

ELLIPTIC THREE-FOLDS II: MULTIPLE FIBRES

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ABSTRACT. Let $f : X \rightarrow S$ be an elliptic fibration with a section, where S is a projective surface and X is a projective threefold. We determine when it is possible to perform a logarithmic transformation along a closed subset $Z \subseteq S$ to obtain a new elliptic fibration $f' : X' \rightarrow S$ which now has multiple fibres along Z . This is done in the setting of Ogg-Shafarevich theory. We find a number of obstructions to performing such a logarithmic transformation, the very last of which takes values in the torsion part of the codimension 2 Chow group of X .

0. INTRODUCTION

Suppose that we would like to classify algebraic elliptic three-folds, i.e. algebraic three-folds X with a fibration $f : X \rightarrow S$ whose general fibre is an elliptic curve. In analogy with the classification of elliptic surfaces, there are three steps:

- (1) Classify elliptic fibrations with section. These can all be obtained birationally using Weierstrass models.
- (2) Classify elliptic fibrations $f' : X \rightarrow S$ without multiple fibres which have a given jacobian fibration $f : J \rightarrow S$. This has already been carried out in [8]; the set of such elliptic fibrations is parametrized up to birational equivalence by the Tate-Shafarevich group $\text{III}_S(A)$, where A is the generic fibre of f .
- (3) Understand how to perform logarithmic transformations. In the case of complex elliptic surfaces, a logarithmic transformation is an operation for producing multiple fibres over points in the base curve. It is a purely local operation, which can then be patched to give a new complex elliptic surface, which may or may not be algebraic. However, it is well understood when the resulting surface will be algebraic. In the case of threefolds, this is a considerably more complicated task. Now one should expect to perform logarithmic transformations along curves in the base surface, and so when dealing with a global base, one cannot expect a purely local description for a logarithmic transformation.

It is step 3 which this paper carries out. Instead of the local analytic description of logarithmic transformations given by Kodaira in [14] and extended to higher dimensions by N. Nakayama in [19], we take the algebraic approach of Ogg-Shafarevich theory. (See [20] and [21], and also [4].) In this point of view,

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we begin with a jacobian elliptic fibration $f : J \rightarrow S$ with generic fibre A , and we compute $\text{III}_{S-Z}(A)$, which is essentially the set of elliptic fibrations with multiple fibres only over Z .

More explicitly, as explained in §1, we can always assume J and S are non-singular, and $f : J \rightarrow S$ is flat with discriminant locus Σ such that the one dimensional component of $Z \cup \Sigma_{red}$ is a simple normal crossings (s.n.c.) divisor. This can be accomplished using [16] by blowing up the base, if necessary. If A is the generic fibre of f , we can think of A as an étale sheaf on η . If $i : \eta \rightarrow S$ is the inclusion of the generic point of η in S , then $\text{III}_S(A) = H^1(S, i_*A)$ (étale cohomology) is the Tate-Shafarevich group which essentially classifies, in this case, fibrations over S with Jacobian $J \rightarrow S$ which have no multiple fibres. More precisely, it was shown in [8] that $\text{III}_S(A)$ is the set of isomorphism classes E of curves of genus one over η with jacobian A , such that there exists a model $f' : X \rightarrow S$ with f' relatively minimal in the sense of Mori with $X_\eta = E$ and $f' : X \rightarrow S$ having only so-called locally trivial multiple fibres, i.e. multiple fibres for which there exists locally a rational section of the fibration. Such multiple fibres are necessarily isolated, and can only occur at very special places; for the most part, we ignore them. Thus, what we wish to calculate is the quotient

$$\text{III}_{S-Z}(A) / \text{III}_S(A).$$

This will tell us which new fibrations can occur when we allow multiple fibres over Z . To do this, we use the exact sequence of local cohomology:

$$0 \rightarrow H^1(S, i_*A) \rightarrow H^1(S - Z, i_*A) \rightarrow H_Z^2(S, i_*A) \rightarrow H^2(S, i_*A).$$

We first calculate $H_Z^2(S, i_*A)$ and obtain a group of possible global invariants. The obstruction to realising these global invariants lives in $H^2(S, i_*A)$. For example, if Z is a smooth curve such that $f^{-1}(Z)$ is also smooth, then $H_Z^2(S, i_*A)$ is the torsion part of the Mordell-Weil group of the elliptic fibration $f^{-1}(Z) \rightarrow Z$. The situation becomes more complicated when Z contains components of Σ , but in this case it is essentially a local calculation. We complete these calculations in §2.

The next problem is the obstruction in $H^2(S, i_*A)$. Without going into details now, there is an exact sequence

$$0 \rightarrow G_1 \rightarrow H^2(S, i_*A) \rightarrow G_2$$

for certain groups G_1 and G_2 which can be calculated. The obstruction in G_2 , which we call the first obstruction, is essentially discrete and can be completely calculated. Intuitively, this obstruction detects the following problem. If, say, Z is a smooth curve which intersects some component Σ only once, say a component Σ_1 for which $f^{-1}(\Sigma_1)$ consists of several components, we would expect that having multiple fibres along Z would force us to rearrange those components in such a way that there is now some ramification in the map of these components to Σ_1 possibly violating Hurwitz's formula. This is not a precise statement, but gives the flavor of the first obstruction.

The obstruction in G_1 is more subtle. Given an element of $H_Z^2(S, i_*A)$ whose first obstruction is zero, one can construct a torsion codimension 2 algebraic cycle on X corresponding to it. We show, using the Bloch map for torsion algebraic cycles, that this element of $H_Z^2(S, i_*A)$ can be realised only if the correspond-

ing torsion cycle is rationally equivalent to zero, modulo a set of cycles contained in fibres of f . In fact, this obstruction in a sense has already appeared in the case that S is a curve, except that in that case, the obstruction turns out to be always zero unless $J \rightarrow S$ is trivial, and thus any logarithmic transformation on a non-trivial fibration is algebraic. This is not the case once S is a surface.

The calculations are all performed using local cohomology calculations on J and S . In order to do this, we need very explicit descriptions of the geometry of J . For this, we rely heavily on the explicit resolutions of Weierstrass models given by Miranda in [16]. Without these explicit resolutions, there appears to be no way to obtain useful global information. One unpleasant side effect of the need to use such explicit resolutions is that some of the computations for I_M fibres, $M \geq 1$, become very long. We leave many of these details to an appendix.

I would like to stress that the final results can be applied without using étale cohomology, the information that one needs being completely geometric. To calculate $H_{\mathbb{Z}}^2(S, i_*A)$ one only needs to know the torsion part of certain Mordell-Weil groups of elliptic surfaces, as mentioned above, and a description of the discriminant locus of the elliptic fibration in question. To calculate the first obstruction one needs no additional information. The last obstruction requires information about rational equivalence on J . In many practical applications of these results, one can determine everything one needs to know. Examples are given throughout the text, and the reader can consult [12] for further examples.

The paper is organized as follows. §1 reviews basic results on models of elliptic fibrations and the Tate-Shafarevich groups. It closes with a number of basic computations of local cohomology which will be used throughout the paper. §2 then computes $H_{\mathbb{Z}}^2(S, i_*A)$, and §§3 and 4 detail the first and second obstructions, respectively. The appendix contains details about type I_M fibres, $M \geq 1$.

There are two works which have considered this problem in more restricted circumstances. [9] considers the problem of logarithmic transformations if the elliptic fibration is the product of the base with an elliptic curve, but also constructs non-algebraic examples. As already mentioned, [19] considers these questions locally, in all dimensions. Nakayama manages to treat all dimensions by relying on monodromy arguments rather than by using a model for the original elliptic fibration as we do. As a result, his methods do not extend to the global situation. We reproduce some of his results in dimension 3 using our own techniques to save some effort in the global calculations.

Finally, I would like to mention one important application of these results to the classification of three-folds. In [12], the results of this paper play a crucial role in giving a theoretical classification of elliptic Calabi-Yau manifolds. There all aspects of the current work are used in order to show, up to birational equivalence, that there are only a finite number of families of elliptic Calabi-Yau three-folds. Since this paper has been revised since [12] has appeared in print, I would like to note how some references to this paper have changed. References to §4 should now be references to §3, and references to §5 should now be references to §4. The reference to Theorem 4.5 should now be to Theorem 3.4, and the reference to Example 5.2 should now be to Theorem 4.3 or Example 4.5.

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1. PRELIMINARIES

We adopt the following notation:

S : a normal integral excellent scheme. In this paper S will always be a variety over an algebraically closed field k of characteristic 0, or an open subscheme of the spectrum of the strict henselization of a local ring of such a variety.

$S^{(n)}$: the set of points in S of codimension n , i.e. such that $\dim \mathcal{O}_{S,s} = n$.

η : the generic point of S .

$i : \eta \rightarrow S$: the natural inclusion morphism.

$K = K(\eta)$: the field of rational functions on S .

A : a fixed abelian variety of dimension 1 over K .

$\mathcal{O}_{S,\bar{s}}$: the strict henselization of the local ring $\mathcal{O}_{S,s}$.

$K_{\bar{s}}$: the field of fractions of $\mathcal{O}_{S,\bar{s}}$.

$\eta_{\bar{s}}$: $\text{Spec } K_{\bar{s}}$

$A_{\bar{s}} = A \times_{\eta} \eta_{\bar{s}}$.

$k(s)$: the residue field of a point $s \in S$.

${}_n G$: the elements of an abelian group G killed by multiplication by n .

All cohomology is étale unless otherwise stated.

We first recall some results from [8] on Ogg-Shafarevich theory for higher dimensional varieties. See [8] for details. We recall that given an elliptic curve A over a field K (in our case the function field of S), we put $WC(A) = H^1(\text{Spec } K, A)$, which can be identified with the set of isomorphism classes of curves of genus one over K which have A as their jacobian. Here we think of A as defining an étale sheaf over $\text{Spec } K$.

If $s \in S$, there is a natural localization map

$$WC(A) \rightarrow WC(A_{\bar{s}})$$

given by $E \mapsto E \times_{\eta} \eta_{\bar{s}}$, and we define $\text{III}_S(A) = \ker(WC(A) \rightarrow \prod_{s \in S} WC(A_{\bar{s}}))$. The cohomological interpretation is $\text{III}_S(A) = H^1(S, i_* A)$. We recall how to calculate $\text{III}_S(A)$ given the following hypothesis:

Hypothesis. *There exists a flat proper fibration $f : X \rightarrow S$ with generic fibre equal to A , with X and S non-singular, and f has a regular section $\sigma_0 : S \rightarrow X$.*

Then we have

Theorem 1.1. *Let $P_{X/S} = R^1 f_* \mathbf{G}_m$. Then there is an exact sequence*

$$(1.1) \quad 0 \rightarrow \mathcal{E} \rightarrow P_{X/S} \rightarrow i_* i^* P_{X/S} \rightarrow 0$$

such that \mathcal{E} has support contained in the set

$$\{s \in S \mid X_{\bar{s}} \text{ is reducible}\}.$$

Furthermore,

$$H^i(S, P_{X/S}) = H^{i+1}(X, \mathbf{G}_m) / H^{i+1}(S, \mathbf{G}_m),$$

$$H^1(S, \mathcal{E}) = 0, \quad H^1(S, i_* i^* P_{X/S}) = H^1(S, i_* A),$$

and there is an exact sequence

$$0 \rightarrow H^2(S, i_* A) \rightarrow H^2(S, i_* i^* P_{X/S}) \rightarrow H^1(S, \mathbf{Q}/\mathbf{Z})$$

coming from the exact sequence

$$(1.2) \quad 0 \rightarrow i_* A \rightarrow i_* i^* P_{X/S} \rightarrow \mathbf{Z} \rightarrow 0.$$

This yields an exact sequence

$$0 \rightarrow \frac{H^2(X, \mathbf{G}_m)}{H^2(S, \mathbf{G}_m)} \rightarrow \text{III}_S(A) \rightarrow H^2(S, \mathcal{E}) \rightarrow \frac{H^3(X, \mathbf{G}_m)}{H^3(S, \mathbf{G}_m)} \rightarrow H^2(S, i_* i^* P_{X/S}) \rightarrow H^3(S, \mathcal{E}).$$

Furthermore, if each reduced component of the discriminant locus Σ of f is normal, then

$$H^i(S, \mathcal{E}) = \prod_{t \in S^{(1)}} \text{coker}(H^{i-1}(C(t), \mathbf{Q}/\mathbf{Z}) \rightarrow H^{i-1}(\bar{C}(t), \mathbf{Q}/\mathbf{Z})), \quad i \geq 2.$$

Here, $C(t) = \{t\}^-$. To define $\bar{C}(t)$, let X_t^i be the components of the fibre X_t , \tilde{X}_t^i the normalization of X_t^i , and $\tilde{X}_t^i \rightarrow t_i \rightarrow t$ the Stein factorization. Let $C(t_i)$ be the normalization of $C(t)$ in t_i , and set $\bar{C}(t)$ to be the disjoint union of the $C(t_i)$.

Proof. [8] , §1. •

We recall some definitions from [8] :

Definition. A projective morphism $f : X \rightarrow S$ is called an *elliptic fibration* if its generic fibre is a smooth elliptic curve and all fibres are geometrically connected. The closed subset

$$\Sigma = \{s \in S \mid X_s \text{ is not regular}\}$$

is called the *discriminant locus*. If $t \in S^{(1)}$, the *fibre type* of t is the Kodaira fibre type (${}_m I_n, I_n^*, II, II^*, III, III^*, IV, IV^*$) of the central fibre of a relative minimal model (or Néron model) of $X(\bar{t}) = X \times_S \text{Spec } \mathcal{O}_{S, \bar{t}}$. (See [1] , page 150, for a description of these singular fibres.) A *collision* is a singular point of Σ . The closed subset

$$\Sigma^m = \{s \in S \mid f \text{ is not smooth for any } x \in f^{-1}(s)\}$$

is called the *multiple locus* of f . A fibre over a point $s \in \Sigma^m$ is called *multiple*. A fibre is called an *isolated multiple fibre* if it is over a zero-dimensional component of Σ^m . A *section* (resp. a *birational section*) of f is a closed subscheme Y of X for which the restriction of f to Y is an isomorphism (resp. a birational isomorphism).

Next we review the definition of a Weierstrass model. See [7], [17], [18] for details. Let \mathcal{L} be a line bundle on a scheme S , $a \in H^0(S, \mathcal{L}^{\otimes 4})$, and $b \in H^0(S, \mathcal{L}^{\otimes 6})$ such that $4a^3 + 27b^2$ is a non-zero section of $\mathcal{L}^{\otimes 12}$. Let $\mathbf{P} = \mathbf{P}(\mathcal{O}_S \oplus \mathcal{L}^{\otimes -2} \oplus \mathcal{L}^{\otimes -3})$, $\pi : \mathbf{P} \rightarrow S$ the natural projection, and $\mathcal{O}_{\mathbf{P}}(1)$ be the tautological line bundle on \mathbf{P} . We define the scheme $W(\mathcal{L}, a, b)$ as a closed subscheme of \mathbf{P} given by the equation $Y^2Z = X^3 + aXZ^2 + bZ^3$ where X, Y and Z are given by the sections of $\mathcal{O}_{\mathbf{P}}(1) \otimes \mathcal{L}^{\otimes 2}$, $\mathcal{O}_{\mathbf{P}}(1) \otimes \mathcal{L}^{\otimes 3}$, and $\mathcal{O}_{\mathbf{P}}(1)$ which correspond to the natural injections of $\mathcal{L}^{\otimes -2}$, $\mathcal{L}^{\otimes -3}$ and \mathcal{O}_S into $\pi_* \mathcal{O}_{\mathbf{P}}(1) = \mathcal{O}_S \oplus \mathcal{L}^{\otimes -2} \oplus \mathcal{L}^{\otimes -3}$, respectively.

The structure morphism $W(\mathcal{L}, a, b) \rightarrow S$ is a flat elliptic fibration, called a *Weierstrass fibration*. It has a section $\sigma_0 : S \rightarrow W(\mathcal{L}, a, b)$ defined by the S -point $(X, Y, Z) = (0, 1, 0)$. It is easy to see that $\sigma_0(S)$ lies in the smooth locus of $W(\mathcal{L}, a, b)$. We will call this section the *section at infinity*. Its conormal bundle is isomorphic to \mathcal{L} .

The discriminant locus of $W(\mathcal{L}, a, b) \rightarrow S$ is equal to the support of the Cartier divisor defined by the section Σ of $\mathcal{L}^{\otimes 12}$ given by $4a^3 + 27b^2$. This gives the discriminant locus a scheme structure. One can define the J -invariant J of $W(\mathcal{L}, a, b)$ as a section of \mathbf{P}^1_S given by $J(s) = 4a^3/(4a^3 + 27b^2)$. If a and b do not have any common zeroes, then J defines a morphism $J : S \rightarrow \mathbf{P}^1$. This is true in particular if the reduced discriminant locus Σ_{red} is s.n.c.

For an interpretation of $\text{III}_S(A)$, we restrict to $\dim X = 3$, $\text{char } k = 0$, in which case we restrict to a special sort of model, a Miranda model. This is obtained by blowing up the base S until only certain sorts of collisions occur, in which case [16] gives an explicit flat resolution of the Weierstrass model.

Definition. A *Miranda elliptic fibration* is an elliptic fibration $f : X \rightarrow S$ such that

- a) X and S are regular and f is flat and has a regular section;
- b) the discriminant locus Σ_{red} has simple normal crossing;
- c) All collisions are of type $I_{M_1} + I_{M_2}$, $I_{M_1} + I_{M_2}^*$, $II + IV$, $II + I_0^*$, $II + IV^*$, $IV + I_0^*$ or $III + I_0^*$.

Theorem 1.2. *Let $f : X \rightarrow S$ be a Miranda model, and $E \in \text{III}_S(A)$. Then if $f' : X' \rightarrow S$ is a relatively minimal model (in the sense of Mori) with $X'_\eta = E$, then f' has no multiple fibres, except possibly over collisions of type $IV + I_0^*$.*

Proof. This is a part of Theorem 2.21 and Remark 2.22 of [8] . •

For our purposes, it will be useful to modify this model slightly. First, as noted in [8], we have to avoid collisions of type $I_{M_1} + I_{M_2}$ where both M_1 and M_2 are odd, as these may not have a small resolution in the category of schemes. This may be done by blowing up the base. We also note that Miranda's resolution is obtained by resolving the singularities of the Weierstrass model. In resolving a collision of type $I_{M_1} + I_{M_2}$, where one has fibre type I_{M_1} over one branch of Σ meeting another branch of Σ with fibre type I_{M_2} , one first blows up the singular curves over one branch, and then over the other branch, and one finds one has a non-singular model. The order, however, will affect the final result. We will assume that we resolve the curve with fibre type I_M , M even, before any curve of fibre type I_M , M odd. This means we have only one sort of resolution to worry about, and the details will be given in Table 1, at the end of the paper.

Another problem arises when M is odd. If $C \subseteq S$ is a curve of fibre type I_M , $M = 1$, then $f^{-1}(C)$ definitely is not normal; if $M > 1$, some component of $f^{-1}(C)$ may not be normal. Indeed, the Miranda model is constructed as a double cover, and a fibre of type I_M is a double cover of the dotted lines in Figure 1.1 if M is even, where the solid lines represent the branch locus; and in Figure 1.2 if M is odd (or in Figure 1.3 if $M = 1$). In the odd case, the last component may or may not split into two surfaces over C ; if it does not split, that component will not be normal, the singular locus being traced out by the point P . In any case, we construct a *modified Miranda model* by blowing up all curves traced out by the point P ; this will remove any singularities, and $f^{-1}(C)$ becomes a s.n.c. divisor. We do this over all curves C with I_M odd, so that we can treat the odd case uniformly. Table 1 gives information about these modified models. Notice the modified model is still flat, so we can still apply Theorem 1.1.

The above discussion also shows that there are two cases if C is a curve of type I_M : $f^{-1}(C)$ may split into M components, or into $\lfloor M/2 \rfloor + 1$ components (before

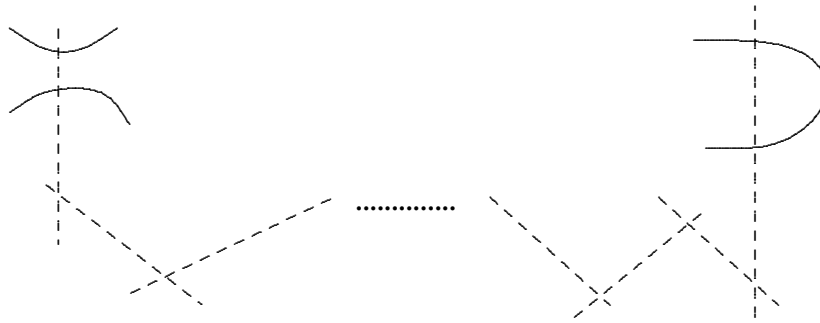


FIGURE 1.1

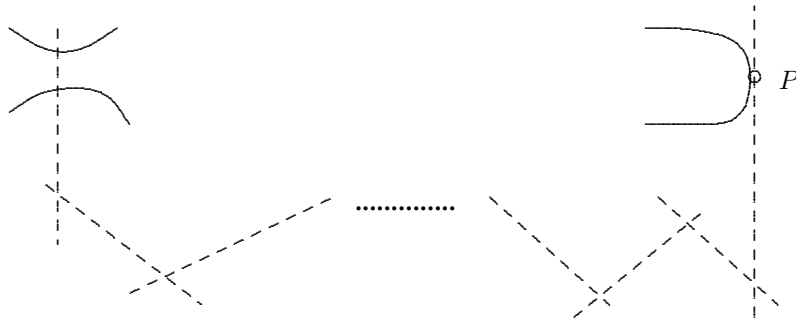


FIGURE 1.2



FIGURE 1.3

blowing up in the odd case). We refer to these as case *I* and case *I** respectively. After blowing up, we can describe these two cases as follows, putting

$$M' = \begin{cases} M & \text{if } M \text{ is even,} \\ M + 1 & \text{if } M \text{ is odd.} \end{cases}$$

Case I. $f^{-1}(C)$ splits into M' components, $Y_1, \dots, Y_{M'}$, each a ruled surface over C , Y_1 intersecting the zero-section of $f : X \rightarrow S$. In addition, $Y_1 \cap \text{Sing}(f^{-1}(C))$ splits into two curves.

Case I.* $f^{-1}(C)$ splits into $M'/2 + 1$ components, $Y_1, \dots, Y_{M'/2+1}$, with Y_1 and $Y_{M'/2+1}$ ruled surfaces over C and $Y_2, \dots, Y_{M'/2}$ ruled surfaces over a double cover

of C . In addition, $Y_1 \cap \text{Sing}(f^{-1}(C))$ is irreducible. (When $M = 1$ or 2 , this is the only way to distinguish between Case I and I^* . For fibre type I_1 , the only way to distinguish between these two cases is to perform the blowing up to arrive at the modified Miranda model. We shall see that these two cases present very different behavior when trying to make logarithmic transformations along C .)

Finally, we note that if the curve C intersects a curve of fibre type I_N^* for some N , then it is necessarily of case I^* , as can be seen from Miranda’s description of $I_M + I_N^*$ collisions ([16] or Table 2). Thus the notation I and I^* .

Thus we will adopt the following for the remainder of the paper:

Additional Hypothesis. $f : X \rightarrow S$ is a modified Miranda model, $\dim X = 3$, $\text{char } k = 0$, and $Z \subseteq S$ is a closed set such that $Z \cup \Sigma_{red}$ is a simple normal crossing divisor plus a finite set of points.

Our goal is to calculate

$$\frac{\text{III}_{S-Z}(A)}{\text{III}_S(A)}.$$

Note that the assumption that $Z \cup \Sigma_{red}$ is s.n.c. is not problematic. Indeed, if it is not so, then we can blow up S to obtain $f' : X' \rightarrow S'$ such that $Z' \cup \Sigma'_{red}$ is s.n.c. where Z' is the total transform of Z . Then, assuming that we only blew up points in Z , we have $S' - Z' = S - Z$, and by [8], Theorem 2.21, $\text{III}_{S'}(A) = \text{III}_S(A)$. Thus we have not changed our final result.

We end this section with a few results on local cohomology.

Let X be an arbitrary scheme, and let Z be a closed subset of X , $U = X - Z$, $i : Z \rightarrow X$, $j : U \rightarrow X$ the inclusions. The functor $i^!$ which takes sheaves on X to sheaves on Z is defined by

$$i^! \mathcal{F} = \ker(i^* \mathcal{F} \xrightarrow{\phi} i^* j_* j^* \mathcal{F})$$

where ϕ is the natural functorial map. (See [15], pg. 76) Equivalently, $(i_* i^! \mathcal{F})(V) = \ker(\mathcal{F}(V) \rightarrow \mathcal{F}(V \times_X U))$. $H_Z^p(X, \mathcal{F})$ is the p th derived functor of $\mathcal{F} \mapsto \Gamma(Z, i^! \mathcal{F}) = \ker(\mathcal{F}(X) \rightarrow \mathcal{F}(U))$, and $\underline{H}_Z^p(X, \mathcal{F}) = (R^p i^!) \mathcal{F}$. One has the long exact sequence

$$\dots \rightarrow H_Z^p(X, \mathcal{F}) \rightarrow H^p(X, \mathcal{F}) \rightarrow H^p(U, \mathcal{F}) \rightarrow \dots$$

We will need information about the cohomology of the sheaf \mathbf{G}_m . Recall from [13] the following facts: $H^1(X, \mathbf{G}_m) = \text{Pic } X$, and $H^i(X, \mathbf{G}_m)$ is torsion for $i \geq 2$ if X is regular. From the Kummer sequence

$$0 \rightarrow \mu_n \rightarrow \mathbf{G}_m \rightarrow \mathbf{G}_m \rightarrow 0,$$

we see that we have

$$0 \rightarrow \text{Pic } X/n\text{Pic } X \rightarrow H^2(X, \mu_n) \rightarrow {}_n H^2(X, \mathbf{G}_m) \rightarrow 0$$

which, upon taking the direct limit over all n and choosing identifications $\mu_n \cong \mathbf{Z}/n\mathbf{Z}$ (assuming we are working over an algebraically closed ground field of characteristic 0) we obtain a sequence for X regular

$$(1.3) \quad 0 \rightarrow \text{Pic } X \otimes \mathbf{Q}/\mathbf{Z} \rightarrow H^2(X, \mathbf{Q}/\mathbf{Z}) \rightarrow H^2(X, \mathbf{G}_m) \rightarrow 0.$$

Similarly we have exact sequences

$$0 \rightarrow H^{i-1}(X, \mathbf{G}_m) \otimes \mathbf{Q}/\mathbf{Z} \rightarrow H^i(X, \mathbf{Q}/\mathbf{Z}) \rightarrow H^i(X, \mathbf{G}_m) \rightarrow 0$$

for $i \geq 3$, and since $H^{i-1}(X, \mathbf{G}_m)$ is torsion, this yields an isomorphism

$$(1.4) \quad H^i(X, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\cong} H^i(X, \mathbf{G}_m), \quad i \geq 3.$$

Proposition 1.3 (Purity of the Brauer group). *Let X be a regular excellent scheme over a field of characteristic 0, Z a closed regular subscheme of pure codimension d . If $d \geq 2$, then*

$$\underline{H}_Z^i(X, \mathbf{G}_m) = \begin{cases} 0 & \text{for } i \neq 2d; \\ (\mathbf{Q}/\mathbf{Z})_Z & \text{for } i = 2d. \end{cases}$$

If $d = 1$, then

$$\underline{H}_Z^i(X, \mathbf{G}_m) = 0, \quad i = 0, i \geq 2,$$

and

$$\underline{H}_Z^1(X, \mathbf{G}_m) = \bigoplus_i \mathbf{Z}_{Z_i}$$

where the Z_i are the codimension one components of Z .

Proof. [13], p. 134. •

We will need to calculate, given a closed set $Z \subseteq S$, the group $H_Z^2(S, P_{X/S})$ and the sheaf $\underline{H}_Z^2(S, P_{X/S})$ ($P_{X/S} = R^1 f_* \mathbf{G}_m$ for a map $f : X \rightarrow S$). We will show how to calculate these under the assumption that Z and $f^{-1}(Z)$ are s.n.c. divisors.

Proposition 1.4. *Given a map $f : X \rightarrow S$ satisfying the Hypothesis, and $Z \subseteq S$ any closed subset, then*

$$H_Z^2(S, P_{X/S}) \cong \frac{H_{f^{-1}(Z)}^3(X, \mathbf{G}_m)}{H_Z^3(S, \mathbf{G}_m)}.$$

Proof. There is a spectral sequence

$$H_Z^p(S, R^q f_* \mathcal{F}) \Rightarrow H_{f^{-1}(Z)}^{p+q}(X, \mathcal{F})$$

because

$$H_Z^0(S, f_* \mathcal{F}) = H_{f^{-1}(Z)}^0(X, \mathcal{F}).$$

Since $R^q f_* \mathbf{G}_m = 0 \forall q > 1$ by [8, 1.3], this spectral sequence yields the long exact sequence

$$\begin{aligned} \dots &\rightarrow H_Z^3(S, \mathbf{G}_m) \rightarrow H_{f^{-1}(Z)}^3(X, \mathbf{G}_m) \rightarrow H_Z^2(S, P_{X/S}) \\ &\rightarrow H_Z^4(S, \mathbf{G}_m) \rightarrow H_{f^{-1}(Z)}^4(X, \mathbf{G}_m) \rightarrow \dots \end{aligned}$$

Now, as $f : X \rightarrow S$ has a section $\sigma : S \rightarrow X$, one has a composition of natural maps

$$H_Z^i(S, \mathbf{G}_m) \xrightarrow{f^*} H_{f^{-1}(Z)}^i(X, \mathbf{G}_m) \xrightarrow{\sigma^*} H_Z^i(S, \mathbf{G}_m)$$

which must be an isomorphism; thus f^* is injective for all i and one obtains the desired result. •

Proposition 1.5. *Let X be a regular excellent scheme over an algebraically closed field of characteristic 0, Z a s.n.c. divisor with components Z_1, \dots, Z_n . Then there is an exact sequence*

$$0 \rightarrow \bigoplus_i H^1(Z_i, \mathbf{Q}/\mathbf{Z}) \rightarrow H_Z^3(X, \mathbf{G}_m) \rightarrow \bigoplus_{i < j} H^0(Z_i \cap Z_j, \mathbf{Q}/\mathbf{Z}) \xrightarrow{d} \bigoplus_k H^2(Z_k, \mathbf{Q}/\mathbf{Z}) \rightarrow H_Z^4(X, \mathbf{G}_m)$$

where the map d is defined as follows: let $d_{ijk} : H^0(Z_i \cap Z_j, \mathbf{Q}/\mathbf{Z}) \rightarrow H^2(Z_k, \mathbf{Q}/\mathbf{Z})$ be the natural Gysin map, with $k = i$ or j . Then $(d)_{ij} = d_{iji} - d_{ijj}$.

Proof. If $i : Z \rightarrow X$ is a closed immersion, define $\mathbf{Z}_Z = i_* i^* \mathbf{Z}$. If $Z \subseteq Z' \subseteq X$, then there is a natural restriction map $\rho_{Z', Z} : \mathbf{Z}_{Z'} \rightarrow \mathbf{Z}_Z$. Now, if Z is s.n.c, there is a natural resolution

$$(1.5) \quad 0 \rightarrow \mathbf{Z}_Z \rightarrow A_0 \rightarrow A_{-1} \rightarrow A_{-2} \rightarrow \dots$$

where

$$A_{-n} = \bigoplus_{i_0 < \dots < i_n} \mathbf{Z}_{Z_{i_0 \dots i_n}},$$

with $Z_{i_0 \dots i_n} = Z_{i_0} \cap \dots \cap Z_{i_n}$, and the map $d_{-n} : A_{-n} \rightarrow A_{-(n+1)}$ is defined by

$$(d_{-n}s)_{i_0 \dots i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j \rho_{Z_{i_0 \dots \hat{i}_j \dots i_{n+1}}, Z_{i_0 \dots i_n}} (s_{i_0 \dots \hat{i}_j \dots i_{n+1}}).$$

Exactness of (1.5) can be checked on stalks. One has $H_0(A.) = \mathbf{Z}_Z, H_i(A.) = 0, i \neq 0$. Now if \mathcal{F} is a sheaf on X , then

$$\text{Ext}^i(\mathbf{Z}_Z, \mathcal{F}) = H_Z^i(X, \mathcal{F}).$$

Let

$$0 \rightarrow \mathcal{F} \rightarrow I$$

be an injective resolution for \mathcal{F} , and consider the bicomplex $M^{\cdot\cdot} = \text{Hom}(A., I)$, with differentiation maps $d_I : \text{Hom}(A_p, I^q) \rightarrow \text{Hom}(A_{p+1}, I^q)$ and $d_{II} : \text{Hom}(A_p, I^q) \rightarrow \text{Hom}(A_p, I^{q+1})$. One thus obtains two spectral sequences

$$\begin{aligned} {}_I E_1^{p,q} &= \text{Ext}^q(A_p, \mathcal{F}) \Rightarrow H^{p+q}(\text{Tot}(M^{\cdot\cdot})) \\ {}_{II} E_1^{p,q} &= \text{Hom}(H_q(A.), I^p) \Rightarrow H^{p+q}(\text{Tot}(M^{\cdot\cdot})) \end{aligned}$$

From the second, we deduce that $H^p(\text{Tot}(M^{\cdot\cdot})) = \text{Ext}^p(\mathbf{Z}_Z, \mathcal{F}) = H_Z^p(X, \mathcal{F})$, so we obtain from the first a second quadrant spectral sequence

$$E_1^{p,q} = \bigoplus_{i_0 < \dots < i_{-p}} H_{Z_{i_0 \dots i_{-p}}}^q(X, \mathcal{F}) \Rightarrow H_Z^{p+q}(X, \mathcal{F}).$$

Calculating the $E_1^{p,q}$ terms in our situation, using Proposition 1.3 and the spectral sequence

$$E_2^{p,q} = H^p(Z, \underline{H}_Z^q(X, \mathbf{G}_m)) \Rightarrow H_Z^{p+q}(X, \mathbf{G}_m),$$

one can calculate all the relevant terms to obtain the desired exact sequence, and, keeping in mind that $d : \bigoplus_{i < j} H_{Z_{ij}}^4(X, \mathbf{G}_m) \rightarrow \bigoplus_k H_{Z_k}^4(X, \mathbf{G}_m)$ is induced by the map $d_0 : A_0 \rightarrow A_{-1}$, we obtain the description of the map d . •

Proposition 1.6. *Let $f : X \rightarrow S$ satisfy the Hypothesis. If $Z \subseteq S$ is a s.n.c. divisor with $f^{-1}(Z)$ a s.n.c. divisor in X , with $Z = Z_1 \cup \dots \cup Z_m$, $f^{-1}(Z_i) = Y_i^1 \cup \dots \cup Y_i^{n_i}$, and $k : f^{-1}(Z) \rightarrow Z$ the restriction of f , then we have a sheaf \mathcal{F} defined by*

$$0 \longrightarrow \bigoplus_{i < j} (\mathbf{Q}/\mathbf{Z})_{Z_i \cap Z_j} \xrightarrow{\alpha} \bigoplus_{(i,j) < (i',j')} k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j \cap Y_{i'}^{j'}} \longrightarrow \mathcal{F} \longrightarrow 0$$

giving

$$0 \longrightarrow \bigoplus_{i,j} R^1 k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j} \longrightarrow \underline{H}_Z^2(S, P_{X/S}) \longrightarrow \mathcal{F} \xrightarrow{d} \bigoplus_{i,j} R^2 k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j}$$

where $(i, j) < (i', j')$ if $i < i'$ or $i = i'$ and $j < j'$, and the map α , over a point P of $Z_i \cap Z_j$, can be identified with

$$\mathbf{Q}/\mathbf{Z} \rightarrow (\mathbf{Q}/\mathbf{Z})^c$$

where c is the number of irreducible components of the fibre over P , given by a $\imath \mapsto (m_i a)$ with m_i being the multiplicity of the i th component of the fibre. The map d is the relative version of the map d given in Proposition 1.5.

Proof. We have the diagram

$$\begin{array}{ccc} f^{-1}(Z) & \xrightarrow{l} & X \\ \downarrow k & & \downarrow f \\ Z & \xrightarrow{g} & S \end{array}$$

Note that $k_* l^! = g^! f_*$. Indeed, if \mathcal{F} is a sheaf on X , U étale over S , then

$$\begin{aligned} (g_* k_* l^! \mathcal{F})(U) &= (f_* l_* l^! \mathcal{F})(U) \\ &= (l_* l^! \mathcal{F})(U \times_S X) \\ &= \ker(\mathcal{F}(U \times_S X) \rightarrow \mathcal{F}(U \times_S X - U \times_S f^{-1}(Z))) \\ &= \ker(f_* \mathcal{F}(U) \rightarrow f_* \mathcal{F}(U - U \times_S Z)) \\ &= (g_* g^! f_* \mathcal{F})(U). \end{aligned}$$

Since g_* is fully faithful ([15, II 3.14 d]), the claim follows.

We have the spectral sequence

$$(R^p g^!)(R^q f_*) \mathbf{G}_m \Rightarrow R^{p+q}(g^! f_*) \mathbf{G}_m = R^{p+q}(k_* l^!) \mathbf{G}_m,$$

and since $f_* \mathbf{G}_m = \mathbf{G}_m$ and $R^q f_* \mathbf{G}_m = 0 \forall q > 1$ ([8], 1.4), the spectral sequence degenerates, giving, in part, the exact sequence

$$(1.6) \quad \underline{H}_Z^3(S, \mathbf{G}_m) \rightarrow R^3(g^! f_*) \mathbf{G}_m \rightarrow \underline{H}_Z^2(S, R^1 f_* \mathbf{G}_m) \rightarrow \underline{H}_Z^4(S, \mathbf{G}_m).$$

Now by the exact sequence (1.5) for Z and the fact that

$$\underline{\text{Ext}}^i(\mathbf{Z}_Z, \mathcal{F}) = g_* \underline{\text{Ext}}^i(\mathbf{Z}, g^! \mathcal{F}) = g_* \underline{H}_Z^i(S, \mathcal{F})$$

([15], pg. 242) we see that we have, by Proposition 1.3,

$$\underline{H}_Z^3(S, \mathbf{G}_m) = \bigoplus_{i < j} \underline{H}_{Z_i \cap Z_j}^4(S, \mathbf{G}_m) = \bigoplus_{i < j} (\mathbf{Q}/\mathbf{Z})_{Z_i \cap Z_j} \quad \text{and} \quad \underline{H}_Z^4(S, \mathbf{G}_m) = 0.$$

Now

$$g_* R^q(g^! f_*) \mathbf{G}_m = R^q(g_* k_* l^!) \mathbf{G}_m = R^q(f_* l_* l^!) \mathbf{G}_m = R^q(f_* \underline{\text{Hom}}(\mathbf{Z}_{f^{-1}(Z)}, \cdot)) \mathbf{G}_m.$$

If $0 \rightarrow \mathbf{G}_m \rightarrow I'$ is an injective resolution of \mathbf{G}_m , then as in the proof of Proposition 1.5 we obtain the bicomplex $M'' = f_* \underline{\mathbf{Hom}}(A, I')$ and thus obtain a spectral sequence

$$E_1^{p,q} = \bigoplus_{i_0 < \dots < i_{-p}} R^q(f_* \underline{\mathbf{Hom}}(\mathbf{Z}_{Y_{i_0 \dots i_{-p}}}, \cdot)) \mathbf{G}_m = \bigoplus_{i_0 < \dots < i_{-p}} g_* R^q(k_* l_{i_0 \dots i_{-p}}^!) \mathbf{G}_m \Rightarrow g_* R^{p+q}(k_* l^!) \mathbf{G}_m$$

where $l_{i_0 \dots i_n} : Y_{i_0 \dots i_n} \rightarrow X$ is the inclusion, $Y_{i_0 \dots i_n}$ denoting the intersection of $n+1$ components of $f^{-1}(Z)$. Now one proceeds as before, using the spectral sequence

$$(R^p k_*)(R^q l_{i_0 \dots i_n}^!) \mathbf{G}_m \Rightarrow R^{p+q}(k_* l_{i_0 \dots i_n}^!) \mathbf{G}_m$$

to compute the various terms. We see that

$$R^q(k_* l_{i_0}^!) \mathbf{G}_m = \begin{cases} 0, & q = 0, \\ (R^{q-1} k_*)(l_{i_0}^* \mathbf{Z}), & q \geq 1, \end{cases}$$

with $(R^{q-1} k_*)(l_{i_0}^* \mathbf{Z}) = (R^{q-2} k_*)(l_{i_0}^* \mathbf{Q}/\mathbf{Z})$ for $q \geq 3$, and

$$R^q(k_* l_{i_0 i_1}^!) \mathbf{G}_m = \begin{cases} 0, & q \leq 3, \\ (R^{q-4} k_*)(l_{i_0 i_1}^* \mathbf{Q}/\mathbf{Z}), & q \geq 4. \end{cases}$$

Dropping the g_* 's by [15], II Cor 3.14 d), one thus obtains an exact sequence

$$0 \rightarrow \bigoplus_{i,j} R^1 k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j} \rightarrow R^3(g^! f_*) \mathbf{G}_m \rightarrow \bigoplus_{(i,j) < (i',j')} k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j \cap Y_{i'}^{j'}} \rightarrow \bigoplus_{i,j} R^2 k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j}.$$

Take α to be the composed map

$$\bigoplus_{i < j} (\mathbf{Q}/\mathbf{Z})_{Z_i \cap Z_j} = \underline{H}_Z^3(S, \mathbf{G}_m) \rightarrow R^3(g^! f_*) \mathbf{G}_m \rightarrow \bigoplus_{(i,j) < (i',j')} k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j \cap Y_{i'}^{j'}}$$

where the first arrow is from (1.6). This map α takes the desired form, and is then injective since $f : X \rightarrow S$ has a section. This yields a diagram of exact sequences

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \bigoplus_{i < j} (\mathbf{Q}/\mathbf{Z})_{Z_i \cap Z_j} & \xrightarrow{=} & \bigoplus_{i < j} (\mathbf{Q}/\mathbf{Z})_{Z_i \cap Z_j} & & \\ & & \downarrow & & \downarrow \alpha & & \\ \bigoplus_{i,j} R^1 k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j} & \hookrightarrow & R^3(g^! f_*) \mathbf{G}_m & \rightarrow & \bigoplus_{(i,j) < (i',j')} k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j \cap Y_{i'}^{j'}} & \rightarrow & \bigoplus_{(i,j)} R^2 k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j} \\ \downarrow = & & \downarrow & & \downarrow & & \downarrow = \\ \bigoplus_{i,j} R^1 k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j} & \hookrightarrow & \underline{H}_Z^2(S, P_{X/S}) & \rightarrow & \mathcal{F} & \rightarrow & \bigoplus_{(i,j)} R^2 k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j} \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

This yields the desired sequences. •

2. THE CALCULATION OF $H_Z^2(S, i_*A)$

We continue to work in the setting of the Additional Hypothesis of §1.

Lemma 2.1. *Let $i' : \eta \rightarrow S - Z$ be the inclusion of the generic point in $S - Z$. Then*

$$(i_*A)|_{S-Z} = i'_*A,$$

$$P_{X/S}|_{S-Z} = P_{(X-f^{-1}(Z))/S-Z},$$

$$(i_*i^*P_{X/S})|_{S-Z} = i'_*i'^*(P_{X/S}|_{S-Z}).$$

In particular, $\text{III}_{S-Z}(A) = H^1(S - Z, i_*A)$ and we have an exact sequence

$$0 \rightarrow \text{III}_S(A) \rightarrow \text{III}_{S-Z}(A) \rightarrow H_Z^2(S, i_*A) \rightarrow H^2(S, i_*A).$$

Proof. Obvious. •

Proposition 2.2.

- a) $\underline{H}_Z^0(S, i_*A) = \underline{H}_Z^1(S, i_*A) = 0$, and $H_Z^2(S, i_*A) = H^0(\underline{H}_Z^2(S, i_*A))$.
- b) $\underline{H}_Z^i(S, i_*A) = \underline{H}_Z^i(S, i_*i^*P_{X/S})$ for $i \leq 2$, and in addition for $i = 3$ if Z is pure codimension 2.
- c) There is an exact sequence

$$\begin{aligned} 0 \rightarrow \underline{H}_Z^2(S, \mathcal{E}) \rightarrow \underline{H}_Z^2(S, P_{X/S}) \rightarrow \underline{H}_Z^2(S, i_*A) \\ \rightarrow \underline{H}_Z^3(S, \mathcal{E}) \rightarrow \underline{H}_Z^3(S, P_{X/S}) \rightarrow \underline{H}_Z^3(S, i_*i^*P_{X/S}) \end{aligned}$$

Proof. a) If $j : Z \rightarrow S$ is the inclusion, then $j_*\underline{H}_Z^i(S, i_*A)$ is the sheaf associated with the presheaf $U \mapsto H_{U \times_S Z}^i(U, (i_*A)|_U)$. Thus it is sufficient to show that $H_Z^i(S, i_*A) = 0$, $i = 0, 1$ for any Z and S , S irreducible. Indeed, $H^0(S, i_*A) = H^0(S - Z, i_*A) = A(K)$, and $\text{III}_S(A) \rightarrow \text{III}_{S-Z}(A)$ is necessarily an injection. This proves the claim. The last statement then follows from the spectral sequence

$$H^p(Z, \underline{H}_Z^q(S, i_*A)) \Rightarrow H_Z^{p+q}(S, i_*A).$$

b) This follows from the exact sequence (1.2). Taking the long exact sequence obtained by applying the functor $j^!$, and using the fact that $\underline{H}_Z^i(S, \mathbf{Z}) = \underline{H}_Z^{i-1}(S, \mathbf{Q}/\mathbf{Z}) = 0$ for $i - 1 < 2 \text{ codim}(Z)$ by purity, we obtain the desired isomorphisms.

c) This follows by applying $j^!$ to (1.1) and using a) and b). •

We will first calculate $\underline{H}_Z^2(S, i_*A)$ and $\underline{H}_Z^3(S, i_*A)$ when Z is codimension 2. In this case, these sheaves are supported on a finite set of closed points and thus are entirely determined by their stalks at these points. These stalks have already been calculated in [8].

Theorem 2.3. *Suppose $\bar{f} : \bar{X} \rightarrow \bar{S}$ is a flat proper morphism of regular schemes with a section with \bar{S} strictly local, of dimension 2, with closed point s , and generic fibre A an elliptic curve. Then there is an exact sequence, applying Theorem 1.1 to $f : X \rightarrow S$ where $S = \bar{S} - \{s\}$ and $X = \bar{X} \times_{\bar{S}} S$,*

$$0 \rightarrow \text{III}_S(A) \rightarrow H^2(S, \mathcal{E}) \rightarrow H^2(S, P_{X/S}) \rightarrow H^2(S, i_*A) \rightarrow 0.$$

The groups $H^2(S, \mathcal{E})$ and $H^2(S, P_{X/S})$ and the map between them are defined by the commutative diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & \bigoplus_t \mathbf{Q}/\mathbf{Z} & \rightarrow & \bigoplus_t (\mathbf{Q}/\mathbf{Z})^{c(t)} & \rightarrow & H^2(S, \mathcal{E}) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \mathbf{Q}/\mathbf{Z} & \rightarrow & (\mathbf{Q}/\mathbf{Z})^c & \rightarrow & H^2(S, P_{X/S}) \rightarrow 0
 \end{array}$$

where the sum is over all $t \in S^{(1)}$ such that X_t is not geometrically irreducible, $c(t)$ is the number of components of X_t , and c is the number of components of the fibre X_0 over the closed point of \bar{S} . The first arrow in the first row is given for a given t by

$$a \mapsto (m_i r_i a),$$

where m_i is the multiplicity of X_t^i and r_i is the ramification index of $C(t_i) \rightarrow C(t)$ at s . The first arrow in the second row is given by $a \mapsto (m_i a)$ where m_i is the multiplicity of the i th component of X_0 . The first vertical arrow is summation and the second takes X_t^i to the sum of components of the central fibre of $\bar{X}_t^i \rightarrow C(t_i)$ taken with the appropriate multiplicities, where \bar{X}_t^i is the normalization of the closure of X_t^i in \bar{X} .

Proof. See [8], Theorem 3.1. (One can reconstruct the proof efficiently by using Propositions 1.3, 1.4 and 3.1 of this paper.) •

Theorem 2.4. *Given the hypothesis of Theorem 2.3, assume furthermore that Σ consists of two components meeting transversally, of fibre type α and β respectively. Then the values of $H^1(S, i_* A) = H^2_s(\bar{S}, i_* A)$ and $H^2(S, i_* A) = H^3_s(\bar{S}, i_* A)$ are given by the following table, depending on α and β . This covers all possible collisions occurring in a Miranda model.*

α, β	$H^1(S, i_* A)$	$H^2(S, i_* A)$
$I_{M_1}, I_{M_2}, M_1, M_2 \geq 1$ (not both M_1 and M_2 odd)	0	\mathbf{Q}/\mathbf{Z}
$I_{M_1}, I_{M_2}^*, M_1$ odd	0	0
$I_{M_1}, I_{M_2}^*, M_1 > 0$, even	$\mathbf{Z}/2\mathbf{Z}$	0
II, IV	0	0
II, I_0^*	0	0
II, IV^*	0	0
IV, I_0^*	0	0
III, I_0^*	$\mathbf{Z}/2\mathbf{Z}$	0

Proof. One calculates using Theorem 2.3 and our modification of the explicit resolution given by Miranda. In the cases $I_{M_1} + I_{M_2}$ and $I_{M_1} + I_{M_2}^*$ we give some details of the resolution, because they will be vital cases later. In Table 1, we give some details for $I_{M_1} + I_{M_2}$. The central fibre consists of a cycle of \mathbf{P}^1 's, $f_1, \dots, f_{M_1+M_2}$, $M_1' = M_1$ if M_1 is even and $M_1 + 1$ if M_1 is odd. Each Y_i or Y_j' specialises to a sum of these curves, with no multiplicities. This is given by the f_i 's listed in the second column of the table. The rest of the information in the table can be ignored for now. So, for example, we work out the calculation for the collision $I_2 + I_2$. We

can give the matrix of the map $(\mathbf{Q}/\mathbf{Z})^4 \rightarrow (\mathbf{Q}/\mathbf{Z})^4$ by

$$\begin{matrix} & Y_1 & Y_2 & Y'_1 & Y'_2 \\ \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

In other words, Y_1 specialises to $f_1 + f_3 + f_4$, Y_2 specialises to f_2 , etc. From this one obtains the desired results. The information for $I_{M_1} + I_{M_2}^*$ is presented in Table 2 in a similar fashion, except that now some of the central fibres occur with multiplicities in the specialisation. These are represented by coefficients in the entries in column 2. One such example has already been worked out in [8] .

We also calculate here the $I_0^* + III$ example. In Miranda’s resolution, if $\Sigma = \Sigma_1 \cup \Sigma_2$ with Σ_1 of fibre type I_0^* and Σ_2 of fibre type III , one has $f^{-1}(\Sigma_1) = Y_1 + Y_2 + 2Y_3 + Y_4$ where Y_1, Y_3 and Y_4 are ruled over Σ_1 and Y_2 is ruled over a double cover of Σ_1 branched at the closed point of \bar{S} . $f^{-1}(\Sigma_2) = Y'_1 + Y'_2$, both ruled surfaces over Σ_2 . The central fibre is

$$\bar{X}_m = f_1 + 2f_2 + 3f_3 + 2f_4 + f_5$$

with specialization matrix

$$\begin{matrix} & Y_1 & Y_2 & Y_3 & Y_4 & Y'_1 & Y'_2 \\ \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{matrix}$$

from which we see that we obtain

$$(2.1) \quad (0 \quad 1/2 \quad 0 \quad 1/2 \quad 0 \quad 1/2)$$

as a representative of the non-zero element of $H_s^2(\bar{S}, i_*A) \subseteq H_s^3(\bar{S}, \mathcal{E})$. •

Definition. A point $P \in S$ is *obstructed* if $\underline{H}_P^3(S, i_*A) \neq 0$.

Corollary 2.5. *If $X \rightarrow S$ is a modified Miranda model, then the obstructed points are precisely the points which are at collisions of type $I_{M_1} + I_{M_2}$.*

Proof. By Theorem 2.3, it is easy to see that if P is not a collision point, then $\underline{H}_P^2(S, i_*A) = \underline{H}_P^3(S, i_*A) = 0$. Otherwise, we refer to Theorem 2.4. •

We now proceed to the case that $\dim Z = 1$. The very simplest case proves to be virtually enough to complete the calculations, in conjunction with the above calculations.

Proposition 2.6. *Let $Z \subseteq S$ be a smooth curve with $Z \cap \Sigma = \emptyset$. Then*

$$\underline{H}_Z^2(S, i_*A) = R^1 k_* \mathbf{Q}/\mathbf{Z},$$

where $k : f^{-1}(Z) \rightarrow Z$ is the restriction of $f : X \rightarrow S$.

Proof. Because the support of \mathcal{E} is contained in Σ , the condition that $Z \cap \Sigma = \emptyset$ implies that $\underline{H}_Z^i(S, \mathcal{E}) = 0 \forall i$. Thus, by Proposition 2.2 c), $\underline{H}_Z^2(S, i_*A) \cong \underline{H}_Z^2(S, P_{X/S})$. The claim then follows immediately from Proposition 1.6. •

Recall that the Mordell-Weil group of a jacobian elliptic surface (i.e., an elliptic surface having a section) is the group of sections, or alternatively, the group of points of the generic fibre rational over the function field of the base. We shall always take the given section $\sigma_0 : S \rightarrow X$ of the Hypothesis as the zero section.

Proposition 2.7. *Let $k : Y \rightarrow C$ be a jacobian elliptic fibration with no reducible fibres, $\dim C = 1$. Then*

$$H^0(C, R^1k_*\mathbf{Q}/\mathbf{Z}) = MW(Y/C)_{tors},$$

the torsion part of the Mordell-Weil group of Y over C .

Proof. $R^1k_*\mathbf{G}_m$ is the relative Picard sheaf, the sheaf associated to the presheaf

$$U \mapsto \text{Pic}(U \times_C Y)/\text{Pic}(U).$$

Since $k : Y \rightarrow C$ has a section, this is in fact already a sheaf. By the Kummer sequence, we have

$$0 \rightarrow R^1k_*\mu_n \rightarrow R^1k_*\mathbf{G}_m \rightarrow R^1k_*\mathbf{G}_m,$$

where the last arrow is given by multiplication by n . Thus $R^1k_*\mu_n$ is the sheaf given by

$$U \mapsto {}_n(\text{Pic}(U \times_C Y)/\text{Pic}U) = {}_nMW(U \times_C Y/U),$$

the last equality since all fibres are irreducible. Finally, we have

$$R^1k_*\mathbf{Q}/\mathbf{Z} = \varinjlim R^1k_*\mu_n.$$

•

Proposition 2.8. *Suppose that $Z \subseteq S$ consists of a union of non-singular curves Z_1, \dots, Z_n such that no Z_i is contained in Σ and Z contains no obstructed points or points P with $H_P^2(S, i_*A) \neq 0$. Then*

$$H_Z^2(S, i_*A) = \bigoplus_{i=1}^n MW(f^{-1}(Z_i)/Z_i)_{tors}.$$

Proof. Let $Z' = (\text{sing}Z) \cup (Z \cap \Sigma)$. By the hypothesis, $H_{Z'}^2(S, i_*A) = H_{Z'}^3(S, i_*A) = 0$, and thus, by [15, III, 1.26], we have an exact sequence

$$0 = H_{Z'}^2(S, i_*A) \rightarrow H_Z^2(S, i_*A) \rightarrow H_{Z-Z'}^2(S - Z', i_*A) \rightarrow H_{Z'}^3(S, i_*A) = 0,$$

so

$$H_Z^2(S, i_*A) \cong H_{Z-Z'}^2(S - Z', i_*A).$$

The latter is calculated by Propositions 2.6 and 2.7:

$$\begin{aligned} H_{Z-Z'}^2(S - Z', i_*A) &= \bigoplus_{i=1}^n H^0(Z_i - Z', R^1k_{i*}\mathbf{Q}/\mathbf{Z}) \\ &= \bigoplus_{i=1}^n MW(f^{-1}(Z_i - Z')/Z_i - Z')_{tors} \\ &= \bigoplus_{i=1}^n MW(f^{-1}(Z_i)/Z_i)_{tors} \end{aligned}$$

where $k_i : f^{-1}(Z_i - Z') \rightarrow Z_i - Z'$ is the restriction. •

In other words, by removing unobstructed points from S , we do not change the calculation at all.

Since the $I_{M_1} + I_{M_2}$ collisions are obstructed, in order to understand what happens there, we need to do a local calculation at the collision point itself; we cannot hide the collisions as we did above. We note that the following local calculation agrees with that performed in [19] using different methods.

Proposition 2.9. *Let \bar{S} be strictly local of dimension 2, $f : \bar{X} \rightarrow \bar{S}$ a modified Miranda model, $\Sigma = \Sigma_1 \cup \Sigma_2$, Σ_1 and Σ_2 the two irreducible components of the discriminant locus, Σ_1 and Σ_2 meeting transversally with fibre type I_{M_1} and I_{M_2} respectively, $M_1, M_2 > 0$, and M_2 even. Then if $S = \bar{S} - \Sigma_1 \cup \Sigma_2$, $X = S \times_{\bar{S}} \bar{X}$, we have*

$$\text{III}_S(A) \subseteq \mathbf{Q}/\mathbf{Z} \oplus \mathbf{Q}/\mathbf{Z}$$

with

$$\text{III}_S(A) = \{(b, c) | b, c \in \mathbf{Q}/\mathbf{Z} \text{ and } M_2b = M_1c\}.$$

Proof. By Theorem 1.1, $\text{III}_S(A) = H^2(X, \mathbf{G}_m)/H^2(S, \mathbf{G}_m)$, and, since $H^i(\bar{X}, \mathbf{G}_m) = H^i(\bar{S}, \mathbf{G}_m) = 0, i \geq 2$, putting $Z = \Sigma_1 \cup \Sigma_2$,

$$H^2(X, \mathbf{G}_m)/H^2(S, \mathbf{G}_m) = H_{f^{-1}(Z)}^3(\bar{X}, \mathbf{G}_m)/H_Z^3(\bar{S}, \mathbf{G}_m).$$

Put

$$M'_1 := \begin{cases} M_1 & \text{if } M_1 \text{ is even,} \\ M_1 + 1 & \text{if } M_1 \text{ is odd.} \end{cases}$$

Now $f^{-1}(Z)$ is a s.n.c. divisor consisting of divisors $Y_1, \dots, Y_{M'_1}, Y'_1, \dots, Y'_{M_2}$, following the notation of Table 1. For notational convenience, set

$$Y_{M'_1+j} := Y'_j, \quad 1 \leq j \leq M_2.$$

Noting that the Y_i are ruled surfaces over a strictly local base, we have $H^1(Y_i, \mathbf{Q}/\mathbf{Z}) = 0, \forall i$, and thus by Prop. 1.5, we have an exact sequence

$$0 \rightarrow H_{f^{-1}(Z)}^3(\bar{X}, \mathbf{G}_m) \rightarrow \bigoplus_{i < j} H^0(Y_i \cap Y_j, \mathbf{Q}/\mathbf{Z}) \xrightarrow{d} \bigoplus_k H^2(Y_k, \mathbf{Q}/\mathbf{Z}).$$

Now, by proper base change, $H^2(Y_k, \mathbf{Q}/\mathbf{Z}) = H^2((Y_k)_0, \mathbf{Q}/\mathbf{Z}) = \bigoplus_{i=1}^{c_k} H^2(f_k^i, \mathbf{Q}/\mathbf{Z})$, where $(Y_k)_0$ is the central fibre and the $f_k^i, 1 \leq i \leq c_k$, its components. The Gysin map $H^0(Y_i \cap Y_j, \mathbf{Q}/\mathbf{Z}) \rightarrow H^2(Y_i, \mathbf{Q}/\mathbf{Z})$ then is obtained by $a[Y_i \cap Y_j] \rightarrow (a[Y_i \cap Y_j] \cdot f_i^k)$. These intersection numbers are given in the third column of Table 1. Following the notation there, $f_1, \dots, f_{M_1+M_2}$ are the components of the central fibre of $\bar{X} \rightarrow \bar{S}$, the C_i are the curves of intersection between the components of $f^{-1}(\Sigma_1)$, and the C'_j are the curves of intersection between the components of $f^{-1}(\Sigma_2)$. Column 2 of Table 1 shows those curves among these listed contained in each component of $f^{-1}(Z)$.

Let

$$\begin{aligned} (a_1, \dots, a_{M'_1+M_2}, b_1, \dots, b_{M'_1}, c_1, \dots, c_{M_2}) &\in \bigoplus_{i < j} H^0(Y_i \cap Y_j, \mathbf{Q}/\mathbf{Z}) \\ &= \bigoplus_{i=1}^{M'_1+M_2} H^0(f_i, \mathbf{Q}/\mathbf{Z}) \oplus \bigoplus_{i=1}^{M'_1} H^0(C_i, \mathbf{Q}/\mathbf{Z}) \oplus \bigoplus_{i=1}^{M_2} H^0(C'_i, \mathbf{Q}/\mathbf{Z}). \end{aligned}$$

The image under d will be an element of

$$\bigoplus_{i=1}^{M_1+M_2} H^2(Y_i, \mathbf{Q}/\mathbf{Z}) = \bigoplus_{i=1}^{M_1+M_2} H^2(f_i, \mathbf{Q}/\mathbf{Z}) \oplus \bigoplus_{i=1}^{M_1+M_2} H^2(f_i, \mathbf{Q}/\mathbf{Z}).$$

We first consider the case M_1 even. Setting the image of this element under d equal to zero, we obtain $2(M_1 + M_2)$ equations, using the intersection numbers in the third column of Table 1. The row labelled f_i in the following table gives the values of the image of (a_1, \dots, c_{M_2}) in $H^2(f_i, \mathbf{Q}/\mathbf{Z}) \subseteq \bigoplus_{i=1}^{M_1} H^2(Y_i, \mathbf{Q}/\mathbf{Z})$ and $\bigoplus_{i=1}^{M_2} H^2(Y'_i, \mathbf{Q}/\mathbf{Z})$ respectively. Since we want this image to be zero, we set each of these values to zero.

	In $\bigoplus_{i=1}^{M_1} H^2(Y_i, \mathbf{Q}/\mathbf{Z})$	In $\bigoplus_{i=1}^{M_2} H^2(Y'_i, \mathbf{Q}/\mathbf{Z})$
f_1	$b_1 - a_1 + a_{M_1+M_2} = 0$	$-c_{M_2/2} + a_1 - a_2 = 0$
f_2	$-b_1 + b_2 = 0$	$-a_1 + 2a_2 - a_3 = 0$
f_3	$-b_2 + b_3 = 0$	$-a_2 + 2a_3 - a_4 = 0$
\vdots	\vdots	\vdots
f_{M_1}	$-b_{M_1-1} - b_{M_1} = 0$	$-a_{M_1-1} + 2a_{M_1} - a_{M_1+1} = 0$
f_{M_1+1}	$b_{M_1} - a_{M_1+1} + a_{M_1+2} = 0$	$c_{M_2/2+1} - a_{M_1} + a_{M_1+1} = 0$
f_{M_1+2}	$a_{M_1+1} - 2a_{M_1+2} + a_{M_1+3} = 0$	$-c_{M_2/2+1} + c_{M_2/2+2} = 0$
\vdots	\vdots	\vdots
		$-c_{M_2-1} - c_{M_2} = 0$
		$c_{M_2} + c_1 = 0$
		$-c_1 + c_2 = 0$
\vdots	\vdots	\vdots
$f_{M_1+M_2}$	$a_1 + a_{M_1+M_2-1} - 2a_{M_1+M_2} = 0$	$-c_{M_2/2-1} + c_{M_2/2} = 0$

From this one sees that if a_1, b_1 and c_1 are chosen with arbitrary values, all other values are determined: by the right-hand column, one sees that

$$\begin{aligned} a_2 &= -c_1 + a_1 \\ a_3 &= -2c_1 + a_1 \\ &\vdots \\ a_{M_1+1} &= -M_1c_1 + a_1 \end{aligned}$$

and by the left-hand column,

$$\begin{aligned} a_{M_1+M_2} &= -b_1 + a_1 \\ a_{M_1+M_2-1} &= -2b_1 + a_1 \\ &\vdots \\ a_{M_1+1} &= -M_2b_1 + a_1, \end{aligned}$$

from which we see we require the additional relationship that

$$M_2b_1 = M_1c_1.$$

If this relationship is fulfilled, then all the equations are satisfied. Thus

$$H^2(X, \mathbf{G}_m) = \{(a, b, c) \mid a, b, c \in \mathbf{Q}/\mathbf{Z} \text{ and } M_2b = M_1c\}.$$

A similar but much simpler calculation shows that

$$H^2(S, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z},$$

and the map $H^2(S, \mathbf{G}_m) \rightarrow H^2(X, \mathbf{G}_m)$ is given by $a \mapsto (a, 0, 0)$. This proves the theorem in this case.

In the odd case, one gets a very similar table, and one finds that

$$\begin{aligned} a_2 &= -c_1 + a_1 \\ a_3 &= -2c_1 + a_1 \\ &\vdots \\ a_{(M_1+1)/2} &= -\frac{M_1-1}{2}c_1 + a_1 \\ a_{(M_1+3)/2} &= -M_1c_1 + 2a_1 \\ a_{(M_1+5)/2} &= -\frac{M_1+1}{2}c_1 + a_1 \\ &\vdots \\ a_{M_1+2} &= -M_1c_1 + a_1 \end{aligned}$$

(which reduces to

$$\begin{aligned} a_2 &= -c_1 + 2a_1 \\ a_3 &= -c_1 + a_1 \end{aligned}$$

if $M_1 = 1$.) and

$$\begin{aligned} a_{M_1+M_2+1} &= -b_1 + a_1 \\ &\vdots \\ a_{M_1+2} &= -M_2b_1 + a_1, \end{aligned}$$

and we obtain similar results. •

We now move to the global situation. The next case to consider is that Z is a non-singular curve contained in the discriminant locus Σ , and such that the fibre over a point of Z is of type I_N .

Theorem 2.10. *Let $Z \subseteq S$ be a smooth component of Σ of fibre type I_M which does not intersect any other component of Σ . Then if Z is case I (see §1), then*

$$H_Z^2(S, i_*A) = \mathbf{Q}/\mathbf{Z};$$

if Z is case I^* , then

$$H_Z^2(S, i_*A) = \mathbf{Z}/2\mathbf{Z}.$$

Proof. First note that $\underline{H}_Z^2(S, \mathcal{E}) = \underline{H}_Z^3(S, \mathcal{E}) = 0$, since Z intersects no other component of Σ . Thus $\underline{H}_Z^2(S, i_*A) = \underline{H}_Z^2(S, P_{X/S})$ by Prop. 2.2 c). Write $f^{-1}(Z)$ as a union of components $Y_1, \dots, Y_{M'}$ if we are in Case I and $Y_1, \dots, Y_{M'/2+1}$ if we are in Case I^* . ($M' = M$ if M is even, $M' = M + 1$ if M is odd.) Applying Prop. 1.6, one obtains an exact sequence

$$0 \rightarrow \underline{H}_Z^2(S, i_*A) \rightarrow \bigoplus_{i < j} k_*(\mathbf{Q}/\mathbf{Z})_{Y_i \cap Y_j} \rightarrow \bigoplus_k R^2k_*(\mathbf{Q}/\mathbf{Z})_{Y_i}.$$

Taking global sections gives us a sequence

$$0 \rightarrow H_{\mathbf{Z}}^2(S, i_*A) \rightarrow \bigoplus_{i < j} H^0(Y_i \cap Y_j, \mathbf{Q}/\mathbf{Z}) \xrightarrow{d} \bigoplus_i \mathbf{Q}/\mathbf{Z}.$$

In Case I , the matrix of d is given by

$$\begin{matrix} & C_1 & C_2 & \cdots & C_{M'} \\ \begin{matrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_{M'} \end{matrix} & \begin{pmatrix} 1 & 0 & \cdots & 1 \\ -1 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix} \end{matrix}$$

where $C_i = Y_i \cap Y_{i+1}$, $C_{M'} = Y_1 \cap Y_{M'}$. The kernel of this map is clearly \mathbf{Q}/\mathbf{Z} . In the case I^* , the matrix is

$$\begin{matrix} & C_1 & C_2 & \cdots & C_{M'/2} \\ \begin{matrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_{M'/2+1} \end{matrix} & \begin{pmatrix} 2 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \\ 0 & 0 & \cdots & -2 \end{pmatrix} \end{matrix}$$

where $C_i = Y_i \cap Y_{i+1}$. Here, the kernel is clearly $\mathbf{Z}/2\mathbf{Z}$. •

We can now put all of our calculations together:

Theorem 2.11. *Let $Z \subseteq S$ consist of a union of a finite number of isolated points P_1, \dots, P_{n_1} , a finite number of non-singular curves Z_1, \dots, Z_{n_2} not contained in Σ , intersecting Σ transversally, and a finite number of components of Σ , $Z_{n_2+1}, \dots, Z_{n_4}$ consisting of all components of Σ of fibre type I_M numbered so that Z_i is Case I for $n_2 + 1 \leq i \leq n_3$ and Case I^* for $n_3 + 1 \leq i \leq n_4$. Then*

$$H_{\mathbf{Z}}^2(S, i_*A) = \bigoplus_{i=1}^{n_1} H_{P_i}^2(S, i_*A) \oplus \bigoplus_{i=1}^{n_2} MW(f^{-1}(Z_i)/Z_i)_{tors} \oplus G,$$

where

1)

$$H_{P_i}^2(S, i_*A) = \begin{cases} \mathbf{Z}/2\mathbf{Z} & \text{if } P_i \text{ is a collision point of type } III + I_0^*, \\ 0 & \text{otherwise.} \end{cases}$$

2) There is an exact sequence

$$(2.2) \quad 0 \rightarrow (\mathbf{Z}/2\mathbf{Z})^{N^*} \rightarrow G \rightarrow (\mathbf{Q}/\mathbf{Z})^{n_3-n_2} \oplus (\frac{1}{2}\mathbf{Z}/\mathbf{Z})^{n_4-n_3} \xrightarrow{\alpha} (\mathbf{Q}/\mathbf{Z})^N,$$

where N^* is the number of collision points of type $I_a + I_b^*$ with $a > 0$, a even, and N is the number of collision points of type $I_a + I_b$. Denoting these latter collision points Q_1, \dots, Q_n , the map α is defined by

$$(\alpha(a_{n_2+1}, \dots, a_{n_4}))_i = a_j M_k - a_k M_j,$$

if Q_i is at the collision of Z_j and Z_k , of fibre type I_{M_j} and I_{M_k} respectively.

Proof. Let

$$Z' = \{Q_1, \dots, Q_N, Q_1^*, \dots, Q_{N'}^*\} \cup ((Z_1 \cup \dots \cup Z_{n_2}) \cap (Z_{n_2+1} \cup \dots \cup Z_{n_4})),$$

the Q_i^* being the collision points of type $I_a + I_b^*$. As in Proposition 2.8, we have the sequence

$$\begin{aligned} 0 = H_{Z-Z'}^1(S - Z', i_*A) &\rightarrow H_{Z'}^2(S, i_*A) \rightarrow H_Z^2(S, i_*A) \\ &\rightarrow H_{Z-Z'}^2(S - Z', i_*A) \rightarrow H_{Z'}^3(S, i_*A) \end{aligned}$$

which yields

$$\begin{aligned} 0 &\rightarrow (\mathbf{Z}/2\mathbf{Z})^{N^*} \rightarrow H_Z^2(S, i_*A) \\ &\rightarrow \bigoplus_{i=1}^{n_1} H_{P_i}^2(S, i_*A) \oplus \bigoplus_{i=1}^{n_2} MW(f^{-1}(Z_i)/Z_i)_{tors} \oplus (\mathbf{Q}/\mathbf{Z})^{n_3-n_2} \oplus (\frac{1}{2}\mathbf{Z}/\mathbf{Z})^{n_4-n_3} \\ &\hspace{20em} \rightarrow (\mathbf{Q}/\mathbf{Z})^N. \end{aligned}$$

The part $\bigoplus_{i=1}^{n_1} H_{P_i}^2(S, i_*A) \oplus \bigoplus_{i=1}^{n_2} MW(f^{-1}(Z_i)/Z_i)_{tors}$ clearly factors out of this sequence, leaving the exact sequence (2.2). Using Theorem 2.9, one sees that the map α can be written as described. •

Remark 2.12. Note that if $s \in S$, there is a natural map

$$H_Z^2(S, i_*A) \xrightarrow{\cong} H^0(S, \underline{H}_Z^2(S, i_*A)) \longrightarrow (\underline{H}_Z^2(S, i_*A))_{\bar{s}} \subseteq WC(A_{\bar{s}}).$$

If $\gamma \in H_Z^2(S, i_*A)$, and $t \in S^{(1)}$ is the generic point of a curve C , then *the invariant of γ along C* is the image of γ in $WC(A_{\bar{t}}) = (\mathbf{Q}/\mathbf{Z})^r$, $r = 2, 1$ or 0 depending on whether the geometric fibre $X_{\bar{t}}$ is non-singular, multiplicative (I_M , $M > 0$) or additive (any other singular Kodaira fibre-type).

In Proposition 2.11, given $\gamma \in H_Z^2(S, i_*A)$, the invariant of γ along Z_i is the image of γ under the natural projection $H_Z^2(S, i_*A) \rightarrow MW(f^{-1}(Z_i)/Z_i)_{tors} \subseteq (\mathbf{Q}/\mathbf{Z})^{\oplus 2}$ if $1 \leq i \leq n_2$, and the image of γ under the natural projection $H_Z^2(S, i_*A) \rightarrow G \rightarrow \mathbf{Q}/\mathbf{Z}$ if $n_2 + 1 \leq i \leq n_4$.

If γ maps to 0 in $H^2(S, i_*A)$, then there exists an element $E \in WC(A)$ which realises these invariants. If one finds a relatively minimal model $f' : X' \rightarrow S$ for E with $X'_\eta = E$, and if $t \in S^{(1)}$ is the generic point of a curve $C \subseteq S$, then $X'(\bar{t}) \rightarrow \text{Spec } \mathcal{O}_{S, \bar{t}}$ (here $X'(\bar{t}) = X' \times_S \text{Spec } \mathcal{O}_{S, \bar{t}}$) is relatively minimal, and hence the geometric fibre $X'_{\bar{t}}$ is of Kodaira fibre type ${}_m I_n$, where m is the order of the invariant of γ along C , with $X_{\bar{t}}$ being of fibre type I_n .

Finally, note that there is no need to consider $Z \subseteq S$ in Proposition 2.11 containing other components of Σ , not of fibre type I_m , as there would be no additional contribution to $H_Z^2(S, i_*A)$, as $WC(A_{\bar{t}}) = 0$ if $t \in S^{(1)}$ and $X_{\bar{t}}$ not of type I_m , $m \geq 0$.

Example 2.13. We shall introduce here four examples which we shall come back to several times later on.

1) Take any elliptic curve E , and put $X_1 = \mathbf{P}^2 \times E$, $f_1 : X_1 \rightarrow \mathbf{P}^2$ the first projection. If $Z = Z_1 \cup \dots \cup Z_n$ is a union of non-singular, irreducible curves, then $H_{Z_i}^2(\mathbf{P}^2, i_*A) = (\text{Pic } E)_{tors}$, $H_Z^2(\mathbf{P}^2, i_*A) = (\text{Pic } E)_{tors}^{\oplus n}$. Of course, $(\text{Pic } E)_{tors} = (\mathbf{Q}/\mathbf{Z})^{\oplus 2}$. (We do not actually need the Z_i to be non-singular.)

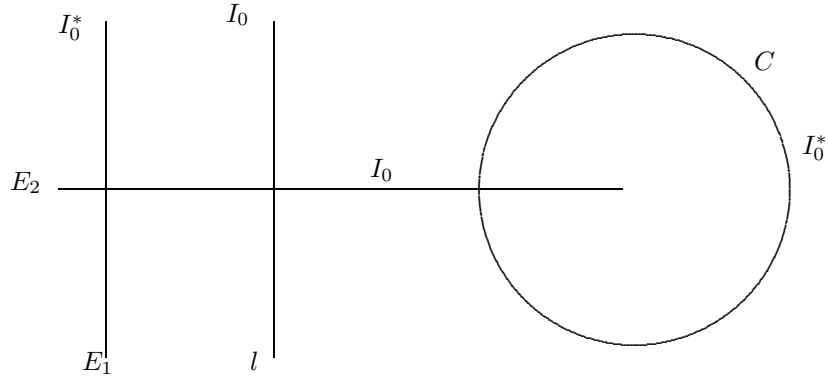


FIGURE 2.1

2) Let p_1 and p_2 be the projections of $\mathbf{P}^2 \times \mathbf{P}^1$ onto \mathbf{P}^2 and \mathbf{P}^1 respectively. Let $B \subseteq \mathbf{P}^2 \times \mathbf{P}^1$ be $p_1^{-1}(C) \cup p_2^{-1}\{P_1, P_2, P_3, P_4\}$ with $C \subseteq \mathbf{P}^2$ a smooth conic and $P_1, \dots, P_4 \in \mathbf{P}^1$ four distinct points. Consider the double cover of $\mathbf{P}^2 \times \mathbf{P}^1$ branched over B . This has a singular locus, consisting of four curves, which can be resolved by blowing those curves up once, and we obtain in this way an elliptic fibration $f_2 : X_2 \rightarrow \mathbf{P}^2$ induced by the first projection. This has $\Sigma = C$, of fibre type I_0^* .

If $Z_1 \subseteq \mathbf{P}^2$ is a non-singular curve intersecting C transversally, it is not hard to see that $MW(f_2^{-1}(Z_1)/Z_1)_{tors} = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Thus this group is the value of $H_{\mathbf{Z}_1}^2(\mathbf{P}^2, i_*A)$.

Let l be a line tangent to C . Here we blow up \mathbf{P}^2 twice to get a fibration $f'_2 : X'_2 \rightarrow S_2$ to obtain a picture on S_2 as shown in Figure 2.1. Put $Z_2 = l \cup E_1 \cup E_2$. We see that $f'^{-1}_2(l)$ is a trivial elliptic fibration over l , and thus

$$\begin{aligned} H_{\mathbf{Z}_2}^2(S_2, i_*A) &= MW(f'^{-1}_2(l)/l)_{tors} \oplus MW(f'^{-1}_2(E_2)/E_2)_{tors} \\ &= (\mathbf{Q}/\mathbf{Z})^{\oplus 2} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}. \end{aligned}$$

3) Take a general net of quadrics in \mathbf{P}^3 , blow up the eight base-points to obtain X , and thus obtain a morphism $X \rightarrow \mathbf{P}^2$. The discriminant locus Σ is the dual curve to the quartic of singular quadrics in the net and thus has 24 cusps and 28 nodes. Of course $\text{III}_{\mathbf{P}^2}(A) = 0$; we want to know what $\text{III}_{\mathbf{P}^2 - \Sigma}(A)$ is.

To discover this, we first blow up \mathbf{P}^2 to obtain Σ s.n.c. A node gets blown up once to obtain Figure 2.2 while a cusp must be blown up three times (see [11] for details), to obtain Figure 2.3. We see the curve of fibre type I_1 is case I^* . Let $f_3 : X_3 \rightarrow S$ be the resulting modified Miranda model, and let $Z = \Sigma$. Then

$$H_{\mathbf{Z}}^2(S, i_*A) = (\mathbf{Z}/2\mathbf{Z})^{\oplus 24} \oplus \mathbf{Z}/2\mathbf{Z}.$$

The first term comes from the 24 $I_0^* + III$ collisions, the second along the curve of type I_1 . There can be no multiple fibres over the curves of fibre type I_2 .

4) Consider over \mathbf{P}^2 with coordinates $(x : y : z)$ the Weierstrass model obtained by the equation

$$s^2 = t^3 + xyz^2t + xyz^3(x - y)/\sqrt{27}.$$

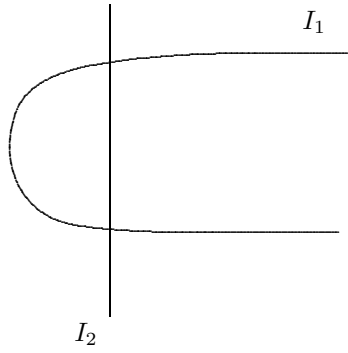


FIGURE 2.2

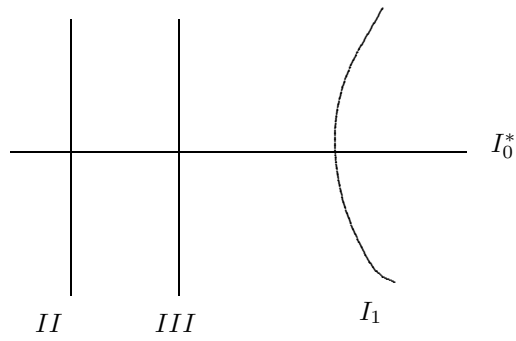


FIGURE 2.3

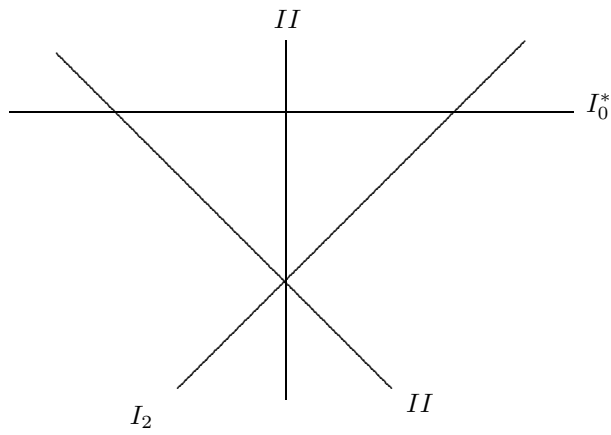


FIGURE 2.4

This has discriminant curve given by the equation

$$x^2y^2z^6(x+y)^2 = 0,$$

which looks like Figure 2.4. Blowing up the point $(0 : 0 : 1)$, we obtain a Weierstrass

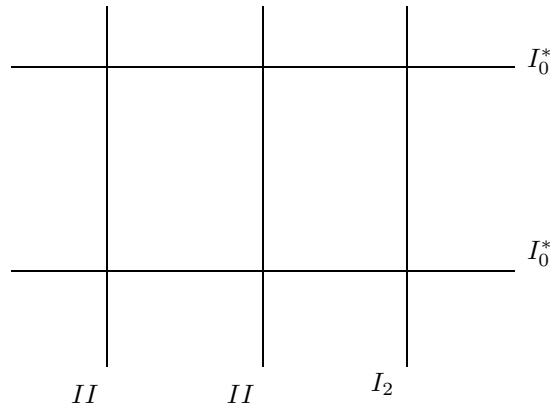


FIGURE 2.5

model over F_1 with discriminant curve as shown in Figure 2.5. Let $f_4 : X_4 \rightarrow F_1$ be the Miranda model of this Weierstrass fibration, and let Z be the curve of fibre type I_2 . Then $H_{\mathbb{Z}}^2(S, i_*A) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We have the possibility of multiplicity 2 fibres along Z or isolated multiple fibres at P_1 or P_2 . (We will show in Theorem 3.5 that this is the appropriate extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.)

Remark 2.14. The reader may note that in the calculations of Propositions 2.6–2.10, we did not need to assume that $\dim X = 3, \dim S = 2$, so long as we had $f : X \rightarrow S$ satisfying our hypothesis. If one could prove without the use of a model that $H_{\mathbb{Z}}^2(S, i_*A) = H_{\mathbb{Z}}^3(S, i_*A) = 0$ for all $Z \subseteq S$ of codimension ≥ 3 , then one can remove codimension 3 sets without changing the results, and thus extend the results of this section to all dimensions, as [16] can always give a good model over an open set $U \subseteq S$ with $S - U$ codimension ≥ 3 .

3. THE FIRST OBSTRUCTION

Our next goal is to understand the map $H_{\mathbb{Z}}^2(S, i_*A) \rightarrow H^2(S, i_*A)$; in particular, we wish to know its kernel. We have (Theorem 1.1) an inclusion

$$0 \rightarrow H^2(S, i_*A) \rightarrow H^2(S, i_*i^*P_{X/S})$$

and an exact sequence

$$0 \rightarrow G(X/S) \rightarrow H^2(S, i_*i^*P_{X/S}) \rightarrow H^3(S, \mathcal{E})$$

where

$$G(X/S) := \text{coker}(H^2(S, \mathcal{E}) \rightarrow H^2(S, P_{X/S})).$$

We will first study the composed map

$$H_{\mathbb{Z}}^2(S, i_*A) \xrightarrow{\phi} H^3(S, \mathcal{E}),$$

which we call the first obstruction, and then the induced map

$$\ker \phi \xrightarrow{\psi} G(X/S)$$

which we call the second obstruction. We then have

$$\frac{\text{III}_{S-Z}(A)}{\text{III}_S(A)} = \ker \psi.$$

The map $\phi : H_Z^2(S, i_*A) \rightarrow H^3(S, \mathcal{E})$ is a composition of the maps

$$H_Z^2(S, i_*A) \cong H_Z^2(S, i_*i^*P_{X/S}) \xrightarrow{\phi_1} H_Z^3(S, \mathcal{E}) \xrightarrow{\phi_2} H^3(S, \mathcal{E}),$$

the first isomorphism by (1.2), as $H_Z^2(S, \mathbf{Z}) = H_Z^1(S, \mathbf{Q}/\mathbf{Z}) = 0$.

We will first clarify exactly what $H_Z^3(S, \mathcal{E})$ and the map ϕ_2 are. Recall from [8, 1.8] that, since we are assuming that all components of Σ are normal,

$$\mathcal{E} = \bigoplus_{t \in S^{(1)}} \mathcal{E}_t,$$

where $\mathcal{E}_t = i_{t*}i_t^*\mathcal{E}$, with $i_t : t \rightarrow S$ the inclusion. Furthermore, we have an exact sequence

$$0 \rightarrow j_{t*}\mathbf{Z}_{C(t)} \rightarrow j_{t*}q_{t*}\mathbf{Z}_{\bar{C}(t)} \rightarrow \mathcal{E}_t \rightarrow 0,$$

where $j_t : C(t) \rightarrow S$ is the inclusion and $q_t : \bar{C}(t) \rightarrow C(t)$ the natural projection. The first map is given by $a \mapsto (m_i a)$ where m_i are the multiplicities of the components X_t^i of X_t . From this, one sees easily

Proposition 3.1.

1) If $t \notin Z$, then

$$H_Z^3(S, \mathcal{E}_t) = \text{coker}(H^0(C(t) \cap Z, \mathbf{Q}/\mathbf{Z}) \rightarrow H^0(q_t^{-1}(C(t) \cap Z), \mathbf{Q}/\mathbf{Z}))$$

where the map is given as follows. If $Q \in C(t) \cap Z$, $q_t^{-1}(Q) = \{P_1, \dots, P_n\}$, with P_i in $C(t_{j_i})$, then the map for $a \in H^0(Q, \mathbf{Q}/\mathbf{Z})$ is

$$a \mapsto (r_i m_{j_i} a)$$

where r_i is the ramification of $C(t_{j_i}) \rightarrow C(t)$ at P_i and m_{j_i} is the multiplicity of the component $X_t^{j_i}$. The map $\phi_2 : H_Z^3(S, \mathcal{E}_t) \rightarrow H^3(S, \mathcal{E}_t)$ is given by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(C(t) \cap Z, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^0(q_t^{-1}(C(t) \cap Z), \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H_Z^3(S, \mathcal{E}_t) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \phi_2 \\ 0 & \longrightarrow & H^2(C(t), \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^2(\bar{C}(t), \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(S, \mathcal{E}_t) \longrightarrow 0 \end{array}$$

where the first and second vertical arrows are the natural Gysin maps.

2) If $t \in Z$, then $H_Z^3(S, \mathcal{E}_t) \rightarrow H^3(S, \mathcal{E}_t)$ is an isomorphism.

We can describe $\ker \phi$ as

$$\ker(H_Z^2(S, i_*A) \xrightarrow{\phi} \bigoplus_{t \in S^{(1)}} H^3(S, \mathcal{E}_t)) = \bigcap_{t \in S^{(1)}} \ker(H_Z^2(S, i_*A) \xrightarrow{\phi_t} H^3(S, \mathcal{E}_t)),$$

so we can treat each $t \in \Sigma$ separately.

By Theorem 2.11, we can split the study of ϕ_t into three cases:

- (1) Describe $\phi_t : H_P^2(S, i_*A) \rightarrow H^3(S, \mathcal{E}_t)$ for P a collision point of type $III + I_0^*$. Here the only values for t we need to consider are the generic points of the two components of Σ meeting at P .
- (2) Describe $\phi_t : H_Z^2(S, i_*A) \rightarrow H^3(S, \mathcal{E}_t)$, where Z is a curve transversal to Σ . Here we are interested in those t which are generic points of components of Σ which Z intersects.
- (3) Describe $\phi_t : H_Z^2(S, i_*A) \rightarrow H^3(S, \mathcal{E}_t)$ when $Z \subseteq \Sigma$ consists of all components of Σ of type I_M , $M \geq 1$.

First, suppose that P is an isolated point of Z , at a collision point of type $I_0^* + III$, with generic point t of the curve of fibre type I_0^* and t' of the curve of fibre type III . The map $H_P^2(S, i_*A) \rightarrow H_P^3(S, \mathcal{E})$ corresponds exactly to the map $H^1(S, i_*A) \rightarrow H^2(S, \mathcal{E})$ in the local situation in Theorem 2.3. This map has been calculated for this example in Theorem 2.4:

$$H_P^3(S, \mathcal{E}) = \text{coker}(\mathbf{Q}/\mathbf{Z} \xrightarrow{\alpha_1} (\mathbf{Q}/\mathbf{Z})^4) \oplus \text{coker}(\mathbf{Q}/\mathbf{Z} \xrightarrow{\alpha_2} (\mathbf{Q}/\mathbf{Z})^2),$$

where by Prop. 3.1 α_1 is given by

$$a \mapsto (a, 2a, 2a, a)$$

and α_2 by

$$a \mapsto (a, a),$$

with components corresponding to the inverse image of P on curves $C(t_1), \dots, C(t_4)$ and $C(t'_1), C(t'_2)$ over which the normalizations of Y_1, \dots, Y_4 and Y'_1, Y'_2 are fibred. (Here we adopt the notation of the proof of Theorem 2.4.) We have $q_t^{-1}(P) = \{P_1, \dots, P_4\}$, $P_i \in C(t_i)$, $q_{t'}^{-1}(P) = \{P'_1, P'_2\}$, $P'_i \in C(t'_i)$. In this case, $H_P^2(S, i_*A) = \mathbf{Z}/2\mathbf{Z}$, and writing the non-zero element of $\mathbf{Z}/2\mathbf{Z} \subseteq \mathbf{Q}/\mathbf{Z}$ as $1/2$, we have $\phi_1(1/2) \in H_P^3(S, \mathcal{E})$ is represented by the element which was given by (2.1). Writing down the value of this representative on each point, we get:

$$(3.1) \quad \begin{array}{ll} P_1 : 0 & P'_1 : 0 \\ P_2 : 1/2 & P'_2 : 1/2 \\ P_3 : 0 & \\ P_4 : 1/2 & \end{array}$$

This describes completely the map $\phi_1 : H_P^2(S, i_*A) \rightarrow H_P^3(S, \mathcal{E})$, and we compose this with ϕ_2 to obtain ϕ in this case.

Next consider a curve $C \subseteq Z$ transversal to Σ . The calculation is easiest if we assume C is complete.

Lemma 3.2. *Let $f : Y \rightarrow C$ be an elliptic surface with a section and at least one singular fibre, C a smooth, irreducible, complete curve. Then $f^* : H^1(C, \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(Y, \mathbf{Q}/\mathbf{Z})$ is an isomorphism.*

Proof. f^* is clearly injective. Now $H^1(C, \mathbf{Q}/\mathbf{Z}) = (\text{Pic } C)_{\text{tors}}$ and $H^1(Y, \mathbf{Q}/\mathbf{Z}) = (\text{Pic } Y)_{\text{tors}}$, so by [4], Cor. 5.2.2, $\text{coker}(f^*) = NS(X)_{\text{tors}}$. But this is zero, say by [6] (1.48). •

Proposition 3.3. *Let $C \subseteq S$ be a smooth irreducible complete curve meeting Σ transversally, $Y = f^{-1}(C)$ a smooth surface. Then the map $\phi_1 : H_C^2(S, i_*A) \rightarrow H_C^3(S, \mathcal{E})$ is given as follows: if $\sigma \in {}_m MW(Y/C) = {}_m H_C^2(S, i_*A)$, σ_0 the zero section, then on Y one can write $m(\sigma - \sigma_0) \sim E$, E a divisor supported in the fibres of $f : Y \rightarrow C$. $E \otimes (1/m)$ then gives the appropriate element of $H_C^3(S, \mathcal{E})$, identifying $H_C^3(S, \mathcal{E})$ in the natural way with $((\text{Pic } Y)_{\text{fib}}/f^*\text{Pic } C) \otimes \mathbf{Q}/\mathbf{Z}$, where $(\text{Pic } Y)_{\text{fib}} \subseteq \text{Pic } Y$ is the subgroup of divisors supported on fibres of f .*

Proof. We have an exact sequence

$$0 \rightarrow H_C^2(S, P_{X/S}) \rightarrow H_C^2(S, i_*A) \rightarrow H_C^3(S, \mathcal{E}) \rightarrow H_C^3(S, P_{X/S}).$$

By Propositions 1.4, 1.5, 3.1 and Lemma 3.2,

$$H_C^2(S, P_{X/S}) = H^1(Y, \mathbf{Q}/\mathbf{Z})/H^1(C, \mathbf{Q}/\mathbf{Z}) = 0,$$

$$H_C^3(S, P_{X/S}) = H^2(Y, \mathbf{Q}/\mathbf{Z})/H^2(C, \mathbf{Q}/\mathbf{Z})$$

and

$$H_C^3(S, \mathcal{E}) = \bigoplus_{1 \leq i \leq n} \frac{(\mathbf{Q}/\mathbf{Z})^{c(f_i)}}{\mathbf{Q}/\mathbf{Z}} = ((\text{Pic } Y)_{fib}/f^*\text{Pic } C) \otimes \mathbf{Q}/\mathbf{Z},$$

where f_i , $1 \leq i \leq n$, runs over the reducible fibres of $f : Y = f^{-1}(C) \rightarrow C$, and $c(f_i)$ is the number of components of f_i . The map

$$H_C^3(S, \mathcal{E}) \rightarrow H_C^3(S, P_{X/S})$$

is clearly the Gysin map.

Now let $U \subseteq C$ be the largest open subset such that $f|_{f^{-1}(U)}$ is smooth, $C - U$ consisting of n points. Then there is a commutative diagram

$$\begin{array}{ccccccccc} & & & 0 & & 0 & & 0 & & \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & R & \rightarrow & \mathbf{Z}^n & \rightarrow & \text{Pic } C & \rightarrow & \text{Pic } U & \rightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \rightarrow & R & \rightarrow & \bigoplus_{1 \leq i \leq n} \mathbf{Z}^{c(f_i)} & \rightarrow & \text{Pic } Y & \rightarrow & \text{Pic } f^{-1}(U) & \rightarrow 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & \\ & & & & \bigoplus_{1 \leq i \leq n} \mathbf{Z}^{c(f_i)}/\mathbf{Z} & \rightarrow & \text{Pic } Y/\text{Pic } C & \rightarrow & \text{Pic } f^{-1}(U)/\text{Pic } U & \rightarrow 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & \\ & & & & 0 & & 0 & & 0 & \end{array}$$

the equality because there are no relations on divisors on Y supported in the singular fibres other than those which come from C .

Applying the torsion functor to the bottom row, one obtains

$$(3.2) \quad 0 \rightarrow \left(\frac{\text{Pic } f^{-1}(U)}{\text{Pic } U} \right)_{tors} \xrightarrow{\partial} \bigoplus_{1 \leq i \leq n} \frac{(\mathbf{Q}/\mathbf{Z})^{c(f_i)}}{\mathbf{Q}/\mathbf{Z}} \xrightarrow{\alpha_1} \frac{(\text{Pic } Y) \otimes \mathbf{Q}/\mathbf{Z}}{(\text{Pic } C) \otimes \mathbf{Q}/\mathbf{Z}}.$$

The first term is $MW(Y/C)_{tors}$ by the proof of 2.7, and there is a natural injective map

$$\frac{(\text{Pic } Y) \otimes \mathbf{Q}/\mathbf{Z}}{(\text{Pic } C) \otimes \mathbf{Q}/\mathbf{Z}} \xrightarrow{\alpha_2} \frac{H^2(Y, \mathbf{Q}/\mathbf{Z})}{H^2(C, \mathbf{Q}/\mathbf{Z})},$$

by (1.3). $\alpha_2 \circ \alpha_1$ clearly coincides with the map

$$H_C^3(S, \mathcal{E}) \rightarrow H_C^3(S, P_{X/S}).$$

Thus we have seen in another way that $H_C^2(S, i_*A) = MW(Y/C)_{tors}$, but the point is that now we can interpret the map

$$H_C^2(S, i_*A) \rightarrow H_C^3(S, \mathcal{E})$$

as the map ∂ in (3.2). In the long exact sequence associated with the torsion functor, this map is defined as follows. If

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is an exact sequence of groups, one obtains a long exact sequence

$$0 \rightarrow A_{tors} \rightarrow B_{tors} \rightarrow C_{tors} \xrightarrow{\partial} A \otimes \mathbf{Q}/\mathbf{Z}.$$

If $c \in C_{tors}$ with $mc = 0$, then c is the image of an element $b \in B$ with $mb = a \in A$. Then $\partial(c) = a \otimes (1/m)$. This agrees with the description given in the statement of the theorem. •

The values for the map described in Theorem 3.3 are well-known: see, for example, [6] for a complete table of values.

We move on to those components of Z contained in Σ . All details of the global calculations are given in the appendix; here we just state the results and perform a local calculation.

Theorem 3.4. *Let $C \subseteq Z$ be a complete curve of fibre type I_M and case I, with generic point t . We have*

$$H^3_Z(S, \mathcal{E}_t) = H^3(S, \mathcal{E}_t) = \text{coker}(\mathbf{Q}/\mathbf{Z} \rightarrow \bigoplus_{i=1}^{M'} \mathbf{Q}/\mathbf{Z}),$$

$$M' = \begin{cases} M & \text{if } M \text{ is even,} \\ M + 1 & \text{if } M \text{ is odd,} \end{cases}$$

with the summands of $\bigoplus_{i=1}^{M'} \mathbf{Q}/\mathbf{Z}$ corresponding to the components $Y_1, \dots, Y_{M'}$ over C respectively. Let $\gamma \in H^2_Z(S, i_*A)$ have invariants b along C and invariants c_i along the curves of Σ intersecting C , of fibre type I_{M_i} , $1 \leq i \leq s$ (see Remark 2.12 for the definition of invariants). Put $\alpha = -bC^2 - \sum_{j=1}^s c_j$. Then $\phi_t(\gamma) \in H^3(S, \mathcal{E}_t)$ is given by the class of the element $\beta = (\beta_i) \in (\mathbf{Q}/\mathbf{Z})^{M'}$, with

$$\beta_i = \begin{cases} (i - 1)\alpha & \text{if } 1 \leq i \leq (M + 1)/2; \\ M\alpha & \text{if } i = (M + 3)/2; \\ (i - 2)\alpha & \text{if } (M + 5)/2 \leq i \leq M + 1 \end{cases}$$

if M is odd and

$$\beta_i = (i - 1)\alpha$$

if M is even, for all values of i .

Proof. See the Appendix. •

For Case I^* , we first want to do local calculations for a $I_{M_1} + I_{M_2}^*$ collision. We note that the values for $H^2_C(\bar{S}, i_*A)$ given below agree with those of [19].

Theorem 3.5. *Let $f : \bar{X} \rightarrow \bar{S}$ be a modified Miranda model with \bar{S} strictly local, with $\Sigma = C \cup C'$, C and C' two curves meeting transversally, C of fibre type I_{M_1} and C' of fibre type $I_{M_2}^*$. Then*

$$H^2_C(\bar{S}, i_*A) = \begin{cases} \mathbf{Z}/2\mathbf{Z}, & M_1 \text{ odd,} \\ \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}, & M_1, M_2 \text{ even,} \\ \mathbf{Z}/4\mathbf{Z}, & M_1 \text{ even, } M_2 \text{ odd.} \end{cases}$$

(This gives more precise information than does Theorem 2.11 in this case.) Following the notation of Table 2 and identifying $H^3_C(\bar{S}, \mathcal{E})$ with $\text{coker}(\mathbf{Q}/\mathbf{Z} \rightarrow (\mathbf{Q}/\mathbf{Z})^{m_2})$

where m_2 is the number of components of $f^{-1}(C')$, the map $\phi_1 : H_C^2(\bar{S}, i_*A) \rightarrow H_C^3(\bar{S}, \mathcal{E})$ is given by representatives

$$M_1 \text{ odd, } b \in \mathbf{Z}/2\mathbf{Z}, (\phi_1(b))_i = \begin{cases} 0 & \text{on } Y'_i, i \text{ odd,} \\ b & \text{on } Y'_i, i \text{ even.} \end{cases}$$

$$M_1, M_2 \text{ even, } (b, c) \in \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}, (\phi_1(b, c))_i = \begin{cases} 0 & \text{on } Y'_i, i \leq M_2 + 3, i \text{ odd,} \\ b & \text{on } Y'_i, i \leq M_2 + 2, i \text{ even,} \\ c & \text{on } Y'_{M_2+4} \\ b + c & \text{on } Y'_{M_2+5}. \end{cases}$$

In these two cases b is the invariant along C .

$$M_1 \text{ even, } M_2 \text{ odd, } b \in \mathbf{Z}/4\mathbf{Z}, (\phi_1(b))_i = \begin{cases} 0 & \text{on } Y'_i, i \leq M_2 + 2, i \text{ odd,} \\ 2b & \text{on } Y'_i, i \leq M_2 + 3, i \text{ even,} \\ b & \text{on } Y'_{M_2+4}, \\ -b & \text{on } Y'_{M_2+5}. \end{cases}$$

Here $2b$ is the invariant along C .

Proof. Removing the curve C of fibre type I_{M_1} , putting $S = \bar{S} - C$ and $X = \bar{X} \times_{\bar{S}} S$, we use the spectral sequence of the proof of Prop. 1.5 to get an exact sequence

$$0 \rightarrow H^1(S, P_{X/S}) \rightarrow \bigoplus_{i < j} H^0(Y_i \cap Y_j, \mathbf{Q}/\mathbf{Z}) \rightarrow \bigoplus_i H^2(Y_i, \mathbf{Q}/\mathbf{Z}) \rightarrow H^2(S, P_{X/S}) \rightarrow 0,$$

where Y_1, \dots, Y_{m_1} are the divisors over C in \bar{X} . Using Table 2 to calculate, one sees then in fact that $H^1(S, P_{X/S}) = 0$, and we have an exact sequence

$$0 \rightarrow \text{III}_S(A) \rightarrow H^2(S, \mathcal{E}) \rightarrow H^2(S, P_{X/S}).$$

The map $H^2(S, \mathcal{E}) \rightarrow H^2(S, P_{X/S})$ factors through the natural specialization map $H^2(S, \mathcal{E}) = H_C^3(\bar{S}, \mathcal{E}) \rightarrow \bigoplus_i H^2(Y_i, \mathbf{Q}/\mathbf{Z})$, and thus $\text{III}_S(A)$ can be calculated as the kernel of the map

$$\bigoplus_{i < j} H^0(Y_i \cap Y_j, \mathbf{Q}/\mathbf{Z}) \oplus H^2(S, \mathcal{E}) \rightarrow \bigoplus_i H^2(Y_i, \mathbf{Q}/\mathbf{Z})$$

with the map $\text{III}_S(A) \rightarrow H^2(S, \mathcal{E})$ corresponding to the map $\phi_1 : H_C^2(\bar{S}, i_*A) \rightarrow H_C^3(\bar{S}, \mathcal{E})$. Consider $b_i \in H^0(C_i, \mathbf{Q}/\mathbf{Z})$ and $(a_i)_{1 \leq i \leq m_2}$ a representative of an element in $H^2(S, \mathcal{E})$, following the notation of Table 2. Proceeding as in the proof of Proposition 2.9, we can identify $\bigoplus_i H^2(Y_i, \mathbf{Q}/\mathbf{Z})$ with $\bigoplus_i H^2(f_i, \mathbf{Q}/\mathbf{Z})$, f_1, \dots, f_m the components of the central fibre, using the intersection pairing given in Table 2. Thus, in order to obtain zero in $\bigoplus_i H^2(Y_i, \mathbf{Q}/\mathbf{Z})$, we must have the b_i and a_i

satisfying the following equations:

$$\begin{aligned}
 f_1 &: -2a_1 + a_3 = 0 \\
 f_2 &: -2a_2 + a_3 = 0 \\
 f_3 &: a_1 + a_2 - 2a_3 + a_4 = 0 \\
 f_4 &: a_3 - 2a_4 + a_5 = 0 \\
 &\vdots \\
 f_{n_1-1} &: a_{n_1-2} - 2a_{n_1-1} + a_{n_1} = 0 \\
 f_{n_1} &: b_1 + a_{n_1-1} - a_{n_1} = 0 \\
 f_{n_1+1} &: -b_1 + b_2 = 0 \\
 &\vdots
 \end{aligned}$$

$ \begin{aligned} &M_1, M_2 \text{ even} \\ f_{M-4} &-b_{m_1-3} + b_{m_1-2} = 0 \\ f_{M-3} &-b_{m_1-2} + b_{m_1-1} = 0 \\ f_{M-2} &-b_{m_1-1} - a_{n_1} + a_{n_1+1} + a_{n_1+2} = 0 \\ f_{M-1} &a_{n_1} - 2a_{n_1+1} = 0 \\ f_M &a_{n_1} - 2a_{n_1+2} = 0 \\ &M_1 \text{ odd}, M_2 \text{ even} \\ f_{M-4} &-b_{m_1-3} + b_{m_1-2} = 0 \\ f_{M-3} &-b_{m_1-2} + b_{m_1-1} = 0 \\ f_{M-2} &-b_{m_1-1} - a_{n_1} + a_{n_1+1} = 0 \\ f_{M-1} &a_{n_1} - a_{n_1+1} + b_{m_1} = 0 \\ f_M &2b_{m_1} = 0 \end{aligned} $	$ \begin{aligned} &M_1 \text{ even}, M_2 \text{ odd} \\ &-b_{m_1-2} + b_{m_1-1} = 0 \\ &-b_{m_1-1} - a_{n_1} + a_{n_1+1} = 0 \\ &a_{n_1} - 2a_{n_1+1} + a_{n_1+2} + a_{n_1+3} = 0 \\ &a_{n_1+1} - 2a_{n_1+2} = 0 \\ &a_{n_1+1} - 2a_{n_1+3} = 0 \\ &M_1, M_2 \text{ odd} \\ &-b_{m_1-2} + b_{m_1-1} = 0 \\ &-b_{m_1-1} - a_{n_1} + a_{n_1+1} = 0 \\ &a_{n_1} - 2a_{n_1+1} + a_{n_1+2} = 0 \\ &b_{m_1} + a_{n_1+1} - a_{n_1+2} = 0 \\ &2b_{m_1} = 0 \end{aligned} $
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Thus, as we can set $a_1 = 0$, we see that $a_2 = 0$ or $1/2$, and

$$a_i = \begin{cases} 0, & i \text{ odd}, \\ a_2, & i \text{ even}, \end{cases}$$

for $1 \leq i \leq n_1$, and that $b_1 = b_2 = \dots = b_{m_1-1} = a_2$. For M_1, M_2 even, we see in addition that

$$a_{n_1+1}, a_{n_1+2} \in \{0, 1/2\}$$

and

$$a_{n_1+1} + a_{n_1+2} = a_2.$$

Thus we get $\text{III}_S(A) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. In particular, the invariant along C is a_2 .

For M_1 even, M_2 odd, we have $a_{n_1+1} = a_2$, $a_{n_1+2} + a_{n_1+3} = 0$, and $2a_{n_1+2} = 2a_{n_1+3} = a_2$, so we get $\text{III}_S(A) = \mathbf{Z}/4\mathbf{Z}$. The invariant along C is a_2 .

For M_1 odd, M_2 even, we see that $a_{n_1+1} = a_2$, and we have $\text{III}_S(A) = \mathbf{Z}/2\mathbf{Z}$.

For M_1, M_2 odd, we have $a_{n_1+1} = a_2$, $a_{n_1+2} = 0$ and again, we get $\text{III}_S(A) = \mathbf{Z}/2\mathbf{Z}$.

This gives the desired results. If $M = 1$, one makes a similar calculation and obtains the same results. •

Definition. If $\gamma \in H_Z^2(S, i_*A)$, $s \in Z$ a collision point of type $I_{M_1} + I_{M_2}^*$, M_1 even, we say the *local invariant* of γ at s is (b, c) if M_2 is even and the image of γ in $(\underline{H}_Z^2(S, i_*A))_{\bar{s}} = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ is (b, c) , as in Theorem 3.5; if M_2 is odd, we say the local invariant of γ in $(\underline{H}_Z^2(S, i_*A))_{\bar{s}} = \mathbf{Z}/4\mathbf{Z}$ is b .

Theorem 3.6. *Let $C \subseteq Z$ be a curve of fibre type I_M and case I^* , with generic point t , intersecting curves of fibre type I_{M_i} , $1 \leq i \leq s$, and $I_{N_i}^*$, $1 \leq i \leq u$. Let $\gamma \in H_Z^2(S, i_*A)$.*

- 1) *If M is odd then $\phi_t(\gamma) \in H^3(S, \mathcal{E}_t)$ is zero.*
- 2) *If M is even, the invariant of γ along C is b , and the local invariants of γ at the collision $I_M + I_{N_i}^*$ are (b, d_i) or b_i with $b = 2b_i$ depending on whether N_i is even or odd, then $\phi_t(\gamma) \in H^3(S, \mathcal{E}_t) = \text{coker}(\mathbf{Q}/\mathbf{Z} \rightarrow (\mathbf{Q}/\mathbf{Z})^{M/2+1})$ is represented by*

$$(0, \dots, -ba + \sum_{j=1}^u \alpha^j - \sum_{j=r+1}^s \frac{Mc_j}{2}),$$

with the components corresponding to $Y_1, \dots, Y_{M/2+1}$ respectively, a is an integer which depends only on X and α^j depends on the value of the local invariants at the collisions, with

$$\alpha^j = \begin{cases} b + d_j & \text{if } N_j \text{ is even,} \\ -b_j & \text{if } N_j \text{ is odd.} \end{cases}$$

Proof. See the Appendix. •

Remark 3.7. The parity of the integer a in the above theorem does not seem to be determinable from purely numeric data, but see the proof of Theorem 3.6 if you wish to compute it from the geometry of X . However, for many applications, it doesn't matter: see Example 3.8, (4).

Example 3.8. We continue the examples of 2.13.

- 1) Here $\mathcal{E} = 0$ so $\ker \phi = H_Z^2(\mathbf{P}^2, i_*A)$.
- 2) We have $H_{Z_1}^2(\mathbf{P}^2, i_*A) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Let $\gamma \in H_Z^2(S, i_*A)$. Using Propositions 3.1 and 3.3 and the fact that Z intersects C an even number of times and that γ is 2-torsion, it is easy to see that $\phi(\gamma) = 0$. Thus $\ker \phi = H_{Z_1}^2(\mathbf{P}^2, i_*A)$.

For Z_2 we see from Proposition 3.3 that if we choose a non-zero section over E_2 which represents a non-zero element of the $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ part of $H_{Z_2}^2(S_2, i_*A)$, the contribution to at least one of the components over C will be $1/2$ and never zero. Thus one sees that $\ker \phi = (\mathbf{Q}/\mathbf{Z})^{\oplus 2} = H_l^2(S_2, i_*A)$, as l does not intersect the discriminant locus.

3) We check the contributions for ϕ over the various components of Σ . Let $\gamma \in H_Z^2(S, i_*A)$. γ has, in particular, invariant 0 along the curves of type I_2 and invariant 0 or $1/2$ along the curve of type I_1 . If t is the generic point of a curve of type I_2 , then $\phi_t(\gamma) = 0$ by Theorem 3.4, and if t is the generic point of the curve of type I_1 , then $\phi_t(\gamma) = 0$ by Theorem 3.6. Over a curve C of type I_0^* , one can check that $f_3^{-1}(C)$ consists of two reduced components and one multiple component. One of the reduced components intersects σ_0 and the other factors through a curve mapping 3-1 to C (because of the collision $II + I_0^*$ [16]). Thus adding up all the local contributions from Theorem 3.5 and (3.1), we obtain $\phi_t(\gamma) = 0$ in $H^3(S, \mathcal{E}_t)$ where t is the generic point of C .

Finally we have a curve D of fibre type III , with $f^{-1}(D)$ consisting of two components Y_1 and Y_2 . By (3.1) Y_1 gets a contribution of 0 and Y_2 0 or 1/2 depending on whether we want an isolated multiple fibre at the $I_0^* + III$ collisions. This shows that it is not possible to have an isolated multiple fibre there, and so $\ker \phi = \mathbf{Z}/2\mathbf{Z}$, the only possibility being to have multiplicity two fibres over the I_1 curve.

4) Here over each curve of type I_0^* there are two reduced components, one factoring through a 3-1 cover of the curve, again because of the $II + I_0^*$ collisions. Theorems 3.5 and 3.6 then tell us $\ker \phi = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$, the possibilities being to have isolated multiple fibres at P_1 and P_2 at the same time, or else to have fibre-type ${}_2I_2$ along Z . Notice that we haven't computed the integer a of Theorem 3.6; this would tell us precisely which subgroup of $H_Z^2(S, i_*A)$ the kernel of ϕ is. However, this information is not particularly helpful.

4. THE SECOND OBSTRUCTION

We will now consider the nature of the second obstruction. Recall from §3 that we have maps

$$H_Z^2(S, i_*A) \xrightarrow{\phi} H^3(S, \mathcal{E})$$

and

$$\ker \phi \xrightarrow{\psi} G(X/S) := \text{coker}(H^2(S, \mathcal{E}) \rightarrow H^2(S, P_{X/S})),$$

with

$$\ker \psi \cong \text{III}_{S-Z}(A) / \text{III}_S(A).$$

To understand the map ψ , we need

Definition. Let $\{Y_i^j\}$ be the set of prime divisors of X mapping to curves Z_i on S contained in the discriminant locus, and let $i : \tilde{Y}_i^j \rightarrow X$ be the map of the normalization of Y_i^j to X , and $\tilde{Y}_i^j \xrightarrow{\pi} \bar{Z}_i^j \rightarrow Z_i$ be the Stein factorization of the map $\tilde{Y}_i^j \rightarrow Z_i$. We put

$$C(X/S) = \frac{CH^2(X)_{tors}}{f^*CH^2(S)_{tors} + \sum_{i,j} i_*\pi^*\text{Pic}(\bar{Z}_i^j)_{tors}}.$$

Theorem 4.1. *Suppose S (and hence X) is projective. Then there is an injective map $\lambda : C(X/S) \rightarrow G(X/S)$ through which the map ψ factors, giving a map $\psi' : \ker \phi \rightarrow C(X/S)$ with $\ker \psi' = \ker \psi$.*

To prove this, we will need

Theorem 4.2. *Let X be a projective variety over an algebraically closed field k of characteristic zero, $Z \subseteq X$ a closed subset, l a prime number. Then there is a factorization of the natural map*

$$H_Z^3(X, \mathbf{Q}_l/\mathbf{Z}_l(2)) \rightarrow H^3(X, \mathbf{Q}_l/\mathbf{Z}_l(2))$$

via

$$H_Z^3(X, \mathbf{Q}_l/\mathbf{Z}_l(2)) \rightarrow CH^2(X)(l) \xrightarrow{\lambda_2^l} H^3(X, \mathbf{Q}_l/\mathbf{Z}_l(2))$$

where $CH^2(X)(l)$ is the l -primary part of the second Chow group of X , and λ_2^l is the Bloch cycle map. (See [2] and later in this section for the construction and properties of the Bloch map.) Furthermore, λ_2^l is injective. Over an algebraically

closed field, as usual we can choose an identification of μ_n with $\mathbf{Z}/n\mathbf{Z}$ for each n , and this yields by (1.4) a factorization

$$H_Z^3(X, \mathbf{G}_m) \rightarrow CH^2(X)_{tors} \xrightarrow{\lambda_X} H^3(X, \mathbf{G}_m)$$

with λ_X injective.

This theorem will be proved in the discussion of the Bloch map later in this section.

Proof of Theorem 4.1. Since $H^2(S, P_{X/S}) = H^3(X, \mathbf{G}_m)/H^3(S, \mathbf{G}_m)$, by Theorem 4.2 we get a natural map $CH^2(X)_{tors}/f^*CH^2(S)_{tors} \xrightarrow{\lambda'} H^2(S, P_{X/S})$. This map is in fact an injection. Indeed, suppose $\alpha \in CH^2(X)_{tors}$ is such that $\lambda_X(\alpha) = f^*(\beta)$ for some $\beta \in H^3(S, \mathbf{G}_m)$. Recall we have a section $\sigma_0 : S \rightarrow X$ inducing maps $\sigma_0^* : CH^2(X)_{tors} \rightarrow CH^2(S)_{tors}$ and $\sigma_0^* : H^3(X, \mathbf{G}_m) \rightarrow H^3(S, \mathbf{G}_m)$, which are left inverses to f^* . So

$$\begin{aligned} \lambda_X(\alpha) &= f^*\sigma_0^*\lambda_X(\alpha) \\ &= f^*\lambda_S\sigma_0^*(\alpha) \\ &= \lambda_X f^*\sigma_0^*(\alpha). \end{aligned}$$

Thus

$$\begin{aligned} \lambda_X(\alpha - f^*\sigma_0^*(\alpha)) &= \lambda_X(\alpha) - \lambda_X(\alpha) \\ &= 0. \end{aligned}$$

Since λ_X is injective, $\alpha = f^*\sigma_0^*(\alpha)$, and so the class of α in

$$CH^2(X)_{tors}/f^*CH^2(S)_{tors}$$

is zero. We conclude that λ' is injective.

The existence and injectivity of λ follows from the commutative diagram

$$\begin{array}{ccccc} \bigoplus_{i,j} i_*\pi^*\text{Pic}(\bar{Z}_j^i)_{tors} & \longrightarrow & H^2(S, \mathcal{E}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 \longrightarrow CH^2(X)_{tors}/f^*CH^2(S)_{tors} & \xrightarrow{\lambda'} & H^2(S, P_{X/S}) & & \\ \downarrow & & \downarrow & & \\ C(\bar{X}/S) & \xrightarrow{\lambda} & G(\bar{X}/S) & & \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

To show that ψ factors through λ , we can always enlarge Z , for if $Z \subseteq Z'$, we have $H_{Z'}^2(S, i_*A) \subseteq H_Z^2(S, i_*A)$, so we can assume that $\Sigma \subseteq Z$. Then $\ker \phi =$

$\ker(H_Z^2(S, i_*A) \rightarrow H_Z^3(S, \mathcal{E}))$ since $H_Z^3(S, \mathcal{E}) = H^3(S, \mathcal{E})$. Thus we have a diagram

$$\begin{array}{ccc}
 0 & & \\
 \downarrow & & \\
 H_Z^2(S, \mathcal{E}) & \xrightarrow{\cong} & H^2(S, \mathcal{E}) \\
 \downarrow & & \downarrow \\
 H_Z^2(S, P_{X/S}) & \xrightarrow{g} & H^2(S, P_{X/S}) \\
 \downarrow & & \downarrow \\
 \ker \phi & \xrightarrow{\psi} & G(X/S) \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

By Theorem 4.2, g factors through $\frac{CH^2(X)_{tors}}{f^*CH^2(S)_{tors}} \subseteq H^2(S, P_{X/S})$, and so ψ factors as desired. •

This is not so helpful unless we understand the nature of the factorization of Theorem 4.2. Our goal then for the remainder of this paper is to understand this factorization, and to use this understanding to sketch how to explicitly construct elements of $C(X/S)$ from elements of $\ker \phi$. We will then give some examples and some important special cases.

We begin by reviewing Bloch-Ogus theory. Let X be a quasi-projective variety of finite type over an algebraically closed field k of characteristic 0. Denote by $\underline{H}^i(\mu_n^{\otimes r})$ the Zariski sheaf on X associated to the presheaf

$$U \mapsto H^i(U, \mu_n^{\otimes r});$$

i.e. $\underline{H}^i(\mu_n^{\otimes r}) = R^i j_* \mu_n^{\otimes r}$ where j is the morphism from the étale to the Zariski site on X . By the Leray spectral sequence, we have a spectral sequence

$$E_2^{p,q} = H_{Zar}^p(X, \underline{H}^q(\mu_n^{\otimes r})) \Rightarrow H^{p+q}(X, \mu_n^{\otimes r}).$$

We have the following resolution of the sheaf $\underline{H}^q(\mu_n^{\otimes r})$ by flasque sheaves in the Zariski topology, given in [3] :

$$\begin{aligned}
 (4.1) \quad 0 \rightarrow \underline{H}^q(X, \mu_n^{\otimes r}) &\rightarrow \bigoplus_{x \in X^{(0)}} i_x H^q(x, \mu_n^{\otimes r}) \rightarrow \bigoplus_{x \in X^{(1)}} i_x H^{q-1}(x, \mu_n^{\otimes(r-1)}) \rightarrow \dots \\
 &\rightarrow \bigoplus_{x \in X^{(q)}} i_x H^0(x, \mu_n^{\otimes(r-q)}) \rightarrow 0
 \end{aligned}$$

where

$$H^i(x, \mu_n^{\otimes r}) := \varinjlim_{U \subseteq \{x\}^-} H^i(U, \mu_n^{\otimes r})$$

and $i_x G$ is the constant sheaf G on $\{x\}^-$ extended by zero outside of this set, G an abelian group.

One can compute Zariski cohomology using the resolution (4.1), and in particular,

$$H_{Zar}^p(X, \underline{H}^q(\mu_n^{\otimes r})) = 0 \quad \forall p > q.$$

The above spectral sequence tells us that we have an exact sequence

$$0 \rightarrow H_{Zar}^1(X, \underline{H}^2(\mu_n^{\otimes r})) \rightarrow H^3(X, \mu_n^{\otimes r}) \rightarrow H^0(X, \underline{H}^3(\mu_n^{\otimes r})),$$

the last group being contained in $H^3(K(X), \mu_n^{\otimes r})$ by the resolution (4.1) for $q = 3$. Thus $H_{Zar}^1(X, \underline{H}^2(\mu_n^{\otimes r}))$ is the set of classes in $H^3(X, \mu_n^{\otimes r})$ with support on a closed subset of X . In particular, the image of $H_Z^3(X, \mu_n^{\otimes r})$ in $H^3(X, \mu_n^{\otimes r})$ is contained in $H_{Zar}^1(X, \underline{H}^2(\mu_n^{\otimes r}))$. In fact, using the spectral sequence

$$E_2^{p,q} = H_Z^p(X, \underline{H}^q(\mu_n^{\otimes r})) \Rightarrow H_Z^{p+q}(X, \mu_n^{\otimes r})$$

and the Bloch-Ogus resolution (4.1) which shows $H_Z^0(X, \underline{H}^3(\mu_n^{\otimes 2})) = 0$, we see that

$$H_Z^3(X, \mu_n^{\otimes 2}) = H_Z^1(X, \underline{H}^2(\mu_n^{\otimes 2})),$$

where the latter local cohomology group is in the Zariski topology. Recalling that Z is a divisor, this is then the cohomology of the complex

$$H_Z^0(X, i_x H^2(K(X), \mu_n^{\otimes 2})) = 0 \rightarrow \bigoplus_{x \in Z^{(0)}} k(x)^*/k(x)^{*n} \xrightarrow{\partial_n^Z} \bigoplus_{x \in Z^{(1)}} \mathbf{Z}/n\mathbf{Z}$$

($k(x)$ is the residue field of the point $x \in X$) where ∂_n^Z is reduction modulo n of the natural map ∂^Z in

$$\bigoplus_{x \in Z^{(0)}} k(x)^* \xrightarrow{\partial^Z} \bigoplus_{x \in Z^{(1)}} \mathbf{Z} \rightarrow CH^1(Z) \rightarrow 0,$$

where we take this exact sequence as the suitable definition for $CH^1(Z)$ if Z is singular (i.e. the Fulton Chow groups [10]). Thus

$$H_Z^3(X, \mu_n^{\otimes 2}) = \ker \partial_n^Z.$$

We also have similar sequences for X : $H_{Zar}^1(\underline{H}^2(X, \mu_n^{\otimes 2}))$ is the cohomology of the complex

$$H^2(K(X), \mu_n^{\otimes 2}) \rightarrow \bigoplus_{x \in X^{(1)}} k(x)^*/k(x)^{*n} \xrightarrow{\partial_n^X} \bigoplus_{x \in X^{(2)}} \mathbf{Z}/n\mathbf{Z}$$

where ∂_n^X is reduction modulo n of the natural map ∂^X in

$$\bigoplus_{x \in X^{(1)}} k(x)^* \xrightarrow{\partial^X} \bigoplus_{x \in X^{(2)}} \mathbf{Z} \rightarrow CH^2(X) \rightarrow 0.$$

Finally, the snake lemma for the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{x \in Z^{(0)}} k(x)^*/k^* & \xrightarrow{n} & \bigoplus_{x \in Z^{(0)}} k(x)^*/k^* & \longrightarrow & \bigoplus_{x \in Z^{(0)}} k(x)^*/k(x)^{*n} \longrightarrow 0 \\ & & \downarrow \partial^Z & & \downarrow \partial^Z & & \downarrow \partial_n^Z \\ 0 & \longrightarrow & \bigoplus_{x \in Z^{(1)}} \mathbf{Z} & \xrightarrow{n} & \bigoplus_{x \in Z^{(1)}} \mathbf{Z} & \longrightarrow & \bigoplus_{x \in Z^{(1)}} \mathbf{Z}/n\mathbf{Z} \longrightarrow 0 \end{array}$$

gives us an exact sequence

$$(4.2) \quad 0 \rightarrow \ker \partial^Z/n \ker \partial^Z \rightarrow H_Z^3(X, \mu_n^{\otimes 2}) \rightarrow {}_nCH^1(Z) \rightarrow 0.$$

The similar diagram for X [2, (2.1)] gives

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker \partial^X/n \ker \partial^X & \longrightarrow & \ker \partial_n^X & \longrightarrow & {}_nCH^2(X) \longrightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & H^1(X, \underline{H}^2(\mu_n^{\otimes 2})) & & \\
 & & & & \downarrow & & \\
 & & & & H^3(X, \mu_n^{\otimes 2}) & &
 \end{array}$$

(only the row is exact) and the map $H_Z^3(X, \mu_n^{\otimes 2}) \rightarrow H^3(X, \mu_n^{\otimes 2})$ is clearly compatible with the map $\ker \partial_n^Z \rightarrow \ker \partial_n^X$.

The map $\ker \partial_n^X \rightarrow {}_nCH^2(X)$ has the following explicit description: given $(\bar{f}_x)_{x \in X^{(1)}} \in \ker \partial_n^X$ with $f_x \in k(x)^*$ representatives for \bar{f}_x , we obtain the associated principal algebraic 2-cycle

$$(4.3) \quad 0 \sim \sum n_i C_i = \sum_{x \in X^{(1)}} (f_x),$$

and we have $n|n_i \forall i$. Thus $\sum \frac{n_i}{n} C_i$ is an n -torsion cycle, which is well-defined up to rational equivalence.

Bloch proves, using the Weil conjectures and assuming X is projective, that if one considers the induced map

$$\rho : \ker \partial^X \rightarrow \varinjlim_{l^\nu} H^3(X, \mu_{l^\nu}^{\otimes 2}) = H^3(X, \mathbf{Z}_l(2)),$$

the image of ρ is torsion, and hence the map obtained by taking the direct limit over l^ν

$$\ker \partial^X \otimes \mathbf{Q}_l/\mathbf{Z}_l \rightarrow H^3(X, \mathbf{Q}_l/\mathbf{Z}_l(2))$$

is zero. The induced map $\lambda_2^l : CH^2(X)(l) \rightarrow H^3(X, \mathbf{Q}_l/\mathbf{Z}_l(2))$ is, up to sign, the Bloch map. Thus we see, in particular, that if X is projective then the map

$$H_Z^3(X, \mathbf{Q}_l/\mathbf{Z}_l(2)) \rightarrow H^3(X, \mathbf{Q}_l/\mathbf{Z}_l(2))$$

factors as

$$H_Z^3(X, \mathbf{Q}_l/\mathbf{Z}_l(2)) \rightarrow CH^1(Z)(l) \rightarrow CH^2(X)(l) \xrightarrow{\lambda_2^l} H^3(X, \mathbf{Q}_l/\mathbf{Z}_l(2)).$$

Furthermore, by [5] Cor. 4, the Bloch map is injective for codimension 2 cycles. Taking the limit over l , working over an algebraically closed field of characteristic zero and choosing identifications of μ_n with $\mathbf{Z}/n\mathbf{Z}$, one obtains

$$H_Z^3(X, \mathbf{Q}/\mathbf{Z}) \rightarrow CH^1(Z)_{tors} \rightarrow CH^2(X)_{tors} \rightarrow H^3(X, \mathbf{Q}/\mathbf{Z})$$

which by (1.4) gives the desired factorization

$$(4.4) \quad H_Z^3(X, \mathbf{G}_m) \rightarrow CH^1(Z)_{tors} \rightarrow CH^2(X)_{tors} \rightarrow H^3(X, \mathbf{G}_m).$$

This proves Theorem 4.2.

We now make several other comments which will be useful in understanding these maps.

Note that we have the diagram, for $U = X - Z$,

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & \ker \partial_n^Z & \xrightarrow{\cong} & H_Z^3(X, \mu_n^{\otimes 2}) & \\
 & & & \downarrow & & \downarrow & \\
 (4.5) & H^2(K(X), \mu_n^{\otimes 2}) & \longrightarrow & \ker \partial_n^X & \longrightarrow & H^1(X, \underline{H}^2(\mu_n^{\otimes 2})) & \longrightarrow 0 \\
 & \downarrow = & & \downarrow & & \downarrow & \\
 & H^2(K(U), \mu_n^{\otimes 2}) & \longrightarrow & \ker \partial_n^U & \longrightarrow & H^1(U, \underline{H}^2(\mu_n^{\otimes 2})) & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & \text{coker } \partial_n^Z & \xrightarrow{\cong} & H_Z^2(X, \underline{H}^2(\mu_n^{\otimes 2})) &
 \end{array}$$

A similar argument where C is a projective non-singular curve and $Z \subseteq C$ a closed subset provides a diagram

$$\begin{array}{ccccccc}
 (4.6) & & & 0 & & & \\
 & & & \downarrow & & & \\
 0 & \longrightarrow & H^1(C, \mu_n) & \longrightarrow & K(C)^*/K(C)^{*n} & \xrightarrow{\partial_n^C} & \bigoplus_{x \in C^{(1)}} \mathbf{Z}/n\mathbf{Z} \\
 & & \downarrow & & \downarrow = & & \downarrow \\
 0 & \longrightarrow & H^1(C - Z, \mu_n) & \longrightarrow & K(C)^*/K(C)^{*n} & \xrightarrow{\partial_n^{C-Z}} & \bigoplus_{x \in (C-Z)^{(1)}} \mathbf{Z}/n\mathbf{Z} \\
 & & \downarrow \delta & & & & \\
 & & \bigoplus_{z \in Z} \mathbf{Z}/n\mathbf{Z} & & & &
 \end{array}$$

where the map δ is the projection of ∂_n^C restricted to $H^1(C - Z, \mu_n)$ onto the factor $\bigoplus_{z \in Z} \mathbf{Z}/n\mathbf{Z}$.

We now return to understanding the map ψ' in Theorem 4.1. How do we calculate the map ψ' to find our final obstruction? We will sketch the method in general here and give a number of examples. In the proof of Theorem 4.1, Z was enlarged in such a way that $f^{-1}(Z)$ might not be s.n.c., which makes calculations harder, so it is easiest to actually split $\ker \phi$ up into two parts. Thus we work with the assumption that $f^{-1}(Z)$ is s.n.c., which holds if Z is s.n.c. and the only components of Σ it contains are of fibre type I_M . Let

$$G_1 = \ker(H_Z^2(S, i_*A) \rightarrow H_Z^3(S, \mathcal{E}));$$

from this one obtains an exact sequence

$$(4.7) \quad 0 \rightarrow G_1 \rightarrow \ker \phi \rightarrow G_2 \rightarrow 0$$

fitting into a diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K_1 & \longrightarrow & \ker \psi & \longrightarrow & K_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 (4.8) \quad 0 & \longrightarrow & G_1 & \longrightarrow & \ker \phi & \longrightarrow & G_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & G'_1 & \longrightarrow & G(X/S) & \longrightarrow & C_2 \longrightarrow 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

with $K_1 = G_1 \cap \ker \psi$. We calculate K_1 and K_2 .

As far as the map $G_1 \rightarrow G(X/S)$ is concerned, we have a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & H_Z^2(S, \mathcal{E}) & \longrightarrow & H^2(S, \mathcal{E}) & & \\
 & & \downarrow & & \downarrow & & \\
 & & H_Z^2(S, P_{X/S}) & \xrightarrow{g} & H^2(S, P_{X/S}) & \longrightarrow & C_1 \longrightarrow 0 \\
 0 & \longrightarrow & K_1 & \longrightarrow & G_1 & \longrightarrow & G(X/S) \longrightarrow C_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where g is interpreted by factoring through the Bloch map, as before.

For G_2 we have a diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & H^2(S, \mathcal{E}) & \xrightarrow{\beta} & C_1 & \xrightarrow{\alpha_5} & C_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow^{\alpha_4} & & \\
 & & H^2(S - Z, \mathcal{E}) & \xrightarrow{\alpha_3} & H^2(S - Z, P_{X/S}) & & \\
 & & \downarrow^{\alpha_2} & & \downarrow & & \\
 G_2 & \xrightarrow{\alpha_1} & \ker(H_Z^3(S, \mathcal{E}) \rightarrow H^3(S, \mathcal{E})) & \longrightarrow & H_Z^3(S, P_{X/S}) & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

The kernel of the map $G_2 \rightarrow C_2$ given by the snake lemma is then K_2 .

To interpret these maps, we use our discussion of Bloch-Ogus theory. By Theorem 1.1 and (4.6), $H^2(S, \mathcal{E})$, $H^2(S - Z, \mathcal{E})$ and $\ker(H_Z^3(S, \mathcal{E}) \rightarrow H^3(S, \mathcal{E}))$ are subquotients of

$$\bigoplus_{t \in \Sigma^{(0)}} \bigoplus_i \ker(K(C(t_i))^* \otimes \mathbf{Q}/\mathbf{Z}) \rightarrow \bigoplus_{x \in C(t_i)} \mathbf{Q}/\mathbf{Z},$$

$$\bigoplus_{t \in (\Sigma - Z)^{(0)}} \bigoplus_i \ker(K(C(t_i))^* \otimes \mathbf{Q}/\mathbf{Z}) \rightarrow \bigoplus_{x \in C(t_i) - q_t^{-1}(Z)} \mathbf{Q}/\mathbf{Z},$$

and

$$\bigoplus_{t \in (\Sigma - Z)^{(0)}} \bigoplus_{x \in q_t^{-1}(Z \cap C(t))} \mathbf{Q}/\mathbf{Z}$$

respectively, while by (4.5) and (4.9), the images of β and α_3 are subquotients of $\varinjlim_n \ker \partial_n^X$ and $\varinjlim_n \ker \partial_n^U$ respectively. These groups fit into the following diagram, compatible with (4.10):

$$(4.11) \quad \begin{array}{ccc} \bigoplus_{t \in \Sigma^{(0)}} \bigoplus_i \ker(K(C(t_i))^* \otimes \mathbf{Q}/\mathbf{Z}) \rightarrow \bigoplus_{x \in C(t_i)} \mathbf{Q}/\mathbf{Z} & \longrightarrow & \varinjlim_n \ker \partial_n^X \\ \downarrow & & \downarrow \alpha'_4 \\ \bigoplus_{t \in (\Sigma - Z)^{(0)}} \bigoplus_i \ker(K(C(t_i))^* \otimes \mathbf{Q}/\mathbf{Z}) \rightarrow \bigoplus_{x \in C(t_i) - q_t^{-1}(Z)} \mathbf{Q}/\mathbf{Z} & \xrightarrow{\alpha'_3} & \varinjlim_n \ker \partial_n^U \\ \downarrow \alpha'_2 & & \\ \bigoplus_{t \in (\Sigma - Z)^{(0)}} \bigoplus_{x \in q_t^{-1}(Z \cap C(t))} \mathbf{Q}/\mathbf{Z} & & \end{array}$$

The map α_1 can be understood using §3. If $\gamma \in G_2$, we choose some suitable representative γ_1 of $\alpha_1(\gamma)$ in $\bigoplus_{t \in (\Sigma - Z)^{(0)}} \bigoplus_{x \in q_t^{-1}(Z \cap C(t))} \mathbf{Q}/\mathbf{Z}$. γ_1 can be pulled back via α'_2 , using the precise interpretation of the map in (4.6) corresponding to δ , and then pushed forward to γ_3 via α'_3 , considering the rational functions on $C(t_i)$ as rational functions on the corresponding divisors in X . γ_3 can in turn be pulled back via α'_4 to γ_4 by finding suitable rational functions on the components of $f^{-1}(Z)$, using the explicit maps of (4.5). Finally, via (4.3), this gives a 2-cycle on X , and the class of this cycle in $C(X/S)$ is the desired element.

We now give a representative special case of this construction which is quite important. If the reader should need to compute the obstruction in a more complicated case when $C(X/S) \neq 0$, then this case along with those of the examples should provide a sufficient guide.

Theorem 4.3. *Given the hypothesis of Theorem 4.1, assume furthermore that Z is an irreducible non-singular curve disjoint from Σ . Let $\gamma \in H^2_Z(S, i_*A)$ be represented by $\sigma \in MW(f^{-1}(Z)/Z)$, with σ_0 the zero section. If $\iota : f^{-1}(Z) \rightarrow X$ is the inclusion, then $\phi(\gamma) = 0$ and $\psi'(\gamma)$ is the class of $\iota_*[\sigma - \sigma_0]$ in $C(X/S)$.*

Proof. In fact, since $Z \cap \Sigma = \emptyset$, $H^2_Z(S, \mathcal{E}) = H^3_Z(S, \mathcal{E}) = 0$ and $G_1 = \ker \phi = H^2_Z(S, i_*A) = H^2_Z(S, P_{X/S})$, and we have the map of (4.9)

$$G_1 = H^2_Z(S, P_{X/S}) \xrightarrow{g} H^2(S, P_{X/S}).$$

Now

$$\begin{aligned} H^2_Z(S, P_{X/S}) &= H^3_Z(X, \mathbf{G}_m) / H^3_Z(S, \mathbf{G}_m) \\ &= H^1(f^{-1}(Z), \mathbf{Q}/\mathbf{Z}) / H^1(Z, \mathbf{Q}/\mathbf{Z}) \\ &= CH^1(f^{-1}(Z))_{tors} / f^*CH^1(Z)_{tors}, \end{aligned}$$

and g is induced via the factorization

$$CH^1(f^{-1}(Z))_{tors} \xrightarrow{\iota_*} CH^2(X)_{tors} \xrightarrow{\lambda_X} H^3(X, \mathbf{G}_m).$$

The class in $CH^1(f^{-1}(Z))_{tors}$ corresponding to σ is the torsion cycle $[\sigma - \sigma_0]$, and thus $\psi'(\gamma) = \iota_*[\sigma - \sigma_0] \in C(X/S)$.

Remark 4.4. More generally, suppose Z is an irreducible curve which intersects Σ transversally, and an element $\gamma \in H^2_Z(S, i_*A)$ is given by $\sigma \in MW(f^{-1}(Z)/Z)$, with $\phi(\gamma) = 0$. Then using (4.10) and (4.11), one finds that $\psi'(\gamma) = \iota_*[\sigma - \sigma_0] + \alpha$, where α is a cycle supported on $f^{-1}(\Sigma)$. See the second example below for a specific calculation.

Example 4.5. We continue the examples of 2.13 and 3.8.

1) Here $CH^2(X_1) = \mathbf{Z}p_1^*P \oplus (p_2^*\text{Pic } E).p_1^*l$ where $p_1, p_2 : \mathbf{P}^2 \times E \rightarrow \mathbf{P}^2, E$ are the first and second projections, $P \in \mathbf{P}^2$ the class of a point and $l \subseteq \mathbf{P}^2$ the class of a line. Thus $C(X_1/\mathbf{P}^2) = CH^2(X_1)_{tors} = (\text{Pic } E)_{tors}$. Thus by Theorem 4.3, ψ' takes $\sigma_i - \sigma_0 \in (\text{Pic } (Z_i \times E))_{tors}$, where σ_i is a torsion section of $p_1^{-1}(Z_i)$ with σ_0 the zero section, to the 2-cycle $\sigma_i - \sigma_0$, which can be written as $p_1^*Z_i.p_2^*(P_i - P_0)$, $P_i, P_0 \in E$. This is rationally equivalent to

$$(\text{deg } Z_i)p_1^*l.p_2^*(P_i - P_0).$$

Thus, if $\gamma \in H^2_Z(S, i_*A) = H^2_Z(S, P_{X/S})$ is determined by points $P_i \in E, 1 \leq i \leq n$, ($Z = Z_1 \cup \dots \cup Z_n$) with $P_i - P_0 \in (\text{Pic } E)_{tors}$, then we have

$$\psi'(\gamma) = \sum_{i=1}^n \text{deg } Z_i(P_i - P_0) \in C(X_1/\mathbf{P}^2) = (\text{Pic } E)_{tors}.$$

Thus we have the following special cases:

$n = 1$: Z is an irreducible non-singular curve of degree d . Then

$$\text{III}_{\mathbf{P}^2-Z}(A) / \text{III}_{\mathbf{P}^2}(A) = {}_d\text{Pic } E = (\mathbf{Z}/d\mathbf{Z})^{\oplus 2}.$$

Thus the multiplicity along Z must divide d . (See [9] for the first proof of this fact.)

$n = 2$: Let Z be the union of two lines in \mathbf{P}^2 . Then

$$\text{III}_{\mathbf{P}^2-Z}(A) / \text{III}_{\mathbf{P}^2}(A) = (\mathbf{Q}/\mathbf{Z})^{\oplus 2}.$$

We can obtain any desired multiplicity along one of the lines, but the same multiplicity is then forced along the other line, with invariant the negative of the invariant along the first line.

2) By construction, there is a map $g : X_2 \rightarrow \mathbf{P}^1$ via $X_2 \rightarrow \mathbf{P}^2 \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$. The general fibre is a non-singular quadric surface and there are four singular fibres consisting each of two components, a \mathbf{P}^2 and a scroll. The four \mathbf{P}^2 's can be contracted to smooth points, leaving $g' : X' \rightarrow \mathbf{P}^1$, a relatively minimal quadric bundle. It is well-known that $CH^2(X') = \mathbf{Z} \oplus \text{Pic } E$, where E is the curve (elliptic in this case) obtained by taking the branched cover $\xi : E \rightarrow \mathbf{P}^1$, branched at the four points $\{P_1, \dots, P_4\}$ where the fibres of g' are singular. For a point $P \in E$, the corresponding cycle is a line l_P in the appropriate family of lines in the quadric $X'_{\xi(P)}$. The curve E is of course isomorphic to any of the non-singular fibres of $f_2 : X_2 \rightarrow \mathbf{P}^2$. We denote the components of $f_2^{-1}(C)$ by Y_1, \dots, Y_5, Y_3 the multiple component, with Y_1, Y_2, Y_4 and Y_5 mapped to P_1, P_2, P_3 and P_4 respectively by g .

Now consider $\gamma \in H^2_{Z_1}(S, i_*A) = \ker \phi$. In the notation of (4.7), $G_1 = 0$ and $G_2 = \ker \phi$. Let $\sigma_0 : \mathbf{P}^2 \rightarrow X_2$ be the section whose image is the \mathbf{P}^2 contained in $g^{-1}(P_1)$, and let $\sigma : \mathbf{P}^2 \rightarrow X_2$ be the section whose image is the \mathbf{P}^2 contained in $g^{-1}(P_2)$. Let $S_0 = \sigma_0(\mathbf{P}^2) \cap f^{-1}(Z_1)$ be the zero section in $MW(f^{-1}(Z_1)/Z_1)$, and without loss of generality, suppose $\gamma \in H^2_{Z_1}(\mathbf{P}^2, i_*A)$ is given by the section

$S = \sigma(\mathbf{P}^2) \cap f^{-1}(Z_1)$. Then as one can see either by direct calculation or by referring to [6], we find on $f^{-1}(Z_1)$ that

$$2(S - S_0) \sim Y_1 \cap f^{-1}(Z_1) - Y_2 \cap f^{-1}(Z_1).$$

Thus, by Theorem 3.3, γ maps in $H_{Z_1}^3(S, \mathcal{E})$ to the element represented by $(1/2, 1/2, 0, 0, 0)$ at each point in $Z_1 \cap C = \{Q_1, \dots, Q_n\}$. Following diagram (4.11), this pulls back to an element

$$(h \otimes 1/2, h \otimes 1/2, 0, 0, 0) \in \bigoplus_{i=1}^5 \ker(K(C(t_i))^* \otimes \mathbf{Q}/\mathbf{Z}) \rightarrow \bigoplus_{x \in C(t_i) - q_i^{-1}(Z)} \mathbf{Q}/\mathbf{Z}$$

where $h \in K(C(t))^* = K(C(t_i))^*$ is a rational function whose associated principal divisor is

$$(h) \equiv \sum_{i=1}^n Q_i \pmod{2}.$$

(For example, we can choose h so that $(h) = \sum_{i=1}^n (-1)^i Q_i$, since n is even.) This then maps to $\ker \partial_2^U$ by considering h as a rational function on Y_i via the pull-back $Y_i \rightarrow C(t_i)$. To pull this back to an element of $\ker \partial_2^X$, we need a rational function $g \in K(f^{-1}(Z_1))^*$ which will together with h in $K(Y_1)^*$ and $K(Y_2)^*$ yield an element of $\ker \partial_2^X$. To do this, on $f^{-1}(Z_1)$ we have, as we have already noted,

$$2(S - S_0) \sim \sum_{i=1}^n C_1^i - C_2^i$$

with $C_j^i = Y_j \cap f^{-1}(Q_i)$, so we take $g \in K(f^{-1}(Z_1))^*$ so that

$$(g) = 2(S - S_0) + \sum_{i=1}^n C_2^i - C_1^i,$$

and $g \otimes 1/2$ then cancels the contributions of $h \otimes 1/2$ on Y_1 and Y_2 in $\bigoplus_{x \in X^{(2)}} \mathbf{Z}/2\mathbf{Z}$.

The recipe of (4.3) tells us that the torsion cycle corresponding to the collection of functions $h \in K(Y_1)^*$, $h \in K(Y_2)^*$ and $g \in K(f^{-1}(Z_1))^*$ representing an element of $\ker \partial_2^X$ is

$$S - S_0 + nC_2/2 - nC_1/2,$$

where C_2 is the class of C_2^i for any i . Of course, $n = 2 \deg Z_1 = 2d$, so we obtain

$$\psi'(\gamma) = [S - S_0 + dC_2 - dC_1] \in C(X/S) = (\text{Pic } E)_{tors},$$

where square brackets denote the class of the cycle in $C(X/S)$. Let l_i be a line on the cone $g'^{-1}(P_i)$, and $\pi_1 : X_2 \rightarrow X'$ be the blowing-down. Then

$$\pi_{1*}([S - S_0 + dC_2 - dC_1]) = dl_2 - dl_1$$

on X' . In fact, $l_2 - l_1$ is a 2-torsion cycle, so we see that $\psi'(\gamma) = 0$ if and only if d is even.

Finally, we obtain

$$\text{III}_{\mathbf{P}^2 - Z_1}(A) / \text{III}_{\mathbf{P}^2}(A) = \begin{cases} \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } \deg Z_1 \text{ is even,} \\ 0 & \text{if } \deg Z_1 \text{ is odd.} \end{cases}$$

We next consider Z_2 . Since for this situation we showed $\ker \phi = H_l^2(S, i_*A)$, we can replace Z_2 with l . We have $f^{-1}(l) = l \times E$, $p_2 : l \times E \rightarrow E$ the second

projection. We have $\sigma_0(S_2) \cap f^{-1}(l) = p_2^{-1}(P_0)$ for some $P_0 \in E$. Given a section $S \in MW(f^{-1}(l)/l)_{tors}$, we have $S = p_2^{-1}(P)$ for some $P \in E$, and the corresponding cycle class in $C(X'_2/S_2) = CH_2(X'_2)_{tors}$ is $[S - S_0]$ by Theorem 4.3. It is easy to see that $[S - S_0]$ is not rationally equivalent to zero unless $S = S_0$ by restricting this cycle to $f^{-1}(l)$. Thus $\ker \phi = 0$, and

$$\text{III}_{\mathbf{P}^2-l}(A) = \text{III}_{\mathbf{P}^2}(A).$$

We obtain no multiple fibres along l .

3) By running Mori’s minimal model algorithm on X_3 , one finds that X_3 is obtained by blowing up a \mathbf{P}^1 bundle over the scroll F_1 in a series of points and \mathbf{P}^1 ’s. Thus $CH^2(X)_{tors} = 0$, and $C(X/S) = 0$. Hence there is no further obstruction. We omit the details of this analysis.

4) Similarly, X_4 can be obtained by blowing up points and \mathbf{P}^1 ’s and some flops from \mathbf{P}^3 blown up in 8 points, except for a final blow-up of the singular curve in the divisor over the I_1 curve, this being a curve C of genus 3. From this description, we see that $CH^2(X_4)_{tors} = (\text{Pic } C)_{tors}$, and $C(X/S) = 0$. Again there is no obstruction. For details on the blowups, see [11], where the blowups of cuspidal points are analysed.

5) We add another example. Suppose $f : X \rightarrow S$ is a modified Miranda model with a curve $C \subseteq S$ with $C^2 = 0$ of fibre type I_M , $M > 0$, and $f^{-1}(C)$ a product of C and a cycle of M' rational curves. Then it is easy to check that $H^2_C(S, i_*A) = \ker \phi = \mathbf{Q}/\mathbf{Z}$ and that $CH^1(f^{-1}(C))_{tors} = (CH^1(C)_{tors})^{M'}$. In fact sequence (4.2) gives

$$0 \rightarrow \mathbf{Q}/\mathbf{Z} \rightarrow H^3_{f^{-1}(C)}(X, \mathbf{Q}/\mathbf{Z}) \rightarrow CH^1(f^{-1}(C))_{tors} \rightarrow 0$$

and this \mathbf{Q}/\mathbf{Z} term maps to zero in $H^3(X, \mathbf{Q}/\mathbf{Z})$. Thus we see that

$$\text{III}_{S-C}(A) / \text{III}_S(A) = \mathbf{Q}/\mathbf{Z}$$

completely independently of any other geometry of X .

APPENDIX: THE CALCULATIONS FOR I_m , $m \geq 1$

We begin with some auxiliary geometric results about the models of elliptic threefolds we use. We are assuming that $f : X \rightarrow S$ is a modified Miranda model, which, as we mentioned in §1, has Σ s.n.c. and contains only a small number of possible collisions. In particular, we have no collisions $I_{M_1} + I_{M_2}$ with M_1 and M_2 odd, and, as mentioned earlier, we have blown up the minimal model so as to avoid divisors which are not normal over curves of fibre type I_M , M odd. Under these conditions, if S is projective, then we have strong conditions on these curves of the discriminant locus.

We also note that the Miranda model is hardly unique; it depends on the order in which one resolves the singular loci of the Weierstrass model over various curves of Σ . In our case, the resolution we described assumes that all singularities over curves of type I_M with M even are resolved before those with M odd; in Table 1, we resolve over Σ_2 before Σ_1 .

Theorem A.1. *Let $C \subseteq \Sigma$ be a complete curve of fibre type I_M , case I, colliding with curves D_1, \dots, D_r of fibre type I_{M_i} , $1 \leq i \leq r$, which are resolved before C , and curves D_{r+1}, \dots, D_s of fibre type I_{M_i} , $r + 1 \leq i \leq s$, which are resolved after*

C. Then one obtains

$$\sum_{j=1}^s M_j = -MC^2,$$

and, for M even, with $f^{-1}(C) = Y_1 \cup \dots \cup Y_M$ as in Table 1,

$$(C_i^2)_{Y_{i+1}} = \begin{cases} iC^2 + \sum_{j=1}^r \frac{M_j}{2}, & 1 \leq i \leq M/2, \\ iC^2 + \sum_{j=1}^r \frac{M_j}{2} + \sum_{j=r+1}^s M_j, & M/2 + 1 \leq i \leq M - 1 \end{cases}$$

(here $(C_i^2)_{Y_{i+1}}$ denotes the self-intersection of the curve C_i considered as a curve on the surface Y_{i+1}) and

$$(C_1^2)_{Y_1} = (C_M^2)_{Y_1} = -\sum_{j=1}^r \frac{M_j}{2}.$$

For M odd, with $f^{-1}(C) = Y_1 \cup \dots \cup Y_{M+1}$, one has

$$(C_i^2)_{Y_{i+1}} = \begin{cases} iC^2 + \sum_{j=1}^r \frac{M_j}{2}, & 1 \leq i \leq (M-1)/2, \\ MC^2 + \sum_{j=1}^r M_j, & i = (M+1)/2, \\ (i-1)C^2 + \sum_{j=1}^r \frac{M_j}{2}, & (M+3)/2 \leq i \leq M, \end{cases}$$

and

$$(C_1^2)_{Y_1} = (C_{M+1}^2)_{Y_1} = -\sum_{j=1}^r \frac{M_j}{2}.$$

Proof. Observe that by our assumptions on ordering, M_i is even for $1 \leq i \leq r$, and if M is odd, then $r = s$. Put

$$M' := \begin{cases} M, & M \text{ even,} \\ M + 1, & M \text{ odd.} \end{cases}$$

Consulting Table 1, the geometry of Y_1 is as shown in Figure A.1 where σ_0 is the zero-section of $f : X \rightarrow S$. From this, it is clear that on Y_1

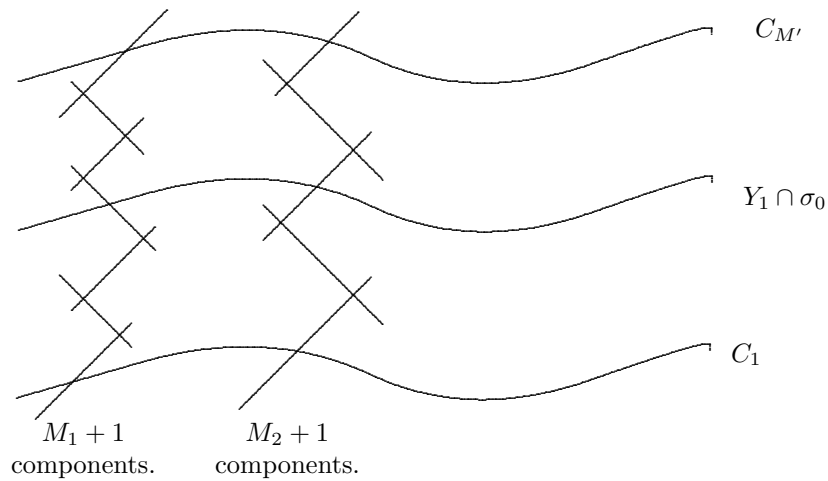


FIGURE A.1

$$(A.1) \quad C_{M'}^2 = C_1^2 = - \sum_{i=1}^r \frac{M_i}{2},$$

since if we contract all components of $f : Y_1 \rightarrow C$ not intersecting $Y_1 \cap \sigma_0$, we obtain a scroll with three disjoint sections, each of which must then have self-intersection 0. We can write

$$f^*C = \sum_{i=1}^{M'} a_i Y_i$$

where

$$a_i = \begin{cases} 2, & M \text{ odd and } i = (M + 3)/2, \\ 1, & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} C^2 = f^*C.C_i &= \sum_{j=1}^{M'} a_j Y_j.C_i \\ &= a_i Y_i.C_i + a_{i+1} Y_{i+1}.C_i \\ &= a_i Y_i.(s_{i+1})_* C_i + a_{i+1} Y_{i+1}.(s_i)_* C_i \\ &= a_i ((s_{i+1}^* Y_i).C_i)_{Y_{i+1}} + a_{i+1} ((s_i^* Y_{i+1}).C_i)_{Y_i} \end{aligned}$$

where $s_i : Y_i \rightarrow X$ is the inclusion. Now $Y_i.Y_{i+1} = C_i$ (or, if $M' = 2$, $Y_1.Y_2 = C_1 + C_2$, with C_1, C_2 disjoint), so we get

$$(A.2) \quad C^2 = a_i (C_i^2)_{Y_{i+1}} + a_{i+1} (C_i^2)_{Y_i}.$$

We also have

$$(A.3) \quad (C_{i-1}^2)_{Y_i} + (C_i^2)_{Y_i} = 0$$

whenever Y_i is a geometrically ruled surface (which is true for $i \neq 1$ if M is odd and $i \neq 1, M/2 + 1$ if M is even), since C_{i-1} and C_i are disjoint. By using (A.2) and (A.3) repeatedly, for M odd, we see that

$$(A.4) \quad (C_i^2)_{Y_{i+1}} = \begin{cases} iC^2 + \sum_{j=1}^r \frac{M_j}{2}, & 1 \leq i \leq (M - 1)/2, \\ MC^2 + \sum_{j=1}^r M_j, & i = (M + 1)/2, \\ (i - 1)C^2 + \sum_{j=1}^r \frac{M_j}{2}, & (M + 3)/2 \leq i \leq M, \end{cases}$$

and

$$(C_{M'}^2)_{Y_1} = MC^2 + \sum_{j=1}^r \frac{M_j}{2} = - \sum_{j=1}^r \frac{M_j}{2}$$

by (A.1), and hence $MC^2 = - \sum_{j=1}^r M_j$ as promised.

When M is even, (A.4) remains valid for $1 \leq i \leq M/2$. The geometry of $Y_{M/2+1}$ is then as shown in Figure A.2. From this, one sees on $Y_{M/2+1}$ that

$$(A.5) \quad C_{M/2}^2 + \sum_{j=r+1}^s M_j + C_{M/2+1}^2 = 0.$$

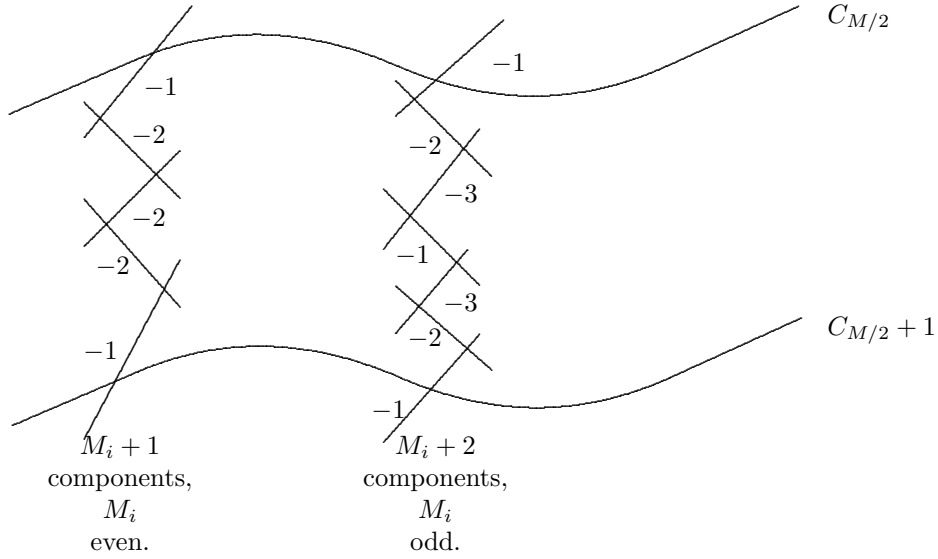


FIGURE A.2

Using this, one continues, seeing that

$$(C_i^2)_{Y_{i+1}} = iC^2 + \sum_{j=1}^r \frac{M_j}{2} + \sum_{j=r+1}^s M_j$$

and

$$(C_M^2)_{Y_1} = MC^2 + \sum_{j=1}^r \frac{M_j}{2} + \sum_{j=r+1}^s M_j = -\sum_{j=1}^r \frac{M_j}{2},$$

so again one has

$$MC^2 = -\sum_{j=1}^s M_j.$$

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Theorem A.2. *Let $C \subseteq \Sigma$ be a complete curve of fibre type I_M , of case I^* , colliding with curves D_1, \dots, D_s of fibre type I_{M_i} , $1 \leq i \leq s$, with D_1, \dots, D_r resolved before C and D_{r+1}, \dots, D_s resolved after C , and also colliding with curves B_1, \dots, B_u of fibre type $I_{N_i}^*$, $1 \leq i \leq u$. Then u is even and*

$$-MC^2 = \sum_{j=1}^s M_j + \sum_{j=1}^u N_j.$$

In addition, if one puts

$$\eta_i = \begin{cases} N_i + 2 & \text{if } N_i \text{ is even, } M > 1; \\ N_i + 1 & \text{if } N_i \text{ is odd, } M > 1; \\ N_i + 3 & \text{if } M = 1, \end{cases}$$

and $f^{-1}(C) = Y_1 \cup \dots \cup Y_{M/2+1}$ if M is even and $f^{-1}(C) = Y_1 \cup \dots \cup Y_{(M+3)/2}$ if M is odd as in Table 2, then one has

$$(C_1^2)_{Y_1} = 2u - \sum_{j=1}^r M_j - \sum_{j=1}^u \eta_j$$

and

$$(C_i^2)_{Y_{i+1}} = 2iC^2 - 2u + \sum_{j=1}^r M_j + \sum_{j=1}^u \eta_j$$

for

$$\begin{cases} 1 \leq i \leq M/2 & \text{if } M \text{ is even,} \\ 1 \leq i \leq (M-1)/2 & \text{if } M \text{ is odd,} \end{cases}$$

and if M is odd,

$$(C_{(M+1)/2}^2)_{Y_{(M+3)/2}} = 2u.$$

If M is odd, then $Y_{(M+3)/2} \cong \mathbf{P}_C(\mathcal{E})$ where \mathcal{E} is a rank 2 vector bundle on C with $c_1(\mathcal{E}) = C^2 - u/2$.

Proof. Letting $\sigma_0 \subseteq X$ be the zero section of $f : X \rightarrow S$, one sees that

$$\begin{aligned} ((\sigma_0 \cap Y_1)^2)_{Y_1} &= \sigma_0^2 \cdot Y_1 \\ &= \sigma_0^2 \cdot f^*C \\ &= f_*\sigma_0^2 \cdot C \\ &= -L \cdot C \end{aligned}$$

where $\mathcal{O}_S(L)$ is the line bundle \mathcal{L} defining the Weierstrass model (see §1).

Now $\Sigma \sim 12L$, and thus

$$12C \cdot L = MC^2 + \sum_{j=1}^s M_j + \sum_{j=1}^u (N_j + 6),$$

as C appears with multiplicity M in Σ , D_j with multiplicity M_j and B_j with multiplicity $N_j + 6$ (see for example [16]). But since Σ is s.n.c., the J function induces a morphism $J : S \rightarrow \mathbf{P}^1$, and $\Sigma \sim S_\infty + \sum n_i \Sigma_i$ where S_∞ is the fibre over $\infty \in \mathbf{P}^1$ and the Σ_i are the components of Σ , $n_i = 0, 2, 3, 4, 6, 8, 9$ or 10 in the cases that Σ_i is of fibre type $I_a, II, III, IV, I_a^*, IV^*, III^*$ or II^* respectively. We thus have

$$C \cdot \Sigma = S_\infty \cdot C + \sum n_i \Sigma_i \cdot C = 0 + \sum_{j=1}^u 6 = 6u.$$

Thus

$$6u = MC^2 + \sum_{j=1}^n M_j + \sum_{j=1}^u (N_j + 6),$$

and

$$-MC^2 = \sum_{j=1}^s M_j + \sum_{j=1}^u N_j,$$

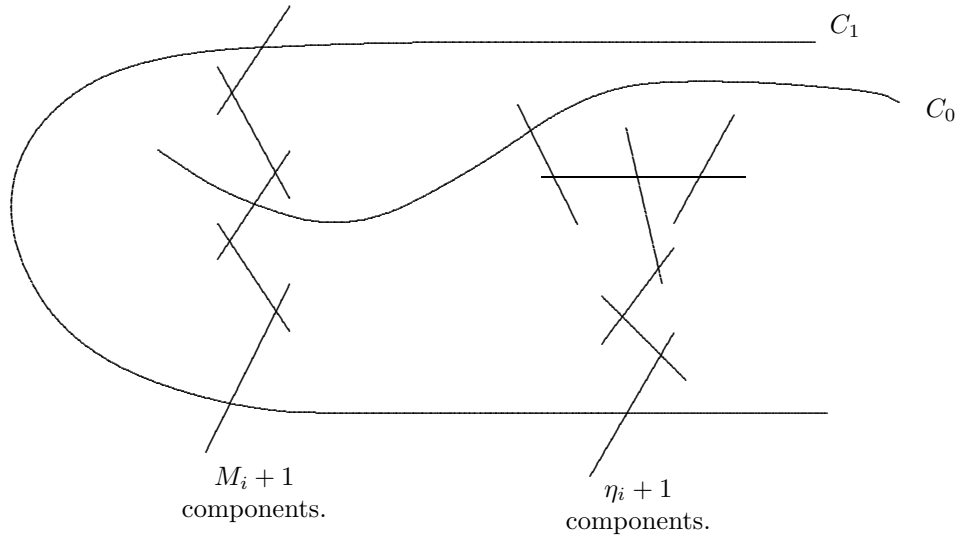


FIGURE A.3

proving the first claim. In particular,

$$((\sigma_0 \cap Y_1)^2)_{Y_1} = -u/2,$$

and u is even.

We now look at the picture for Y_1 (Figure A.3).

From this diagram one sees that, if \bar{Y}_1 is the surface obtained by contracting all components of fibres of $Y_1 \rightarrow C$ not intersecting $C_0 = \sigma_0 \cap Y_1$, and if \bar{C}_1 is the image of C_1 in \bar{Y}_1 , we can write

$$\bar{C}_1 \equiv 2C_0 + af,$$

where \equiv denotes numerical equivalence, f the class of a fibre. Since $\bar{C}_1 \cdot C_0 = 0$, we see that $a = -2C_0^2$, whence $\bar{C}_1^2 = -4C_0^2 = 2u$, and so

$$(C_1^2)_{Y_1} = 2u - \sum_{j=1}^r M_j - \sum_{j=1}^u \eta_j.$$

One then obtains, analogously to the derivation of (A.2), keeping in mind that the curves C_i map 2-1 to C , that

$$(A.6) \quad 2C^2 = a_i(C_i)_{Y_{i+1}}^2 + a_{i+1}(C_i)_{Y_i}^2.$$

Using (A.6) and (A.3), we get

$$(C_i^2)_{Y_{i+1}} = 2iC^2 - 2u + \sum_{j=1}^r M_j + \sum_{j=1}^u \eta_j$$

for

$$\begin{cases} 1 \leq i \leq M/2 & \text{if } M \text{ is even,} \\ 1 \leq i \leq (M-1)/2 & \text{if } M \text{ is odd,} \end{cases}$$

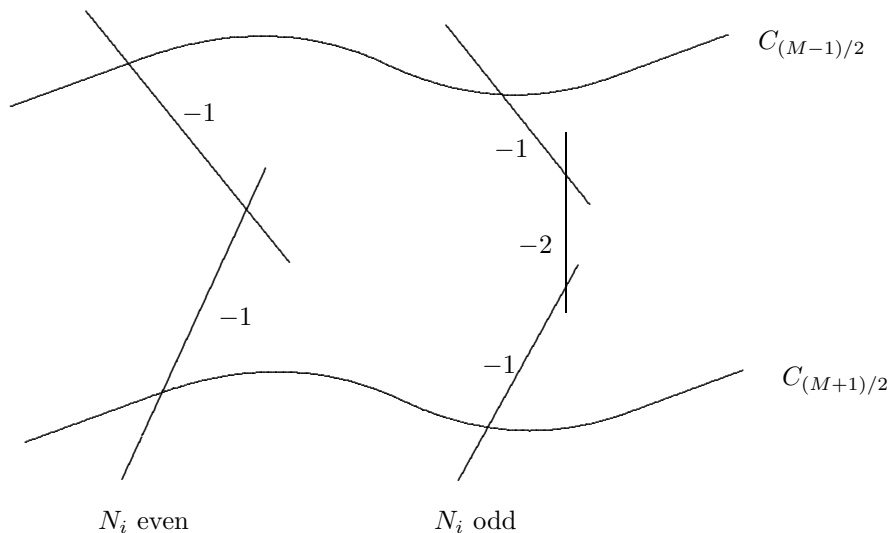


FIGURE A.4

and

$$\begin{aligned}
 (C_1^2)_{Y_2} &= 2C^2 - 4u + 2\left(\sum_{j=1}^r M_j + \sum_{j=1}^u \eta_j\right) \\
 &= 2u
 \end{aligned}$$

if $M = 1$, using the first equality of the theorem. If M is odd, $M > 1$, then $Y_{(M+1)/2}$ looks like Figure A.4 so

$$(C_{(M+1)/2}^2)_{Y_{(M+1)/2}} = -(\# N_i \text{ even} + 2(\# N_i \text{ odd})) - (C_{(M-1)/2}^2)_{Y_{(M+1)/2}}$$

and thus

$$\begin{aligned}
 (C_{(M+1)/2}^2)_{Y_{(M+3)/2}} &= 2C^2 - 2(C_{(M+1)/2}^2)_{Y_{(M+1)/2}} \\
 &= 2MC^2 - 4u + 2\left(\sum_{j=1}^r M_j + \sum_{j=1}^u (\eta_j + 2) - \# N_j \text{ even}\right) \\
 &= 2MC^2 - 4u + 2\left(\sum_{j=1}^r M_j + \sum_{j=1}^u (N_j + 3)\right)
 \end{aligned}$$

by the definition of η_j . Using the first equality of the theorem, this reduces to the form given.

For the last statement, $Y_{(M+3)/2}$ was the exceptional divisor of the blowup of a curve C' in X' , the original Miranda model, with $f' : X' \rightarrow S$ mapping C' isomorphically to C . We thus have $Y_{(M+3)/2} \cong \mathbf{P}_C(\mathcal{N}_{C'/X'})$, where $\mathcal{N}_{C'/X'}$ is the

normal bundle of C' in X' . Now

$$\begin{aligned} c_1(\mathcal{N}_{C'/X'}) &= c_1(\mathcal{T}_{X'}|_{C'}) - c_1(\mathcal{T}_{C'}) \\ &= -K_{X'}.C' + 2g_C - 2 \\ &= -f'^*(K_S + L).C' + 2g_C - 2 \\ &= -K_S.C - L.C + 2g_C - 2 \\ &= 2 - 2g_C + C^2 - C.L + 2g_C - 2 \\ &= C^2 - u/2. \end{aligned}$$

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We now move on to the proofs of Theorems 3.4 and 3.6. We first explain how to carry out the calculation in general. Let $f : X \rightarrow S$ satisfy the additional Hypothesis. We will assume that Z consists of precisely those components of Σ of fibre type I_M . Put $U = S - \Sigma'$ where Σ' consists of all components of Σ contained in S_0 and S_1 , the fibres of the J -map $J : S \rightarrow \mathbf{P}^1$ over $0, 1 \in \mathbf{P}^1$. If we put $\Delta = \Sigma - \Sigma'$, it follows from Miranda's analysis that $f^{-1}(\Delta)$ has simple normal crossings in $X_U = X \times_S U$. Furthermore, the only components of Δ which are complete are of fibre type I_M , $M \geq 1$, or I_M^* , $M \geq 1$, as these are the only components of Σ contained in fibres of J other than S_0 and S_1 . We thus have $H^3(U, \mathcal{E}) = H^3_\Delta(U, \mathcal{E}) = \bigoplus_{t \in I} H^3(U, \mathcal{E}_t)$, where $I \subseteq S^{(1)}$ is the set of generic points of curves of fibre type I_M or I_M^* , $M \geq 1$. Furthermore, by Theorem 2.11 and Remark 2.12, $H^2_Z(S, i_*A) = H^2_\Delta(U, i_*A)$. Thus we need to interpret the map

$$H^2_\Delta(U, i_*A) \xrightarrow{\phi_1} H^3_\Delta(U, \mathcal{E}).$$

By Prop. 2.2, since $\underline{H}^i_\Delta(U, \mathcal{E}) = 0, \forall i > 0$, we have $\underline{H}^2_\Delta(U, i_*A) = \underline{H}^2_\Delta(U, P_{X/S})$ and thus $H^2_\Delta(U, i_*A) = H^0(U, \underline{H}^2_\Delta(U, P_{X/S}))$. By Proposition 1.6, we then have exact sequences

$$0 \longrightarrow \bigoplus_{i < j} (\mathbf{Q}/\mathbf{Z})_{Z_i \cap Z_j} \xrightarrow{\alpha} \bigoplus_{(i,j) < (i',j')} k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j \cap Y_{i'}^{j'}} \longrightarrow \mathcal{F} \longrightarrow 0$$

and

$$0 \rightarrow H^2_\Delta(U, i_*A) \xrightarrow{\beta} H^0(\mathcal{F}) \rightarrow \bigoplus_{i,j} H^0(U, R^2k_*(\mathbf{Q}/\mathbf{Z})_{Y_i^j})$$

where the Y_i^j are the components over the Z_i 's, the components of Δ . Of course,

$$H^3_\Delta(U, \mathcal{E}) = H^3(U, \mathcal{E}) = \bigoplus_i \frac{\bigoplus_j H^2(C_i^j, \mathbf{Q}/\mathbf{Z})}{H^2(Z_i, \mathbf{Q}/\mathbf{Z})}$$

where $Y_i^j \rightarrow C_i^j \rightarrow Z_i$ is the Stein factorization, and thus there is a natural inclusion of $H^3_\Delta(U, \mathcal{E})$ in

$$\bigoplus_i \frac{\bigoplus_j H^2(Y_i^j, \mathbf{Q}/\mathbf{Z})}{H^2(Z_i, \mathbf{Q}/\mathbf{Z})}.$$

The diagram

$$(A.7) \quad \begin{array}{ccc} 0 & \longrightarrow & H^2_{\Delta}(U, i_*A) & \xrightarrow{\beta} & H^0(\mathcal{F}) \\ & & \downarrow & & \downarrow d \\ 0 & \longrightarrow & H^3_{\Delta}(U, \mathcal{E}) & \longrightarrow & \bigoplus_i \frac{\bigoplus_j H^2(Y_i^j, \mathbf{Q}/\mathbf{Z})}{H^2(Z_i, \mathbf{Q}/\mathbf{Z})} \end{array}$$

is commutative, where d is the map induced by

$$d: \bigoplus_{(i,j) < (i',j')} H^0(Y_i^j \cap Y_{j'}^{i'}, \mathbf{Q}/\mathbf{Z}) \rightarrow \bigoplus_{(i,j)} H^2(Y_i^j, \mathbf{Q}/\mathbf{Z})$$

of Proposition 1.5. Thus we use the map d to understand ϕ_1 .

Proof of Theorem 3.4. We retain the assumptions on ordering of resolution of the Weierstrass model made in Theorem A.1. To calculate $\phi_t(\gamma)$, we need to know what the image of γ is in

$$H^0(\mathcal{F}) = \frac{\bigoplus_{(i,j) < (i',j')} H^0(Y_i^j \cap Y_{j'}^{i'}, \mathbf{Q}/\mathbf{Z})}{\bigoplus_{i < j} H^0(Z_i \cap Z_j, \mathbf{Q}/\mathbf{Z})}$$

under the map β . This we already know from our local calculations. Indeed, the curves $Y_i^j \cap Y_{i'}^{j'}$ of concern to us are $C_1, \dots, C_{M'}$ with $C_i = Y_i \cap Y_{i+1}$, $C_{M'} = Y_{M'} \cap Y_1$, and the curves $f_1^j, \dots, f_{M'+M_j}^j$ over the collision points $I_M + I_{M_j}$, $1 \leq j \leq s$. For our purposes, we can assume that there are no other curves of the form $Y_i^j \cap Y_{i'}^{j'}$ than those mentioned, as we are only interested in what is going on over C . A representative of $\beta(\gamma)$ in the H^0 of these curves can be easily read off from the proofs of Theorems 2.9 and 2.10. Let a_i^j , $1 \leq j \leq s$, $1 \leq i \leq M' + M_j$, be the image of a representative of $\beta(\gamma)$ in $H^0(f_i^j, \mathbf{Q}/\mathbf{Z})$; these values are given in the proof of Theorem 2.9, where they are written a_i , since there we only deal with one collision. Since we are obtaining an element in $H^0(\mathcal{F})$, we can choose a representative with $a_1^j = 0 \forall j$. Thus we obtain an element γ' in $\bigoplus_{i,j} H^0(f_i^j, \mathbf{Q}/\mathbf{Z}) \oplus \bigoplus_i H^0(C_i, \mathbf{Q}/\mathbf{Z})$ of the form $(a_1^1, \dots, a_{M'+M_j}^s, b, \dots, b, -b)$, which is the representative for $\beta(\gamma)$ in $\bigoplus_{(i,j) < (i',j')} H^0(Y_i^j \cap Y_{j'}^{i'}, \mathbf{Q}/\mathbf{Z})$.

Fix one of the components of $f^{-1}(C)$, say, Y_1 . We want to calculate the image of γ' in $H^2(Y_1, \mathbf{Q}/\mathbf{Z})$ under the first component of the map d . Call this image γ'_1 . We know that γ'_1 must be a pull-back of an element in $H^2(C, \mathbf{Q}/\mathbf{Z}) = \mathbf{Q}/\mathbf{Z}$ via the map $k : Y_1 \rightarrow C$, and if we choose a section of k , say C_1 , then if $C_1 \cdot \gamma'_1 = \gamma''_1$, then $\gamma'_1 = k^* \gamma''_1$, thinking of γ''_1 as an element of $H^2(C, \mathbf{Q}/\mathbf{Z})$. Thus, since C_1 intersects only C_1 and the f_1^j , $1 \leq j \leq r$, among the curves listed above (we're doing the odd case first), we get that

$$\gamma''_1 \in H^2(C, \mathbf{Q}/\mathbf{Z}) \subseteq H^2(Y_1, \mathbf{Q}/\mathbf{Z})$$

is

$$\begin{aligned} \gamma''_1 &= C_1 \cdot (bC_1 + \sum_{j=1}^r a_1^j f_1^j) = b(C_1^2)_{Y_1} + \sum_{j=1}^r a_1^j \\ &= - \sum_{j=1}^r \frac{M_j b}{2}, \end{aligned}$$

by Theorem A.1 and $a_1^j = 0$ for all j .

Continuing the same strategy, using the section C_i on the surface Y_{i+1} to intersect the image in $H^2(Y_{i+1}, \mathbf{Q}/\mathbf{Z})$, we get

$$\begin{aligned} Y_{i+1}, 1 \leq i \leq (M-1)/2 : \gamma''_{i+1} &= C_i \cdot (-bC_i + \sum_{j=1}^r a_{i+1}^j f_{i+1}^j) \\ &= -b(C_i^2)_{Y_{i+1}} + \sum_{j=1}^r a_{i+1}^j \\ &= -ibC^2 - \sum_{j=1}^r \frac{M_j b}{2} - \sum_{j=1}^r ic_j. \end{aligned}$$

Here C_i has a coefficient of $-b$ by our sign convention for the map d in Proposition 1.5, and the value for a_i^j is obtained from the proof of Proposition 2.9.

$$\begin{aligned} Y_{(M+3)/2} : \gamma''_{(M+3)/2} &= -b(C_{(M+1)/2}^2)_{Y_{(M+3)/2}} + \sum_{j=1}^r a_{(M+3)/2}^j \\ &= 2 \left(-\frac{bM}{2} C^2 - \sum_{j=1}^r \frac{M_j b}{2} - \sum_{j=1}^r \frac{M c_j}{2} \right). \end{aligned}$$

$$\begin{aligned} Y_{i+1}, (M+3)/2 \leq i \leq M : \gamma''_{i+1} &= -b(C_i^2)_{Y_{i+1}} + \sum_{j=1}^r a_{i+1}^j \\ &= -(i-1)bC^2 - \sum_{j=1}^r \frac{M_j b}{2} - \sum_{j=1}^r (i-1)c_j. \end{aligned}$$

Thus we get an element $(\gamma''_i)_{1 \leq i \leq M'} \in (\mathbf{Q}/\mathbf{Z})^{M'} = \bigoplus_{i=1}^{M'} H^2(C, \mathbf{Q}/\mathbf{Z})$; since

$$H^2(S, \mathcal{E}_t) = \text{coker}(H^2(C, \mathbf{Q}/\mathbf{Z}) \rightarrow \bigoplus_{i=1}^{M'} H^2(C, \mathbf{Q}/\mathbf{Z})),$$

we are dividing this out by \mathbf{Q}/\mathbf{Z} with the map $\mathbf{Q}/\mathbf{Z} \rightarrow (\mathbf{Q}/\mathbf{Z})^{M'}$ given by

$$a \mapsto (a, a, \dots, a, 2a, a, \dots, a)$$

where the $2a$ occurs in the $Y_{(M+3)/2}$ component. Thus we can adjust γ'' by any such amount. Using $a = \sum_{j=1}^r M_j b/2$, we obtain, with $\alpha = -bC^2 - \sum_{j=1}^r c_j$,

$$\begin{aligned} Y_i, 1 \leq i \leq (M+1)/2 : &(i-1)\alpha, \\ Y_{(M+3)/2} : &M\alpha, \\ Y_i, (M+5)/2 \leq i \leq M+1 : &(i-2)\alpha. \end{aligned}$$

For M even, we follow the same strategy, but have to keep in mind that for $r+1 \leq j \leq s$, the Y_i 's play the role of the Y_i' 's in Table 1. We have

$$\begin{aligned} Y_1 : &b(C_1^2)_{Y_1} + \sum_{j=1}^r a_1^j - \sum_{j=r+1}^s a_{1+M/2+M'_j}^j \\ &= -\sum_{j=1}^r \frac{bM_j}{2} + \sum_{j=r+1}^s \frac{M c_j}{2}. \end{aligned}$$

Here we use that

$$a_{(i+1)+M/2+M'_j}^j = a_{(M'_j+M+1)-(M/2-i)}^j = -(M/2 - i)c_j$$

for $0 \leq i \leq M/2 - 1$, keeping in mind that in this case the roles of the b 's and c 's are reversed.

$$\begin{aligned} Y_{i+1} : 1 \leq i \leq M/2 - 1 : & -b(C_i^2)_{Y_{i+1}} + \sum_{j=1}^r a_{i+1}^j - \sum_{j=r+1}^s a_{(i+1)+M/2+M'_j}^j \\ & = -ibC^2 - \sum_{j=1}^r \frac{M_j b}{2} - \sum_{j=1}^r ic_j + \sum_{j=r+1}^s (M/2 - i)c_j \\ & = -ibC^2 - \sum_{j=1}^s ic_j - \sum_{j=1}^r \frac{M_j b}{2} + \sum_{j=r+1}^s \frac{Mc_j}{2}. \end{aligned}$$

$$\begin{aligned} Y_{M/2+1} : & -b(C_{M/2}^2)_{Y_{M/2+1}} + \sum_{j=1}^r a_{M/2+1}^j - \sum_{j=r+1}^s a_1^j \\ & = -\frac{bM}{2}C^2 - \sum_{j=1}^r \frac{M_j b}{2} - \sum_{j=1}^r \frac{Mc_j}{2} \\ & = -\frac{bM}{2}C^2 - \sum_{j=1}^s \frac{Mc_j}{2} - \sum_{j=1}^r \frac{M_j b}{2} + \sum_{j=r+1}^s \frac{Mc_j}{2}. \end{aligned}$$

Now we use

$$a_{(i+1)-M/2+M'_j}^j = a_{(M+M'_j+1)-(3M/2-i)}^j = -(3M/2 - i)c_j$$

for $M/2 + 1 \leq i \leq M - 1$.

$$\begin{aligned} Y_{i+1} : M/2 + 1 \leq i \leq M - 1 : & \\ & -b(C_i^2)_{Y_{i+1}} + \sum_{j=1}^r a_{i+1}^j - \sum_{j=r+1}^s a_{(i+1)-M/2+M'_j}^j \\ & = -ibC^2 - \sum_{j=1}^r \frac{M_j b}{2} - \sum_{j=r+1}^s M_j b - \sum_{j=1}^r ic_j + \sum_{j=r+1}^s (3M/2 - i)c_j \\ & = -ibC^2 - \sum_{j=1}^s ic_j - \sum_{j=1}^r \frac{M_j b}{2} + \sum_{j=r+1}^s \frac{Mc_j}{2} \end{aligned}$$

using the fact that $M_j b = Mc_j$. Again, with $\alpha = -bC^2 - \sum_{j=1}^s c_j$, we get, after choosing a different representative by adding $\alpha + \sum_{j=1}^r M_j b/2 - \sum_{j=r+1}^s Mc_j/2$ to each component,

$$Y_i : i\alpha.$$

•

Proof of Theorem 3.6. First the case that M is odd.

We follow the same procedure and notation as in Theorem 3.4. The only additional complication is to take into account what the image of γ is in $H^0(\mathcal{F})$ on the

curves over the collision points of type $I_M + I_{N_i}^*$. But this is given by Theorem 3.5. For each collision $I_M + I_{N_i}^*$, set

$$\alpha_j^i = \begin{cases} (\phi_1(b))_j, & M \text{ odd,} \\ (\phi_1(b, d_i))_j, & M, N_i \text{ even,} \\ (\phi_1(b_i))_j, & M \text{ even, } N_i \text{ odd,} \end{cases}$$

using the notation in the statement of Theorem 3.5. If f_1^i, \dots, f_m^i are the components of the fibre over the collision point, then a representative of $\beta(\gamma)$ in the components $\bigoplus_j H^0(f_j^i, \mathbf{Q}/\mathbf{Z})$ of $H^0(\mathcal{F})$ comes from specialising the α_i 's, as in the proof of Theorem 3.5 using the last part of Table 2.

On Y_1 , using instead the section $\sigma_0 \cap Y_1$ (the section C_0 in Figure A.3), we get a contribution of

$$Y_1 : \sum_{j=1}^r a_{M'+1+M_j/2}^j + \sum_{j=1}^u \alpha_1^j = \sum_{j=1}^r -\frac{M_j b}{2}.$$

On Y_{i+1} , $1 \leq i \leq (M-1)/2$, we use the section C_i , keeping in mind that C_i now intersects both the curves f_{i+1}^j and $f_{M'+1-i}^j$:

$$\begin{aligned} & Y_{i+1}, 1 \leq i \leq (M-1)/2 : \\ & -b(C_i^2)_{Y_{i+1}} + \sum_{j=1}^r a_{i+1}^j + \sum_{j=1}^r a_{M'+1-i}^j + \sum_{j=1}^u \alpha_{n_1}^j \\ & = -2ibC^2 + 2bu - \sum_{j=1}^r M_j b - \sum_{j=1}^u \eta_j b - \sum_{j=1}^r ic_j - \sum_{j=1}^r (M-i)c_j \\ & = -\sum_{j=1}^r 2M_j b \\ & = 0 \end{aligned}$$

keeping in mind that $b = 0$ or $1/2$, that the η_j 's are even, ($M > 1$ in this range), and $Mc_j = M_j b$. (Remember this is all in \mathbf{Q}/\mathbf{Z} , where $2b = 0$.)

For $Y_{(M+3)/2}$, the contribution is calculated as follows. On $Y_{(M+3)/2}$,

$$C_{(M+1)/2} \equiv 2C_0 + af$$

for some integer a , with C_0 a section of $Y_{(M+3)/2} \rightarrow C$, and since $Y_{(M+3)/2} \cong \mathbf{P}_C(\mathcal{E})$ with $c_1(\mathcal{E}) = C^2 - u/2$ (Theorem 3.2), C_0^2 must have the same parity as $C^2 - u/2$. Now

$$(C_{(M+1)/2}^2)_{Y_{(M+3)/2}} = 4C_0^2 + 4a = 2u,$$

so $C_0^2 + a \equiv u/2 \pmod{2}$, and hence a has the same parity as C^2 since u is even. Since b is 0 or $1/2$, the class of $-bC_{(M+1)/2}$ in $H^2(Y_{(M+3)/2}, \mathbf{Q}/\mathbf{Z})$ is $bC^2 f$, where

f is a fibre of $Y_{(M+3)/2} \rightarrow C$. Thus we get, in total,

$$\begin{aligned} Y_{(M+3)/2} : bC^2 + \sum_{j=1}^r a_{(M+3)/2}^j + \sum_{j=1}^u \alpha_{N_j+4}^j \\ = bC^2 - \sum_{j=1}^r Mc_j + (\#N_j \text{ even})b \\ = bC^2 + (\#N_j \text{ even})b, \end{aligned}$$

since $Mc_j = M_j b$ and the M_j 's are all even. Notice that we have

$$C^2 \equiv MC^2 \equiv \sum_{j=1}^r M_j + \sum_{j=1}^u N_j \equiv \# N_j \text{ odd} \pmod{2}$$

so we get for $Y_{(M+3)/2}$

$$bC^2 + (\#N_j \text{ even})b = (\#N_j \text{ odd})b + (\#N_j \text{ even})b = ub = 0.$$

Now $H^3(S, \mathcal{E}_t) = \text{coker}(\mathbf{Q}/\mathbf{Z} \rightarrow (\mathbf{Q}/\mathbf{Z})^{(M+3)/2})$ and the latter map is given by $a \mapsto (a, 2a, \dots, 2a)$, so we can adjust our representative for $\phi_1(\gamma) \in H^3(S, \mathcal{E}_t)$ by $(\sum_{j=1}^r \frac{M_j b}{2}, 0, \dots, 0)$. Thus we obtain

$$Y_i, 1 \leq i \leq (M+3)/2 : 0.$$

Next, the case that M is even, following the same procedure:

$$\begin{aligned} Y_1 : \sum_{j=1}^r a_{M+1+M_j/2} - \sum_{j=r+1}^s a_{1+M/2+M'_j} + \sum_{j=1}^u \alpha_1^j \\ = - \sum_{j=1}^r \frac{M_j b}{2} + \sum_{j=r+1}^s \frac{Mc_j}{2}. \end{aligned}$$

$$\begin{aligned} Y_{i+1}, 1 \leq i \leq M/2 - 1 : -b(C_i^2)_{Y_{i+1}} + \sum_{j=1}^r a_{i+1}^j + \sum_{j=1}^r a_{M+1-i}^j \\ - \sum_{j=r+1}^s a_{(i+1)+M/2+M'_j}^j - \sum_{j=r+1}^s a_{M/2-i+1+M'_j}^j + \sum_{j=1}^u \alpha_{n_1}^j \\ = -2ibC^2 + 2bu - \sum_{j=1}^r bM_j - \sum_{j=1}^u b\eta_j - \sum_{j=1}^r ic_j - \sum_{j=1}^r (M-i)c_j \\ + \sum_{j=r+1}^s (M/2-i)c_j + \sum_{j=r+1}^s (M/2+i)c_j \\ = - \sum_{j=1}^r 2bM_j + \sum_{j=r+1}^s Mc_j \\ = \sum_{j=r+1}^s Mc_j. \end{aligned}$$

For $Y_{M/2+1}$, we use the section σ' where σ' is any section of $k : Y_{M/2+1} \rightarrow C$ intersecting the reducible fibres in the $Y'_{N_i+5} \cap Y_{M/2+1}$ component at the $I_M + I_{N_i}^*$ collisions and the component f_1^j at the $I_M + I_{M_j}$ collisions, $r + 1 \leq j \leq s$. This gives us

$$\begin{aligned} Y_{M/2+1} &: -bC_{M/2} \cdot \sigma' + \sum_{j=1}^r a_{M/2+1}^j - \sum_{j=r+1}^s a_1^j + \sum_{j=1}^u \alpha_{N_j+5}^j \\ &= -bC_{M/2} \cdot \sigma' - \sum_{j=1}^r \frac{Mc_j}{2} + \sum_{j=1}^u \alpha^j. \end{aligned}$$

Choosing the appropriate representative by subtracting

$$- \sum_{j=1}^r \frac{M_j b}{2} + \sum_{j=r+1}^s \frac{Mc_j}{2}$$

from the values for Y_1 and $Y_{M/2+1}$ and twice that for the others, keeping in mind M_j is even for $1 \leq j \leq r$, we get the desired results for Y_i , $i \leq M/2$. For $Y_{M/2+1}$, we have

$$\begin{aligned} -bC_{M/2} \cdot \sigma' - \sum_{j=r+1}^s \frac{Mc_j}{2} + \sum_{j=1}^u \alpha^j &= -bC_{M/2} \cdot \sigma' + \sum_{j=1}^r \frac{M_j b}{2} - \sum_{j=1}^s \frac{Mc_j}{2} + \sum_{j=1}^u \alpha^j \\ &= -b(C_{M/2} \cdot \sigma' + \sum_{j=1}^r \frac{M_j}{2}) - \sum_{j=1}^s \frac{Mc_j}{2} + \sum_{j=1}^u \alpha^j. \end{aligned}$$

If we remember that M_j is even for $1 \leq j \leq r$, then $a = C_{M/2} \cdot \sigma' + \sum_{j=1}^r \frac{M_j}{2}$ is the desired integer. •

TABLE 2. The analysis of Miranda’s resolution of $I_{M_1} + I_{M_2}^*$ collisions. (See [16].)

Here, we have Σ_1 a curve of fibre type I_{M_1} and Σ_2 a curve of fibre type $I_{M_2}^*$. The fibre over the collision point is

$$\begin{cases} f_1 + f_2 + 2f_3 + \cdots + 2f_{M-2} + f_{M-1} + f_M, & M_1 \text{ even, } M = M_2 + M_1/2 + 5, \\ f_1 + f_2 + 2f_3 + \cdots + 2f_M, & M_1 \text{ odd, } M = M_2 + (M_1 - 1)/2 + 5. \end{cases}$$

$$f^{-1}(\Sigma_1) = Y_1 \cup \cdots \cup Y_{m_1}, m_1 = \begin{cases} M_1/2 + 1, & M_1 \text{ even,} \\ (M_1 + 3)/2, & M_1 \text{ odd.} \end{cases}$$

$$f^{-1}(\Sigma_2) = Y'_1 \cup \cdots \cup Y'_{m_2}, m_2 = \begin{cases} M_2 + 5, & M_1 \text{ even,} \\ M_2 + 4, & M_1 \text{ odd.} \end{cases}$$

Set

$$n_1 = \begin{cases} M_2 + 3 & \text{if } M_2 \text{ is even;} \\ M_2 + 2 & \text{if } M_2 \text{ is odd.} \end{cases}$$

$$C_i = Y_i \cap Y_{i+1}$$

The zero section σ_0 intersects Y_1, Y'_1 and f_1 . Y_1 and Y_{m_1} are ruled over Σ_1 ; $Y_i, 2 \leq i \leq m_1 - 1$ are ruled over a double cover of Σ_1 , branched at the collision point. Y'_1, \dots, Y'_{m_2-1} are ruled over Σ_2 , as well as Y'_{m_2} unless M_1 is odd, in which case Y'_{m_2} is ruled over a double cover of Σ_2 branched at the collision point.

Component	Curves	Intersection Matrix
$Y_1,$	$C_1, f_1, f_2, 2f_3$ \dots $2f_{n_1}$ if $M_1 > 1$; $C_1, f_1, f_2, 2f_3,$ \dots $2f_{M-1}$ if $M_1 = 1.$	$\begin{pmatrix} 0 & -2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & & & & & & \vdots & \vdots & \vdots \\ 0 & 0 & -2 & 1 & 0 & & & & \\ 0 & 1 & 1 & -2 & 1 & & & & \\ 0 & 0 & 0 & 1 & -2 & & & & \\ \vdots & \vdots & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}$
Y_i $2 \leq i \leq m_1 - 1$ M_1 even; $2 \leq i \leq m_1 - 2$ M_1 odd.	$C_{i-1}, C_i, f_{n_1-1+i}$	$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$
Y_{m_1} M_1, M_2 even	$C_{m_1-1},$ $2f_{M-2}, f_{M-1}, f_M$	$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{pmatrix}$
Y_{m_1} M_1 even M_2 odd	$C_{m_1-1}, 2f_{M-3},$ $2f_{M-2}, f_{M-1}, f_M$	$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{pmatrix}$
Y_{m_1-1} M_1 odd M_2 even	C_{m_1-2}, C_{m_1-1} f_{M-2}, f_{M-1}	$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$
Y_{m_1-1} M_1, M_2 odd	$C_{m_1-2}, C_{m_1-1},$ $f_{M-3}, f_{M-2}, f_{M-1}$	$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 1 & -1 \end{pmatrix}$
Y_{m_1} M_1 odd	C_{m_1-1}, f_M	$\begin{pmatrix} 2 & 0 \end{pmatrix}$

TABLE 2. continued

Component	Curves
$Y'_i, i = 1, 2$	f_i
Y'_i $3 \leq i \leq n_1 - 1$	f_i
Y'_{n_1} M_2 even	f_{n_1}, \dots, f_{M-2}
Y'_{n_1} M_2 odd	f_{n_1}, \dots, f_{M-3}
Y'_i $i = m_2 - 1, m_2$ M_1 even	f_{M-1} or f_M
Y'_{M_2+3} M_2 odd	f_{M-2}
Y'_{M_2+4} M_1 odd	f_{M-1}, f_M

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