THE NONEXISTENCE OF EXPANSIVE HOMEOMORPHISMS OF A CLASS OF CONTINUA WHICH CONTAINS ALL DECOMPOSABLE CIRCLE-LIKE CONTINUA

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Abstract. A homeomorphism \( f : X \to X \) of a compactum \( X \) with metric \( d \) is expansive if there is \( c > 0 \) such that if \( x, y \in X \) and \( x \neq y \), then there is an integer \( n \in \mathbb{Z} \) such that \( d(f^n(x), f^n(y)) > c \). It is well-known that \( p \)-adic solenoids \( S_p \) (\( p \geq 2 \)) admit expansive homeomorphisms, each \( S_p \) is an indecomposable continuum, and \( S_p \) cannot be embedded into the plane. In case of plane continua, the following interesting problem remains open: For each \( 1 \leq n \leq 3 \), does there exist a plane continuum \( X \) so that \( X \) admits an expansive homeomorphism and \( X \) separates the plane into \( n \) components? For the case \( n = 2 \), the typical plane continua are circle-like continua, and every decomposable circle-like continuum can be embedded into the plane. Naturally, one may ask the following question: Does there exist a decomposable circle-like continuum admitting expansive homeomorphisms? In this paper, we prove that a class of continua, which contains all chainable continua, some continuous curves of pseudo-arcs constructed by W. Lewis and all decomposable circle-like continua, admits no expansive homeomorphisms. In particular, any decomposable circle-like continuum admits no expansive homeomorphism. Also, we show that if \( f : X \to X \) is an expansive homeomorphism of a circle-like continuum \( X \), then \( f \) is itself weakly chaotic in the sense of Devaney.

1. Introduction

All spaces considered in this paper are assumed to be separable metric spaces. Maps are continuous functions. By a compactum we mean a nonempty compact metric space. A continuum is a connected compactum. A homeomorphism \( f : X \to X \) of a compactum \( X \) with metric \( d \) is called expansive ([20], [1] and [2]) if there is \( c > 0 \) such that for any \( x, y \in X \) with \( x \neq y \), there is an integer \( n \in \mathbb{Z} \) such that

\[
d(f^n(x), f^n(y)) > c.
\]

A homeomorphism \( f : X \to X \) of a compactum \( X \) is continuum-wise expansive [8] if there is \( c > 0 \) such that if \( A \) is a nondegenerate subcontinuum of \( X \), then there is an integer \( n \in \mathbb{Z} \) such that

\[
diam f^n(A) > c,
\]

where \( diam B = \sup\{d(x, y) | x, y \in B\} \) for a set \( B \). Such a positive number \( c \) is called an expansive constant for \( f \). Note that each expansive homeomorphism

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is continuum-wise expansive, but the converse assertion is not true. There are many continuum-wise expansive homeomorphisms which are not expansive (e.g., see [8], [9] and [11]). In fact, there are many decomposable circle-like continua admitting continuum-wise expansive homeomorphisms. By the definitions, we see that expansiveness and continuum-wise expansiveness do not depend on the choice of the metric $d$ of $X$. These notions have been extensively studied in the area of topological dynamics, ergodic theory and continuum theory (e.g., see [1], [2], [7]–[12], [20], and [21]).

Let $f : X \to X$ be a homeomorphism of a compactum $X$. A (nonempty) closed subset $M$ of $X$ is a minimal set of $f$ if $M$ is $f$-invariant, i.e., $f(M) = M$, and for any $x \in M$, the orbit $O(f) = \{f^n(x)| n \in \mathbb{Z}\}$ is dense in $M$. Note that every homeomorphism of a compactum has a minimal set. For a point $x \in X$, the $\omega$-limit set $\omega f(x)$ of $x$ is the set

$$\omega f(x) = \{y \in X | \text{there is a sequence } n_1 < n_2 < \ldots \text{ of natural numbers such that } \lim_{i \to \infty} f^{n_i}(x) = y\}.$$ 

Similarly, the $\alpha$-limit set $\alpha f(x) = \{x\}$ of $x$ is the set $\omega f^{-1}(x)$.

Let $X$ be a compactum. Let $2^X$ be the set of all nonempty closed sets of $X$ and $C(X)$ the set of all nonempty subcontinua of $X$. Suppose that $U_1, \ldots , U_n$ are nonempty open sets of $X$. Put

$$\langle U_1, \ldots , U_n \rangle = \{A \in 2^X | A \cap U_i \neq \emptyset, A \subset \bigcup_{i=1}^n U_i\}.$$ 

Then

$$\beta = \{\langle U_1, U_2, \ldots , U_n \rangle | n \geq 1 \text{ and } U_i (i \leq n) \text{ are nonempty open sets of } X\}$$

is a base of $2^X$, and it is called the Vietoris topology. Then $2^X$ and $C(X)$ are compacta. The spaces $2^X$ and $C(X)$ are called the hyperspaces of $X$. For a map $f : X \to X$, we define a map $f_* : 2^X \to 2^X$ by $f_*(A) = f(A) = \{f(a) | a \in A\}$ for $A \in 2^X$. Also, put $C(f) = f_*|C(X) : C(X) \to C(X)$. Then $X$ is identified with the closed invariant subset of singletons, i.e., degenerate subcontinua.

For the map $C(f) : C(X) \to C(X)$, we shall deal with $\omega(E) = \omega(C(f))(E)$ and $\alpha(E) = \alpha(C(f)^{-1})(E)$ for $E \subset C(X)$.

For a homeomorphism $f : X \to X$, if $Z \subset X$ is a closed invariant subset for $X$, then $Z$ is isolated if for some neighborhood $U$ of $Z$ in $X$ any orbit lying entirely in $U$ is in fact in $Z$, i.e., $Z = \bigcap_{n=1}^\infty f^n(U)$. Then $f$ is expansive (resp. continuum-wise expansive) if and only if $X$ is isolated in $2^X$ for $f_*$ (resp. in $C(X)$ for $C(f)$) (see [1]).

Let $A$ and $B$ be closed $C(f)$-invariant sets in $C(X)$. Then we define the orderings $<, <_*$ as follows: Define $A <_* B$ (resp. $A < B$) iff for any $A \subset A$ there is $B \subset B$ (resp. for any $B \subset B$ there is $A \subset A$) such that $A \subset B$. Also, define $A <_* B$ if $A <_* B$ and $A <_* B$. Example: for $E_0, E_1 \subset C(X)$, $E_0 \subset E_1$ implies $\omega(E_0) <_* \omega(E_1)$ and $\alpha(E_0) <_* \alpha(E_1)$. 


For a homeomorphism \( f : X \to X \), we define sets of stable and unstable nondegenerate subcontinua of \( X \) as follows (see [9]):

\[
\begin{align*}
V^s &= \{ A \mid A \text{ is a nondegenerate subcontinuum of } X \text{ such that } \\
& \quad \lim_{n \to \infty} \text{diam } f^n(A) = 0 \}, \\
V^u &= \{ A \mid A \text{ is a nondegenerate subcontinuum of } X \text{ such that } \\
& \quad \lim_{n \to \infty} \text{diam } f^{-n}(A) = 0 \}.
\end{align*}
\]

For each \( 0 < \delta < \epsilon \), put

\[
\begin{align*}
V^s(\delta; \epsilon) &= \{ A \in C(X) \mid \text{diam } A \geq \delta, \text{ and diam } f^n(A) \leq \epsilon \text{ for each } n \geq 0 \}, \\
V^u(\delta; \epsilon) &= \{ A \in C(X) \mid \text{diam } A \geq \delta, \text{ and diam } f^{-n}(A) \leq \epsilon \text{ for each } n \geq 0 \}.
\end{align*}
\]

Then \( V^s(\delta; \epsilon) \) (\( \sigma = u, s \)) is closed in \( C(X) \). Note that if \( f : X \to X \) is a continuum-wise expansive homeomorphism with an expansive constant \( c > 0 \), then for each \( 0 < \delta < \epsilon < c \) we have \( V^s(\delta; \epsilon) \subset V^s(\epsilon) \), and \( V^s(\delta; \epsilon) \) is an \( F_\sigma \)-set in \( C(X) \).

A chain \( C = [C_1, C_2, \ldots, C_m] \) of \( X \) is a finite collection of open sets of \( X \) satisfying the following property:

\[
\text{Cl}(C_i) \cap \text{Cl}(C_j) \neq \emptyset \text{ if and only if } |i - j| \leq 1.
\]

Each \( C_i \) is called a link of the chain \( C \). Moreover, if for each \( i = 1, \ldots, m \), \( \text{diam } (C_i) < \epsilon \), i.e., mesh\((C_i) < \epsilon \), then we say that the chain \( C \) is an \( \epsilon \)-chain.

For a chain \( C = [C_1, C_2, \ldots, C_m] \) and two points \( p, q \in X \), if \( p \in C_1 \) and \( q \in C_m \), we say that \( C = [C_1, C_2, \ldots, C_m] \) is a chain from \( p \) to \( q \). A continuum \( X \) is chainable if for any \( \epsilon > 0 \) there is an \( \epsilon \)-chain covering of \( X \).

If \( n \) is a natural number, let \( I(n) = \{1, 2, \ldots, n\} \). A surjective function \( f : I(m) \to I(n) \) is called a pattern provided that \( |f(i + 1) - f(i)| \leq 1 \) for each \( i = 1, \ldots, m - 1 \). Let \( C = [C_1, C_2, \ldots, C_n] \) and \( D = [D_1, D_2, \ldots, D_m] \) be chain coverings of \( X \) and let \( f : I(m) \to I(n) \) be a pattern. We say that \( D \) follows the pattern \( f \) in \( C \) provided that \( D_i \subset C_{f(i)} \) for each \( i \in I(m) \).

Let \( \mathcal{P} \) be a family of compact polyhedra. A continuum \( X \) is called a \( \mathcal{P} \)-like continuum if for any \( \epsilon > 0 \) there is an onto map \( g : X \to P \) such that \( P \in \mathcal{P} \) and \( \text{diam } g^{-1}(y) < \epsilon \) for each \( y \in P \). Note that \( X \) is chainable if and only if \( X \) is arc-like. A circular chain differs from a chain in that the first and last links intersect. Then a continuum \( X \) is circle-like if and only if for any \( \epsilon > 0 \) there is an \( \epsilon \)-circular chain covering of \( X \).

Concerning expansive homeomorphisms, we have the following general problem:

**Problem 1.1.** What kinds of (plane) continua admit expansive homeomorphisms?

Note that \( p \)-adic solenoids \( S_p \) \( (p \geq 2) \) are indecomposable circle-like continua admitting expansive homeomorphisms (see [21]), and they cannot be embedded into the plane \( R^2 \). On the other hand, each decomposable circle-like continuum \( X \) can be embedded into \( R^2 \), and \( R^2 - X \) has at most 2 components. It is known that for each \( n \geq 4 \) there is a plane continuum \( X \) which is called a Lake of Wada, such that \( X \) admits an expansive homeomorphism and \( R^2 - X \) has \( n \) components. It is not known whether there exists a plane continuum \( X \) such that \( X \) admits an expansive homeomorphism and \( X \) separates the plane \( R^2 \) into \( n \) components \( (n \leq 3) \), or not. For the case \( n = 2 \), the typical continua are circle-like continua.

In [7, 8], we proved that if \( X \) is a tree-like continuum admitting a continuum-wise expansive homeomorphism, it must contain an indecomposable subcontinuum.
Also, in [10], we proved that chainable continua admit no expansive homeomorphisms. Naturally, we are interested in the following problem:

**Problem 1.2.** Does there exist a decomposable circle-like continuum admitting an expansive homeomorphism?

In this paper, we prove that some kinds of continua, including all chainable continua, some continuous curves of pseudo-arcs constructed by W. Lewis and all decomposable circle-like continua, admit no expansive homeomorphisms. In particular, any decomposable circle-like continuum admits no expansive homeomorphism. For example, we know that a solenoid of pseudo-arcs and the circle of pseudo-arcs isomorphic to each chainable continuum (see [6] and [14]). From the proof of [8, Proposition 2.3] we have

\[
\text{Let } A \subseteq X, \text{ a positive number } \delta < \epsilon, \text{ and let } \phi : X \to X. \text{ Then the following are true.}
\]

(a) For every nondegenerate subcontinuum \( A \) of \( X \) with \( \dim A = \delta \), exactly one of the following assertions holds:

1. For all \( n \geq 0 \), \( \text{diam } f^n(A) \leq \epsilon \), in which case \( A \subseteq V^s \) and \( \omega(A) \subseteq C(X) \).
2. For \( n \geq 0 \) sufficiently large, \( \text{diam } f^n(A) \geq \delta \).

(b) For every subcontinuum \( A \), either \( \omega(A) \subseteq X \subseteq C(X) \) or \( \text{diam } E \geq \delta \) for all \( E \in \omega(A) \).

For \( n \leq 0 \), \( V^u \) and \( \alpha(A) \), the similar properties are satisfied.

**Lemma 2.3** ([8, Corollary 2.4]). Let \( f : X \to X \) be a continuum-wise expansive homeomorphism of a compactum \( X \) with \( \dim X > 0 \). Then the following are true.

1. \( V^u \neq \phi \) or \( V^s \neq \phi \).
2. If \( \delta > 0 \) is as in the above lemma, then for each \( \gamma > 0 \) there is a natural number \( N(\gamma) \) such that if \( A \) is a subcontinuum of \( X \) with \( \text{diam} \, A \geq \gamma \), then either \( \text{diam} \, f^n(A) \geq \delta \) for each \( n \geq N(\gamma) \) or \( \text{diam} \, f^{-n}(A) \geq \delta \) for each \( n \geq N(\gamma) \).

From the above lemma, we see that \( V^s \cap V^u = \emptyset \) and moreover if \( A \in V^u, B \in V^s \), then \( \dim (A \cap B) \leq 0 \).

**Lemma 2.4.** Under the same hypothesis as in Lemma 2.3, let \( E_0, E_1 \) be nondegenerate subcontinua of \( X \) with \( E_1 \in \omega(E_0) \). Then one of the following holds:

1. Every nondegenerate subcontinuum \( A_0 \) of \( E_0 \) with \( \text{diam} \, A_0 < \delta \) lies in \( V^s \).
2. There is a subcontinuum \( A_1 \) of \( E_1 \) with \( \text{diam} \, A_1 = \delta \) lying in \( V^u \). Moreover, if \( E_0 \in V^u \), then for any \( x \in E_1 \) there is a subcontinuum \( A_1 \) of \( E_1 \) such that \( \text{diam} \, A_1 = \delta \) and \( x \in A_1 \in V^u \).

**Proof.** If the first condition is not true, then there is a subcontinuum \( B \) of \( E_0 \) with \( 0 < \gamma = \text{diam} \, B < \delta \) and a natural number \( n \) such that \( \text{diam} \, f^n(B) > \epsilon \). Choose a sequence \( 0 = n_0 < n_1 < \ldots \) of natural numbers such that \( n_{i+1} - n_i \geq N(\gamma) \) (see Lemma 2.3) and \( \lim_{i \to \infty} f^n(E_0) = E_1 \). By using Lemmas 2.1 inductively, we can construct a sequence \( B_0, B_1, \ldots \) of subcontinua with \( B_0 = B, \text{diam} \, B_0 = \gamma < \delta, \text{diam} \, B_{i+1} = f^{n_{i+1}-n_i}(B_i), \text{diam} \, B_i = \delta (i \geq 1), \) and \( \text{diam} \, f^{-j}(B_i) \leq \epsilon \) for each \( 0 \leq j \leq n_i \). We may assume that \( \lim_{i \to \infty} B_i = A_1 \). Then \( A_1 \in V^u \) and \( A_1 \subseteq E_1 \).

Moreover, suppose that \( E_0 \in V^u \). For any \( x \in E_1 \), we choose a sequence \( x_0, x_1, \ldots \) of points such that \( x_i \in f^n(E_0) \) and \( \lim_{i \to \infty} x_i = x \). Choose a subcontinuum \( B \) of \( E_0 \) such that \( x_0 \in B \) and \( \text{diam} \, B = \gamma < \delta \). By Lemma 2.1, we can choose a sequence \( B_0, B_1, \ldots \) satisfying the above conditions with \( x_i \in B_i \) for each \( i \). Then \( x \in A_1 \in V^u \).

**Corollary 2.5.** Under the same hypothesis as in Lemma 2.3, let \( A \) be a minimal set of \( C(f) \). Assume that there is a nondegenerate subcontinuum \( A \in A \).

(a) For all \( A \in A \), \( \text{diam} \, A \geq \delta \).

(b) Exactly one of the three following conditions holds for \( A \):

1. For all \( A \in A \) and all subcontinua \( B \) of \( A \) with \( \text{diam} \, B < \delta \), \( B \in V^s \).
2. For all \( A \in A \) and all subcontinua \( B \) of \( A \) with \( \text{diam} \, B \leq \delta \), \( B \in V^u \).
3. For all \( A \in A \) there are subcontinua \( B_0, B_1 \) of \( A \) with \( \text{diam} \, B_0 = \text{diam} \, B_1 = \delta \) and \( B_0 \in V^s, B_1 \in V^u \).

(c) If \( A \in A \) and \( B \) is a nondegenerate subcontinuum of \( A \) with \( B \not\in V^s \), then \( \text{diam} \, E \geq \delta \) for each \( E \in \omega(B) \), \( \omega(B) \not<_s A \), and if \( \mathcal{A}_0 \) is a minimal set in \( \omega(B) \), then \( \mathcal{A}_0 \not<_s A \) as well.

The following propositions are used in the sequel.

**Proposition 2.6.** Let \( f : X \to X \) be a continuum-wise expansive homeomorphism of a compactum \( X \) with \( \dim X > 0 \). Suppose that \( B \) is a \( C(f) \)-invariant set such that some element of \( B \) is nondegenerate. Then there exists a minimal set \( A \prec B \) of \( C(f) \) such that each element of \( A \) is nondegenerate, and such that for each \( A \in A \) and each nondegenerate subcontinuum \( B \) of \( A \) either \( B \in V^s \) or \( A \subseteq \omega(B) \), and either \( B \in V^u \) or \( A \subseteq \alpha(B) \).

**Proof.** For pairs \( (A, A) \) such that \( A \) is minimal, \( A \prec B, A \in A \) and \( A \) is a nondegenerate subcontinuum, we consider the order by inclusion of the \( A \)’s. By Corollary 2.2.(b), there exists such a pair. If \( \{(A_\alpha, A_\alpha)\} \) is a totally ordered family,
then \( B = \bigcap_{\alpha} A_{\alpha} \) is a nondegenerate subcontinuum and so either \( \omega(B) \) or \( \alpha(B) \) contains a minimal subset \( A \) such that its elements are nondegenerate subcontinua and \( A \prec_{a} A_{\alpha} \) for all \( \alpha \). For each \( \alpha \) choose \( B_{\alpha} \subset A_{\alpha} \) with \( B_{\alpha} \in A \). Then any limit point \( A \) of the net \( \{ B_{\alpha} \} \) is an element of \( A \) contained in all the \( A_{\alpha} \)'s. So Zorn’s lemma applies to the pairs. If \((A, A)\) is minimal with respect to this ordering, then \( A \) satisfies the conclusion. In fact, if \( A \in A \) and \( B \) is a nondegenerate subcontinuum of \( A \) not in \( V^a \), then \( \omega(B) \prec_{a} A \) and \( A_0 \prec_{a} A \) for any minimal subset \( A_0 \) of \( \omega(B) \). Then there is \( A_0 \in A_0 \) such that \( A_0 \subset A \), and so by minimality we see that \( A_0 = A \) and so \( A = \omega(A_0) = A_0 \subset \omega(B) \).

This completes the proof.

**Proposition 2.7.** Under the same assumption as in the above proposition, the minimal set \( A \) satisfies one of the following conditions:

1. If some \( A_0 \in A \) contains an element of \( V^a \), then for any \( x \in A \in A \), there is a nondegenerate subcontinuum \( A_x \) of \( A \) such that \( x \in A_x \in V^a \), and if \( A' \) is a nondegenerate subcontinuum of \( A \in A \) with \( A' \notin V^a \), then for each \( H \in A \) there is a sequence \( n_1 < n_2 < \ldots \) of natural numbers such that
   \[
   \lim_{i \to \infty} f^{n_i}(A) = \lim_{i \to \infty} f^{n_i}(A') = H.
   \]

2. If some \( A_0 \in A \) contains an element of \( V^s \), then for any \( x \in A \in A \), for any \( x \in A \in A \), there is a nondegenerate subcontinuum \( A_x \) of \( A \) such that \( x \in A_x \in V^s \), and if \( A' \) is a any nondegenerate subcontinuum of \( A \in A \) with \( A' \notin V^s \), then for each \( H \in A \) there is a sequence \( n_1 < n_2 < \ldots \) of natural numbers such that
   \[
   \lim_{i \to \infty} f^{-n_i}(A) = \lim_{i \to \infty} f^{-n_i}(A') = H.
   \]

**Proof.** We shall show the first case. Let \( B \in V^u \) and \( B \subset A_0 \in A \). By the above proposition, we see that \( A \subset \omega(B) \). By Lemma 2.4, we see that for any \( x \in A \in A \), there is \( A_x \in V^u \) such that \( x \in A_x \subset A \). Since \( A \) is closed in \( C(X) \), \( A \) contains an maximal element in order by inclusion. In fact, for a Whitney map \( \mu : C(X) \to [0,1] \) (see \[18\]), we can choose an element \( T \) of \( A \) such that \( \mu(T) = \max(\mu(E) | E \in A) \). Then \( T \) is a maximal element of \( A \). Suppose that \( A' \) is a nondegenerate subcontinuum of \( A \in A \) with \( A' \notin V^u \). Let \( H \in A \). Since \( \omega(A') \supset A \) (see Proposition 2.6), \( T \in \omega(A') \). Hence there is a sequence \( i_1 < i_2 < \ldots \) of natural numbers such that \( \lim_{k \to \infty} f^{i_k}(A') = T \). We may assume that \( \{ f^{i_k}(A) \}_{k=1}^{\infty} \) is convergent. Since \( T \) is maximal in \( A \), we see that \( \lim_{k \to \infty} f^{i_k}(A) = T \). Since \( A \) is minimal, \( H = \omega(T) \). Then we can choose a sequence \( n_1 < n_2 < \ldots \) of natural numbers such that
   \[
   \lim_{i \to \infty} f^{n_i}(A') = \lim_{i \to \infty} f^{n_i}(A) = H.
   \]

This completes the proof.

Let \( f : X \to X \) be a continuum-wise expansive homeomorphism of a compactum \( X \) with \( \dim X > 0 \). Note that every minimal set of \( f \) is 0-dimensional (see \[8, Theorem 5.2\]). Consider the following sets (see \[12\]):

1. \( I(f) = \{ A \in 2^X | A \text{ is } f \text{-invariant} \} \).
2. \( I^+(f) = \{ A \in I(f) | \dim A > 0 \} \).
3. \( D(f) \) is the set of all minimal elements of \( I^+(f) \) in the order by inclusion.
Note that $\mathcal{D}(f) \neq \emptyset$ (see [12, Proposition 2.4]) and if $Y \in \mathcal{D}(f)$, then $f_Y = f|Y : Y \to Y$ is weakly chaotic in the sense of Devaney, i.e., $f_Y$ has sensitive dependence on initial conditions, $f_Y$ is topologically transitive and the union of all minimal sets of $f_Y$ is dense in $Y$ ([12, Theorem 2.7]), i.e., the min-center of $f_Y$ is $Y$ (see [1, p. 70]).

**Proposition 2.8.** Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum $X$ with $\dim X > 0$. If $Y \in \mathcal{D}(f)$, then there is a minimal set $A$ of $C(f)$ satisfying one of the conditions (1) and (2) as in Proposition 2.7 and $\bigcup \{A | A \in A\} = Y$.

**Proof.** Consider the map $f|Y : Y \to Y$. Then there is a minimal set $A$ of $C(f|Y)$ as in Proposition 2.7. Put $Y' = \bigcup \{A | A \in A\}$. Then $Y'$ is $f$-invariant and $\dim Y' > 0$. Hence $Y = Y'$.

The following lemma follows from [3, Theorem 6] (see also [15, Lemmas 2 and 1.1]).

**Lemma 2.9.** Let $P$ be the pseudo-arc. Let $C = [C_1, C_2, \ldots, C_n]$ be a chain covering of $P$ and $f : I(m) \to I(n)$ a pattern with $f(1) = 1$. Let $p \in C_1$. Then there is a chain covering $D = [D_1, D_2, \ldots, D_m]$ such that $D$ refines the chain $C$, $p \in D_1$ and $D$ follows the pattern $f$ in $C$.

The following lemma is a simple modification of the uniformization theorem of Mioduszewski (see [17] and [19]). For completeness, we give the proof.

**Lemma 2.10.** Let $I = [0, 1]$ be the unit interval. Suppose that $f, g : I \to I$ are piecewise linear onto maps. If $f(0) = g(0) = 0$, then there are onto maps $a, b : I \to I$ such that $f \cdot a = g \cdot b$ and $a(0) = b(0) = 0$.

**Proof.** Let $\psi : I^2 \to R$ be the map defined by $\psi(x, y) = f(x) - g(y)$. Note that $I^2$ is unicoherent (i.e., if $A$ and $B$ are continua with $A \cup B = I^2$, then $A \cap B$ is connected). In [17], Mioduszewski proved that there is a component $K$ of $\psi^{-1}(0)$ such that $K$ meets all four sides of $I^2$ (see also [19]). Note that each component of $\psi^{-1}(0)$ is a polyhedron. Let $L$ be a component of $\psi^{-1}(0)$ containing the point $p = (0, 0) \in I^2$. Suppose, on the contrary, that $L \cap (I \times \{1\} \cup \{1\} \times I) = \emptyset$. Then there is an arc $\alpha : I \to I^2$ such that $\alpha(0) = (x_1, 0) \in I \times \{0\}$, $\alpha(1) = (0, y_1) \in \{0\} \times I$, and $\psi^{-1}(0) \cap \alpha(I) = \emptyset$. Note that $g(0) = 0 < f(x_1)$ and $g(y_1) > f(0)$. Hence we can see that there is a point $q = (q_1, q_2) \in \alpha(I)$ such that $f(q_1) = g(q_2)$, which implies that $q \in \phi^{-1}(0)$. This is a contradiction. Hence $K$ contains $L$. By using this fact, we can choose desired maps $a, b : I \to I$.

3. The nonexistence of expansive homeomorphisms of certain continua

The following is the main theorem in this paper.

**Theorem 3.1.** Let $f : X \to X$ be a homeomorphism of a compactum $X$. If there is a minimal set $A$ of $C(f)$ such that some element $A$ of $A$ is a (nondegenerate) chainable continuum, then $f$ is not expansive.

**Proof.** Suppose, on the contrary, that $f$ is expansive. Replace $A$ if necessary by an $A_0 \prec A$ which satisfies the condition (1) of Proposition 2.7. Since every subcontinuum of a chainable continuum is also chainable, we may assume that $A$ satisfies the conditions of Proposition 2.7,(1).
Let $c > 0$ be an expansive constant for $f$ and $c/2 > \epsilon > 0$. Now, we shall prove the following property

(3.1.1) For any $0 < \tau < \epsilon$ there are two points $x, y$ of $X$ and a natural number $n(\tau)$ such that $d(x, y) \leq \tau$, $d(f^{n(\tau)}(x), f^{n(\tau)}(y)) \leq \tau$, and

$$\epsilon \leq \text{sup}\{d(f^j(x), f^j(y)) | 0 \leq j \leq n(\tau)\} \leq 2\epsilon.$$

Let $A \in \mathbf{A}$ be a chainable continuum. Since $A$ is chainable, there is a $\tau/4$-chain $C = [C_1, C_2, \ldots, C_r]$ in $X$ which is an open covering of $A$. We can choose a subcontinuum $B_1$ of $A$ such that $B_1 \in \mathbf{V}^n(\tau; \epsilon)$ (see (1) of Proposition 2.7), and we may assume that $\text{diam}(B_1) = \tau$. Since $B_1 \in \mathbf{V}^n$ and $f$ is expansive, we can choose a natural number $N_1$ such that if $x, y \in B_1$ and $d(x, y) \geq \tau/4$, then

$$\sup\{d(f^i(x), f^i(y)) | 0 \leq i \leq N_1\} > \tau/2.$$

Choose a subcontinuum $B_2$ of $B_1$ such that $\text{diam}(B_2) = \tau/2$. By the assumption, there is a sequence $n_1 < n_2 < \ldots$ of natural numbers such that $\lim_{i \to \infty} f^{n_i}(B_2) = \lim_{i \to \infty} f^{n_i}(A) = A$ (see Proposition 2.7). Hence, we can choose a natural number $N > N_1$ such that $f^N(B_2), f^N(B_2) \in (C_1, \ldots, C_r)$. Choose a point $e \in B_2$ such that $f^N(e) \in C_1$. Since $B_1, B_2$ are chainable, by [6] or [14] there are onto maps $\psi_k : P \to B_k$ $(k = 1, 2)$ from the pseudo-arc $P$ onto $B_k$. Let $p \in P$. Since $P$ is homogeneous [3], we may assume that $\psi_k(p) = e$ for each $k = 1, 2$. Choose a chain covering $D = [D_1, \ldots, D_s]$ of $P$ such that its mesh is sufficiently small. We may assume that if $x, y \in D_i \cup D_{i+1}$, then

$$\sup\{d(f^i(\psi_k(x)), f^i(\psi_k(y))) | 0 \leq i \leq N\} < \epsilon/2$$

for each $k = 1, 2$. We may assume that $p \in D_1$ (see the proof of [3, Theorem 13]). Also we may assume that $D$ is a refinement of the chains $C_k = \langle f^N \cdot \psi_k \rangle^{-1}(C)$ $(k = 1, 2)$. Let $f_k : I(s) \to I(r)$ $(k = 1, 2)$ be patterns such that $D$ follows the patterns $f_k$ in $C_k$ $(k = 1, 2)$. Then the patterns $f_k$ $(k = 1, 2)$ induce maps $f_k : N(D) = N((D_1, \ldots, D_s)) \to N(C) = N(C_1, \ldots, C_r)$ which are natural simplicial maps from the nerve $N(D)$ of $D$ to $N(C)$ of $C$ such that $f_k(D_j) = C_{f_k(j)}$ for each $j$. Since the above nerves are arcs, we can consider that $f_k$ is a map from the unit interval $I$ onto $I$ such that $f_k(0) = 0$ $(k = 1, 2)$. By Lemma 2.10, there are onto maps $g_k : I \to I$ such that $f_1 \cdot g_1 = f_2 \cdot g_2$ and $g_k(0) = 0$.

By using $g_k = g_1$ $(k = 1, 2)$, we obtain patterns $g_1 : I(I) \to I(s)$ satisfying the inequality $|f_1(g_1(j) - f_2g_2(j))| \leq 1$ for each $j = 1, 2, \ldots, l$. By Lemma 2.9, we can choose chain coverings $E = \{E_1, E_2, \ldots, E_l\}$ and $F = \{F_1, F_2, \ldots, F_l\}$ of $P$ such that $E$ follows the pattern $g_1$ in $D$ and $F$ follows the pattern $g_2$ in $D$. We may assume that $p \in E_1 \cap F_1$.

Choose points $a_1, \ldots, a_l, b_1, \ldots, b_l$ of $P$ beginning with $p = a_1 = b_1$ and such that $a_j \in E_j, b_j \in F_j$. Note that

$$d(f^N(\psi_1(a_j)), f^N(\psi_2(b_j))) \leq \tau.$$

For each $i = 1, 2, \ldots, l$, put

$$r_i = \text{sup}\{d(f^j(\psi_1(a_i)), f^j(\psi_2(b_i))) | 0 \leq j \leq N\}.$$

Since the chain cover $D$ is sufficiently small (see (3.1.2)), we may assume that

$$|r_i - r_{i+1}| < \epsilon.$$
Note that $r_1 = 0$. Since $\psi_1$ is surjective, there is a point $a_n$ ($u \leq l$) such that $d(\psi_1(a_n), B_2) \geq \tau/4$, and hence $d(\psi_1(a_n), \psi_2(a_n)) \geq \tau/4$. Thus $r_2 > 2\epsilon$. Then we can choose $i \leq u$ such that $\epsilon \leq r_i \leq 2\epsilon$. The two points $a_i, b_i$ satisfy the condition (3.1.1).

Let $\{\epsilon_i\}_{i=1}^\infty$ be a sequence of positive numbers such that $\lim_{i \to \infty} \epsilon_i = 0$. By the condition (3.1.1), there are two points $x_i, y_i \in X$ and a natural number $n(i)$ such that

$$d(x_i, y_i) < \epsilon_i, \quad d(f^{n(i)}(x_i), f^{n(i)}(y_i)) < \epsilon_i$$

and

$$\epsilon \leq \sup \{d(f^j(x_i), f^j(y_i)) | 0 \leq j \leq n(i)\} \leq 2\epsilon.$$  

Choose $0 < m(i) < n(i)$ such that $d(f^{m(i)}(x_i), f^{m(i)}(y_i)) > \epsilon$. We may assume that $\{f^{m(i)}(x_i)\}$ and $\{f^{m(i)}(y_i)\}$ are convergent to $x_0$ and $y_0$, respectively. Note that

$$\lim_{i \to \infty} (n(i) - m(i)) = \infty = \lim_{i \to \infty} m(i).$$

Then $x_0 \neq y_0$ and $d(f^n(x_0), f^n(y_0)) \leq 2\epsilon < c$ for all $n \in \mathbb{Z}$. This is a contradiction.

**Corollary 3.2.** If $X$ is a decomposable circle-like continuum, then $X$ admits no expansive homeomorphism.

**Proof.** Suppose, on the contrary, that there is an expansive homeomorphism $f : X \to X$. Since $X$ is decomposable, there are two proper nonempty subcontinua $A, B$ of $X$ such that $A \cup B = X$. Since $X$ is circle-like, $A$ and $B$ are chainable. Note that $A \cap B$ has at most 2 components [5, Theorem 5]. By [11, Theorem 3.6], there is a $\sigma$-chaotic continuum $C$ of $f$. We may assume that $\sigma = u$. Then $C$ is indecomposable (see [11, Corollary 5.3]) and is a proper subcontinuum of $X$. Note that $f^n(C) \cap A$ and $f^n(C) \cap B$ have at most 2 components. Since $f^n(C)$ is indecomposable, we can easily see that for each $n = 0, 1, \ldots, f^n(C) \subset A$ or $f^n(C) \subset B$. Hence we see that there is a minimal set $A$ of $C(f)$ satisfying the condition of Theorem 3.1. By Theorem 3.1, $f$ is not expansive.

**Corollary 3.3.** Let $f : X \to X$ be a homeomorphism of a compactum $X$. Suppose that there are maps $\psi : X \to Y$ and $g : Y \to Y$ such that $\psi \cdot f = g \cdot \psi$ and for each $y \in Y \psi^{-1}(y)$ is a (nondegenerate) chainable continuum. Then $f$ is not expansive.

**Proof.** Let $y_0 \in Y$. By Corollary 2.2, we may assume each element of $\omega(\psi^{-1}(y_0))$ is nondegenerate. Since $\psi \cdot f = g \cdot \psi$ and the collection $\{\psi^{-1}(y) | y \in Y\}$ is an upper semi-continuous decomposition of $X$, each element of $\omega(\psi^{-1}(y_0))$ is contained in some $\psi^{-1}(y_0)$, and hence it is chainable.

Take a minimal set $A$ of $\omega(\psi^{-1}(y_0))$. Then each element of $A$ is a chainable continuum. By Theorem 3.1, $f$ is not expansive.

**Corollary 3.4.** Let $f : X \to X$ be an expansive homeomorphism of a circle-like continuum $X$, and let $\delta > 0$ be a positive number as in Lemma 2.1. Then one of the following conditions is satisfied:

(i) If $A \subset C(X)$ and $0 < \text{diam } A < \delta$, then $A \subset V^u$, and if $B$ is a nondegenerate subcontinuum of $X$, then $X \in \omega(B)$.

(ii) If $A \subset C(X)$ and $0 < \text{diam } A < \delta$, then $A \subset V^s$, and if $B$ is a nondegenerate subcontinuum of $X$, then $X \in \alpha(B)$.  

Remark. In [16], Lewis showed that, for every 1-dimensional continuum weakly chaotic in the sense of Devaney, every nondegenerate subcontinuum. By Corollary 3.4, we see that $X \in \omega(A)$, by Lemma 2.4 we see that for each $x \in X$ there is $x \in A_x \in V^u(\delta; \epsilon)$, where $\delta, \epsilon$ are as in Lemma 2.1. Suppose, on the contrary, that $V^s \neq \phi$. Then we see also that for each $x \in X$ there is $x \in B_x \in V^s(\delta; \epsilon)$. Since $f$ is expansive, we know that $A_x \cap B_x = \{x\}$. Choose $A_x$ and two points $y, z \in A_x$ such that $x, y$ and $z$ are different. Then there are three subcontinua $B_x, B_y, B_z$ such that their diameters are small and $B_x, B_y$ and $B_z$ are mutually disjoint. Then $T = A_x \cup B_x \cup B_y \cup B_z$ is a triod. Since $X$ is atriodic, this is a contradiction. Hence $V^s = \phi$. Let $A$ be a nondegenerate subcontinuum of $X$ with $\dim A = \gamma < \delta$. Suppose, on the contrary, that $\sup\{\dim f^{-n}(A) | n \geq 0\} \geq \epsilon$. By using Lemmas 2.1 and 2.3 inductively, we have a sequence $n_1 < n_2 < \ldots$ of natural numbers and a sequence $B_1, B_2, \ldots$ of subcontinua such that $B_i \subset f^{-n_i}(A), \dim B_i = \delta$ and $\dim f(B_i) \leq \epsilon$ for each $0 \leq j \leq n_i$. We may assume that $\lim_{i \to \infty} B_i = B \in V^s$. This is a contradiction. Hence we see that $A \in V^u$. Clearly, if $B$ is a nondegenerate subcontinuum, then $X \in \omega(B)$, because $B \notin V^s$.

**Corollary 3.5.** If $f : X \to X$ is an expansive homeomorphism of a circle-like continuum $X$, then $f$ is itself weakly chaotic in the sense of Devaney.

**Proof.** Consider the set $D(f) \neq \phi$. Let $Y \in D(f)$. Since $\dim Y > 0$, $Y$ contains a nondegenerate subcontinuum. By Corollary 3.4, we see that $Y = X$. Hence $f$ is weakly chaotic in the sense of Devaney.

Remark. In [16], Lewis showed that, for every 1-dimensional continuum $M$ there exists a 1-dimensional continuum $\hat{M}$ such that $M$ has a continuous decomposition $\psi : \hat{M} \to M$ into pseudo-arcs such that the decomposition space is homeomorphic to $M$ and the decomposition elements are all terminal continua in $\hat{M}$, i.e., every subcontinuum of $\hat{M}$ either is contained in a single decomposition element or is a union of decomposition elements. More generally, let $\hat{N}$ be a compactum that has an upper semi-continuous decomposition $\varphi$ into indecomposable chainable continua such that the decomposition elements are all terminal, and let $N$ be the decomposition space. Moreover, if each proper subcontinuum of $\hat{N}$ is decomposable, then for any homeomorphism $\hat{h} : \hat{N} \to \hat{N}$ there is a homeomorphism $h : N \to N$ such that $\varphi \cdot \hat{h} = h \cdot \varphi$. By Corollary 3.3, $\hat{N}$ admits no expansive homeomorphism. The typical continua are solenoids of pseudo-arcs and hence they admit no expansive homeomorphisms.

**Problem 3.6.** Does there exist an indecomposable plane circle-like continuum which admits an expansive homeomorphism? In particular, does the pseudo-circle admit an expansive homeomorphism?

**Problem 3.7.** Does there exist a hereditarily indecomposable continuum which admits an expansive homeomorphism?

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REFERENCES


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