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TAME COMBINGS OF GROUPS

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Abstract. In this paper, we introduce the idea of tame combings for finitely presented groups. If M is a closed irreducible 3-manifold and $\pi_1(M)$ is tame combable, then the universal cover of M is homeomorphic to $\mathbb{R}^3$. We show that all asynchronously automatic and all semihyperbolic groups are tame combable.

Introduction

In [9], V. Poénaru proved that if a closed irreducible 3-manifold M has an *almost convex* fundamental group, then the universal cover of M is homeomorphic to $\mathbb{R}^3$. A. Casson has devised a group theoretic condition that implies certain closed irreducible 3-manifolds are covered by $\mathbb{R}^3$. In particular, Casson's ideas can be used to show that a closed irreducible 3-manifold with infinite word hyperbolic fundamental group is covered by $\mathbb{R}^3$. This result is also obtained in [2] using different techniques. In this paper, we develop the notion of a tame combing for finitely presented groups, and show that all asynchronously automatic and all semihyperbolic groups are tame combable. If a closed irreducible 3-manifold M has tame combable fundamental group, then we show that the universal cover of M is homeomorphic to $\mathbb{R}^3$. The main theorem of [9] easily follows from our Theorems 1 and 2. (See the remark following Theorem 2 for more on this.) In [3], Brick and Mihalik examine an idea related to Casson's work, the quasi-simply-filtered (QSF) property, which can also be used to show certain 3-manifolds are covered by $\mathbb{R}^3$. Our Theorem 3 states that any tame combable group is QSF.

For G a finitely presented group, X any compact polyhedron with $\pi_1(X) = G$, and $\tilde{X}$ the universal cover of X, our combing condition on G is equivalent to the following geometric property for $\tilde{X}$: If C is a finite connected subcomplex of $\tilde{X}$, then $\pi_1(\tilde{X} - C)$ is finitely generated. In [10], T. Tucker shows that if M is a non-compact P^2-irreducible 3-manifold, and for each finite subcomplex C of M, $\pi_1(M - C)$ is finitely generated, then M is a missing boundary manifold (i.e., there exist a compact 3-manifold N and a closed subset K of the boundary of N such that $N - K$ is homeomorphic to M). If, additionally, M is the universal cover of a closed 3-manifold, then results in [5] imply that M is homeomorphic to $\mathbb{R}^3$ (bypassing the QSF theory).

For technical reasons, the CW-complexes considered in [3] are those where attaching maps on 2-cells are piecewise linear (PL). For the most part we are interested in polyhedra, since all of our applications are directed toward the (covering)

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conjecture — All closed irreducible 3-manifolds with infinite fundamental group are covered by $\mathbb{R}^3$.

The following definition is due to Stephen Brick.

Definition. A finitely presented group G is *quasi-simply-filtered* (QSF) if for some (equivalently any) finite CW-complex X with $\pi_1(X) = G$, the universal cover $\tilde{X}$ of X has the following property: If C is a finite, connected subcomplex of $\tilde{X}$, then there exist a finite, simply connected complex K and a cellular map $f : K \to \tilde{X}$ such that $f|_{f^{-1}(C)}$ is a homeomorphism of $f^{-1}(C)$ onto C.

The notion of bounded and asynchronously bounded combings of groups has gained notoriety in connection with the study of automatic and asynchronously automatic groups (see [6]).

Definition. Suppose X is a 1-complex with fixed basepoint $*$ and edge path metric d. A *discrete path* in X is a map $p : [0, T_p] \cap \mathbb{N} \to X^0$, where $T_p \in \mathbb{N}$ is the *length* of p, such that $d(p(t), p(t+1)) \leq 1$ for all $t < T_p$. For simplicity, if p is a discrete path and $t > T_p$, interpret $p(t)$ to be $p(T_p)$. A *combing* of X is a map Ψ which assigns to each $x \in X^0$ a discrete path $p(t) = \Psi(x, t)$ such that $p(0) = *$ and $p(T_p) = x$. A combing Ψ of X is *bounded* if there exists a constant K such that, for all adjacent $x, y \in X^0$ and all t, $d(\Psi(x, t), \Psi(y, t)) < K$. A combing Ψ of X is *asynchronously bounded* if there exists a constant K such that, for all adjacent $x, y \in X^0$, there exist non-decreasing surjections $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ such that, for all t, $d(\Psi(x, \alpha(t)), \Psi(y, \beta(t))) < K$. A finitely presented group G is said to have a bounded or asynchronously bounded combing if there exists a bounded or asynchronously bounded combing of the 1-skeleton of the universal cover of some (equivalently any) finite complex X with $\pi_1(X) \cong G$.

The definition of combings in terms of discrete paths proves to be somewhat inconvenient for our purposes. We offer the obvious reformulation of combings in terms of continuous paths and a crucial generalization of this notion.

Definition. Suppose X is a 2-complex. A 0-*combing* of X is a homotopy $\Psi : X^0 \times [0, 1] \to X^1$ such that $\Psi(x, 1) = x$, for all $x \in X^0$, and $\Psi|_{X^0 \times \{0\}}$ is constant. A 1-*combing* of X is a homotopy $\Psi : X^1 \times [0, 1] \to X$ such that $\Psi(x, 1) = x$, for all $x \in X^1$, $\Psi|_{X^1 \times \{0\}}$ is constant, and $\operatorname{im}(\Psi|_{X^0 \times [0,1]}) \subseteq X^1$ (i.e., so $\Psi|_{X^0 \times [0,1]}$ is a 0-combing). A 0-combing Ψ of X is *bounded* if there exists $K > 0$ such that for all adjacent $x, y \in X^0$, there are orientation preserving homeomorphisms $\alpha, \beta : [0, 1] \to [0, 1]$ such that $d(\Psi(x, \alpha(t)), \Psi(y, \beta(t))) \leq K$ for all $t \in [0, 1]$.

Clearly, any combing can be made into a 0-combing by connecting successive vertices in a combing path by edges and parameterizing. Conversely, a 0-combing gives rise to a combing by taking the sequence of (successively distinct) vertices in each 0-combing path. Moreover, an asynchronously bounded combing will correspond to a bounded 0-combing (we omit the term "asynchronously" to simplify our terminology and note that 0-combing paths are all parameterized by $[0, 1]$). A 1-combing contains homotopies showing that the loop formed by any edge e and the 0-combing paths to the endpoints of e is homotopically trivial. Hence, while any connected 2-complex is 0-combable, a connected 2-complex will be 1-combable iff it is also simply connected. One can also give an analogous definition of what it would mean for a complex to be 2-combable, but this notion would only become interesting in considering complexes of dimension 3 or higher, since a 2-complex is (essentially by definition) 2-combable iff it is contractible.

Definition. If Ψ is a 0-combing of X, then Ψ is *tame* if for each compact set $C \subseteq X$ there exists a compact set $D \subseteq X$ such that, for all $x \in X^0$, $\Psi^{-1}(C) \cap (\{x\} \times [0,1])$ is contained in one path component of $\Psi^{-1}(D) \cap (\{x\} \times [0,1])$. If Ψ is a 1-combing of X, then Ψ is *tame* if $\Psi|_{X^0 \times [0,1]}$ is a tame 0-combing and, for each compact set $C \subseteq X$, there exists a compact set $D \subseteq X$ such that, for each edge e of X, $\Psi^{-1}(C) \cap (e \times [0,1])$ is contained in one path component of $\Psi^{-1}(D) \cap (e \times [0,1])$. (Observe that if E is a compact subset of X containing a D as above, then one path component of $\Psi^{-1}(E)$ contains $\Psi^{-1}(C)$ as well; hence it suffices to take D sufficiently large. Furthermore, if the condition is satisfied for a compact $C' \supseteq C$ then it will also be satisfied for C, so we need only that the condition holds for all sufficiently large C. Thus we could have taken the compact sets above to be subcomplexes.)

We now list our results.

Theorem 1. *If X and Y are finite, connected 2-dimensional CW-complexes and $\pi_1(X) \cong \pi_1(Y)$, then the universal cover of X has a tame 0-combing or tame 1-combing iff the universal cover of Y does.*

Definition. A finitely presented group G has a *tame* 0-combing (resp. 1-combing) if for some (equivalently any) finite 2-dimensional CW-complex X with $\pi_1(X) \cong G$, the universal cover of X has a tame 0-combing (resp. 1-combing).

Theorem 2. *Let $\tilde{X}$ be the universal cover of a finite 2-dimensional polyhedra X. Then $\tilde{X}$ has a tame 1-combing iff, for each finite subcomplex $C \subseteq \tilde{X}$, $\pi_1(\tilde{X} - C)$ is finitely generated (i.e., each component of $\tilde{X} - C$ has finitely generated fundamental group).*

Remark. If C is a finite set of generators for a group G, the *Cayley graph* $\Gamma(G, C)$ of G with respect to C is the directed labeled graph with vertex set G, and with a directed edge with label e from g to ge for each $g \in G$ and $e \in C$. A metric d is defined on $\Gamma(G, C)$ by declaring each edge to be isometric to the unit interval. The graph $\Gamma(G, C)$ is *k-almost convex* if there exists an integer N such that any two vertices v_1 and v_2 in $S(n)$ (the n-sphere centered at 1) with $d(v_1, v_2) \leq k$ can be joined by a path in $B(n)$ (the n-ball at 1) of length $\leq N$. For $k \geq 3$ it is an easy exercise to check that if P is a finite presentation of a group G (say with generating set C) and $\Gamma(G, C)$ is k-almost convex, then if X is the standard finite 2-complex corresponding to P and $\tilde{X}$ is the universal cover of X, $\pi_1(\tilde{X} - D)$ is finitely generated for any finite subcomplex $D \subseteq \tilde{X}$. It follows from Tucker's theorem and our theorems 1 and 2 that if M is a closed irreducible 3-manifold, $\pi_1(M)$ is infinite and for some generating set C, $\Gamma(\pi_1(M), C)$ is k-almost convex, $k \geq 3$, then the universal cover of M is homeomorphic to $\mathbb{R}^3$. This is the main theorem of [9].

Theorem 3. *If G is a finitely presented group having a tame 1-combing, then G is QSF.*

The following class of groups is defined in [1].

Definition. If X is a 1-complex and $\lambda, \epsilon > 0$, a discrete path $p : [0, T_p] \cap \mathbb{N} \to X^0$ is a (λ, ϵ)-*quasigeodesic* if for any $x, y \in [0, T_p] \cap \mathbb{N}$ we have $\frac{1}{\lambda}|x - y| - \epsilon \leq d(p(x), p(y)) \leq \lambda |x - y| + \epsilon$. A finitely presented group G is in C_+ if for some $\lambda, \epsilon > 0$, G has a bounded combing by (λ, ϵ)-quasigeodesics.

The following is a list of some of the groups in C_+: automatic groups, semi-hyperbolic groups, Coxeter groups, fundamental groups of closed 3-manifolds with everywhere non-positive sectional curvature, small cancellation groups, and any group which acts properly and cocompactly on a Tits building of Euclidean type (see [1]).

Theorem 4. *If $G \in C_+$, then G has a tame 1-combing.*

Theorem 5. *Asynchronously automatic groups have tame 1-combings.*

Corollary 6. *If M is a closed irreducible 3-manifold with infinite fundamental group which is asynchronously automatic or in C^+, then the universal cover of M is homeomorphic to $\mathbb{R}^3$.*

In [8], some of our results have been extended to show that certain intermediate coverings of 3-manifolds are missing boundary manifolds. The results of [8] do not apply to all groups in C^+, but to an overlapping class of groups – those with almost prefix closed combings. The intermediate coverings considered are those corresponding to subgroups which are quasiconvex with respect to the almost prefix closed combing.

Proofs of theorems

Proof of Theorem 1. It suffices to show that if the universal cover of X has a tame 0-combing or a tame 1-combing, then the universal cover of Y does also, since the same argument will prove the converse. We do the tame 1-combing version, as the proof of the tame 0-combing version is then easily seen. Fix vertices $*_1$ and $*_2$ of $X_1 = X$ and $X_2 = Y$ respectively. For $i \in \{1, 2\}$, let $(\tilde{X}_i, \tilde{*}_i)$ be the universal cover of $(X_i, *_i)$ with covering projection m_i. The proof makes use of the following two facts basic to understanding the relationship between finite 2-complexes with the same fundamental group.

Lemma 1.1. *If $(X_1, *_1)$ and $(X_2, *_2)$ are finite 2-complexes with isomorphic fundamental groups, then there are cellular maps $f_1 : (X_1, *_1) \to (X_2, *_2)$ and $f_2 : (X_2, *_2) \to (X_1, *_1)$ such that $f_2 \circ f_1$ and $f_1 \circ f_2$ induce the identity on $\pi_1(X_1, *_1)$ and $\pi_1(X_2, *_2)$ respectively.*

Lemma 1.2. *Assume $f_1 : (X_1, *_1) \to (X_2, *_2)$ and $f_2 : (X_2, *_2) \to (X_1, *_1)$ are cellular maps such that $f_1 \circ f_2$ induces the identity on $\pi_1(X_2, *_2)$. Let $\tilde{f}_1$ be the lift of f_1 to $(\tilde{X}_1, \tilde{*}_1)$ taking $\tilde{*}_1$ to $\tilde{*}_2$, and let $\tilde{f}_2$ be the lift of f_2 to $(\tilde{X}_2, \tilde{*}_2)$ taking $\tilde{*}_2$ to $\tilde{*}_1$ (see Figure 1). Then there exists an integer N such that, for all $x \in \tilde{X}_2$, $\tilde{f}_1 \circ \tilde{f}_2(x) \in St^N(x)$.*

$$\begin{tikzcd}
(\tilde{X}_2, \tilde{*}_2) \arrow[r, "\tilde{f}_2"] \arrow[d, "m_2"] & (\tilde{X}_1, \tilde{*}_1) \arrow[r, "\tilde{f}_1"] \arrow[d, "m_1"] & (\tilde{X}_2, \tilde{*}_2) \arrow[d, "m_2"] \\
(X_2, *_2) \arrow[r, "f_2"] & (X_1, *_1) \arrow[r, "f_1"] & (X_2, *_2)
\end{tikzcd}$$

Figure 1

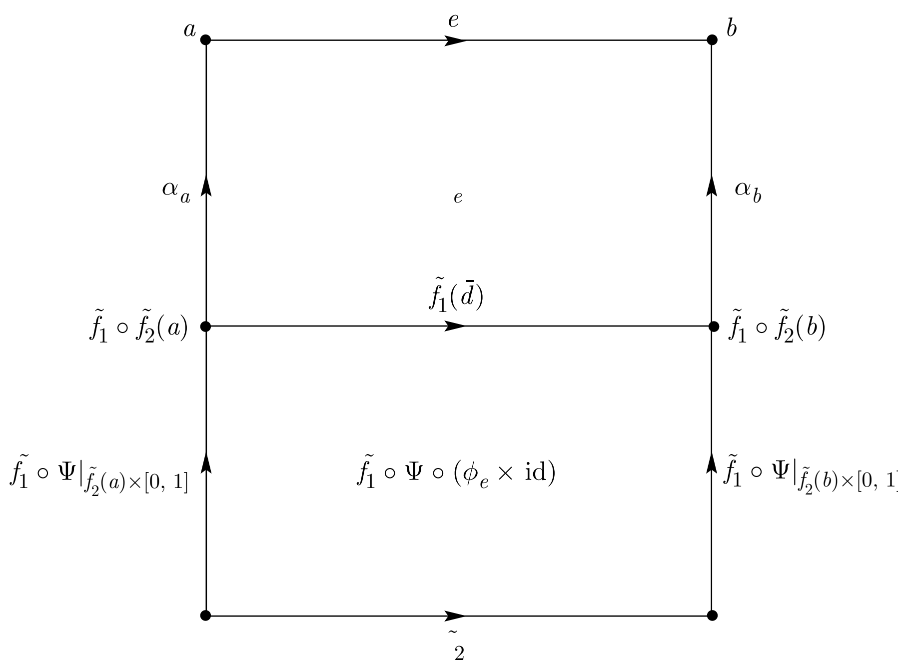

FIGURE 2

Given these two lemmas, to complete the proof of Theorem 1, take $f_1 : (X_1, *_1) \to (X_2, *_2)$ and $f_2 : (X_2, *_2) \to (X_1, *_1)$ such that $f_1 \circ f_2$ induces the identity on $\pi_1(X_2, *_2)$, and $f_2 \circ f_1$ induces the identity on $\pi_1(X_1, *_1)$ by Lemma 1.1. For $i = 1, 2$, let $\tilde{f}_i$ be the lift of f_i as in Lemma 1.2, and take N to be sufficiently large so that, for all $x \in \tilde{X}_2$, $\tilde{f}_1 \circ \tilde{f}_2(x) \in \mathrm{St}^N(x)$. For each vertex v of $\tilde{X}_2$, let α_v be an edge path from $\tilde{f}_1 \circ \tilde{f}_2(v)$ to v, of length $\leq N$.

Let Ψ be a tame 1-combing for $\tilde{X}_1$, say, for the sake of simplicity and without loss of generality, with $\Psi(x, 0) = \tilde{*}_1$. Define a 0-combing $\hat{\Psi}$ of $\tilde{X}_2$ and then extend this map to a 1-combing as follows. For each vertex v of $\tilde{X}_2$ define $\hat{\Psi}$ on $\{v\} \times [0, 1]$ to be the path $\tilde{f}_1 \circ \Psi|_{\{\tilde{f}_2(v)\}\times[0,1]}$ followed by the path α_v. Suppose e is an edge of $\tilde{X}_2$ with initial point a and endpoint b. If $\tilde{f}_2(a) \neq \tilde{f}_2(b)$, choose a simple edge path $\bar{d} = \langle d_1, d_2, \ldots, d_n\rangle$ in $\tilde{X}_1$ from $\tilde{f}_2(a)$ to $\tilde{f}_2(b)$, contained in $\tilde{f}_2(e)$, and homeomorphic to e by $\phi_e : e \to \bar{d}$, and otherwise take $\bar{d}$ a constant path at $\tilde{f}_2(a)$, so $\phi_e : e \to \bar{d}$ is a constant map. Then $\tilde{f}_1(\bar{d})$ is a path from $\alpha_a(0)$ to $\alpha_b(0)$, with image in $\tilde{f}_1 \circ \tilde{f}_2(e) \subseteq \mathrm{St}^N(e) \subseteq \mathrm{St}^{N+1}(a)$. There exists a fixed M such that if v is any vertex of $\tilde{X}_2$ and β is a loop in $\mathrm{St}^{N+1}(v)$, then β is homotopically trivial in $\mathrm{St}^M(v)$. Take a homotopy H_e killing the loop $\langle \alpha_a^{-1}, \tilde{f}_1(\bar{d}), \alpha_b, e^{-1}\rangle$ in $\mathrm{St}^M(a)$ and define $\hat{\Psi}|_{e\times[0,1]}$ by patching H_e to $\tilde{f}_1 \circ \Psi \circ (\phi_e \times \mathrm{id})$ as in Figure 2.

To see that $\hat{\Psi}$ is tame, let C be any compact subset of $\tilde{X}_2$. Then $C_1 = \tilde{f}_1^{-1}(C) \cup \{\tilde{*}_1\}$ is compact in $\tilde{X}_1$, since $\tilde{f}_1$ is proper (see [7]). Since Ψ is tame, there exists a compact $D_1 \subseteq \tilde{X}_1$ such that, if c is any edge or vertex of $\tilde{X}_1$, then $\Psi^{-1}(C_1) \cap (c \times [0, 1])$ is contained in one component of $\Psi^{-1}(D_1) \cap (c \times [0, 1])$. There are only finitely many edges e of $\tilde{X}_2$ such that $\mathrm{im}(H_e) \cap C \neq \emptyset$ (since $\mathrm{im}(H_e) \subseteq \mathrm{St}^M(a)$, where a is the initial point of e). Take D to be the union of $\tilde{f}_1(D_1)$ and the $\mathrm{im}(\hat{\Psi}|_{e\times[0,1]})$ for these finitely many e. Then D is compact, and for edges e with $\mathrm{im}(H_e)\cap C \neq \emptyset$ we trivially have $\hat{\Psi}^{-1}(C)\cap(e\times[0,1])$ contained in one component of $\hat{\Psi}^{-1}(D)\cap(e\times[0,1]) = e\times[0,1]$ since $\mathrm{im}(\hat{\Psi}|_{e\times[0,1]})\subseteq D$. For e with $\mathrm{im}(H_e)\cap C=\emptyset$ we need only check that $(\tilde{f}_1\circ\Psi\circ(\phi_e\times\mathrm{id}))^{-1}(C)$ is contained in one component of $(\tilde{f}_1\circ\Psi\circ(\phi_e\times\mathrm{id}))^{-1}(D)$. With $\mathrm{im}(\phi_e)=\bar{d}=\langle d_1,\dots,d_n\rangle$ as before, for each d_i by the choice of D_1, we have that $\Psi^{-1}(C_1)\cap(d_i\times[0,1])$ is contained in one component of $\Psi^{-1}(D_1)\cap(d_i\times[0,1])$. But $\tilde{*}_1\in C_1$ so $\Psi(d_i\times\{0\})=\tilde{*}_1\in D_1$ and the one component of $\Psi^{-1}(D_1)\cap(d_i\times[0,1])$ of interest contains $d_i\times\{0\}$. Thus $\Psi^{-1}(C_1)\cap(\bar{d}\times[0,1])$ is contained in one component of $\Psi^{-1}(D_1)\cap(\bar{d}\times[0,1])$. Since $\tilde{f}_1^{-1}(C)\subseteq C_1$ and $\tilde{f}_1(D_1)\subseteq D$, we have

$$(\tilde{f}_1\circ\Psi\circ(\phi_e\times\mathrm{id}))^{-1}(C)\subseteq(\phi_e\times\mathrm{id})^{-1}(\Psi^{-1}(C_1)\cap(\bar{d}\times[0,1]))$$

is contained in one component of

$$(\phi_e\times\mathrm{id})^{-1}(\Psi^{-1}(D_1)\cap(\bar{d}\times[0,1]))\subseteq(\tilde{f}_1\circ\Psi\circ(\phi_e\times\mathrm{id}))^{-1}(D)$$

as required (and this also works if $\bar{d}$ is simply a constant path, since D_1 also witnesses that $\Psi|_{X_1^0\times[0,1]}$ is tame). A similar but simpler argument applies to show that the restriction of $\hat{\Psi}$ to $\tilde{X}_2^0\times[0,1]$ is a tame 0-combing, hence $\hat{\Psi}$ is tame. □

Remark on Theorem 2. The spaces of this theorem are polyhedra rather than CW-complexes. The reason for this is that if A is a finite polyhedron and B is a subcomplex of A, then $\pi_1(A-B)$ is finitely generated. The corresponding result is not true for A a CW-complex. R. Geoghegan has exhibited an elementary example of a finite CW-complex A with one vertex v such that $\pi_1(A-\{v\})$ is not finitely generated.

Proof of Theorem 2. First suppose that, for each finite subcomplex $C\subseteq\tilde{X}$, the group $\pi_1(\tilde{X}-C)$ is finitely generated. Define a nested sequence $C_0\subseteq C_1\subseteq C_2\subseteq\dots$ of finite connected subcomplexes of $\tilde{X}$ whose union is $\tilde{X}$ as follows. Let $C_0=\emptyset$, and take C_1 to be a finite connected subcomplex of $\tilde{X}$ containing the vertex $*$ such that $\tilde{X}-C_1$ is a union of unbounded path components (where a set is *unbounded* in $\tilde{X}$ if it is contained in no compact subset of $\tilde{X}$). For $i\geq 2$, take C_i to be a connected finite subcomplex of $\tilde{X}$ such that

- a_i) C_i contains $\mathrm{St}(C_{i-1})$,
- b_i) each path component of $\tilde{X}-C_i$ is unbounded,
- c_i) if Γ is a path component of $\tilde{X}-C_{i-1}$, then $\Gamma\cap C_i$ is path connected and C_i contains loops representing generators of some finite generating set of $\pi_1(\Gamma)$, and
- d_i) if $i=2$, then C_2 is such that any loop in C_1 is homotopically trivial in C_2, and for $i\geq 3$, any loop based in C_{i-2} and contained in $C_{i-1}-C_{i-3}$ is homotopic rel$\{0,1\}$ in C_i-C_{i-3} to a loop in $C_{i-2}-C_{i-3}$.

(To see this, construct the C_i recursively. Take C' containing $\mathrm{St}(C_1)$ such that any loop in C_1 is homotopically trivial in C', add to C' enough paths to make c_2) hold for C', and then throw in any bounded path components of the complement of that to get C_2. Suppose $i\geq 3$ and that, for $j<i$, C_j is a finite connected subcomplex of $\tilde{X}$ satisfying a_j)–d_j). If D is a connected finite subcomplex of $\tilde{X}$ satisfying a_i) or d_i) and E is a finite connected subcomplex of $\tilde{X}$ containing D, then E trivially satisfies a_i) or d_i) respectively. If D is a connected finite subcomplex of

$\tilde{X}$ containing $\mathrm{St}(C_{i-1})$ and satisfying c_i), and E is a finite connected subcomplex of $\tilde{X}$ containing D, then an elementary path connectedness argument shows that for Γ as in c_i), $\Gamma \cap E$ is path connected. Also E trivially satisfies the second condition of c_i). Hence if we find D_a, D_c, and D_d satisfying a_i), c_i), and d_i) respectively, then the union of D_a, D_c, D_d and all bounded path components of $\tilde{X} - (D_a \cup D_c \cup D_d)$ will be a connected finite subcomplex of $\tilde{X}$ satisfying $a(i)$–$d(i)$. Let D_a be $\mathrm{St}(C_{i-1})$. Let D'_c be the union of $\mathrm{St}(C_{i-1})$ and, for each path component Γ of $\tilde{X} - C_{i-1}$ (there are only finitely many), finitely many loops based in $\mathrm{St}(C_{i-1}) - C_{i-1}$ representing generators of some finite generating set of $\pi_1(\Gamma)$, together with paths in Γ joining any two components of $\Gamma \cap \mathrm{St}(C_{i-1})$. As D'_c is compact, there is a finite connected subcomplex D_c of $\tilde{X}$ containing D'_c. For each path component Γ of $\tilde{X} - C_{i-3}$, $\pi_1(\Gamma)$ is finitely generated and, by c_{i-2}, any loop based in C_{i-2} representing a generator of $\pi_1(\Gamma)$ is homotopic rel$\{0,1\}$ in $\tilde{X} - C_{i-3}$ to a loop in $C_{i-2} - C_{i-3}$. Take a finite subcomplex C' containing $\mathrm{St}(C_{i-1})$ and large enough to contain homotopies showing that for each such Γ, each loop based in C_{i-2} in a finite generating set of $\pi_1(\Gamma \cap C_{i-1})$ is homotopic rel$\{0,1\}$ to a loop in $C_{i-2} - C_{i-3}$. Then any loop based in C_{i-2} and contained in $C_{i-1} - C_{i-3}$ is homotopic rel$\{0,1\}$ in $C' - C_{i-3}$ to a loop in $C_{i-2} - C_{i-3}$.)

For each vertex $v \in \tilde{X}$, fix α_v to be an edge path from $*$ to v such that, for all i, $\alpha_v^{-1}(C_i)$ is connected (since for each $i > 0$ and each component Γ of $\tilde{X} - C_{i-1}$, $\Gamma \cap C_i$ is path connected). Suppose e is an edge of $\tilde{X}$ from a vertex x to a vertex y. Let k be the first integer such that $e \subseteq C_k$. For $0 < i < k$, take $t_i, s_i \in [0,1]$ such that $\alpha_x^{-1}(C_i) = [0, t_i]$ and $\alpha_y^{-1}(C_i) = [0, s_i]$ (note that, by a), the $\{t_i\}$ and $\{s_i\}$ sequences are strictly increasing) and take $\beta(e,i)$ to be a path in $C_i - C_{i-1}$ from $\alpha_x(t_i)$ to $\alpha_y(s_i)$ (again since, for Γ a path component of $\tilde{X} - C_{i-1}$, $\Gamma \cap C_i$ is path connected). The loop

$$\langle \alpha_x|_{[t_{k-1},1]}, e, \alpha_y^{-1}|_{[s_{k-1},1]}, \beta^{-1}(e, k-1)\rangle$$

has image in $C_k - C_{k-2}$ and hence is homotopic rel$\{0,1\}$ to a loop $\gamma(e, k-1)$ in $C_{k-1} - C_{k-2}$ by a homotopy $H(e, k-1)$ in $C_{k+1} - C_{k-2}$ (by d) above). Suppose $1 < i < k$ and we have defined a loop $\gamma(e,i)$ in $C_i - C_{i-1}$ based at $\alpha_x(t_i)$. Then $\langle \alpha_x|_{[t_i,t_{i-1}]}, \gamma(e,i), \beta(e,i), \alpha_y^{-1}|_{[s_i,s_{i-1}]}, \beta^{-1}(e,i-1)\rangle$ is a loop in $C_i - C_{i-2}$ and therefore is homotopic rel$\{0,1\}$ to a loop $\gamma(e,i-1)$ in $C_{i-1} - C_{i-2}$ by a homotopy $H(e,i-1)$ in $C_{i+1} - C_{i-2}$ (again, by d) above). Continuing until $\gamma(e,1)$ and $H(e,1)$ are defined, at the final step $\langle \alpha_x|_{[0,t_1]}, \gamma(e,1), \beta(e,1), \alpha_y^{-1}|_{[0,s_1]}\rangle$ is a loop in C_1, and therefore is homotopically trivial by a homotopy $H(e,0)$ in C_2.

Define a 1-combing Ψ of $\tilde{X}$ by defining, for each edge e of $\tilde{X}$, $\Psi|_{e\times[0,1]}$ to be the homotopy combining the $H(e,i)$ for $i \in \{0,1,\ldots,k-1\}$ as in Figure 3 (clearly $\Psi|_{\tilde{X}\times\{0\}}$ is constant and $\Psi|_{\tilde{X}\times\{1\}}$ is the identity, and this is continuous since on the overlaps of these the $\Psi|_{\{v\}\times[0,1]}$ are the fixed paths α_v). Given any compact set C in $\tilde{X}$, there exists n sufficiently large so that $C \subseteq C_n$. Given any edge e of $\tilde{X}$, for each i such that $H(e,i)$ is defined, $H(e,i)$ has image in $C_{i+2} - C_{i-1}$. Hence if $i > n$, then $H(e,i) \cap C_n = \emptyset$, and so $\Psi^{-1}(C_n) \cap (e \times [0,1]))$ is contained in the union of the domains of $H(e,0), \ldots, H(e,n)$. As $\bigcup_{i=0}^{n} im(H(e,i)) \subseteq C_{n+2}$, $\Psi^{-1}(C) \cap (e \times [0,1]) \subseteq \Psi^{-1}(C_n) \cap (e \times [0,1])$ is contained in one component of $\Psi^{-1}(C_{n+2}) \cap (e \times [0,1])$. By similar, but simpler reasoning, the restriction of Ψ to $\tilde{X}^0 \times [0,1]$ is a tame 0-combing; hence Ψ is a tame 1-combing of $\tilde{X}$.

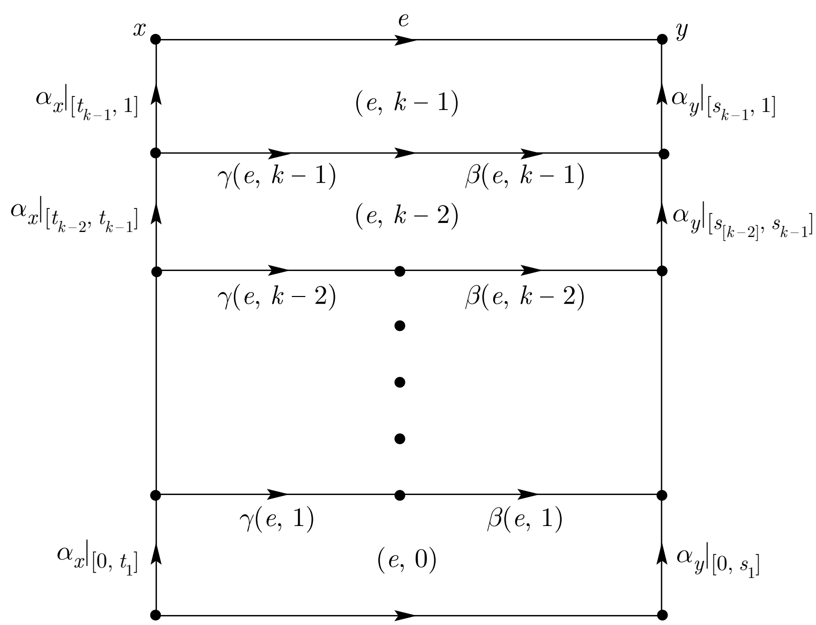

FIGURE 3

Conversely, suppose Ψ is a tame 1-combing of $\tilde{X}$ and let $* = \Psi(\tilde{X} \times \{0\})$. Given any finite subcomplex C of $\tilde{X}$, let Γ be a component of $\tilde{X} - C$. If Γ is bounded, then $\pi_1(\Gamma)$ is finitely generated. Otherwise, let $*' \in \Gamma$ be a base point. Let D be a finite subcomplex of $\tilde{X}$ containing $\{*, *'\} \cup \mathrm{St}(C)$, and such that for any vertex or edge c of $\tilde{X}$, $\Psi^{-1}(\mathrm{St}(C) \cup \{*\}) \cap (c \times [0,1])$ is contained in one path component of $\Psi^{-1}(D) \cap (c \times [0,1])$. Let E be compact in $\tilde{X}$ such that for any vertex or edge c of $\tilde{X}$, $\Psi^{-1}(\mathrm{St}(D)) \cap (c \times [0,1])$ is contained in one path component of $\Psi^{-1}(E) \cap (c \times [0,1])$.

Let e be an edge of $\Gamma - D$ and let $T = e \times [0,1]$. Say that $\hat{D}$ and $\hat{E}$ are the path components of $\Psi^{-1}(D) \cap T$ and $\Psi^{-1}(E) \cap T$ containing $\Psi^{-1}(\mathrm{St}(C) \cup \{*\}) \cap T$ and $\Psi^{-1}(\mathrm{St}(D)) \cap T$ respectively. Clearly, $\hat{D} \cap (e \times \{1\}) = \emptyset$. Let a and b be the initial point and endpoint of e.

Lemma 2.1. *There is an arc β in T such that $im(\beta) \cap (\{a\} \times [0,1]) = \{(a,s)\}$ is the initial point of β, $im(\beta) \cap (\{b\} \times [0,1]) = \{(b,t)\}$ is the endpoint of β, β separates $e \times \{1\}$ from $\Psi^{-1}(C) \cap T$, $im(\beta) \cap \hat{D} = \emptyset$, and $(\{a\} \times [0,s]) \cup (\{b\} \times [0,t]) \cup im(\beta) \subseteq \hat{E}$.*

Proof. The map $\Psi|_T$ is a uniformly continuous map of a rectangle into $\tilde{X}$. Choose $\epsilon_1 > 0$ such that for $x, y \in T$ and $d(x,y) < \epsilon_1$, $\Psi(x) \in \mathrm{St}(\Psi(y))$. Choose $\epsilon > 0$ such that $2\epsilon \leq \epsilon_1$ and $2\epsilon < d(e \times \{1\}, \hat{D})$.

If $x \in \mathrm{bd}(\hat{D}) \cap (\{a,b\} \times [0,1])$, then let D_x be the disk of radius ϵ in T centered at x. If $x \in \mathrm{bd}(\hat{D}) - (\{a,b\} \times [0,1])$, let D_x be a closed disk of radius less than the minimum of ϵ and $d(x, \{a,b\} \times [0,1])$, centered at x. By the compactness of $\mathrm{bd}(\hat{D})$, there are finitely many D_x, call them $D_1, \ldots, D_n$, in T such that $\mathrm{bd}(\hat{D}) \subseteq \bigcup_{i=1}^n \mathrm{int}(D_i)$. By slightly enlarging the radius of some D_i, we may assume that no two D_i intersect at a single point, while retaining the properties that $\Psi(D_i) \subseteq$

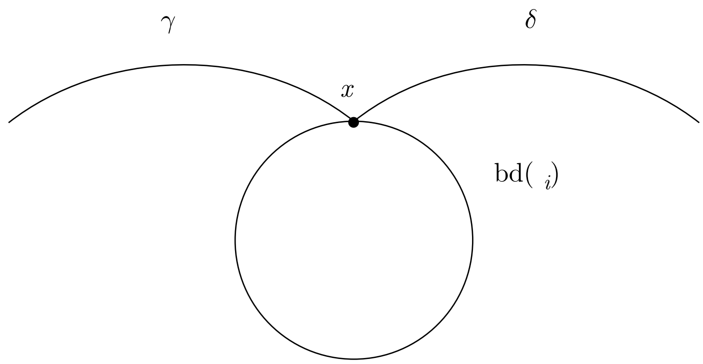

FIGURE 4

St(D), if D_i is not centered at a point of $\{a,b\}\times[0,1]$ then $D_i\cap(\{a,b\}\times[0,1])=\emptyset$, and $(\bigcup_{i=1}^n D_i)\cap(e\times\{1\})=\emptyset$.

The boundary of $\bigcup_{i=1}^n D_i$ is a finite union of arcs of circles. Say that γ and δ are two such arcs and $x\in\gamma\cap\delta$. If $\mathrm{bd}(D_k)$ is a circle containing x but not containing γ or δ, then there are open arcs on either side of x in $\mathrm{bd}(D_k)$ that are subsets of $\bigcup_{i=1}^n \mathrm{int}\, D_i$ (see Figure 4), i.e., each component of $\mathrm{bd}(\bigcup_{i=1}^n D_i)$ is a 1-manifold.

Let s be the largest number with $(a,s)\in\bigcup_{i=1}^n D_i$. Then $(a,s)\in\mathrm{bd}(\bigcup_{i=1}^n D_i)$, so $(a,s)\notin\hat{D}$. Let β be the component of $\mathrm{bd}(\bigcup_{i=1}^n D_i)$ containing (a,s). Observe that β is an arc with initial point (a,s) and, for some k, $(a,s)\in\mathrm{bd}(D_k)$, where D_k is one of the disks centered at an $(a,s')\in\mathrm{bd}(\hat{D})$ with $s'<s$. Also, β does not intersect $\hat{D}$ (as $\mathrm{bd}(\hat{D})\subseteq\mathrm{int}(\bigcup_{i=1}^n D_i)$). If the endpoint of β were in $\{a\}\times[0,1]$, then β would separate $\mathrm{int}(D_k)$ from $(a,0)$ and hence separate points of $\hat{D}$, contrary to the connectedness of $\hat{D}$. By the definition of ϵ, $\mathrm{im}(\beta)\cap(e\times\{1\})=\emptyset$. Thus the endpoint of β is in $\{b\}\times[0,1]$. Since $\{a\}\times\{0,s\}\subseteq\Psi^{-1}(\mathrm{St}(D))\cap(\{a\}\times[0,1])$, we have $\{a\}\times[0,s]\subseteq\hat{E}$. Similarly, if (b,t) is the terminal point of β, then $\{b\}\times[0,t]\subseteq\hat{E}$. Since $\Psi(\bigcup_{i=1}^n D_i)\subseteq\mathrm{St}(D)$, we have $\mathrm{im}(\Psi\circ\beta)\subseteq\mathrm{St}(D)$ so $\mathrm{im}(\beta)\subseteq\hat{E}$. Now β separates $e\times\{1\}$ from $(a,0)$. But $\hat{D}$ is connected and $(a,0)\in\hat{D}$, so β separates $e\times\{1\}$ from $\hat{D}$ and hence $e\times\{1\}$ from $\Psi^{-1}(C)$. □

Let a be a vertex of Γ. Define $\ell(a)\in[0,1]$ to be the largest number such that $\Psi(\{a\}\times[0,\ell(a)])\subseteq D$. Now $\Psi^{-1}(\mathrm{St}(C))\cap(\{a\}\times[0,1])\subseteq[0,\ell(a)]$, and if e is an edge of $\tilde{X}-D$ with initial vertex a, then the s given in Lemma 2.1 is larger than $\ell(a)$. By Lemma 2.1, we have that if e is an edge of $\Gamma-D$ with initial point a and terminal point b, then Ψ gives a homotopy between e and

$$\langle(\Psi|_{\{a\}\times[s,1]})^{-1},\Psi\circ\beta,\Psi|_{\{b\}\times[t,1]}\rangle$$

(where s and t are defined in Lemma 2.1). Furthermore, the image of this homotopy does not intersect C, and so lies in Γ. Combining this homotopy with two homotopies that eliminate backtracking in $\{a\}\times[0,1]$ and $\{b\}\times[0,1]$, we have that e is homotopic to

$$\langle(\Psi|_{\{a\}\times[\ell(a),1]})^{-1},\Psi|_{\{a\}\times[\ell(a),s]},\Psi\circ\beta,(\Psi|_{\{b\}\times[\ell(b),t]})^{-1},\Psi|_{\{b\}\times[\ell(b),1]}\rangle$$

by a homotopy in Γ. By the choice of $\ell(a)$, $\ell(b)$, a, and b, $\mathrm{im}(\Psi|_{\{a\}\times[\ell(a),s]})\cup\mathrm{im}(\Psi|_{\{b\}\times[\ell(b),t]})$ is a subset of $E-C$. Hence, if e is an edge of $\Gamma-D$, with initial point

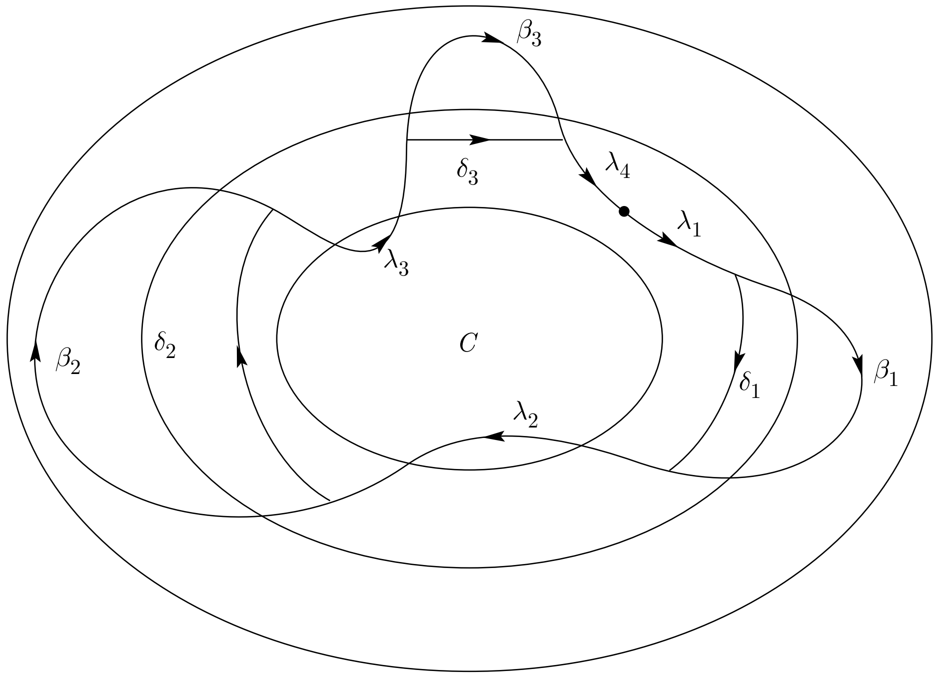

FIGURE 5

a and endpoint b, then e is homotopic to a path $\langle(\Psi|_{\{a\}\times[\ell(a),1]})^{-1}, \alpha, \Psi|_{\{b\}\times[\ell(b),1]}\rangle$ by a homotopy H_e in $\hat{\Gamma}$, where α is a path with $\operatorname{im}(\alpha) \subseteq E$.

Now let τ be any edge loop $\langle e_1, e_2, \ldots, e_n\rangle$ in Γ based at $*'$. If e_i is an edge of D, let H_{e_i} be the trivial homotopy of e_i to itself. If e_i is an edge of $\mathrm{St}(D)$ but not of D, then there are two cases when defining H_{e_i}. If both the initial vertex a and the terminal vertex b of e_i lie in $\mathrm{St}(D) - D$, then let H_{e_i} be the homotopy of e_i to

$$\langle(\Psi|_{\{a\}\times[\ell(a),1]})^{-1}, \Psi|_{\{a\}\times[\ell(a),1]}, e_i, (\Psi|_{\{b\}\times[\ell(b),1]})^{-1}, \Psi|_{\{b\}\times[\ell(b),1]}\rangle$$

that eliminates backtracking in $\{a\} \times [0,1]$ and $\{b\} \times [0,1]$. Similarly define H_{e_i} if one vertex of e_i is in $\mathrm{St}(D) - D$ and the other vertex is in D.

Combining the homotopies H_{e_i} gives a homotopy of τ rel$\{*'\}$ to a loop in $E \cap \Gamma$. As $\pi_1(E\cap\Gamma, *')$ is finitely generated, $\pi_1(\Gamma, *')$ is finitely generated. $\square$

Proof of Theorem 3. Take X to be the standard 2-complex for a finite presentation of G and $\tilde{X}$ to be the universal cover of X. Assume G has a tame 1-combing, or equivalently, by the last theorem, for any finite subcomplex C of $\tilde{X}$, $\pi_1(\tilde{X} - C)$ is finitely generated. (A proof that G is QSF can be given using either of these. We give the version using the latter condition.) Let C be a connected finite subcomplex of $\tilde{X}$. Choose D a connected finite subcomplex of $\tilde{X}$ containing $\mathrm{St}(C)$ such that for each component Γ of $\tilde{X}-C$, $D\cap\Gamma$ is connected and D contains loops (based in $\Gamma\cap D$) representing generators in a finite generating set for $\pi_1(\Gamma)$. Let E be a connected finite subcomplex of X containing D such that any loop in D is homotopically trivial in E. Say that $\alpha_1, \ldots, \alpha_n$ are edge path loops based at $* \in C$ such that $[\alpha_1], \ldots, [\alpha_n]$ generate $\pi_1(E, *)$.

If $\alpha \in \{\alpha_1, \dots, \alpha_n\}$, then α can be written as $\langle \lambda_1, \beta_1, \lambda_2, \beta_2, \cdots, \lambda_{k-1}, \beta_{k-1}, \lambda_k \rangle$, where λ_i is an edge path in D and β_i is an edge path in $E - C$. By the definition of D, there is an edge path δ_i in $D - C$ from $\beta_i(0)$ to $\beta_i(1)$ (see Figure 5). Let

$$\gamma_i = \langle \lambda_1, \delta_1, \dots, \lambda_{i-1}, \delta_{i-1}, \lambda_i, \beta_i, \delta_i^{-1}, \lambda_i^{-1}, \delta_{i-1}^{-1}, \lambda_{i-1}^{-1}, \dots, \delta_1^{-1}, \lambda_1^{-1} \rangle.$$

Then α is homotopic rel$\{0,1\}$ to $\langle \gamma_1, \dots, \gamma_{k-1}, \xi \rangle$ by a homotopy in E, where $\xi = \langle \lambda_1, \delta_1, \dots, \lambda_{k-1}, \delta_{k-1}, \lambda_k \rangle$. As ξ is a loop in D, ξ is homotopically trivial in E. The loop $\langle \beta_i, \delta_i^{-1} \rangle$ has image in $E - C$ with initial point $*'$, a vertex of D. Let Γ be the component of $\tilde{X} - C$ containing $*'$ (and hence $\mathrm{im}\langle \beta_i, \delta_i^{-1} \rangle$). By the definition of D there exist a point $*'' \in \Gamma \cap D$ and finitely many loops in $D \cap \Gamma$, based at $*''$, which represent generators for $\pi_1(\Gamma, *'')$. As $D \cap \Gamma$ is connected, there are finitely many loops in $D \cap \Gamma$ based at $*'$ which represent generators for $\pi_1(\Gamma, *')$. As $\langle \beta_i, \delta_i^{-1} \rangle$ is a loop in Γ passing through $*'$, $\langle \beta_i, \delta_i^{-1} \rangle$ is homotopic rel$\{*'\}$ to a product of loops in $D \cap \Gamma$ based at $*'$, by a homotopy in Γ. Hence there is a map H_i of a 2-disk into Γ such that H_i restricted to the boundary of the disk is the loop $\langle \beta_i, \delta_i^{-1} \rangle$ followed by a loop in $D \cap \Gamma$.

Define a finite 2-complex K and a map of $f : K \to \tilde{X}$ as follows. Take K to be E with a finite number of attached 2-disks, and take $f|_E$ to be the inclusion map of E into $\tilde{X}$. For each $\alpha \in \{\alpha_1, \dots, \alpha_n\}$ and each corresponding $\langle \beta_i, \delta_i^{-1} \rangle$, attach the 2-disk $\mathrm{dom}(H_i)$ to E with attaching map H_i restricted to the boundary of $\mathrm{dom}(H_i)$. Take f on this 2-disk to be defined by H_i. Observe that in K, $\langle \beta_i, \delta_i^{-1} \rangle$ is homotopic to a loop in D, which by the definition of E is homotopically trivial in E. In other words, all γ_i for $i \in \{1, \dots, k-1\}$ are homotopically trivial in K. Hence each α_i is homotopic in K to a loop in D, and this loop is then homotopically trivial in K. As $[\alpha_1], \cdots, [\alpha_n]$ are generators of $\pi_1(K, *)$, we have that K is simply connected. As $\mathrm{im}(H_i) \subseteq \tilde{X} - C$, we have that $f|_{f^{-1}(C)} : f^{-1}(C) \to C$ is a homeomorphism. Hence G is QSF. ☐

Proof of Theorem 4. Suppose $G \in C_+$, let X be the standard 2-complex for some finite presentation of G, and let $\tilde{X}$ be the universal cover of X. Take λ and ϵ such that $\tilde{X}$ has a bounded combing by (λ, ϵ)-quasigeodesics. First define a tame 0-combing Ψ on $\tilde{X}$ as follows. If v is a vertex of $\tilde{X}$, then let $p_v : [0, T_v] \cap \mathbb{N}$ be the (λ, ϵ)-quasigeodesic from a fixed $* \in \tilde{X}$ to v in a bounded combing of $\tilde{X}$. Extend p_v to $[0, T_v]$ by taking edges between successive vertices in the path and define $\Psi|_{\{v\} \times [0,1]}$ by $\Psi(v, t) = p_v(tT_v)$ (a simple reparameterization of p_v). Since the p_v are part of a bounded combing, Ψ is a bounded 0-combing (see the remarks after the definition of bounded 0-combing above).

To see that Ψ is tame, it suffices to show that for any compact $C = \mathrm{St}^N(*) \subseteq \tilde{X}$, there exists a compact $D = \mathrm{St}^M(*) \subseteq \tilde{X}$ such that, for any vertex v, $\Psi^{-1}(C) \cap (\{v\} \times [0,1])$ is contained in one component of $\Psi^{-1}(D) \cap (\{v\} \times [0,1])$. Let $M = \lambda^2 N + \lambda^2 \epsilon + \epsilon$. Then since $\Psi|_{\{v\} \times [0,1]}$ is simply a reparameterization of p_v, it suffices to show that if $0 < t_1 < t_2 \leq T_v$ and $p_v(t_1) \in \tilde{X} - \mathrm{St}^M(*)$, then $p_v(t_2) \in \tilde{X} - \mathrm{St}^N(*)$. Suppose to the contrary that $d(*, p_v(t_1)) > M$ and $d(*, p_v(t_2)) \leq N$. Then $M < \lambda t_1 + \epsilon$ and $\frac{1}{\lambda} t_2 - \epsilon \leq N$, since p_v is a (λ, ϵ)-quasigeodesic. But then

$$\lambda t_2 + \epsilon = \lambda^2 (\frac{1}{\lambda} t_2 - \epsilon) + \lambda^2 \epsilon + \epsilon \leq \lambda^2 N + \lambda^2 \epsilon + \epsilon = M < \lambda t_1 + \epsilon < \lambda t_2 + \epsilon,$$

a contradiction.

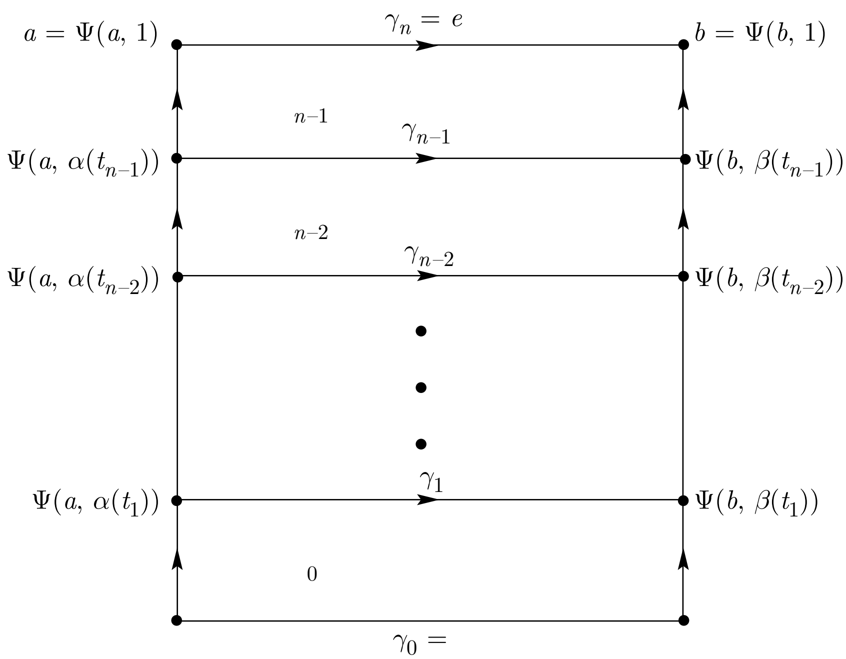

FIGURE 6

Thus there exists a bounded, tame 0-combing of $\tilde{X}$. Theorem 4 is now an immediate consequence of the following lemma.

Lemma 4.1. *Suppose $\tilde{X}$ is the universal cover of a finite 2-complex X. If $\tilde{X}$ has a bounded, tame 0-combing, then $\tilde{X}$ has a tame 1-combing.*

Proof. Suppose Ψ is a bounded, tame 0-combing of $\tilde{X}$, and take K to be a suitable constant in the definition of bounded 0-combing. Let e be an edge of $\tilde{X}$, with initial point a and endpoint b. Take $\alpha, \beta : [0,1] \to [0,1]$ orientation preserving homeomorphisms such that $d(\Psi(a,\alpha(t)), \Psi(b,\beta(t))) < K$. Consider a partition of $[0,1]$, $0 = t_0 < t_1 < \ldots < t_n = 1$, such that, for all $i < n$, $\Psi(\{a\} \times \alpha|_{[t_i,t_{i+1}]}) \subseteq \mathrm{St}(\Psi(a,\alpha(t_i)))$ and $\Psi(\{b\} \times \beta|_{[t_i,t_{i+1}]}) \subseteq \mathrm{St}(\Psi(b,\beta(t_i)))$. Then, for all $i < n$, since $d(\Psi(a,\alpha(t_i)), \Psi(b,\beta(t_i))) \leq K$, there exists a path γ_i from $\Psi(a,\alpha(t_i))$ to $\Psi(b,\beta(t_i))$ of length at most K. Let $\gamma_n = e$ and take γ_0 to be the constant map at $*$. For a fixed integer L, depending only on K, if γ is a loop of length $\leq 2K+2$ in $\tilde{X}$, then γ is homotopically trivial, by a homotopy in $\mathrm{St}^L(v)$, for any v in the image of the homotopy. Extend Ψ to $e \times [0,1]$ by patching together homotopies H_i as in Figure 6, where H_i is a homotopy killing $\langle \gamma_i, \Psi(\{b\} \times \beta|_{[t_i,t_{i+1}]}), \gamma_{i+1}^{-1}, (\Psi(\{a\} \times \alpha|_{[t_{i-1},t_i]}))^{-1} \rangle$ in $\mathrm{St}^L(\Psi(a,\alpha(t_i)))$.

Given a compact $C \subseteq \tilde{X}$, let $C' = \mathrm{St}^L(C)$, and take $D' \subseteq \tilde{X}$ to be a compact set such that, for any vertex v of $\tilde{X}$, $\Psi^{-1}(\mathrm{St}^L(C)) \cap (\{v\} \times [0,1])$ is contained in one path component of $\Psi^{-1}(D') \cap (\{v\} \times [0,1])$. Let $D = \mathrm{St}^L(D')$. Suppose e is an edge with initial point a and endpoint b, and $\Psi|_{e \times [0,1]}$ is defined by the H_i as above. If $x \in \Psi^{-1}(C) \cap (e \times [0,1])$, then $x \in \mathrm{dom}(H_i)$ for an i with $\Psi(a,\alpha(t_i)) \in \mathrm{St}^L(H_i(x)) \subseteq \mathrm{St}^L(C) = C'$. Thus $(a,\alpha(t_i))$ is in the one path component of $\Psi^{-1}(D') \cap (\{a\} \times [0,1])$

containing $\Psi^{-1}(C') \cap (\{a\} \times [0,1])$. If $\Psi(a, \alpha(t_i)) \in D'$, then, for all $y \in \mathrm{dom}(H_i)$, $H_i(y) \in \mathrm{St}^L(\Psi(a, \alpha(t_i))) \in \mathrm{St}^L(D') = D$; so $\mathrm{im}(H_i) \subseteq D$. Thus $\Psi^{-1}(C) \cap (e \times [0,1])$ is contained in a union of the $\mathrm{dom}(H_i) \subseteq D$ for which $(a, \alpha(t_i))$ is in a subinterval of $\{a\} \times [0,1]$ determined by a path component of $\Psi^{-1}(D') \cap (\{a\} \times [0,1])$; but then this union is contained in a one-path component of $\Psi^{-1}(D) \cap (e \times [0,1])$. □

As we said above, this also completes the proof of Theorem 4. □

Proof of Theorem 5. By [6], G has an asynchronously automatic structure with uniqueness, and the combing corresponding to this regular language is asynchronously bounded. Moreover, this combing will have a *departure function* $\Delta : \mathbb{R} \to \mathbb{R}$ such that for any word w in the regular language, any $r, s \geq 0$, and any $t \geq \Delta(r)$ with $s + t$ less than the length of w, the distance in the Cayley graph of G from $w(s)$ to $w(s + t)$ is greater than r (see [6] or [4]). Given any compact C, taking r sufficiently large so $C \subseteq \mathrm{St}^r(*)$, and taking $D = \mathrm{St}^{\Delta(r)}(*)$, we see that a combing with a departure function corresponds to a tame 0-combing (but the converse need not hold). Passing to the continuous combing corresponding to the combing from the asynchronous structure of G, we get that G has a bounded, tame 0-combing. By Lemma 4.1, this 0-combing can then be extended to a tame 1-combing. □

The use of a departure function in the above argument simplifies our original proof of this result. In fact, the proof that the 0-combing derived from an asynchronus automatic structure of a group is tame, is somewhat easier than the proof that there is a departure function, but there seems little reason to include the details here. Automatic groups are included in the class of asynchronously automatic groups and also belong to C_+; hence we have two ways of showing that such groups have asynchronously bounded, tame 0-combings.

Concluding remarks

Suppose X is a finite 2-complex with fundamental group G and universal cover $\tilde{X}$. In the search for the definition of tame 1-combing, a (most likely strictly) weaker combing condition on $\tilde{X}$ was discovered that also implies G is QSF, namely that there exists a 1-combing Ψ of $\tilde{X}$ such that, for any compact $C \subseteq \tilde{X}$, there exists a compact $D \subseteq \tilde{X}$ with $\Psi^{-1}(C)$ contained in one component of $\Psi^{-1}(D)$. The definition of tame 1-combing corresponds to a sort of uniform localization of this property. While this property is simpler to state, the 1-combings we construct are all tame, so there seems little reason at this time to develop this idea further.

Suppose Ψ is a 1-combing of $\tilde{X}$ such that, for all compact $C \subseteq \tilde{X}$, there exists a compact $D \subseteq \tilde{X}$ such that, for any edge e of $\tilde{X}$, $\Psi^{-1}(C) \cap (e \times [0,1])$ is contained in one path component of $\Psi^{-1}(D) \cap (e \times [0,1])$ (i.e., part of the definition of tame 1-combing, but dropping the condition that the restriction of Ψ to $\tilde{X}^0$ is a tame 0-combing). Then (with some effort) it can be shown that there must also exist a tame 1-combing of $\tilde{X}$. Again, there seems little reason to pursue this definition since the most convenient way of constructing a tame 1-combing has been to first construct a tame 0-combing which can be extended.

In terms of pro-groups, the condition that for each finite subcomplex C of $\tilde{X}$, $\pi_1(\tilde{X} - C)$ is finitely generated, is equivalent to the condition that for each end $\mathcal{E}$ of $\tilde{X}$, pro-$\pi_1(\mathcal{E})$ is pro-finitely generated.

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