P.)} \qquad a(f_1 \cdot f_2, \mathcal{F}) = a(f_1, \mathcal{F}) \cdot a(f_2, \mathcal{F})$$

for arbitrary spaces Y and morphisms f_i $(i = 1, 2)$; or, if (M.P.) does not hold, then characterize the correction term $a(f_1 \cdot f_2) \cdot a(f_1)^{-1} \cdot a(f_2)^{-1}$.

It is not very difficult to see that in order to have (M.P.) one needs the vanishing of the Euler–characteristic of G: $\chi(G) = 0$. (Take, for example, $Y = G$, $\mathcal{F} = \mathbf{C}_G$, $f_1 = id_G$, and f_2 =the constant map $g \mapsto g_0$, for a generic g_0.) With this

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topological restriction there are exactly two candidates for G: $G = \mathbf{C}^*$ and G supported by an elliptic curve. Now, for $G = \mathbf{C}^*$, the multiplicativity property again does not hold (see below), but the correction term can be determined: it is related to the monodromies around the "points at infinity" of the non–complete space $\mathbf{C}^*$. On the other hand, for elliptic curves the following holds:

Proposition 1. *Let G be an elliptic curve with an arbitrary fixed group structure $\mu : G \times G \to G$. Then the multiplicativity property (M.P) holds for arbitrary morphisms $f_i : Y \to G$ $(i = 1, 2)$ and a constructible sheaf $\mathcal{F}$ on Y.*

We will give the proof in section 4.

In order to describe the correction term in the case $G = \mathbf{C}^*$, we need one more invariant. For this, we need more information about the sheaves $\{R^k f_! \mathcal{F}\}_k$. It is well known that there exists a finite set $\Gamma(f, \mathcal{F})$ of $\mathbf{C}^*$ such that the restriction of $\bigoplus_k R^k f_! \mathcal{F}$ on its complement determines a flat bundle. Let $R > r > 0$ be two positive numbers such that for any $t \in \Gamma(f, \mathcal{F})$ one has $r < |t| < R$. The monodromy $M_0^* \in Aut\, H^*(f^{-1}(r), \mathcal{F})$ of the fibration over $\{t \in \mathbf{C}^* : \ |t| = r\}$ defines the zeta–function $\zeta_0(\lambda) = \prod_{q \geq 0} \det(1 - \lambda M_0^q)^{(-1)^q}$. Similarly, the monodromy M_∞^* of the fibration over $\{t \in \mathbf{C}^* : \ |t| = R\}$ defines the zeta function $\zeta_\infty(\lambda)$. Since the roots and the poles of ζ_0 and ζ_∞ are roots of unity, these rational functions can be written in the following form:

$$\zeta_0(\lambda) = \prod_{N>0} (1 - \lambda^N)^{\alpha_N} \quad \text{and} \quad \zeta_\infty(\lambda) = \prod_{M>0} (1 - \lambda^M)^{\beta_M}.$$

Let $b_0(f) = \prod_{N>0} N^{N \cdot \alpha_N}$ and $b_\infty(f) = \prod_{M>0} (-M)^{M \cdot \beta_M}$. Our second invariant associated with f and $\mathcal{F}$ is $b(f, \mathcal{F}) = b_0(f)/b_\infty(f)$.

The main result of this paper is the following theorem:

Theorem. *Let Y be a quasi–projective space and $\mathcal{F}$ a constructible sheaf on Y. For any two morphisms $f_i : Y \to \mathbf{C}^*$ $(i = 1, 2)$ one has*

$$\text{(M.T.)} \qquad \frac{a(f_1, \mathcal{F}) \cdot a(f_2, \mathcal{F})}{a(f_1 \cdot f_2, \mathcal{F})} = \frac{b(f_1, \mathcal{F}) \cdot b(f_2, \mathcal{F})}{b(f_1 \cdot f_2, \mathcal{F})}.$$

This means that the invariant $c(f, \mathcal{F}) = a(f, \mathcal{F})/b(f, \mathcal{F})$ is multiplicative: $c(f_1, \mathcal{F}) \cdot c(f_2, \mathcal{F}) = c(f_1 \cdot f_2, \mathcal{F})$.

The multplicativity theorem has the following

Corollary. *Let Y be a quasi–projective space and let $\mathcal{F}$ and $\mathcal{F}'$ be two constructible sheafs on Y such that $\dim \mathcal{F}_Q = \dim \mathcal{F}'_Q$ for any $Q \in Y$. If $f : Y \to \mathbf{C}^*$ is a morphism, the relative b–invariant by $\Delta b(f; \mathcal{F}, \mathcal{F}') := b(f, \mathcal{F})/b(f, \mathcal{F}')$. Then $\Delta b(\cdot) = \Delta b(\cdot; \mathcal{F}, \mathcal{F}')$ is multiplicative; i.e. for any two morphisms f_1 and f_2 one has*

$$\Delta b(f_1) \cdot \Delta b(f_2) = \Delta b(f_1 f_2).$$

Proof. By a Meyer–Vietoris argument, $\chi(f^{-1}(t), \mathcal{F}) = \chi(f^{-1}(t), \mathcal{F}')$ for any $t \in \mathbf{C}^*$. Now apply the multiplicativity formula (M.T.).

A direct proof of the corollary can be extracted from the proof of the main theorem (section 3). □

The proof the main theorem is contained in sections 2 and 3. The key point is a generalization of *Weil's reciprocity law*. This is stated in section 2. (For Weil's result, see, for example, [1], p. 242.)

We end this section with the following:

Problem. Find the analogue of the Multiplicativity Theorem for higher dimensional group varieties G !

Acknowledgement and bibliographical note

The theorem in the special case of $\mathcal{F} = \mathbf{C}_Y$ was first obtained by J. Denef in January 1991 by using arithmetical results of [3] (Théorème 3.2.1.1) and [4]. Directly afterwards F. Loeser noticed that the result (with the same restriction) also follows from Théorème 3.3.1 of [5], by considering the determinant of the Aomoto complex of $f_1^{s_1} f_2^{s_2}$, putting $s_1 = s_2 = s$ and comparing with the determinant of the Aomoto complex of $(f_1 f_2)^s$. The work of Loeser and Sabbah uses the theory of D–modules. (Denef's and Loeser's proofs are both unpublished.)

Since the theorem can be formulated in topological terms, they asked for a more natural proof, which is able to explain the "mystery" of the relation. This paper not only generalizes their result, but, we hope, satisfies this demand as well. (The first version of this note was written in the autumn of 1993.)

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2. The generalized reciprocity law

The main result of this section is a generalization of Weil's reciprocity law. This result is the heart of the proof of the multiplicativity theorem. In fact, the multiplicativity theorem can be considered as a reciprocity law, as we will see later.

If f is a meromorphic function defined on a compact Riemann surface S, then $supp(f)$ denotes its support, and $ord_x(f)$ is the order of f at x.

The Generalized Weil's Reciprocity Law. *Let f_i and g_i be meromorphic functions $(i = 1, \dots, s)$ on the compact Riemann surface S so that there exists a finite set $P = \{x_1, \dots, x_n\}$ with the following properties:*

(a) $\mathrm{supp}(f_i) \cap \mathrm{supp}(g_i) \subset P$ *for each $i = 1, \dots, s$;*

(b) there exist integers $n_1, \dots, n_s$ and $m_1, \dots, m_s$ with $\sum_{i=1}^s n_i m_i = 0$, and integers $o_f(x_j)$ and $o_g(x_j)$ $(j = 1, \dots, n)$ such that

$$\begin{cases} ord_{x_j}(f_i) = n_i \cdot o_f(x_j), \\ ord_{x_j}(g_i) = m_i \cdot o_g(x_j) \quad \text{for any } i \text{ and } j. \end{cases}$$

Then

$$\prod_{p \in S \setminus P} \prod_{i=1}^{s} f_i(p)^{ord_p(g_i)} \cdot \prod_{x \in P} [(\prod_{i=1}^{s} f_i^{m_i})(x)]^{o_g(x)}$$

$$= \prod_{p \in S \setminus P} \prod_{i=1}^{s} g_i(p)^{ord_p(f_i)} \cdot \prod_{x \in P} [(\prod_{i=1}^{s} g_i^{n_i})(x)]^{o_f(x)}.$$

The proof is elementary and similar to the proof of the Weil's reciprocity law [1, p. 242], and it is left to the reader.

The main application of this reciprocity law is an (apparently) particular version of the multiplicativity theorem.

We will fix the homogeneous coordinates $[u_0 : u_1 : u_2]$ in $\mathbf{P}^2(\mathbf{C})$. Let $P(u_0, u_1, u_2)$ be an irreducible homogeneous polynomial. Let $X \in \mathbf{P}^2$ be its zero set $Z(P)$. Let $\pi : \tilde{X} \to X$ be the normalization of X.

The set $A = X \cap \{u_0u_1u_2 = 0\}$ has a natural decomposition into the following subsets: $A_{ij} = A_{ji} = X \cap \{u_i = u_j = 0\}$ for $0 \le i < j \le 2$; and $A_i = X \cap \{u_i = 0\} \setminus (\bigcup_{j\neq i} A_{ij})$ for $0 \le i \le 2$. Their inverse images in $\tilde{A} = \pi^{-1}(A)$ are $\tilde{A}_{ij} = \pi^{-1}(A_{ij})$ and $\tilde{A}_i = \pi^{-1}(A_i)$. The space $X^* = X \setminus A$ can be considered as an affine space in $\mathbf{C}^2 = \mathbf{P}^2 \setminus \{u_2 = 0\}$. Introduce the affine coordinates (x, y) in $\mathbf{C}^2$ by $[x : y : 1] = [u_0 : u_1 : u_2]$ and denote $p(x, y) = P(x, y, 1)$. Obviously $Z(p) \cap (\mathbf{C}^* \times \mathbf{C}^*) = X^*$.

We assume that $X^* \neq \emptyset$, and X^* is not of the form $\{x = \text{constant}\}$ or $\{y = \text{constant}\}$ or $\{xy = \text{constant}\}$.

For any point $Q \in X$, in the local ring $\mathcal{O}_Q(\mathbf{P}^2)$ the germ of P has a prime–decomposition $P = \prod_{j\in J(Q)} P_j$. The space–germs $Z_j = \{P_j = 0\}$ give the local irreducible components of the germ of $Z(P)$ at Q. Corresponding to this decomposition, the inverse image $\pi^{-1}(Q)$ contains exactly $J(Q)$ points $\pi^{-1}(Q) = \bigcup_{j\in J(Q)} \tilde{Q}_j$. We will use the notation t_j for a local coordinate at the smooth points $\tilde{Q}_j$. If f is a meromorphic function on X then the composition $f \circ \pi$ is denoted by $\tilde{f}$. It is a meromorphic function on $\tilde{X}$.

The following lemma will be useful in multiplicity and order computations:

Lemma 1. *Let $Q \in X$ be a point in $X \cap \mathbf{C}^2$ and $\varphi : (\mathbf{C}^2, Q) \to (\mathbf{C}, 0)$ be the germ of an analytic function which does not vanish on any of the space–germs Z_j. Let t_j be a local coordinate at $\tilde{Q}_j$. Then:*

(a) $ord_{t_j} \widetilde{Jac(p_j, \varphi)} = \mu(p_j) + i_Q(\varphi, p_j) - 1$ *for any j;*

(b) $ord_{t_j} \widetilde{Jac(p, \varphi)} = \mu(p_j) + i_Q(p_j, p/p_j) + i_Q(\varphi, p_j) - 1$ *for any j.*

Here $Jac(\psi, \phi)$ denotes the Jacobian $\psi_x\phi_y - \psi_y\phi_x$ of ψ and ϕ (ψ_x and ψ_y are the partial derivatives), $p_j(x, y) = P_j(x, y, 1)$, $\mu(p_j)$ is the Milnor number of the germ p_j, and $i_Q(\cdot, \cdot)$ denotes the intersection multiplicity at Q.

Proof. Consider a good representative $\bar{\varphi} : (\mathcal{X}, Q) \to (D_2, 0)$ of φ such that $Z_j \cap \partial\mathcal{X} \subset \bar{\varphi}^{-1}(\partial D_2)$ and $\bar{\varphi}|Z_j \cap \mathcal{X} : (Z_j \cap \mathcal{X}, Q) \to (D_2, 0)$ is a good representative for the restriction of φ on Z_j. For ϵ sufficiently small, consider the Milnor fiber $F_j = \{p_j = \epsilon\} \cap \mathcal{X}$. Now,

$$ord_{t_j} \widetilde{Jac(p_j, \varphi)} = i_Q(p_j, Jac(p_j, \varphi)) = \sum_T i_T(p_j - \epsilon, Jac(p_j, \varphi)),$$

where the sum is over the intersection points T of the fiber F_j with $Z(Jac(p_j, \varphi))$. In these intersection points T, the fiber $\varphi^{-1}(\varphi(T)) \cap \mathcal{X}$ is smooth. These points T are exactly those points where $\varphi^{-1}(\varphi(T)) \cap \mathcal{X}$ intersects F_j tangentially. Therefore, the restriction $\bar{\varphi} : F_j \to D_2$ on F_j is a branched covering of degree $i = i_Q(\varphi, p_j)$, with branch points of index $i_T + 1 = i_T(p_j - \epsilon, Jac(p_j, \varphi)) + 1$ corresponding to the points T.

By an Euler–characteristic argument, we obtain $1 - \mu(p_j) = \chi(F_j) = i \cdot \chi(D_2) - \sum_T i_T$, which gives (a).

(b) For the second part, write $p = p_j \cdot q_j$. Then the result follows from (a) and the relations $Jac(p, \varphi) = q_j \cdot Jac(p_j, \varphi) + p_j \cdot Jac(q_j, \varphi)$ and $ord_{t_j} \tilde{q}_j = i_Q(p_j, q_j)$. □

Let $X \subset \mathbf{P}^2$ as above and $\tilde{Q}_j \in \pi^{-1}(Q)$ for $j \in J(Q)$. The map–germ $\pi : (\tilde{X}, \tilde{Q}_j) \to (Z_j, Q)$ can be written in the form

$$t_j \mapsto [x(t_j) : y(t_j) : 1] = [t_j^{N_j} a_j(t_j) : t_j^{M_j} b_j(t_j) : 1],$$

where $N_j,\ M_j$ are integers, and $a_j(t_j),\ b_j(t_j)$ are invertible germs at $\tilde{Q}_j$.

Define the set $P_i = \{\tilde{Q}_j \in \tilde{A}_{i2} : N_j + M_j = 0\}$ for $i = 0, 1$. Let $P = P_0 \cup P_1$.

Lemma 2. *Consider the following meromorphic functions on $\tilde{X}$:*

$$f_1 = \tilde{x},\ f_2 = \tilde{y},\ g_1 = (\frac{yp_y}{yp_y - xp_x})\tilde{},\ g_2 = (\frac{xp_x}{xp_x - yp_y})\tilde{}.$$

Then the Riemann surface $S = \tilde{X}$, the set $P \subset \tilde{X}$ and the functions $f_i,\ g_i$ $(i = 1, 2)$ satisfy the hypothesis of the Generalized Reciprocity Law. More precisely:

(a) $supp(f_i) \cap supp(g_i) \subset P$ for $i = 1, 2$; and

(b) $ord_x(g_1) = ord_x(g_2)$ and $ord_x(f_1) = -ord_x(f_2)$ for any $x \in P$.

Therefore, the reciprocity law

$$(*) \qquad \prod_{x \in S \setminus P} f_1(x)^{ord_x g_1} f_2(x)^{ord_x g_2} \prod_{x \in P} [(f_1 f_2)(x)]^{ord_x g_1}$$
$$= \prod_{x \in S \setminus P} g_1(x)^{ord_x f_1} g_2(x)^{ord_x f_2} \prod_{x \in P} [\ \frac{g_1}{g_2}(x)]^{ord_x f_1}$$

can be applied for these functions.

The contributions from the sets $\tilde{X}^* = \pi^{-1}(X^*)$, $\tilde{A}_{ij}$ and $\tilde{A}_i$, in the left, respectively in the right hand side of the reciprocity law are listed in Table 1.

In Table 1 (x_Q, y_Q) are the coordinates of the point Q, and the product $\prod_j$ is over the index set $J(Q)$, where the point Q is clear from the context. The points in $\tilde{A}_{i2}$ (or equivalently, the local irreducible components of $Z(P)$ at A_{i2}) are separated into three groups, which have geometrically different behaviour. In fact, they correspond to those χ–critical values which come from the bad behaviour of the functions at infinity.

Proof. Let R be one of the sets $\tilde{X}^*, \tilde{A}_{ij}$ or $\tilde{A}_i$. The conditions (a) and (b) are equivalent to the set of conditions:

(a_R) $supp(f_i) \cap supp(g_i) \cap R \subset P \cap R$ $(i = 1, 2)$;

(b_R) the identities of (b) are satisfied for any point $x \in P \cap R$, where $R = \tilde{X}^*, \tilde{A}_{ij}$ and $\tilde{A}_i$.

The case $R = \tilde{X}^*$.

Since $\tilde{x}$ and $\tilde{y}$ are invertible holomorphic functions on $\tilde{X}^*$, the conditions (a_R) and (b_R) are satisfied. This also shows that the contribution in the right hand side is trivial. The support of g_i in $\tilde{X}^*$ has the following geometric interpretation. Fix a point $Q \in X^*$ and let φ be one of the functions $x,\ y$ or xy. Then $Jac(\varphi, p)(Q) = 0$ if and only if the tangent line at Q of the smooth curve $\{\varphi = \varphi(Q)\}$ is in the Zariski tangent space of $Z(p)$ at Q. Let $\tilde{Q}_j \in (supp(g_1) \cup supp(g_2)) \cap \tilde{X}^*$. Then by lemma 1 one has $ord_{t_j} g_1 = i_Q(x - x_Q, p_j) - i_Q(xy - x_Q y_Q, p_j)$, and symmetrically for g_2. This gives the result from the table.

The case $R = \tilde{A}_{02}$.

In this case $M_j < 0$ and $M_j < N_j$. Notice that $g_1 + g_2 = 1$. The identity $g_1 = 1/(1 + \frac{xy'}{yx'})$ implies that

$$(\#) \qquad g_1(t_j) = \frac{N_j a_j b_j + t_j b_j a_j'}{(N_j + M_j) a_j b_j + t_j (a_j b_j)'}.$$

Case (i): Assume that $N_j \neq 0$ and $N_j + M_j \neq 0$. Then (#) implies that $\tilde{Q}_j \notin supp(g_i)$ $(i = 1, 2)$ and the contribution follows.

TABLE 1

	contributions in the left hand side of (*)	contributions in the right hand side of (*)
$\tilde{X}^*$	$\prod_{Q\in X^*} \frac{x_Q^{i_Q(x-x_Q,p)-1}\cdot y_Q^{i_Q(y-y_Q,p)-1}}{(x_Qy_Q)^{i_Q(xy-x_Qy_Q,p)-1}}$	1
$\tilde{A}_{01}$	1	$\prod_j \frac{N_j^{N_j}\cdot M_j^{M_j}}{(N_j+M_j)^{N_j+M_j}}$
$\tilde{A}_0$	$\prod_{Q\in A_0} y_Q^{i_Q(y-y_Q,p)}$	1
$\tilde{A}_1$	$\prod_{Q\in A_1} x_Q^{i_Q(x-x_Q,p)}$	1
$\tilde{A}_2$	1	$\prod_{Q\in A_2}\prod_{j\in J(Q)} \frac{1}{4^{N_j}}$
$\tilde{A}_{02}$		
(i) $N_j\neq 0\neq N_j+M_j$	1	$\prod_j \frac{N_j^{N_j}\cdot M_j^{M_j}}{(N_j+M_j)^{N_j+M_j}}$
(ii) $N_j=0$	$\prod_j a_j(0)^{ord_{t_j}[a_j(t_j)-a_j(0)]}$	1
(iii) $N_j+M_j=0$	$\prod_j [a_j(0)b_j(0)]^{-ord_{t_j}[a_jb_j-a_j(0)b_j(0)]}$	$\prod_j (-1)^{N_j}$
$\tilde{A}_{12}$		
(i) $M_j\neq 0\neq N_j+M_j$	1	$\prod_j \frac{N_j^{N_j}\cdot M_j^{M_j}}{(N_j+M_j)^{N_j+M_j}}$
(ii) $M_j=0$	$\prod_j b_j(0)^{ord_{t_j}[b_j(t_j)-b_j(0)]}$	1
(iii) $N_j+M_j=0$	$\prod_j [a_j(0)b_j(0)]^{-ord_{t_j}[a_jb_j-a_j(0)b_j(0)]}$	$\prod_j (-1)^{M_j}$

Case (ii): Assume that $N_j=0$. Then (#) shows that $ord_{t_j}g_1>0$. Since $g_2=1-g_1$ we obtain that $ord_{t_j}g_2=0$. On the other hand, $ord_{t_j}\tilde{x}=0$ (by assumtion). Therefore $\tilde{Q}_j\notin supp(f_i)\cap supp(g_i)$ $(i=1,2)$, i.e. the conditions are satisfied. Since $g_2(\tilde{Q}_j)=1$ and $ord\tilde{x}=0$, this case has no contribution to the left hand side. The right hand side is $\prod_j a_j(0)^{ord_{t_j}g_1}$. But $ord_{t_j}g_1=1+ord_{t_j}a_j'(t_j)=ord_{t_j}[a_j(t_j)-a_j(0)]$.

Case (iii): Assume that $N_j+M_j=0$. These local irreducible components correspond exactly to the set P_0. In particular, condition (a) is satisfied, but we have to verify (b). The identity $ord_{t_j}f_1+ord_{t_j}f_2=0$ is assured by the assumtion. The identity (#) gives that $ord_{t_j}g_1<0$. Since $g_1+g_2=1$ we obtain that $ord_{t_j}g_1=ord_{t_j}g_2$; hence (b) is satisfied too.

The contribution in the right hand side is $\prod_j[-\frac{yp_y}{xp_x}(\tilde{Q}_j)]^{N_j}$. But $[-\frac{yp_y}{xp_x}](\tilde{Q}_j) = \frac{yx'}{xy'}(\tilde{Q}_j)=\frac{N_j}{M_j}=-1$.

The contribution in the left hand side is $\prod_j [a_j(0)b_j(0)]^{ord_{t_j} g_1}$. But, by (#), $ord_{t_j} g_1 = -[1 + ord(a_j b_j)'] = -ord[(a_j b_j)(t_j) - (a_j b_j)(0)]$.

In the other cases the (case-by-case) verification is similar to the above cases (or even simpler), and it is left to the reader. □

The next proposition relates the reciprocity law with the multiplicativity theorem.

Let $Y = Z(p)$ be an irreducible curve in $\mathbf{C}^* \times \mathbf{C}^*$ as above. Any morphism $f : Y \to \mathbf{C}^*$ defines two invariants, $a(f) = a(f, \mathbf{C}_Y)$ and $b(f) = b(f, \mathbf{C}_Y)$, where $\mathbf{C}_Y$ is the constant sheaf on Y (cf. the definitions in the introduction).

Proposition 2. *The expression*

$$\frac{a(x)a(y)}{a(xy)} \quad (\textit{respectively } \frac{b(x)b(y)}{b(xy)})$$

is equal to the left (respectively to the right) hand side of the Generalized Reciprocity Law $(*)$*, applied in Lemma 2 for the meromorphic functions* f_1, f_2, g_1 *and* g_2*.*

In particular, the Reciprocity Law implies the Multiplicativity Theorem in the following case: Y *is an irreducible curve in* $\mathbf{C}^* \times \mathbf{C}^*$*,* $f_1(x, y) = x$*,* $f_2(x, y) = y$*, and* $\mathcal{F}$ *is the constant sheaf* $\mathbf{C}_Y$*.*

Proof. If $f : Y \to \mathbf{C}^*$ is a morphism, we denote the set of critical values $\{f(y) : y \in Y$ satisfies $df(y) = 0\}$ by $\Sigma(f)$.

As a first step, we we will find the χ–critical set $C(x)$ and the invariant $a(x)$.

It is not hard to verify that the "points at infinity" contained in the union $A_0 \cup A_2 \cup A_{01} \cup A_{12}$ have no contribution in $C(x)$.

A point $Q \in X^*$ is χ–critical if and only if it is critical. In this case x_Q will be a χ–critical value of the function x, and the contribution in $a(x)$ is $x_Q^{i_Q(x - x_Q, p) - 1}$.

Any point "at infinity" $Q \in A_1$ gives a χ–critical value x_Q and a contribution equal to $x_Q^{i_Q(x - x_Q, p)}$ in $a(x)$.

The last and more interesting case is given by the points $\tilde{Q}_j \in \tilde{A}_{02}$. A point $\tilde{Q}_j$ is a "bad point at infinity" if the tangent cone of the local irreducible component Z_j at $[0 : 1 : 0]$ is not $u_0 = 0$ or $u_2 = 0$. Consider the parametrization $t_j \mapsto [x(t_j) : y(t_j) : 1]$ of Z_j similarly as above. The line which joins $[0 : 1 : 0]$ and $[u_0 : u_1 : u_2](t_j)$ is $u_0 - t_j^{N_j} a_j(t_j) u_2 = 0$. The tangent cone is given by the limit equation $\lim_{t_j \to 0} [u_0 - t_j^{N_j} a_j(t_j) u_2] = 0$. If $N_j > 0$ then the tangent cone is $u_0 = 0$; if $N_j < 0$ then it is $u_2 = 0$. If $N_j = 0$ then its equation is $u_0 - a_j(0) u_2 = 0$, in affine coordinates $x = a_j(0)$. This means that the vertical line $x = a_j(0)$ is an asymptote of $Z(p)$ and $a_j(0)$ is a χ–critical value.

Consider the set of intersection points of Z_j and the line $u_0 - (a_j(0) + \epsilon) u_2 = 0$ which are in a small neighbourhood of $[0 : 1 : 0]$ and are lying in X^*. Their number is the number of the local, non–zero solutions of the equation $a_j(t_j) - a_j(0) = \epsilon$; in particular, the jump in Euler–characteristic is $ord_{t_j} [a_j(t_j) - a_j(0)]$.

Therefore,

$$C(x) = \Sigma(x : X^* \to \mathbf{C}^*) \cup A_1 \bigcup_{j \in J(A_{02}), N_j = 0} \{a_j(0)\},$$

and

$$a(x) = \prod_{q \in X^*} x_Q^{i_Q(x - x_Q, p) - 1} \cdot \prod_{Q \in A_1} x_Q^{i_Q(x - x_Q, p)} \cdot \prod_{j \in J(A_{02}), N_j = 0} [a_j(0)]^{ord[a_j - a_j(0)]}.$$

We have symmetric identities for $C(y)$ and $a(y)$.

In the case of the function xy the χ–critical values are provided by the critical points in X^* and maybe by some "critical point at infinity" contained in $\tilde{A}_{i2}$ ($i = 0,1$). Corresponding to the point $\tilde{Q}_j \in \tilde{A}_{02}$, we have to compute the number of local, non–zero solutions of the equation $t_j^{N_j+M_j} a_j(t_j)b_j(t_j) = s \in \mathbf{C}^*$. This number is zero for any s if $N_j + M_j \neq 0$. If $N_j + M_j = 0$, then for $s = a_j(0)b_j(0)$ we have a jump of $ord[a_jb_j - a_j(0)b_j(0)]$. Therefore

$$C(xy) = \Sigma(xy) \bigcup_{j\in J(A_{i2}),N_j+M_j=0} \{a_j(0)b_j(0)\},$$

and

$$a(xy) = \prod_{Q\in X^*} [x_Qy_Q]^{i_Q(xy-x_Qy_Q,p)-1} \cdot \prod_{j\in J(A_{i2}),N_j+M_j=0} [a_j(0)b_j(0)]^{ord[a_jb_j-a_j(0)b_j(0)]}.$$

Now using the table it is easy to verify that $a(x)a(y)/a(xy)$ is exactly the left hand side of the reciprocity law.

The expression $b(x)b(y)/b(xy)$ depends only on the points at infinity $X - X^*$.

If $j \in J(A_{01})$, then Z_j contributes with $1 - \lambda^{N_j}$ in $\zeta_0(x)$, with $1 - \lambda^{M_j}$ in $\zeta_0(y)$, with $1 - \lambda^{N_j+M_j}$ in $\zeta_0(xy)$, and with 1 in $\zeta_\infty(x)$, $\zeta_\infty(y)$ and $\zeta_\infty(xy)$. So, the contribution in $b(x)b(y)/b(xy)$ is $N_j^{N_j}M_j^{M_j}/(N_j+M_j)^{N_j+M_j}$, exactly as indicated in the table.

The points in A_0 have no contribution in $\zeta_\infty(x)$, $\zeta_0(y)$, $\zeta_\infty(y)$ and $\zeta_\infty(xy)$; and they have the same contributions in $\zeta_0(x)$ and $\zeta_0(xy)$.

Similarly, in all other cases the result can be verified case by case. For this, the following remark is useful. Look at the equation $t_j^{N_j}a_j = re^{2\pi i\alpha}$. If $N_j > 0$, then this gives $1-\lambda^{N_j}$ in $\zeta_0(x)$ and 1 in ζ_∞. If $N_j < 0$ then the contributions are inverse. If $N_j = 0$ then both contributions are trivial. □

3. The proof of the main theorem and examples

1. Consider the map $\phi = (f_1, f_2) : Y \to \mathbf{C}^* \times \mathbf{C}^*$. Then $f_i = pr_i \circ \phi$ ($i = 1, 2$), where $x = pr_1$ and $y = pr_2$ are the projections. Let $\mathcal{G}^\bullet = \mathbf{R}\phi_!\mathcal{F}$. Then the definition of the invariants a and b can be extended to the case of complexes, and we have the identities $a(g, \mathcal{G}^\bullet) = a(g \circ \phi, \mathcal{F})$ and $b(g, \mathcal{G}^\bullet) = b(g \circ \phi, \mathcal{F})$ for any function $g : \mathbf{C}^* \times \mathbf{C}^*$. Therefore, we can assume that $Y = \mathbf{C}^* \times \mathbf{C}^*$ and $f_1 = x$ and $f_2 = y$. Moreover, by the additivity of the invariants a and b, the complex $\mathcal{G}^\bullet$ can be replaced by only one constructible sheaf, which in the sequel will be denoted again by $\mathcal{F}$. We will use the following notations:

$$A(\mathcal{F}) = \frac{a(x,\mathcal{F})a(y,\mathcal{F})}{a(xy,\mathcal{F})} \text{ and } B(\mathcal{F}) = \frac{b(x,\mathcal{F})b(y,\mathcal{F})}{b(xy,\mathcal{F})}.$$

2. Assume that $supp\mathcal{F}$ is zero–dimensional. Then $a(x)a(y)/a(xy) = b(x) = b(y) = b(xy) = 1$; hence the result follows.

3. Assume that there is an irreducible curve $X \overset{i}{\hookrightarrow} \mathbf{P}^2$ such that $\mathcal{F} = i_*(\mathbf{C}_X)|Y$. If $X^* := X \cap Y$ is not of the form $\{g = \text{constant}\}$, where $g = x,\ y$ or xy, then the result follows by section 2. If X^* is of the exceptional type $\{g = \text{constant}\}$ mentioned above, then the result follows by an easy verification.

4. Assume that $supp\mathcal{F}$ is one–dimensional. Let $X_1^*\cup\cdots\cup X_i^*$ be the irreducible decomposition of $supp\mathcal{F}$. Using the second step (and the additivity of our invariants), we can assume that the restriction of $\mathcal{F}$ to the intersection points of the irreducible components is zero. Now, it is easy to verify that $a(f,\mathcal{F})=\prod_j a(f,\mathcal{F}|X_j^*)$ (and similarly for b) for any f. Therefore we can assume that $X^*=supp\mathcal{F}$ is an irreducible curve.

Now, again using step 2, we can assume that $Q\mapsto \dim_{\mathbf{C}}\mathcal{F}_Q$ is constant on X^*. Denote this common dimension by d. It is easy to verify that $a(f,\mathcal{F})=a(f,\mathbf{C}_{X^*})^d$. In particular,

$$\frac{a(x,\mathcal{F})a(y,\mathcal{F})}{a(xy,\mathcal{F})}=[\ \frac{a(x,\mathbf{C}_{X^*})a(y,\mathbf{C}_{X^*})}{a(xy,\mathbf{C}_{X^*})}\ ]^d,$$
$$A(\mathcal{F})=[\ A(\mathbf{C}_{X^*})\ ]^d.$$

If we establish a similar relation for the B–invariant, then this case follows from step 3.

The invariant $B(\mathcal{F})$ is a product of local contributions corresponding to the points at infinity $X-X^*$. We make the verification explicit in the case of $Q=[0:0:1]$. At all the other points the verification is similar, and is left to the reader. Let Z_j be a local component as above, and consider the space–germ (Z_j,Q). Then $\mathcal{F}$ is flat on $Z_j^*:=Z_j-Q$ (in a neighbourhood of Q); hence it is determined by a representation $\rho_j:\mathbf{Z}=\pi_1(Z_j^*)\to Aut\mathbf{C}^d$, i.e. by $\rho_j(1)=A\in Aut\mathbf{C}^d$. Then this point contributes in $\zeta_0(x,\mathcal{F})$ with $\det(I-\lambda^{N_j}A)$, in $\zeta_0(y,\mathcal{F})$ with $\det(I-\lambda^{M_j}A)$, and in $\zeta_0(xy,\mathcal{F})$ with $\det(I-\lambda^{N_j+M_j}A)$ (and it has no contribution in ζ_∞'s).

Write $\det(I-\lambda A)=c\cdot\prod_k(I-\lambda^k)^{\alpha_k}$. Then

$$\frac{\det(I-\lambda^{N_j}A)\det(I-\lambda^{M_j}A)}{\det(I-\lambda^{N_j+M_j}A)}=c\cdot\prod_k\frac{(1-\lambda^{kN_j})^{\alpha_k}(1-\lambda^{kM_j})^{\alpha_k}}{(1-\lambda^{k(N_j+M_j)})^{\alpha_k}}.$$

Its contribution in $B(\mathcal{F})$ is

$$\prod_k\left[\frac{(kN_j)^{kN_j}(kM_j)^{kM_j}}{[k(N_j+M_j)]^{k(N_j+M_j)}}\right]^{\alpha_k}=\left[\frac{N_j^{N_j}M_j^{M_j}}{(N_j+M_j)^{N_j+M_j}}\right]^{\sum k\alpha_k}.$$

But $\sum_k k\alpha_k=d$; therefore $B(\mathcal{F})=B(\mathbf{C}_{X^*})^d$. This shows that the case when $supp\mathcal{F}$ is one-dimensional (even if the corresponding representation is very complicated) can be reduced to the constant sheaf case. But this case was proved in the third step.

5. Consider now an arbitrary constructible sheaf on $Y=\mathbf{C}^*\times\mathbf{C}^*$. Then there is a stratification of Y such that $\mathcal{F}$ is flat on each stratum. Using step 2 and step 4, we can assume that $\mathcal{F}$ restricted to any zero or one-dimensional stratum is zero. In particular, there exists a curve X^* (maybe not irreducible) such that $\mathcal{F}|X^*=0$ and $\mathcal{F}|(Y-X^*)$ is a flat bundle. Let d be the dimension of its fiber.

Now, by a Mayer–Vietoris argument: $A(\mathcal{F})=A(i_*\mathbf{C}_{X^*})^{-d}$. We wish to establish a similar relation: $B(\mathcal{F})=B(i_*\mathbf{C}_{X^*})^{-d}$. Since a factor of type $(1-\lambda)^\alpha$ in a zeta–function has no contribution in the b–invariants, therefore by a Mayer–Vietoris argument, $B(\mathcal{F})$ can be localized in the neighbourhood of the points $X-X^*$ (where X is the closure of X^*). But at these points, any zeta function is a product of some zeta functions which are associated with constructible sheafs with one-dimensional support ([6]; see the precise argument below). Therefore, by similar computation as in the one–dimensional case (step 4), we obtain the desired relation.

We present the explicit argument in the case when $Q = [0 : 0 : 1]$. Assume that $Q \in X - X^*$. The local situation is the following. We have a constructible sheaf $\mathcal{F}$ defined in $(\mathbf{C}^2, Q)$ with discriminant X. This means that $\mathcal{F}$ on the local complement of the curve germ X is flat. Moreover (by step 4), we can assume that $\mathcal{F}|X = 0$. Consider the embedded resolution $\rho : (U, E) \to (\mathbf{C}^2, Q)$ of the curve–germ X. Let X' be the strict transform of $X \cup \{xy = 0\}$. For each exceptional divisor E_i construct a smooth transversal curve S_i in U, and let $C_i = \rho(S_i)$. Let $E_i^0 = E_i - (\bigcup_{j\neq i} E_i \cup X')$. Let $f = x,\ y$ or xy. We are interested in the zeta function $\zeta_0(f, \mathcal{F})$ of the monodromy M_0^* defined on $H^*(f^{-1}(t) \cap [(\mathbf{C}^2, Q) - \{xy = 0\}], \mathcal{F})$, where $f^{-1}(t)$ is the local Milnor fiber. Then for any of the functions $f = x,\ y$ or xy one has the following "reduction formula" [6]:

$$\zeta_0(f, \mathcal{F}) = \prod_{E_i} \zeta(f|C_i, \mathcal{F}|C_i)^{\chi(E_i^0)}.$$

Since the zeta functions can be computed as a product of zeta functions associated with "one–dimensional" objects, the same computation works as in step 4.

This ends the proof of the multiplicativity theorem. □

Example 1. Let $X^* = \{xy - 1 - x^2 = 0\} \subset \mathbf{C}^* \times \mathbf{C}^*$, and $\mathcal{F} = \mathbf{C}_{X^*}$. Then $C(x) = \{-i, +i\}$ and $a(x) = b(x) = 1$. The invariants of y are: $C(y) = \{-2, 2\}$, $a(y) = -4$, $\zeta_0(y) = \zeta_\infty(y) = (1 - \lambda)^2$, hence $b(y) = 1$. On the other hand, $C(xy) = \{1\}$, $a(xy) = 1$, $\zeta_0(xy) = (1 - \lambda)^2$, but $\zeta_\infty(xy) = 1 - \lambda^2$. Therefore $b(xy) = (-2)^{-2} = -1/4$.

Notice the importance of the sign in the definition of b_∞. This sign correction can be explained as follows. It is clear that the theorem is a residue–type result; in particular, in a local parametrization of type $t_j \mapsto x(t_j) = t_j^{N_j} a_j(t_j)$ the order $ord_{t_j} x = N_j$ is crucial. On the other hand, the zeta function recovers only $1 - \lambda^{|N_j|}$, i.e. only the absolute value $|N_j|$ of the order. Therefore, in case $N_j < 0$, we have to make a sign correction if we want to recover the good object (i.e. the order) from the zeta function ζ_∞.

Example 2. Let $X^* = \{x^3 = (y - u^3)^2\} \subset \mathbf{C}^* \times \mathbf{C}^*$, and let $\mathcal{F}$ be as above. Here $u \in \mathbf{C}^*$ is a free parameter. Then $a(x) = u^6$, $a(y) = u^9$ and $a(xy) = \frac{3^3 2^2}{5^5} u^{15}$. (It is instructive to verify this.) The zeta functions are: $\zeta_0(x) = 1 - \lambda^2$, $\zeta_0(y) = (1 - \lambda)^3$, $\zeta_0(xy) = (1 - \lambda)^3(1 - \lambda^2)$, $\zeta_\infty(x) = 1 - \lambda^2$, $\zeta_\infty(y) = 1 - \lambda^3$, and $\zeta_\infty(xy) = 1 - \lambda^5$. Notice that the final results $A = 5^5/2^2 3^3 = B$ come out in two completely different ways. (This gives a very mysterious flavour to the formula.)

Notice also that if we have some parameters in the definition of the functions f_i or in the definition of the sheaf (see the above example), then the invariants $a(f_i, \mathcal{F})$ can depend strongly on the parameters. But, by our theorem, $A(\mathcal{F})$ has to be discrete and all the variations with respect to the parameters are cancelled. The author knows no other argument which would imply this fact.

4. The proof of Proposition 1

Let G be a smooth elliptic curve and fix a point $P_0 \in G$ on G. This defines a group structure on (G, P_0) with P_0 as its neutral element [2, p. 321]. In this section it is more convenient to use the additive notation $\oplus$ for the group structure: $P \oplus Q = R$ if and only if $P + Q = R + P_0$ in $Div(G)$.

We have to prove that $a(f_1 \oplus f_2, \mathcal{F}) = a(f_1, \mathcal{F}) \oplus a(f_2, \mathcal{F})$.

By similar arguments as in section 3, we can assume that $Y \subset G \times G$ is a projective irreducible curve, $\mathcal{F} = \mathbf{C}_Y$, and f_1 (respectively f_2) is induced by the first (respectively, the second) projection $G \times G \to G$. Let $\pi : S \to Y$ be the normalization map of Y, and K_S, respectively K_G the canonical divisor of S, respectively of G.

Let $f : Y \to G$ be a finite morphism. Since for two arbitrary points Q_1 and Q_2 one has $Q_1 \sim Q_2$ in $Div(G)$ if and only if $Q_1 = Q_2$, the invariant $a(f) = \bigoplus_{t \in C(f)} \Delta\chi(t) \cdot t$ is determined by the bifurcation divisor $B_f := \sum_{t \in C(f)} \Delta\chi(t) \cdot t \in Div(G)$: $a(f)$ is the unique point in G with the property

$$a(f) \sim B_f - (\deg B_f - 1)P_0 \quad \text{in } Div(G). \tag{1}$$

Let $Sing(Y) = \{P_1, \dots, P_s\}$ be the (reduced) singular locus of Y, and let $(r_i + 1)$ be the number of local irreducible components of Y at P_i $(i = 1, \dots, s)$. Define the divisor $S(Y) \in Div(Y)$ by $\sum_{i=1}^{s} r_i P_i$. Then, if $\tilde{f} = f \circ \pi$ then, by [2, p. 301], the ramification divisor $R_{\tilde{f}}$ of $\tilde{f}$ is $R_{\tilde{f}} \sim K_S - \tilde{f}^* K_G \sim K_S$ (because $K_G \sim 0$).

Consider the morphism $\tilde{f}_* : Div(S) \to Div(G)$ defined by

$$\tilde{f}_* \left(\sum n_i Q_i \right) = \sum n_i \tilde{f}(Q_i),$$

and similarly

$$f_*(S(Y)) := \sum_{i=1}^{s} r_i f(P_i).$$

Then

$$B_f \sim \tilde{f}_*(R_{\tilde{f}}) + f_*(S(Y)) \sim \tilde{f}_*(K_S) + f_*(S(Y)) \text{ in } Div(G).$$

By a calculation $B_{f_1 \oplus f_2} \sim B_{f_1} + B_{f_2} - (\deg K_S + \sum_{i=1}^{s} r_i)P_0$, and $\deg B_{f_1 \oplus f_2} = \deg B_{f_1} = \deg B_{f_2} = \deg K_S + \sum_{i=1}^{s} r_i$. Now, from (1), one has $a(f_2 \oplus f_2) + P_0 \sim a(f_1) + a(f_2)$; hence $a(f_1 \oplus f_2) = a(f_1) \oplus a(f_2)$.

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