TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 9, September 1997, Pages 3645–3655
S 0002-9947(97)01850-3

THE NONEXISTENCE OF EXPANSIVE HOMEOMORPHISMS OF A CLASS OF CONTINUA WHICH CONTAINS ALL DECOMPOSABLE CIRCLE-LIKE CONTINUA

HISAO KATO

ABSTRACT. A homeomorphism $f : X \to X$ of a compactum X with metric d is expansive if there is $c > 0$ such that if $x, y \in X$ and $x \neq y$, then there is an integer $n \in \mathbf{Z}$ such that $d(f^n(x), f^n(y)) > c$. It is well-known that p-adic solenoids S_p ($p \geq 2$) admit expansive homeomorphisms, each S_p is an indecomposable continuum, and S_p cannot be embedded into the plane. In case of plane continua, the following interesting problem remains open: For each $1 \leq n \leq 3$, does there exist a plane continuum X so that X admits an expansive homeomorphism and X separates the plane into n components? For the case $n = 2$, the typical plane continua are circle-like continua, and every decomposable circle-like continuum can be embedded into the plane. Naturally, one may ask the following question: Does there exist a decomposable circle-like continuum admitting expansive homeomorphisms? In this paper, we prove that a class of continua, which contains all chainable continua, some continuous curves of pseudo-arcs constructed by W. Lewis and all decomposable circle-like continua, admits no expansive homeomorphisms. In particular, any decomposable circle-like continuum admits no expansive homeomorphism. Also, we show that if $f : X \to X$ is an expansive homeomorphism of a circle-like continuum X, then f is itself weakly chaotic in the sense of Devaney.

1. Introduction

All spaces considered in this paper are assumed to be separable metric spaces. *Maps* are continuous functions. By a *compactum* we mean a nonempty compact metric space. A *continuum* is a connected compactum. A homeomorphism $f : X \to X$ of a compactum X with metric d is called *expansive* ([20], [1] and [2]) if there is $c > 0$ such that for any $x, y \in X$ with $x \neq y$, there is an integer $n \in \mathbf{Z}$ such that

$$d(f^n(x), f^n(y)) > c.$$

A homeomorphism $f : X \to X$ of a compactum X is *continuum-wise expansive* [8] if there is $c > 0$ such that if A is a nondegenerate subcontinuum of X, then there is an integer $n \in \mathbf{Z}$ such that

$$\operatorname{diam} f^n(A) > c,$$

where $\operatorname{diam} B = \sup\{d(x, y) |\, x, y \in B\}$ for a set B. Such a positive number c is called an *expansive constant* for f. Note that each expansive homeomorphism

Received by the editors October 9, 1995 and, in revised form, February 6, 1996.
1991 *Mathematics Subject Classification.* Primary 54H20, 54F50; Secondary 54E50, 54B20.
Key words and phrases. Expansive homeomorphism, decomposable, chainable, circle-like, the pseudo-arc, pattern, hyperspace.

is continuum-wise expansive, but the converse assertion is not true. There are many continuum-wise expansive homeomorphisms which are not expansive (e.g., see [8], [9] and [11]). In fact, there are many decomposable circle-like continua admitting continuum-wise expansive homeomorphisms. By the definitions, we see that expansiveness and continuum-wise expansiveness do not depend on the choice of the metric d of X. These notions have been extensively studied in the area of topological dynamics, ergodic theory and continuum theory (e.g., see [1], [2], [7]–[12], [20], and [21]).

Let $f : X \to X$ be a homeomorphism of a compactum X. A (nonempty) closed subset M of X is a *minimal set* of f if M is f-invariant, i.e., $f(M) = M$, and for any $x \in M$, the orbit $O(f) = \{f^n(x)|\, n \in \mathbf{Z}\}$ is dense in M. Note that every homeomorphism of a compactum has a minimal set. For a point $x \in X$, the *ω-limit set* $\omega f(x)$ of x is the set

$$\begin{aligned}\omega f(x)(= \omega(x)) =&\{y \in X|\, \text{there is a sequence}\, n_1 < n_2 < \dots \\ &\text{of natural numbers such that } \lim_{i\to\infty} f^{n_i}(x) = y\}.\end{aligned}$$

Similarly, the *α-limit set* $\alpha f(x)(= \alpha(x))$ of x is the set $\omega f^{-1}(x)$.

Let X be a compactum. Let 2^X be the set of all nonempty closed sets of X and $C(X)$ the set of all nonempty subcontinua of X. Suppose that $U_1, \dots, U_n$ are nonempty open sets of X. Put

$$\langle U_1, \dots, U_n\rangle = \{A \in 2^X|\, A \cap U_i \neq \phi, A \subset \bigcup_{i=1}^{n} U_i\}.$$

Then

$$\begin{aligned}\beta =&\{\langle U_1, U_2, \dots, U_n\rangle|\, n \geq 1 \text{ and} \\ &U_i\,(i \leq n) \text{ are nonempty open sets of}\, X\}\end{aligned}$$

is a base of 2^X, and it is called the *Vietoris topology*. Then 2^X and $C(X)$ are compacta. The spaces 2^X and $C(X)$ are called the *hyperspaces* of X. For a map $f : X \to X$, we define a map $f_* : 2^X \to 2^X$ by $f_*(A) = f(A)(= \{f(a)|\, a \in A\})$ for $A \in 2^X$. Also, put $C(f) = f_*|C(X) : C(X) \to C(X)$. Then X is identified with the closed invariant subset of singletons, i.e., degenerate subcontinua.

For the map $C(f) : C(X) \to C(X)$, we shall deal with $\omega(E) = \omega C(f)(E)$ and $\alpha(E) = \omega C(f)^{-1}(E)$ for $E \in C(X)$.

For a homeomorphism $f : X \to X$, if $Z \subset X$ is a closed invariant subset for X, then Z is *isolated* if for some neighborhood U of Z in X any orbit lying entirely in U is in fact in Z, i.e., $Z = \bigcap_{-\infty}^{\infty} f^n(U)$. Then f is expansive (resp. continuum-wise expansive) if and only if X is isolated in 2^X for f_* (resp. in $C(X)$ for $C(f)$) (see [1]).

Let $\mathbf{A}$ and $\mathbf{B}$ be closed $C(f)$-invariant sets in $C(X)$. Then we define the orderings ${}_*<, <_*$, and ${}_*<_*$ as follows: Define $\mathbf{A} {}_*< \mathbf{B}$ (resp. $\mathbf{A} <_* \mathbf{B}$) iff for any $A \in \mathbf{A}$ there is $B \in \mathbf{B}$ (resp. for any $B \in \mathbf{B}$ there is $A \in \mathbf{A}$) such that $A \subset B$. Also, define $\mathbf{A} {}_*<_* \mathbf{B}$ iff $\mathbf{A} {}_*< \mathbf{B}$ and $\mathbf{A} <_* \mathbf{B}$. Example: for $E_0, E_1 \in C(X)$, $E_0 \subset E_1$ implies $\omega(E_0) {}_*<_* \omega(E_1)$ and $\alpha(E_0) {}_*<_* \alpha(E_1)$.

For a homeomorphism $f : X \to X$, we define sets of *stable* and *unstable* nondegenerate subcontinua of X as follows (see [9]):

$$\mathbf{V}^s = \{A|\, A \text{ is a nondegenerate subcontinuum of } X \text{ such that } \lim_{n\to\infty} \operatorname{diam} f^n(A) = 0\},$$
$$\mathbf{V}^u = \{A|\, A \text{ is a nondegenerate subcontinuum of } X \text{ such that } \lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0\}.$$

For each $0 < \delta < \epsilon$, put

$$\mathbf{V}^s(\delta;\epsilon) = \{A \in C(X)|\operatorname{diam} A \geq \delta, \text{ and } \operatorname{diam} f^n(A) \leq \epsilon \text{ for each } n \geq 0\},$$
$$\mathbf{V}^u(\delta;\epsilon) = \{A \in C(X)|\operatorname{diam} A \geq \delta, \text{ and } \operatorname{diam} f^{-n}(A) \leq \epsilon \text{ for each } n \geq 0\}.$$

Then $\mathbf{V}^\sigma(\delta;\epsilon)$ ($\sigma = u, s$) is closed in $C(X)$. Note that if $f : X \to X$ is a continuum-wise expansive homeomorphism with an expansive constant $c > 0$, then for each $0 < \delta < \epsilon < c$ we have $\mathbf{V}^\sigma(\delta;\epsilon) \subset \mathbf{V}^\sigma$, and $\mathbf{V}^\sigma$ is an F_σ-set in $C(X)$.

A *chain* $C = [C_1, C_2, \dots, C_m]$ of X is a finite collection of open sets of X satisfying the following property:

$$\mathrm{Cl}(C_i) \cap \mathrm{Cl}(C_j) \neq \phi \text{ if and only if } |i - j| \leq 1.$$

Each C_i is called a *link* of the chain C. Moreover, if for each $i = 1, \dots, m$, $\operatorname{diam}(C_i) < \epsilon$, i.e., $\operatorname{mesh}(C) < \epsilon$, then we say that the chain C is an ϵ-chain. For a chain $C = [C_1, C_2, \dots, C_m]$ and two points $p, q \in X$, if $p \in C_1$ and $q \in C_m$, we say that $C = [C_1, C_2, \dots, C_m]$ is *a chain from* p to q. A continuum X is *chainable* if for any $\epsilon > 0$ there is an ϵ-chain covering of X.

If n is a natural number, let $I(n) = \{1, 2, \dots, n\}$. A surjective function $f : I(m) \to I(n)$ is called a *pattern* provided that $|f(i + 1) - f(i)| \leq 1$ for each $i = 1, \dots, m - 1$. Let $C = [C_1, C_2, \dots, C_n]$ and $D = [D_1, D_2, \dots, D_m]$ be chain coverings of X and let $f : I(m) \to I(n)$ be a pattern. We say that D follows the pattern f in C provided that $D_i \subset C_{f(i)}$ for each $i \in I(m)$.

Let $\mathcal{P}$ be a family of compact polyhedra. A continuum X is called a *$\mathcal{P}$-like continuum* if for any $\epsilon > 0$ there is an onto map $g : X \to P$ such that $P \in \mathcal{P}$ and $\operatorname{diam} g^{-1}(y) < \epsilon$ for each $y \in P$. Note that X is chainable if and only if X is arc-like. A *circular chain* differs from a chain in that the first and last links intersect. Then a continuum X is circle-like if and only if for any $\epsilon > 0$, there is an ϵ-circular chain covering of X.

Concerning expansive homeomorphisms, we have the following general problem:

Problem 1.1. What kinds of (plane) continua admit expansive homeomorphisms?

Note that p-adic solenoids S_p ($p \geq 2$) are indecomposable circle-like continua admitting expansive homeomorphisms (see [21]), and they cannot be embedded into the plane R^2. On the other hand, each decomposable circle-like continuum X can be embedded into R^2, and $R^2 - X$ has at most 2 components. It is known that for each $n \geq 4$ there is a plane continuum X which is called a *Lake of Wada*, such that X admits an expansive homeomorphism and $R^2 - X$ has n components. It is not known whether there exists a plane continuum X such that X admits an expansive homeomorphism and X separates the plane R^2 into n components ($n \leq 3$), or not. For the case $n = 2$, the typical continua are circle-like continua.

In [7, 8], we proved that if X is a tree-like continuum admitting a continuum-wise expansive homeomorphism, it must contain an indecomposable subcontinuum.

Also, in [10], we proved that chainable continua admit no expansive homeomorphisms. Naturally, we are interested in the following problem:

Problem 1.2. Does there exsist a decomposable circle-like continuum admitting an expansive homeomorphism?

In this paper, we prove that some kinds of continua, including all chainable continua, some continuous curves of pseudo-arcs constructed by W. Lewis and all decomposable circle-like continua, admit no expansive homeomorphisms. In particular, any decomposable circle-like continuum admits no expansive homeomorphism. For example, we know that a solenoid of pseudo-arcs and the circle of pseudo-arcs admit no expansive homeomorphisms. Also, we show that if $f : X \to X$ is an expansive homeomorphism of a circle-like continuum X, then f is itself weakly chaotic in the sense of Devaney.

2. Preliminaries

A continuum X is *decomposable* if there are two proper subcontinua A and B of X such that $A \cup B = X$. A continuum X is *indecomposable* if it is not decomposable. A continuum X is *hereditarily indecomposable* if each subcontinuum of X is indecomposable. The *pseudo-arc* P is characterized [4] as a (nondegenerate) hereditarily indecomposable chainable continuum. The pseudo-arc has many remarkable properties in topology and chaotic dynamics (e.g., see [3]–[6] and [13]–[16]). For example, the pseudo-arc P is homogeneous [3], each onto map of the pseudo-arc P is a near homeomorphism [15], and the pseudo-arc P admits chaotic homeomorphisms in the sense of Devaney (see [13]). Also, there is an onto map from the pseudo-arc P to each chainable continuum (see [6] and [14]).

From the proof of [8, Proposition 2.3] we have

Lemma 2.1. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with an expansive constant $c > 0$, and let $0 < \epsilon < c/2$. Then there is a positive number $\delta < \epsilon$ such that if A is a subcontinuum of X with $\operatorname{diam} A \leq \delta$ and $\operatorname{diam} f^m(A) \geq \epsilon$ for some integer $m \geq 0$ (resp. $m < 0$), then for each $n \geq m$ and for each $x \in f^n(A)$, there is a subcontinuum B of A such that $x \in f^n(B), \operatorname{diam} f^j(B) \leq \epsilon$ for $0 \leq j \leq n$ and $\operatorname{diam} f^n(B) = \delta$ (resp. for each $n \geq -m$ and for each $x \in f^{-n}(A)$, there is a subcontinuum B of A such that $x \in f^{-n}(B), \operatorname{diam} f^{-j}(B) \leq \epsilon$ for $0 \leq j \leq n$, and $\operatorname{diam} f^{-n}(B) = \delta$).*

Corollary 2.2. *Let $f : X \to X, c, \epsilon, \delta$ be as in Lemma 2.1.*

(a) *For every nondegenerate subcontinuum A of X with $\operatorname{diam} A \leq \delta$, exactly one of the two following assertions holds:*

1. *For all $n \geq 0, \operatorname{diam} f^n(A) \leq \epsilon$, in which case $A \in \mathbf{V}^s$ and $\omega(A) \subset X \subset C(X)$.*
2. *For $n \geq 0$ sufficiently large, $\operatorname{diam} f^n(A) \geq \delta$.*

(b) *For every subcontinuum A, either $\omega(A) \subset X \subset C(X)$ or $\operatorname{diam} E \geq \delta$ for all $E \in \omega(A)$.*

For $n \leq 0, \mathbf{V}^u$ and $\alpha(A)$, the similar properties are satisfied.

Lemma 2.3 ([8, Corollary 2.4])**.** *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$. Then the following are true.*

1. *$\mathbf{V}^u \neq \phi$ or $\mathbf{V}^s \neq \phi$.*

2. *If $\delta > 0$ is as in the above lemma, then for each $\gamma > 0$ there is a natural number $N(\gamma)$ such that if A is a subcontinuum of X with* $\operatorname{diam} A \geq \gamma$*, then either* $\operatorname{diam} f^n(A) \geq \delta$ *for each $n \geq N(\gamma)$ or* $\operatorname{diam} f^{-n}(A) \geq \delta$ *for each $n \geq N(\gamma)$.*

From the above lemma, we see that $\mathbf{V}^s \cap \mathbf{V}^u = \phi$ and moreover if $A \in \mathbf{V}^u, B \in \mathbf{V}^s$, then $\dim (A \cap B) \leq 0$.

Lemma 2.4. *Under the same hypothesis as in Lemma 2.3, let E_0, E_1 be nondegenerate subcontinua of X with $E_1 \in \omega(E_0)$. Then one of the following holds:*

1. *Every nondegenerate subcontinuum A_0 of E_0 with* $\operatorname{diam} A_0 < \delta$ *lies in $\mathbf{V}^s$.*
2. *There is a subcontinuum A_1 of E_1 with* $\operatorname{diam} A_1 = \delta$ *lying in $\mathbf{V}^u$. Moreover, if $E_0 \in \mathbf{V}^u$, then for any $x \in E_1$ there is a subcontinuum A_1 of E_1 such that* $\operatorname{diam} A_1 = \delta$ *and $x \in A_1 \in \mathbf{V}^u$.*

Proof. If the first condition is not true, then there is a subcontinuum B of E_0 with $0 < \gamma = \operatorname{diam} B < \delta$ and a natural number n such that $\operatorname{diam} f^n(B) > \epsilon$. Choose a sequence $0 = n_0 < n_1 < \dots$, of natural numbers such that $n_{i+1} - n_i \geq N(\gamma)$ (see Lemma 2.3) and $\lim_{i \to \infty} f^{n_i}(E_0) = E_1$. By using Lemmas 2.1 inductively, we can construct a sequence $B_0, B_1, \dots$ of subcontinua with $B_0 = B$, $\operatorname{diam} B_0 = \gamma < \delta$, $B_{i+1} \subset f^{n_{i+1}-n_i}(B_i)$, $\operatorname{diam} B_i = \delta$ $(i \geq 1)$, and $\operatorname{diam} f^{-j}(B_i) \leq \epsilon$ for each $0 \leq j \leq n_i$. We may assume that $\lim_{i\to\infty} B_i = A_1$. Then $A_1 \in \mathbf{V}^u$ and $A_1 \subset E_1$.

Moreover, suppose that $E_0 \in \mathbf{V}^u$. For any $x \in E_1$, we choose a sequence $x_0, x_1, \dots$ of points such that $x_i \in f^{n_i}(E_0)$ and $\lim_{i\to\infty} x_i = x$. Choose a subcontinuum B of E_0 such that $x_0 \in B$ and $\operatorname{diam} B = \gamma < \delta$. By Lemma 2.1, we can choose a sequence $B_0, B_1, \dots$ satisfying the above conditions with $x_i \in B_i$ for each i. Then $x \in A_1 \in \mathbf{V}^u$. □

Corollary 2.5. *Under the same hypothesis as in Lemma 2.3, let $\mathbf{A}$ be a minimal set of $C(f)$. Assume that there is a nondegenerate subcontinuum $A \in \mathbf{A}$.*

(a) *For all $A \in \mathbf{A}$,* $\operatorname{diam} A \geq \delta$*.*
(b) *Exactly one of the three following conditions holds for $\mathbf{A}$:*
   1. *For all $A \in \mathbf{A}$ and all subcontinua B of A with* $\operatorname{diam} B < \delta$*, $B \in \mathbf{V}^s$.*
   2. *For all $A \in \mathbf{A}$ and all subcontinua B of A with* $\operatorname{diam} B < \delta$*, $B \in \mathbf{V}^u$.*
   3. *For all $A \in \mathbf{A}$ there are subcontinua B_0, B_1 of A with* $\operatorname{diam} B_0 = \operatorname{diam} B_1 = \delta$ *and $B_0 \in \mathbf{V}^s, B_1 \in \mathbf{V}^u$.*

(c) *If $A \in \mathbf{A}$ and B is a nondegenerate subcontinuum of A with $B \notin \mathbf{V}^s$, then* $\operatorname{diam} E \geq \delta$ *for each $E \in \omega(B)$, $\omega(B) \;{}_*\!<_* \mathbf{A}$, and if $\mathbf{A}_0$ is a minimal set in $\omega(B)$, then $\mathbf{A}_0 \;{}_*\!<_* \mathbf{A}$ as well.*

The following propositions are used in the sequel.

Proposition 2.6. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with* $\dim X > 0$*. Suppose that $\mathbf{B}$ is a $C(f)$-invariant set such that some element of $\mathbf{B}$ is nondegenerate. Then there exists a minimal set $\mathbf{A} \;{}_*\!< \mathbf{B}$ of $C(f)$ such that each element of $\mathbf{A}$ is nondegenerate, and such that for each $A \in \mathbf{A}$ and each nondegenerate subcontinuum B of A either $B \in \mathbf{V}^s$ or $\mathbf{A} \subset \omega(B)$, and either $B \in \mathbf{V}^u$ or $\mathbf{A} \subset \alpha(B)$.*

Proof. For pairs $(A, \mathbf{A})$ such that $\mathbf{A}$ is minimal, $\mathbf{A} \;{}_*\!< \mathbf{B}$, $A \in \mathbf{A}$ and A is a nondegenerate subcontinuum, we consider the order by inclusion of the A's. By Corollary 2.2,(b), there exists such a pair. If $\{(A_\alpha, \mathbf{A}_\alpha)\}$ is a totally ordered family, then $B = \bigcap_\alpha A_\alpha$ is a nondegenerate subcontinuum and so either $\omega(B)$ or $\alpha(B)$ contains a minimal subset $\mathbf{A}$ such that its elements are nondegenerate subcontinua and $\mathbf{A} \ {}_*{<}_* \ \mathbf{A}_\alpha$ for all α. For each α choose $B_\alpha \subset A_\alpha$ with $B_\alpha \in \mathbf{A}$. Then any limit point A of the net $\{B_\alpha\}$ is an element of $\mathbf{A}$ contained in all the A_α's. So Zorn's lemma applies to the pairs. If $(\tilde{A}, \mathbf{A})$ is minimal with respect to this ordering, then $\mathbf{A}$ satisfies the conclusion. In fact, if $A \in \mathbf{A}$ and B is a nondegenerate subcontinuum of A not in $\mathbf{V}^s$, then $\omega(B) \ {}_*{<}_* \ \mathbf{A}$ and $\mathbf{A}_0 \ {}_*{<}_* \ \mathbf{A}$ for any minimal subset $\mathbf{A}_0$ of $\omega(B)$. Then there is $A_0 \in \mathbf{A}_0$ such that $A_0 \subset \tilde{A}$, and so by minimality we see that $A_0 = \tilde{A}$ and so $\mathbf{A} = \omega(A_0) = \mathbf{A}_0 \subset \omega(B)$.

This completes the proof.

Proposition 2.7. *Under the same assumption as in the above proposition, the minimal set* $\mathbf{A}$ *satisfies one of the following conditions:*

1. *If some* $A_0 \in \mathbf{A}$ *contains an element of* $\mathbf{V}^u$*, then for any* $x \in A \in \mathbf{A}$*, there is a nondegenerate subcontinuum* A_x *of* A *such that* $x \in A_x \in \mathbf{V}^u$*, and if* A' *is a nondegenerate subcontinuum of* $A \in \mathbf{A}$ *with* $A' \notin \mathbf{V}^s$*, then for each* $H \in \mathbf{A}$ *there is a sequence* $n_1 < n_2 < \dots$ *of natural numbers such that*
$$\lim_{i\to\infty} f^{n_i}(A) = \lim_{i\to\infty} f^{n_i}(A') = H.$$
2. *If some* $A_0 \in \mathbf{A}$ *contains an element of* $\mathbf{V}^s$*, then for any* $x \in A \in \mathbf{A}$*, for any* $x \in A \in \mathbf{A}$*, there is a nondegenerate subcontinuum* A_x *of* A *such that* $x \in A_x \in \mathbf{V}^s$*, and if* A' *is a any nondegenerate subcontinuum of* $A \in \mathbf{A}$ *with* $A' \notin \mathbf{V}^u$*, then for each* $H \in \mathbf{A}$ *there is a sequence* $n_1 < n_2 < \dots$ *of natural numbers such that*
$$\lim_{i\to\infty} f^{-n_i}(A) = \lim_{i\to\infty} f^{-n_i}(A') = H.$$

Proof. We shall show the first case. Let $B \in \mathbf{V}^u$ and $B \subset A_0 \in \mathbf{A}$. By the above proposition, we see that $\mathbf{A} \subset \omega(B)$. By Lemma 2.4, we see that for any $x \in A \in \mathbf{A}$, there is $A_x \in \mathbf{V}^u$ such that $x \in A_x \subset A$. Since $\mathbf{A}$ is closed in $C(X)$, $\mathbf{A}$ contains an maximal element in order by inclusion. In fact, for a Whitney map $\mu : C(X) \to [0,1]$ (see [18]), we can choose an element T of $\mathbf{A}$ such that $\mu(T) = \max\{\mu(E) | \, E \in \mathbf{A}\}$. Then T is a maximal element of $\mathbf{A}$. Suppose that A' is a nondegenerate subcontinuum of $A \in \mathbf{A}$ with $A' \notin \mathbf{V}^s$. Let $H \in \mathbf{A}$. Since $\omega(A') \supset \mathbf{A}$ (see Proposition 2.6), $T \in \omega(A')$. Hence there is a sequence $i_1 < i_2 < \dots$ of natural numbers such that $\lim_{k\to\infty} f^{i_k}(A') = T$. We may assume that $\{f^{i_k}(A)\}_{k=1}^\infty$ is convergent. Since T is maximal in $\mathbf{A}$, we see that $\lim_{k\to\infty} f^{i_k}(A) = T$. Since $\mathbf{A}$ is minimal, $H \in \omega(T)$. Then we can choose a sequence $n_1 < n_2 < \dots$ of natural numbers such that
$$\lim_{i\to\infty} f^{n_i}(A') = \lim_{i\to\infty} f^{n_i}(A) = H.$$

This completes the proof.

Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$. Note that every minimal set of f is 0-dimensional (see [8, Theorem 5.2]). Consider the following sets (see [12]):

1. $\mathcal{I}(f) = \{A \in 2^X | \, A \text{ is } f\text{-invariant}\}$.
2. $\mathcal{I}^+(f) = \{A \in \mathcal{I}(f) | \dim A > 0\}$.
3. $\mathcal{D}(f)$ is the set of all minimal elements of $\mathcal{I}^+(f)$ in the order by inclusion.

Note that $\mathcal{D}(f) \neq \phi$ (see [12, Proposition 2.4]) and if $Y \in \mathcal{D}(f)$, then $f_Y = f|Y : Y \to Y$ is *weakly chaotic in the sense of Devaney*, i.e., f_Y has *sensitive dependence on initial conditions*, f_Y is *topologically transitive* and the union of all minimal sets of f_Y is dense in Y ([12, Theorem 2.7]), i.e., the *min-center* of f_Y is Y (see [1, p. 70]).

Proposition 2.8. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$. If $Y \in \mathcal{D}(f)$, then there is a minimal set $\mathbf{A}$ of $C(f)$ satisfying one of the conditions (1) and (2) as in Proposition 2.7 and $\bigcup\{A|\, A \in \mathbf{A}\} = Y$.*

Proof. Consider the map $f|Y : Y \to Y$. Then there is a minimal set $\mathbf{A}$ of $C(f|Y)$ as in Proposition 2.7. Put $Y' = \bigcup\{A|\, A \in \mathbf{A}\}$. Then Y' is f-invariant and $\dim Y' > 0$. Hence $Y = Y'$.

The following lemma follows from [3, Theorem 6] (see also [15, Lemmas 2 and 1.1]).

Lemma 2.9. *Let P be the pseudo-arc. Let $C = [C_1, C_2, \dots, C_n]$ be a chain covering of P and $f : I(m) \to I(n)$ a pattern with $f(1) = 1$. Let $p \in C_1$. Then there is a chain covering $D = [D_1, D_2, \dots, D_m]$ such that D refines the chain C, $p \in D_1$ and D follows the pattern f in C.*

The following lemma is a simple modification of the uniformization theorem of Mioduszewski (see [17] and [19]). For completeness, we give the proof.

Lemma 2.10. *Let $I = [0, 1]$ be the unit interval. Suppose that $f, g : I \to I$ are piecewise linear onto maps. If $f(0) = g(0) = 0$, then there are onto maps $a, b : I \to I$ such that $f \cdot a = g \cdot b$ and $a(0) = b(0) = 0$.*

Proof. Let $\psi : I^2 \to R$ be the map defined by $\psi(x, y) = f(x) - g(y)$. Note that I^2 is *unicoherent* (i.e., if A and B are coutinua with $A \cup B = I^2$, then $A \cap B$ is connected). In [17], Mioduszewski proved that there is a component K of $\psi^{-1}(0)$ such that K meets all four sides of I^2 (see also [19]). Note that each component of $\psi^{-1}(0)$ is a polyhedron. Let L be a component of $\psi^{-1}(0)$ containing the point $p = (0, 0) \in I^2$. Suppose, on the contrary, that $L \cap (I \times \{1\} \cup \{1\} \times I) = \phi$. Then there is an arc $\alpha : I \to I^2$ such that $\alpha(0) = (x_1, 0) \in I \times \{0\}$, $\alpha(1) = (0, y_1) \in \{0\} \times I$, and $\psi^{-1}(0) \cap \alpha(I) = \phi$. Note that $g(0) = 0 < f(x_1)$ and $g(y_1) > f(0)$. Hence we can see that there is a point $q = (q_1, q_2) \in \alpha(I)$ such that $f(q_1) = g(q_2)$, which implies that $q \in \phi^{-1}(0)$. This is a contradiction. Hence K contains L. By using this fact, we can choose desired maps $a, b : I \to I$.

3. The nonexistence of expansive homeomorphisms of certain continua

The following is the main theorem in this paper.

Theorem 3.1. *Let $f : X \to X$ be a homeomorphism of a compactum X. If there is a minimal set $\mathbf{A}$ of $C(f)$ such that some element A of $\mathbf{A}$ is a (nondegenerate) chainable continuum, then f is not expansive.*

Proof. Suppose, on the contrary, that f is expansive. Replace $\mathbf{A}$ if necessary by an $\mathbf{A}_0 \,{}_*\!< \mathbf{A}$ which satisfies the condition (1) of Proposition 2.7. Since every subcontinuum of a chainable continuum is also chainable, we may assume that $\mathbf{A}$ satisfies the conditions of Proposition 2.7,(1).

Let $c > 0$ be an expansive constant for f and $c/2 > \epsilon > 0$. Now, we shall prove the following property

(3.1.1)

For any $0 < \tau < \epsilon$ there are two points x, y of X and a natural number $n(\tau)$

$$\text{such that } d(x,y) \leq \tau, d(f^{n(\tau)}(x), f^{n(\tau)}(y)) \leq \tau, \text{ and}$$
$$\epsilon \leq \sup\{d(f^j(x), f^j(y)) |\, 0 \leq j \leq n(\tau)\} \leq 2\epsilon.$$

Let $A \in \mathbf{A}$ be a chainable continuum. Since A is chainable, there is a $\tau/4$-chain $C = [C_1, C_2, \ldots, C_r]$ in X which is an open covering of A. We can choose a subcontinuum B_1 of A such that $B_1 \in \mathbf{V}^u(\tau; \epsilon)$ (see (1) of Proposition 2.7), and we may assume that $\mathrm{diam}\,(B_1) = \tau$. Since $B_1 \in \mathbf{V}^u$ and f is expansive, we can choose a natural number N_1 such that if $x, y \in B_1$ and $d(x,y) \geq \tau/4$, then

$$\sup\{d(f^i(x), f^i(y)) |\, 0 \leq i \leq N_1\} > 2\epsilon.$$

Choose a subcontinuum B_2 of B_1 such that $\mathrm{diam}\, B_2 = \tau/2$. By the assumption, there is a sequence $n_1 < n_2 < \ldots$ of natural numbers such that $\lim_{i\to\infty} f^{n_i}(B_2) = \lim_{i\to\infty} f^{n_i}(A) = A$ (see Proposition 2.7). Hence, we can choose a natural number $N > N_1$ such that $f^N(B_1), f^N(B_2) \in \langle C_1, \ldots, C_r\rangle$. Choose a point $e \in B_2$ such that $f^N(e) \in C_1$. Since B_1, B_2 are chainable, by [6] or [14] there are onto maps $\psi_k : P \to B_k$ $(k = 1, 2)$ from the pseudo-arc P onto B_k. Let $p \in P$. Since P is homogeneous [3], we may assume that $\psi_k(p) = e$ for each $k = 1, 2$. Choose a chain covering $D = [D_1, \ldots, D_s]$ of P such that its mesh is sufficiently small. We may assume that if $x, y \in D_i \cup D_{i+1}$, then

$$\sup\{d(f^j(\psi_k(x)), f^j(\psi_k(y))) |\, 0 \leq j \leq N\} < \epsilon/2 \tag{3.1.2}$$

for each $k = 1, 2$. We may assume that $p \in D_1$ (see the proof of [3, Theorem 13]). Also we may assume that D is a refinement of the chains $C^k = (f^N \cdot \psi_k)^{-1}(C)$ $(k = 1, 2)$. Let $f_k : I(s) \to I(r)$ $(k = 1, 2)$ be patterns such that D follows the patterns f_k in C^k $(k = 1, 2)$. Then the patterns f_k $(k = 1, 2)$ induce maps $f_k : N(D) = N(\{D_1, \ldots, D_s\}) \to N(C) = N(\{C_1, \ldots, C_r\})$ which are natural simplicial maps from the nerve $N(D)$ of D to $N(C)$ of C with $f_k(D_j) = C_{f_k(j)}$ for each j.

Since the above nerves are arcs, we can consider that f_k is a map from the unit interval I onto I such that $f_k(0) = 0$ $(k = 1, 2)$. By Lemma 2.10, there are onto maps $g_k : I \to I$ such that $f_1 \cdot g_1 = f_2 \cdot g_2$ and $g_k(0) = 0$.

By using g_k $(k = 1, 2)$, we obtain patterns $g_k : I(l) \to I(s)$ satisfying the inequality $|f_1 g_1(j) - f_2 g_2(j)| \leq 1$ for each $j = 1, 2, \ldots, l$. By Lemma 2.9, we can choose chain coverings $E = [E_1, E_2, \ldots, E_l]$ and $F = [F_1, F_2, \ldots, F_l]$ of P such that E follows the pattern g_1 in D and F follows the pattern g_2 in D. We may assume that $p \in E_1 \cap F_1$.

Choose points $a_1, \ldots, a_l, b_1, \ldots, b_l$ of P beginning with $p = a_1 = b_1$ and such that $a_j \in E_j, b_j \in F_j$. Note that

$$d(f^N(\psi_1(a_j)), f^N(\psi_2(b_j))) \leq \tau.$$

For each $i = 1, 2, \ldots, l$, put

$$r_i = \sup\{d(f^j(\psi_1(a_i)), f^j(\psi_2(b_i))) |\, 0 \leq j \leq N\}.$$

Since the chain cover D is sufficiently small (see (3.1.2)), we may assume that

$$|r_i - r_{i+1}| < \epsilon.$$

Note that $r_1 = 0$. Since ψ_1 is surjective, there is a point a_u $(u \leq l)$ such that $d(\psi_1(a_u), B_2) \geq \tau/4$, and hence $d(\psi_1(a_u), \psi_2(b_u)) \geq \tau/4$. Thus $r_u > 2\epsilon$. Then we can choose $i \leq u$ such that $\epsilon \leq r_i \leq 2\epsilon$. The two points a_i, b_i satisfy the condition (3.1.1).

Let $\{\epsilon_i\}_{i=1}^{\infty}$ be a sequence of positive numbers such that $\lim_{i\to\infty} \epsilon_i = 0$. By the condition (3.1.1), there are two points $x_i, y_i \in X$ and a natural number $n(i)$ such that

$$d(x_i, y_i) < \epsilon_i, \qquad d(f^{n(i)}(x_i), f^{n(i)}(y_i)) < \epsilon_i$$

and

$$\epsilon \leq \sup\{d(f^j(x_i), f^j(y_i)) \,|\, 0 \leq j \leq n(i)\} \leq 2\epsilon.$$

Choose $0 < m(i) < n(i)$ such that $d(f^{m(i)}(x_i), f^{m(i)}(y_i)) \geq \epsilon$. We may assume that $\{f^{m(i)}(x_i)\}$ and $\{f^{m(i)}(y_i)\}$ are convergent to x_0 and y_0, respectively. Note that

$$\lim_{i\to\infty}(n(i) - m(i)) = \infty = \lim_{i\to\infty} m(i).$$

Then $x_0 \neq y_0$ and $d(f^n(x_0), f^n(y_0)) \leq 2\epsilon < c$ for all $n \in \mathbf{Z}$. This is a contradiction.

Corollary 3.2. *If X is a decomposable circle-like continuum, then X admits no expansive homeomorphism.*

Proof. Suppose, on the contrary, that there is an expansive homeomorphism $f : X \to X$. Since X is decomposable, there are two proper nonempty subcontinua A, B of X such that $A \cup B = X$. Since X is circle-like, A and B are chainable. Note that $A \cap B$ has at most 2 components [5, Theorem 5]. By [11, Theorem 3.6], there is a σ-chaotic continuum C of f. We may assume that $\sigma = u$. Then C is indecomposable (see [11, Corollary 5.3]) and is a proper subcontinuum of X. Note that $f^n(C) \cap A$ and $f^n(C) \cap B$ have at most 2 components. Since $f^n(C)$ is indecomposable, we can easily see that for each $n = 0, 1, \dots,$ $f^n(C) \subset A$ or $f^n(C) \subset B$. Hence we see that there is a minimal set $\mathbf{A}$ of $C(f)$ satisfying the condition of Theorem 3.1. By Theorem 3.1, f is not expansive.

Corollary 3.3. *Let $f : X \to X$ be a homeomorphism of a compactum X. Suppose that there are maps $\psi : X \to Y$ and $g : Y \to Y$ such that $\psi \cdot f = g \cdot \psi$ and for each $y \in Y$ $\psi^{-1}(y)$ is a (nondegenerate) chainable continuum. Then f is not expansive.*

Proof. Let $y_0 \in Y$. By Corollary 2.2, we may assume each element of $\omega(\psi^{-1}(y_0))$ is nondegenerate. Since $\psi \cdot f = g \cdot \psi$ and the collection $\{\psi^{-1}(y) \,|\, y \in Y\}$ is an upper semi-continuous decomposition of X, each element of $\omega(\psi^{-1}(y_0))$ is contained in some $\psi^{-1}(y)$, and hence it is chainable.

Take a minimal set $\mathbf{A}$ of $\omega(\psi^{-1}(y_0))$. Then each element of $\mathbf{A}$ is a chainable continuum. By Theorem 3.1, f is not expansive.

Corollary 3.4. *Let $f : X \to X$ be an expansive homeomorphism of a circle-like continuum X, and let $\delta > 0$ be a positive number as in Lemma 2.1. Then one of the following conditions is satisfied:*

(i) *If $A \in C(X)$ and $0 < \operatorname{diam} A < \delta$, then $A \in \mathbf{V}^u$, and if B is a nondegenerate subcontinuum of X, then $X \in \omega(B)$.*
(ii) *If $A \in C(X)$ and $0 < \operatorname{diam} A < \delta$, then $A \in \mathbf{V}^s$, and if B is a nondegenerate subcontinuum of X, then $X \in \alpha(B)$.*

Proof. By Lemma 2.3, we may assume that $\mathbf{V}^u \neq \phi$. Let $A \in \mathbf{V}^u$. Suppose, on the contrary, that $\omega(A)$ does not contain X. Then we obtain a minimal set $\mathbf{A} \subset \omega(A)$ of $C(f)$ satisfying the condition of Theorem 3.1. Hence f is not expansive, which is a contradiction. Since $X \in \omega(A)$, by Lemma 2.4 we see that for each $x \in X$ there is $x \in A_x \in \mathbf{V}^u(\delta;\epsilon)$, where δ, ϵ are as in Lemma 2.1. Suppose, on the contrary, that $\mathbf{V}^s \neq \phi$. Then we see also that for each $x \in X$ there is $x \in B_x \in \mathbf{V}^s(\delta;\epsilon)$. Since f is expansive, we know that $A_x \cap B_x = \{x\}$. Choose A_x and two points $y, z \in A_x$ such that x, y and z are different. Then there are three subcontinua B_x, B_y, B_z such that their diameters are small and B_x, B_y and B_z are mutually disjoint. Then $T = A_x \cup B_x \cup B_y \cup B_z$ is a triod. Since X is atriodic, this is a contradiction. Hence $\mathbf{V}^s = \phi$. Let A be a nondegenerate subcontinuum of X with $\operatorname{diam} A = \gamma < \delta$. Suppose, on the contrary, that $\sup\{\operatorname{diam} f^{-n}(A) |\, n \geq 0\} \geq \epsilon$. By using Lemmas 2.1 and 2.3 inductively, we have a sequence $n_1 < n_2 < \dots$ of natural numbers and a sequence $B_1, B_2, \dots$ of subcontinua such that $B_i \subset f^{-n_i}(A), \operatorname{diam} B_i = \delta$ and $\operatorname{diam} f^j(B_i) \leq \epsilon$ for each $0 \leq j \leq n_i$. We may assume that $\lim_{i\to\infty} B_i = B \in \mathbf{V}^s$. This is a contradiction. Hence we see that $A \in \mathbf{V}^u$. Clearly, if B is a nondegenerate subcontinuum, then $X \in \omega(B)$, because $B \notin \mathbf{V}^s$.

Corollary 3.5. *If $f : X \to X$ is an expansive homeomorphism of a circle-like continuum X, then f is itself weakly chaotic in the sense of Devaney.*

Proof. Consider the set $\mathcal{D}(f) \neq \phi$. Let $Y \in \mathcal{D}(f)$. Since $\dim Y > 0$, Y contains a nondegenerate subcontinuum. By Corollary 3.4, we see that $Y = X$. Hence f is weakly chaotic in the sense of Devaney.

Remark. In [16], Lewis showed that, for every 1-dimensional continuum M there exists a 1-dimensional continuum $\hat{M}$ such that $\hat{M}$ has a continuous decomposition $\psi : \hat{M} \to M$ into pseudo-arcs such that the decomposition space is homeomorphic to M and the decomposition elements are all terminal continua in $\hat{M}$, i.e., every subcontinuum of $\hat{M}$ either is contained in a single decomposition element or is a union of decomposition elements. More generally, let $\tilde{N}$ be a compactum that has an upper semi-continuous decomposition φ into indecomposable chainable continua such that the decomposition elements are all terminal, and let N be the decomposition space. Moreover, if each proper subcontinuum of N is decomposable, then for any homeomorphism $\tilde{h} : \tilde{N} \to \tilde{N}$ there is a homeomorphism $h : N \to N$ such that $\varphi \cdot \tilde{h} = h \cdot \varphi$. By Corollary 3.3, $\tilde{N}$ admits no expansive homeomorphism. The typical continua are solenoids of pseudo-arcs and hence they admit no expansive homeomorphisms.

Problem 3.6. Does there exist an indecomposable plane circle-like continuum which admits an expansive homeomorphism? In particular, does the pseudo-circle admit an expansive homeomorphism?

Problem 3.7. Does there exist a hereditarily indecomposable continuum which admits an expansive homeomorphism?

The author wishes to thank the referee for very helpful suggestions and comments, in particular, the proof of Proposition 2.6.

References

1. E. Akin, *The General Topology of Dynamical Systems*, Amer. Math. Soc., Providence, 1993. MR **94f:**58041
2. N. Aoki, Topological dynamics, in: *Topics in general topology* (eds, K. Morita and J. Nagata), North-Holland, Amsterdam, (1989), 625-740. MR **91m:**58120
3. R. H. Bing, A homogeneous indecomposable plane continuum, *Duke Math. J.*, 15 (1948), 729-742. MR **10:**261a
4. R. H. Bing, Concerning hereditarily indecomposable continua, *Pacific J. Math.*, 1 (1951), 43-51. MR **13:**265b
5. C. E. Burgess, Chainable continua and indecomposability, *Pacific J. Math.*, 9 (1959), 653-659. MR **22:**1867
6. L. Fearnley, Characterizations of the continuous images of the pseudo-arc, *Trans. Amer. Math. Soc.*, 111 (1964), 380-399. MR **29:**596
7. H. Kato, Expansive homeomorphisms in continuum theory, *Topology Appl.*, 45 (1992), 223-243. MR **93j:**54023
8. ______,Continuum-wise expansive homeomorphisms, *Canad. J. Math.*, 45 (1993), 576-598. MR **94k:**54065
9. ______,Chaotic continua of (continuum-wise) expansive homeomorphisms and chaos in the sense of Li and Yorke, *Fund. Math.*, 145 (1994), 261-279. MR **95i:**54049
10. ______,The nonexistence of expansive homeomorphisms of chainable continua, *Fund. Math.*, 149 (1996), 119-126. CMP 96:09
11. ______,Chaos of continuum-wise expansive homeomorphisms and dynamical properties of sensitive maps of graphs, *Pacific J. Math.*, 175 (1996), 93–116. CMP 97:04
12. ______,Minimal sets and chaos in the sense of Devaney on continuum-wise expansive homeomorphisms, *Lecture Notes in Pure and Applied Mathematics*, 170 (1995), 265-274. MR **96c:**54065
13. J. Kennedy, The construction of chaotic homeomorphisms on chainable continua, *Topology Appl.*, 43 (1992), 91-116. MR **93b:**54040
14. A. Lelek, On weakly chainable continua, *Fund. Math.*, 51 (1962), 271-282. MR **26:**742
15. W. Lewis, Most maps of the pseudo-arc are homeomorphisms, *Proc. Amer. Math. Soc.*, 91 (1984), 147-154. MR **85g:**54025
16. W. Lewis, Continuous curves of pseudo-arcs, *Houston J. Math.*, 11 (1985), 91-99. MR **86e:**54038
17. J. Mioduszewski, On a quasi-ordering in the class of continuous mappings of the closed interval onto itself, *Colloq. Math.*, 9 (1962), 233-240. MR **26:**741
18. S. B. Nadler, Jr., Hyperspaces of sets, *Pure and Appl. Math.*, 49 (Dekker, New York, 1978). MR **58:**18330
19. L. G. Oversteegen and E. D. Tymchatyn, On span and weakly chainable continua, *Fund. Math.*, 122 (1984), 159-174. MR **85m:**54034
20. W. Utz, Unstable homeomorphisms, *Proc. Amer. Math. Soc.*, 1 (1950), 769-774. MR **12:**344b
21. R. F. Williams, A note on unstable homeomorphisms, *Proc. Amer. Math. Soc.*, 6 (1955), 308-309. MR **16:**846d

Institute of Mathematics, University of Tsukuba, Ibaraki 305, Japan
E-mail address: `hisakato@sakura.cc.tsukuba.ac.jp`