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VIRTUALLY FREE GROUPS WITH FINITELY MANY OUTER AUTOMORPHISMS

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ABSTRACT. Let G be a finitely generated virtually free group. From a presentation of G as the fundamental group of a finite graph of finite-by-cyclic groups, necessary and sufficient conditions are derived for the outer automorphism group of G to be finite. Two versions of the characterization are given, both effectively verifiable from the graph of groups. The more purely group theoretical criterion is expressed in terms of the structure of the normalizers of the edge groups (Theorem 5.10); the other version involves certain finiteness conditions on the associated G-tree (Theorem 5.16). Coupled with an earlier result, this completes a description of the finitely generated groups whose full automorphism groups are virtually free.

1. Introduction

It is a result of J. Alperin [1] that if G is a finitely generated group, then G has only finitely many automorphisms if and only if it is either finite or a finite central extension of an infinite cyclic group. In a similar vein, it was undertaken in [11] to characterize the finitely generated groups G for which $\mathrm{Aut}(G)$ is virtually free. (A well-known example is the free abelian group of rank two but more significant in the present context (and certainly less obvious) is the example of an amalgamated free product of two finite groups [6].) Stripped of a few technical details, the upshot of [11, Theorem 3.3] was that such a group G must have a finitely generated center $Z(G)$ of torsion-free rank at most two and the quotient $G/Z(G)$ must be virtually free and have only finitely many outer automorphisms. The classification problem thus devolved into one of characterizing those finitely generated virtually free groups whose outer automorphism groups are finite. This is the goal of the present paper.

The main results are Theorems 5.10 and 5.16 (although it is more accurate to describe them as one theorem viewed from two perspectives). For the purpose of constructing or analyzing specific examples, the more useful formulation seems to be Theorem 5.10 which (assuming the graph of groups is regarded simply as a convenient device for encoding a particular type of presentation) is a purely group theoretic result. However, the statement of Theorem 5.16 carries less terminological baggage and reflects more explicitly the interaction between groups and trees which motivates much of the proof, and so we shall restrict ourselves here to a description of this latter version.

In view of the Karrass-Pietrowski-Solitar theorem [5] and in the spirit of the theory of Bass and Serre [2, 3, 13], we consider a pair (G, T) consisting of a group

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G and a directed G-tree T on which G has only finitely many vertex and edge orbits and such that each edge stabilizer in G is finite and each vertex stabilizer is finitely generated with only finitely many automorphisms. (Since finitely generated groups with finite automorphism groups are virtually cyclic [1], such a group G is finitely generated and virtually free. Allowing (to this extent) the possibility of infinite groups as vertex stabilizers is simply a convenience which seems to permit a somewhat cleaner statement of the main results.) These vertex stabilizers are finite-by-cyclic (see, for example, [11, Lemma 2.2]) and so the following simple notation will be useful: If H is a finite-by-cyclic group, H^0 denotes the unique maximal finite subgroup of H. Before considering the general problem, it may be helpful to look briefly at the simplest non-trivial case since this turns out to be a model for the "local" structure of G in the general situation.

Because it reflects the number of amalgamated free products or HNN extensions required to construct it from the vertex groups, a rough indicator of the combinatorial complexity of G is the number of orbits it has on the set of edges of T (or equivalently, the number of edges in the corresponding graph of groups). Leaving aside the case that T consists of a single vertex (and $\mathrm{Aut}(G)$ is finite), the smallest configuration of interest thus occurs when the edges of T comprise a single orbit under the action of G. Because there are at most two vertex orbits in this case, the quotient graph T/G is either a segment (with two vertices) or a loop (at a single vertex). Accordingly, the Bass-Serre theory tells us that G is either an amalgamated free product $G_{\mathbf{u}} \underset{G_{\mathbf{e}}}{*} G_{\mathbf{v}}$ (where $G_{\mathbf{u}}$ and $G_{\mathbf{v}}$ are adjacent vertex stabilizers and $G_{\mathbf{e}}$ stabilizes the common edge $\mathbf{e}$) or it is an HNN extension of a vertex stabilizer $G_{\mathbf{u}}$ with associated subgroups $G_{\mathbf{e}}$ and $gG_{\mathbf{e}}g^{-1}$ (where $g \in G$ and $\mathbf{e}$ is an edge with initial vertex $\mathbf{u}$ and terminal vertex $\mathbf{u}^g$). For a fixed group G, there are two situations in which the configuration can be reduced still further. The first of these occurs if G has two vertex orbits (and so is a free product) but $G_{\mathbf{e}} = G_{\mathbf{u}}^0$ (or $G_{\mathbf{v}}^0$). In this case G may be regarded as an HNN extension of $G_{\mathbf{u}}$ and so can be represented on a tree whose vertices lie in just one orbit. The second reducible configuration occurs if G is vertex transitive on T (and so is an HNN extension of $G_{\mathbf{u}}$) but $G_{\mathbf{u}} = G_{\mathbf{e}}$. Here, G is finite-by-(infinite cyclic) and hence, we may replace T by a single vertex. Assuming that any such reductions have been performed and our configuration is "irreducible", we shall see in Section 2 that $\mathrm{Out}(G)$ is finite in all edge transitive cases but one, the exception being when G is vertex transitive with infinite vertex stabilizers (that is, when G is an HNN extension of a finite-by-(infinite cyclic) group). This sets the stage for consideration of the general situation.

Initially, the fact that in the edge-transitive case the finiteness of $\mathrm{Out}(G)$ forces the vertex stabilizers to be finite if G is an HNN extension but not if it is an amalgamated free product threatens to complicate the analysis of the general case. However, it turns out that this anomaly reflects a geometric fact that is no more difficult to describe in the general case than in the edge transitive case: In the free product situation, each vertex of T projects onto a "terminal" vertex (that is, a vertex of degree at most one) of the quotient graph T/G; in the HNN extension case, it does not (a loop being counted once as an incoming edge and once as outgoing). Equivalently, if $\mathbf{u}$ is a vertex of T then in the former case, $G_{\mathbf{u}}$ is transitive on the edges incident with $\mathbf{u}$ while in the latter it is not. What occurs in the general situation is similar: Despite our hypothesis allowing the vertex stabilizers to be finite-by-cyclic, in the event that $\mathrm{Out}(G)$ is finite the stabilizers of those vertices which project onto non-terminal vertices of T/G must actually be finite (Lemma 5.3(i)).

The two irreducibility conditions postulated (without loss of generality) in the edge transitive case will be assumed (equally harmlessly) in the general case. For each edge $\mathbf{e}$ of T, denote by $Fix_T(G_{\mathbf{e}})$ the subgraph of T fixed by the stabilizer $G_{\mathbf{e}}$. If $N = N_G(G_{\mathbf{e}})$, it is easily seen that $Fix_T(G_{\mathbf{e}})$ is an N-finite N-tree (Lemma 3.3 and subsequent remarks). Let $T(\mathbf{e})$ be the N-tree obtained from $Fix_T(G_{\mathbf{e}})$ by contracting to a vertex each connected component of the graph obtained from $Fix_T(G_{\mathbf{e}})$ by deleting the set $\mathbf{e}^N$ of N-conjugates of $\mathbf{e}$. Obviously N acts on and is, a priori, edge-transitive on $T(\mathbf{e})$ but of course, in general, the vertex stabilizers in this action will not be finite-by-cyclic. (In terms of T the vertex stabilizers on $T(\mathbf{e})$ are just the setwise stabilizers of the corresponding components of $Fix_T(G_{\mathbf{e}})\backslash\mathbf{e}^N$.) It will be shown, however, that if $\mathrm{Out}(G)$ is finite, then the components of $Fix_T(G_{\mathbf{e}})\backslash\mathbf{e}^N$ are actually finite subtrees of T and that each component stabilizer in N is the stabilizer of some vertex in this component (and hence, is finite-by-cyclic). Thus, if $\mathrm{Out}(G)$ is finite, the normalizer in G of each edge group $G_{\mathbf{e}}$ acts edge-transitively on a tree $T(\mathbf{e})$ with finite edge groups and finite-by-cyclic vertex groups, precisely the special case discussed earlier. (The pair $(N, T(\mathbf{e}))$ need not be irreducible however, and N need not have a finite outer automorphism group (see the example following Theorem 5.16).)

Suppose that $\mathbf{f}$ is an edge of $Fix_T(G_{\mathbf{e}})$ (so $G_{\mathbf{e}} \le G_{\mathbf{f}}$). If $G_{\mathbf{e}} \ne G_{\mathbf{f}}$, then the tree $Fix_T(G_{\mathbf{f}})$ is contained in a connected component of $Fix_T(G_{\mathbf{e}})\backslash\mathbf{e}^N$ (where $N = N_G(G_{\mathbf{e}})$) and hence, by the remarks of the preceding paragraph, if $\mathrm{Out}(G)$ is finite then $Fix_T(G_{\mathbf{f}})$ must also be finite. However, it will be shown that provided $\mathbf{f}$ is not G-conjugate to $\mathbf{e}$, the finiteness of $\mathrm{Out}(G)$ forces this same conclusion even if $G_{\mathbf{e}} = G_{\mathbf{f}}$.

To summarize, we have described three constraints on (G, T) which purportedly are consequences of the finiteness of $\mathrm{Out}(G)$: (1) stabilizers of vertices of T which project onto non-terminal vertices of T/G must be finite; (2) for any edge $\mathbf{e}$ of T, if $N = N_G(G_{\mathbf{e}})$ then the connected components of $Fix_T(G_{\mathbf{e}})\backslash\mathbf{e}^N$ must be finite; and (3) if an edge $\mathbf{e}$ of T has the same stabilizer in G as another edge which is not G-conjugate to it, then the subtree $Fix_T(G_{\mathbf{e}})$ must be finite. The goal of this paper (as formulated in Theorem 5.16) is to establish that these three conditions are, in fact, both necessary and sufficient.

The organization of the argument is as follows. The edge-transitive case described above is treated in Section 2. Some essential facts about the Bass-Serre theory are reviewed in Section 3 and an algorithm is described for obtaining a presentation for the normalizer of any subgroup of a vertex group (Proposition 3.5). In Section 4, several consequences (involving the normalizers of edge groups) are derived from the assumption that G is virtually free and has only finitely many outer automorphisms (Theorem 4.4). In the concluding section it is shown that two of these consequences actually comprise a sufficient condition for the finiteness of $\mathrm{Out}(G)$ (Theorem 5.10). Several corollaries follow, including a simpler criterion which applies if, for example, all vertex groups are (finite nilpotent)-by-cyclic (Corollary 5.14). Finally, Theorem 5.10 is reformulated (as described above) in terms of the underlying tree (Theorem 5.16).

The problem treated here is related to a much more general question. It is known that the automorphism group and the outer automorphism group of any finitely generated virtually free group G are each finitely presented and virtually of finite cohomological dimension ([4], [7], [10]). What apparently is not known is a procedure for determining the exact value of the virtual cohomological dimension (v.c.d) of either group from the graph of finite groups associated with a presentation of G. Our concern here is only with the case that the v.c.d of $\mathrm{Out}(G)$ is 0 (or equivalently, the v.c.d. of $\mathrm{Aut}(G)$ is at most 1), but for an assault on the general case the reader may consult [8]. There, upper bounds for the v.c.d of $\mathrm{Aut}(G)$ are obtained which, in certain cases, are precise. For example, a conjecture of D. J. Collins that $\mathrm{Out}(G)$ is virtually of cohomological dimension $n-2$ if G is a free product of n finite groups is settled affirmatively.

2. Some preliminary observations and examples

Strictly speaking, since the examples considered in this section are simply generalized free products or HNN extensions of finite-by-cyclic groups, Kurosh's theorem obviates the need here for the following lemma. However, later sections depend heavily on the full generality of this result and so, in the interests of efficiency, we record it here.

Lemma 2.1. *Let G be a finitely generated virtually free group. Then G contains only a finite number of conjugacy classes of finite subgroups. In particular,* $\mathrm{Aut}(G)$ *contains a normal subgroup A of finite index which, in turn, contains* $\mathrm{Inn}(G)$ *and whose elements map each finite subgroup of G to a conjugate of itself. If for some finite subgroup H of G, $C_A(H)$ is finite, then* $\mathrm{Out}(G)$ *is also finite.*

Proof. The first statement is a corollary of the main result of [5] (see also [3, I.4.9 and IV.1.9]). For the second, take A to be the kernel of the action of $\mathrm{Aut}(G)$ on the set of conjugacy classes of finite subgroups of G. For the last statement, observe that by a Frattini argument, $A = \mathrm{Inn}(G)N_A(H)$. □

Next, we consider several special examples.

Lemma 2.2. *Let $G = H \underset{U}{*} K$ where U is finite. If $\alpha \in C_{\mathrm{Aut}(G)}(H)$ and K^α is conjugate in G to K, then $K^\alpha = K^h$ for some $h \in N_H(U)$.*

Proof. This is a special case of [6, Lemma 1]. □

Analogously, for HNN extensions we have

Lemma 2.3. *Suppose that H is finite and $G = H \underset{U}{*} t$ is an HNN extension, where U is a normal subgroup of G. If $\alpha \in C_{\mathrm{Aut}(G)}(H)$, then $t^\alpha = h_0 t^e h_1$ for some $h_0, h_1 \in H$ and $e = \pm 1$. In particular, $C_{\mathrm{Aut}(G)}(H)$ and* $\mathrm{Out}(G)$ *are both finite.*

Proof. We shall treat the case that $U = 1$ (so $G \cong H * \mathbf{Z}$); the general case will then follow from consideration of the quotient G/U (since, by virtue of being normal, U is actually characteristic in G).

Assuming $U = 1$, let $t^\alpha = h_0 t^{e_1} h_1 \cdots t^{e_n} h_n$ where $h_i \in H$, the e_i's are each 1 or -1, and the right side is a reduced word. Then $G = \langle H, w \rangle$ where $w = (h_0^{-1} t h_n^{-1})^\alpha = t^{e_1} h_1 \cdots t^{e_n}$, and so we may write $t = x_0 w^{n_1} \cdots w^{n_s} x_s$ where $x_j \in H$, $x_j \neq 1$ for $1 \le j \le s-1$ and the n_j's are non-zero integers. Note that if N is the normal closure of H in G, then G/N is infinite cyclic and so $t^\alpha \equiv t$ or t^{-1} modulo N. Thus, $\sum_{i=1}^n e_i = \pm 1$ and in particular, n is odd.

If r is the number of t-reductions (see [9, p. 184]) used to transform $w^2 = (t^{e_1} h_1 \cdots t^{e_n})^2$ to its reduced form y_2, then the equation $e_{n-i+1} = -e_i$ must hold for $1 \le i \le r$, whence $r \le (n-1)/2$. More generally, transforming $w^{n_j} = (t^{e_1}h_1 \cdots t^{e_n})^{n_j}$ to its reduced form y_j involves at most $(|n_j|-1)(n-1)/2$ t-reductions and so the number of times that t or t^{-1} appears in y_j is at least $|n_j|n - (|n_j|-1)(n-1) = |n_j| + n - 1$. Because t or t^{-1} must occur at the beginning and end of each of the y_j's and because $x_j \ne 1$ for $1 \le j \le s-1$, the word $x_0y_1x_1 \cdots x_s$ is reduced. But this word represents the element t and so $\sum_{j=1}^{s}(|n_j| + n - 1) \le 1$. It follows that $n = 1$ (and $s = |n_1| = 1$) which proves the first statement. The finiteness of $C_{\mathrm{Aut}(G)}(H)$ is now clear and that of $\mathrm{Out}(G)$ follows from Lemma 2.1. □

With a little extra effort, one can show that, as with Lemma 2.2, the conclusion of Lemma 2.3 holds without the hypothesis that U be normal in G. However, this fact will ultimately emerge as a very special case of the main theorem (Corollary 5.11).

Lemma 2.4. *If G is finitely generated and virtually free, then every periodic subgroup of either $\mathrm{Aut}(G)$ or $\mathrm{Out}(G)$ is finite.*

Proof. In fact, $\mathrm{Aut}(G)$ is virtually of finite cohomological dimension [10] but the conclusion we require is more elementary. Since G is virtually free, so also is $\mathrm{Inn}(G)$ and therefore, it suffices to deal with periodic subgroups of $\mathrm{Out}(G)$. Since G is finitely generated, it contains a free characteristic subgroup F of finite index and finite rank n. If F is cyclic, then by [11, Proposition 3.1 and Theorem 3.4], $\mathrm{Aut}(G)$ is virtually cyclic and $\mathrm{Out}(G)$ is finite. Thus, we may assume that $n \ge 2$. Let $A = \mathrm{Aut}(G)$, $\widehat{F}$ be the subgroup of $\mathrm{Inn}(G)$ induced by conjugation by elements of F (so $\widehat{F}$ is normal in A), and suppose that $P/\mathrm{Inn}(G)$ is a periodic subgroup of $\mathrm{Out}(G)$. Then $PC_A(F)/\widehat{F}C_A(F)$ is isomorphic to a periodic subgroup of $\mathrm{Out}(F)$. Since $C_A(F/F')/\widehat{F}C_A(F)$ is isomorphic to a subgroup of $C_{\mathrm{Aut}(F)}(F/F')/\mathrm{Inn}(F)$ and since the latter group is torsion-free [9, I.4.12], it follows that $C_A(F)C_P(F/F') = PC_A(F) \cap C_A(F/F') = \widehat{F}C_A(F)$. Hence, $P/\widehat{F}C_P(F) = P/C_P(F/F')$ which is isomorphic to a periodic subgroup of $\mathrm{Aut}(F/F') = GL_n(\mathbf{Z})$. It is well-known that $GL_n(\mathbf{Z})$ is virtually torsion-free (see, for example, [12, p. 97, Ex. 8]) and so $P/\widehat{F}C_P(F)$ is finite. But $C_A(F) \cap C_A(G/F) \le C_A(G/Z(F)) = 1$ and so $C_A(F)$ is isomorphic to a subgroup of the finite group $\mathrm{Aut}(G/F)$. It follows that $P/\widehat{F}$ (and hence, $P/\mathrm{Inn}(G)$) is finite. □

Proposition 2.5. *Let $G = H \underset{U}{*} K$ where H and K are each finite-by-cyclic and U is finite. Let H^0 and K^0 denote respectively the unique maximal finite subgroups of H and K. If $H^0 \ne U$ and $K^0 \ne U$, then $\mathrm{Out}(G)$ is finite. In any case, if U is normal in G and H is finite then $C_{\mathrm{Aut}(G)}(H)$ and $\mathrm{Out}(G)$ are both finite.*

Proof. Assume first that $H^0 \ne U \ne K^0$. If A is the subgroup of $\mathrm{Aut}(G)$ consisting of those automorphisms which preserve conjugacy of all finite subgroups of G, then by Lemmas 2.4 and 2.1, it suffices to show that $C_A(H^0)/C_{\mathrm{Inn}(G)}(H^0)$ is periodic.

Let $\alpha \in C_A(H^0)$. Since $H^0 \ne U$, $N_G(H^0) = H$ and so H is invariant under α. But as in [1], $\mathrm{Aut}(H)$ is finite and so some power of α, say $\beta = \alpha^k$, centralizes H. Since $K^0 \ne U$, $K = N_G(K^0)$ and so since $(K^0)^\beta$ is conjugate to K^0, K^β is conjugate to K. By Lemma 2.2, $K^\beta = K^h$ for some $h \in N_H(U)$. Since $H \le C_G(\beta)$, $K^{\beta^n} = K^{h^n}$ for every integer n and so, if $n = |H\colon Z(H)|$, $\beta^n i_h^{-n}$ centralizes H and normalizes K (where i_h denotes the inner automorphism of G induced by h). By [1], $\mathrm{Aut}(K)$ is finite and so, for some non-zero integer m, $\beta^{nm} i_h^{-nm} = (\beta^n i_h^{-n})^m = 1$, whence $\beta^{nm} \in \mathrm{Inn}(G)$. This proves that $C_A(H^0)/C_{\mathrm{Inn}(G)}(H^0)$ is indeed periodic and so, the first conclusion is proved.

Suppose now that U is normal in G and H is finite (so $H = H^0$). In the case that $H \neq U \neq K^0$ considered above, choosing $n = |H|$ yields the fact that β has finite order, whence $C_{\mathrm{Aut}(G)}(H)$ is finite by Lemma 2.4. The only non-trivial case remaining is the one in which $H \neq U$ and $K \neq K^0 = U$ (so K/U is infinite cyclic). Here, K splits over U and consequently, G is an HNN extension $H \underset{U}{*} t$ where $K = [U]\langle t\rangle$, whence Lemma 2.3 yields the desired conclusion. □

3. Graphs of groups and an algorithm

First we review some terminology and a few basic facts from the Bass-Serre theory. Details may be found (with minor notational variations) in [2], [3] or [13].

Let D be a finite connected directed graph with vertex set VD, edge set ED and with incidence maps σ and τ (so e^σ is the source vertex of the edge e and e^τ is the terminal vertex). Let $(G(-), D)$ be a *graph of groups*. Here, the function $G(-)$ assigns to each v in VD a vertex group $G(v)$ and to each edge e in ED an edge group $G(e)$ which we shall assume to be a subgroup of $G(e^\sigma)$. Moreover, for each e in ED, there exists an isomorphism (also denoted by e) from $G(e)$ onto a subgroup $G(e)^e$ of $G(e^\tau)$. Let $F(G(-), D)$ denote the free product of the vertex groups $G(v)$ and the free group on ED, modulo the relations $e^{-1}ge = g^e$ for all $e \in ED$, $g \in G(e)$. For each $v \in VD$, the natural homomorphism from $G(v)$ into $F(G(-), D)$ is a monomorphism and so, for the discussion in the next two paragraphs, we shall regard the $G(v)$'s as subgroups of $F(G(-), D)$.

Having selected a base vertex v_0 in VD, the fundamental group $\pi(G(-), D, v_0)$ is defined to be the subgroup of $F(G(-), D)$ consisting of all elements of the form $x_0 e_1^{\varepsilon_1} x_1 e_2^{\varepsilon_2} \cdots e_n^{\varepsilon_n} x_n$, where $e_1, e_2, \ldots, e_n$ is the edge sequence of a closed path at v_0, the ε_i's are each 1 or -1 according to whether the orientation of e_i agrees or not with the direction of the path, where $x_0 \in G(v_0)$ and where for $i \geq 1$, $x_i \in G(e_i^\tau)$ if $\varepsilon_i = 1$ and $x_i \in G(e_i^\sigma)$ if $\varepsilon_i = -1$. Of course, if v is another vertex of D, $\pi(G(-), D, v)$ and $\pi(G(-), D, v_0)$ are conjugate in $F(G(-), D)$ (and so, if it is the isomorphism class only that concerns us, we may omit reference to the base point and refer simply to the fundamental group as $\pi(G(-), D)$).

If M is a maximal subtree of D, $\pi(G(-), D, M)$ is defined to be the quotient $F(G(-), D)/K_M$, where K_M is the normal closure in $F(G(-), D)$ of the set EM. If $v \in VD$ and $e_1, e_2, \ldots, e_m$ is the edge sequence of the M-geodesic from v_0 to v, let $x = e_1^{\varepsilon_1} e_2^{\varepsilon_2} \cdots e_m^{\varepsilon_m} \in K_M$, where $\varepsilon_i = 1$ or -1 according as e_i is oriented with or opposite to the direction of the geodesic. Regarding $G(v)$ as a subgroup of $F(G(-), D)$, the map $\mu_{M,v} \colon g \mapsto xgx^{-1}$ then defines a monomorphism of $G(v)$ into $\pi(G(-), D, v_0)$. The natural homomorphism $\eta_M \colon F(G(-), D) \to \pi(G(-), D, M)$ induces by restriction a monomorphism from $G(v)$ into $\pi(G(-), D, M)$ and also, an isomorphism φ_M from $\pi(G(-), D, v_0)$ to $\pi(G(-), D, M)$ (so we obtain an explicit presentation for the fundamental group of $(G(-), D)$). Moreover, $g^{\mu_{M,v}\varphi_M} = g^{\eta_M}$ for all $g \in G(v)$ (since, by definition, $x \in K_M$). If M_1 and M_2 are any two maximal trees of D, it follows easily from this that the map $\psi = \varphi_{M_1}^{-1}\varphi_{M_2}$ is an isomorphism from $\pi(G(-), D, M_1)$ to $\pi(G(-), D, M_2)$ with the property that for any vertex $v \in VD$, $(G(v)^{\eta_{M_1}})^\psi$ is conjugate in $\pi(G(-), D, M_2)$ to $G(v)^{\eta_{M_2}}$. The significance of this last remark is the following: It is natural to identify (as we shall) each vertex group $G(v)$ with its image $G(v)^{\eta_M}$ in $\pi(G(-), D, M)$ and, in turn, to suppress M and identify $\pi(G(-), D, M)$ with $\pi(G(-), D)$. With all these identifications, some care must be exercised to ensure that assertions made about $\pi(G(-), D)$ and its subgroups are really independent of the choice of M. The existence of the isomorphism ψ defined above will guarantee that statements about conjugacy of subgroups of the $G(v)$'s and about the structure of their normalizers in $\pi(G(-), D)$ do not, in fact, depend on the choice of M.

If D_0 is a subgraph of D such that $M \cap D_0$ is a maximal subtree of D_0, then there is a natural embedding of $\pi(G(-)|_{D_0}, D_0, M \cap D_0)$ in $\pi(G(-), D, M)$. Whenever we refer to $\pi(G(-)|_{D_0}, D_0)$, it will be identified with a subgroup of $\pi(G(-), D)$ under such an embedding.

In the special case that the underlying graph D contains a single edge e, G is either an HNN extension $G(e^\sigma) \underset{G(e)}{*} e$ (if D is a loop) or an amalgamated free product $G(e^\sigma) \underset{G(e)}{*} G(e^\tau)$ (if D is a segment).

Of course, the relevance of the Bass-Serre machinery to this paper stems from the theorem (of Karrass, Pietrowski and Solitar [5]) alluded to earlier, that a group G is finitely generated and virtually free if and only if it is the fundamental group of a finite graph of finite groups [3, IV.1.9].

Let $G = \pi(G(-), D, M)$. The universal cover (or standard tree) T of $(G(-), D)$ may be constructed with vertex set $VT = (VD \times G)/\sim$ and edge set $ET = (ED \times G)/\sim$ where, if $x, y \in VD \cup ED$ and $g, h \in G$, then $(x, g) \sim (y, h)$ if $x = y$ and $gh^{-1} \in G(x)$. If $x \in VD \cup ED$ and $g \in G$, we denote the equivalence class of (x, g) by $[x, g]$. Incidence maps (which we shall again denote by σ and τ) are defined by $[e, g]^\sigma = [e^\sigma, g]$ and $[e, g]^\tau = [e^\tau, e^{-1}g]$ for all $e \in ED$, $g \in G$ (where in the last expression, e is regarded as an element of G as well as of ED). A right G-action on T is defined by $[y, g]^x = [y, gx]$ for all $y \in VD \cup ED$. Note that the stabilizer $G_{[e,g]}$ of an edge $[e, g]$ of T is $g^{-1}G(e)g$. In particular, $G(e) = G_{[e,1]}$ and $G(e)^e = e^{-1}G(e)e = G_{[e,e]}$. We have a section $s\colon x \mapsto [x, 1]$ from D to T and the projection $\pi\colon [x, g] \mapsto x$ from T onto D, the latter inducing an isomorphism from the quotient T/G to D. The most important (and least trivial) general property of T is that it is actually a tree (see [2, Chap. 8, Theorem 24], [3, I.7.6], or [13, I.5.12]).

Lemma 3.1. *Suppose that $(G(-), D)$ is a graph of groups and let $G = \pi(G(-), D)$. Let $u, v \in VD$ and $g \in G$. If $u \neq v$ or $g \notin G(v)$, then $G(u) \cap G(v)^g \leq G(e)^x$ for some $e \in ED$ with $u \in \{e^\sigma, e^\tau\}$ and some $x \in G(u) \cup eG(u)$.*

Proof. If $u \neq v$ or $g \notin G(v)$, the vertices $[u, 1]$ and $[v, g]$ in the universal cover T are distinct. Let $[e, x]$ be the first edge in the T-geodesic from $[u, 1]$ to $[v, g]$. Then $G(u) \cap G(v)^g = G_{[u,1]} \cap G_{[v,g]} \leq G_{[e,x]} = G(e)^x$. If $[u, 1] = [e, x]^\sigma$, then $u = e^\sigma$ and $x \in G(u)$. Otherwise, $[u, 1] = [e, x]^\tau$, whence $u = e^\tau$, $e^{-1}x \in G(u)$. □

Corollary 3.2. *If $G = \pi(G(-), D)$, $u \in VD$, and $H \leq G(u)$ such that for any edge e of D incident with u, H is not conjugate by an element of $G(u)$ to a subgroup of either $G(e)$ (if $e^\sigma = u$) or $G(e)^e$ (if $e^\tau = u$), then $N_G(H) \leq G(u)$.*

The next lemma is a simple permutation theoretic observation. Recall that a G-set S is said to be *G-finite* if it consists of only a finite number of G-orbits.

Lemma 3.3. *Assume that S is a G-finite G-set with all stabilizers finite-by-(torsion-free). Let $x \in S$, $H \leq G_x$, and $N = N_G(H)$ (so $Fix_S(H)$ is an N-set). Suppose that $y \in Fix_S(H)$ and let $\{H^{g_i}\colon 1 \leq i \leq n\}$ be a complete set of* *representatives of the G_y-conjugacy classes of G-conjugates of H which are contained in G_y. Then $\{y^{g_i^{-1}} : 1 \leq i \leq n\}$ is a complete set of representatives of the N-orbits in $y^G \cap Fix_S(H)$. In particular, $Fix_S(H)$ is an N-finite N-set.*

Proof. Since the stabilizer G_y is finite-by-(torsion-free), it contains only finitely many G-conjugates of H and so certainly, there are only finitely many G_y-conjugacy classes of such subgroups. If $g \in G$ such that $y^g \in Fix_S(H)$, then $H^{g^{-1}} \leq G_y$ and hence, $H^{g^{-1}} = H^{g_i}$ for some i, $1 \leq i \leq n$. Thus, $g \in g_i^{-1}N$ and so $y^{gN} = y^{g_i^{-1}N}$. On the other hand, if $y^{g_i^{-1}N} = y^{g_j^{-1}N}$, then $g_j \in Ng_iG_y$ and so H^{g_i} and H^{g_j} are conjugate in G_y, whence $i = j$. This proves that the N-orbits $y^{g_i^{-1}N}$ cover $y^G \cap Fix_S(H)$ and are distinct. The last statement follows because there are only finitely many G-orbits on S and we have shown above that the intersection of each with $Fix_S(H)$ is the union of finitely many N-orbits. □

Let G be a group which acts on a tree T. A general observation which will be exploited repeatedly in the argument to follow is that if H is any subgroup of G, then $Fix_T(H)$, the subgraph of T fixed by H, is actually a subtree of T (because any two vertices in $Fix_T(H)$ are connected by a unique T-geodesic and thus, all vertices and edges on this geodesic are fixed by H). Moreover, the restriction of the action of G on T defines an action of $N_G(H)$ on $Fix_T(H)$. We will be particularly interested in the case that $H \leq G(u)$, where $u \in VD$. Note that in this case, Lemma 3.3 (with $S = VT \cup ET$) implies that if T is G-finite, then $Fix_T(H)$ is $N_G(H)$-finite. Thus, if G is the fundamental group of a finite graph of finite-by-cyclic groups, then so is $N_G(H)$.

Henceforth, we shall be primarily concerned with graphs of groups in which the edge groups are finite and the vertex groups are finite-by-cyclic or equivalently, finite central extensions of cyclic groups (see, for example, [11, Lemma 2.2]). That is to say, we allow as a vertex group not just any finite group but (in view of Alperin's theorem [1]) any finitely generated group whose automorphism group is finite. As mentioned in the Introduction, allowing certain infinite groups as vertex groups serves only to facilitate the eventual statement of the main results. An infinite finite-by-cyclic group is a split extension of a finite group by an infinite cyclic group and so the fundamental group of a finite graph of finite-by-cyclic groups with finite edge groups may easily be presented as the fundamental group of a finite graph of finite groups. (The underlying graph is embellished by adding a loop to each vertex v for which $G(v)$ is infinite. The new vertex group assigned to v (and the edge group corresponding to the new loop) in the modified graph of groups is the unique maximal finite subgroup of $G(v)$ (denoted in this paper by $G(v)^0$).)

Our eventual goal is, of course, to develop necessary and sufficient conditions for $\text{Out}(G)$ to be finite. In Theorem 5.10 these conditions will involve a knowledge of the structure and action of the normalizers in G of the edge groups and so we shall first demonstrate that this information is, in principle at least, computable from the graph of groups. From now on, $(G(-), D)$ denotes a finite graph of finite-by-cyclic groups with finite edge groups, fundamental group G and universal cover T. The subgroup and conjugacy structure of the vertex groups will be regarded as known.

Proposition 3.4. *If $u, v \in VD$ and $H \leq G(u)$, there is an effective procedure for determining all G-conjugates of H which are contained in $G(v)$.*

Proof. Each vertex group is finite-by-cyclic and hence, central-by-finite, so we may begin by choosing in each vertex group $G(w)$, $w \in VD$, a transversal R_w of the center. For each pair of edges e and f which are incident in D at vertex w, let the subset $S_{e,f,w}$ of G be defined by

$$S_{e,f,w} = \begin{cases} R_w & \text{if } e^\sigma = w = f^\sigma, \\ eR_w & \text{if } e^\tau = w = f^\sigma, \\ R_w f^{-1} & \text{if } e^\sigma = w = f^\tau, \\ eR_w f^{-1} & \text{if } e^\tau = w = f^\tau, \end{cases}$$

and let S be the union of all the $S_{e,f,w}$'s (so S is finite).

If $H^g \le G(v)$ for some $g \in G$, then $H^g \le G_{[u,g]} \cap G_{[v,1]}$ and so, if $[e_1, g_1], [e_2, g_2], \dots, [e_n, g_n]$ is the edge sequence of the T-geodesic from $[u, g]$ to $[v, 1]$, then $H^g \le G_{[e_i, g_i]}$, whence $H^{gg_i^{-1}} \le G(e_i)$ for all i. Let $g_0 = g$, $g_{n+1} = 1$, and $H_{n+1} = H^g$. For $0 \le i \le n$, define $H_i = H^{gg_i^{-1}}$. Then for $1 \le i \le n$, $H_i \le G(e_i)$ and for $0 \le i \le n$, $H_i^{g_i g_{i+1}^{-1}} = H_{i+1}$.

Since $[u, g] = [e_1, g_1]^\sigma$ or $[e_1, g_1]^\tau$, we have either $u = e_1^\sigma$ and $g_0 g_1^{-1} = x_0$ or $u = e_1^\tau$ and $g_0 g_1^{-1} = x_0 e_1^{-1}$ for some $x_0 \in G(u)$. Similarly, $[v, 1] = [e_n, g_n]^\sigma$ or $[e_n, g_n]^\tau$ and so either $v = e_n^\sigma$ and $g_n g_{n+1}^{-1} = x_n$ or $v = e_n^\tau$ and $g_n g_{n+1}^{-1} = e_n x_n$ for some $x_n \in G(v)$. If $1 \le i \le n-1$, there are four possibilities:

(i) $[e_i, g_i]^\sigma = [e_{i+1}, g_{i+1}]^\sigma$ and so $g_i g_{i+1}^{-1} = x_i \in G(e_i^\sigma)$.

(ii) $[e_i, g_i]^\tau = [e_{i+1}, g_{i+1}]^\sigma$ and so $g_i g_{i+1}^{-1} = e_i x_i$ for some $x_i \in G(e_i^\tau)$.

(iii) $[e_i, g_i]^\sigma = [e_{i+1}, g_{i+1}]^\tau$ and so $g_i g_{i+1}^{-1} = x_i e_{i+1}^{-1}$ for some $x_i \in G(e_i^\sigma)$.

(iv) $[e_i, g_i]^\tau = [e_{i+1}, g_{i+1}]^\tau$ and so $g_i g_{i+1}^{-1} = e_i x_i e_{i+1}^{-1}$ for some $x_i \in G(e_i^\tau)$.

For each i, $0 \le i \le n$, let y_i be the element of G obtained by replacing x_i in the appropriate formula for $g_i g_{i+1}^{-1}$ above by its representative in the appropriate transversal R_w (where $w = e_i^\sigma$ or e_i^τ). Then the y_i's are all elements of the finite set S constructed at the outset of the proof and moreover, $H_i^{y_i} = H_i^{g_i g_{i+1}^{-1}} = H_{i+1}$. Thus, we have shown that if $H \le G(u)$, then $H^g \le G(v)$ if and only if there are a sequence $\{H_i : 0 \le i \le n+1\}$ of conjugates of H beginning at H and ending at H^g and a corresponding path in D from u to v with edge sequence $\{e_i : 1 \le i \le n\}$ such that for each i, $H_i \le G(e_i)$ and each H_i is conjugate to its successor by an element of S. Of course, this sequence may be chosen so that no term is repeated. Since the vertex groups are finite-by-cyclic, bounds on the number of finite subgroups of the vertex groups are computable and thus, so is an upper bound on the length of such a sequence. Hence, a bound on the number of such sequences is computable and so, finding all G-conjugates of H in $G(v)$ is a finite problem. □

Remark. In the notation above, if $x = y_0 e_1^{\varepsilon_1} y_1 e_2^{\varepsilon_2} \cdots e_n^{\varepsilon_n} y_n$ where $\varepsilon_i = 1$ or -1 according as e_i is oriented with or opposite to the path from u to v, then $H^x = H^g$. For future reference, we note that if $e, f \in ED$ such that $G(e)^g \le G(f)$, then g may be chosen such that, subject to this condition, the T-geodesic connecting a vertex of $[e, g]$ to one of $[f, 1]$ is of shortest possible length. Since $G(e)^g$ fixes every edge in this geodesic, the geodesic can contain no edge of the form $[e, z]$ or $[f, z]$, $z \in G$. The path in D corresponding to g (that is, the projection of the geodesic) then contains neither e nor f. The upshot of this is that if D_0 is the connected component of $D \backslash \{e, f\}$ containing this path and H_0 is the subgroup of G generated by the vertex groups $G(v)$, $v \in VD_0$ and by ED_0, then g belongs to one of H_0, eH_0, H_0f^{-1} or eH_0f^{-1} (depending on which ends of e and f the path connects).

Proposition 3.5. *If $u \in VD$ and $H \leq G(u)$, then there is an effective procedure for obtaining a presentation of $N_G(H)$ as the fundamental group of a finite graph of finite-by-cyclic groups.*

Proof. As remarked earlier, if $N = N_G(H)$, then $Fix_T(H)$ is an N-tree and by Lemma 3.3, it is N-finite so N is certainly the fundamental group of a finite graph of finite-by-cyclic groups. The problem is to determine the corresponding graph of groups which we denote by $(N(-), D_N)$.

For any edge $f \in ED$ and $g \in G$, $[f, g^{-1}] \in Fix_T(H)$ if and only if $H \leq G_{[f,g^{-1}]}$ and so, if and only if $H^g \leq G(f)$. Using Proposition 3.4, one may determine all of the G-conjugates of H in $G(f)$ (if any) and from this, find a complete set of representatives of the $G(f)$-conjugacy classes of such G-conjugates. From Lemma 3.3, one then obtains a complete set of representatives of the orbits of N on the set of G-conjugates of $[f, 1]$ in $Fix_{ET}(H)$. Repeating this for all $f \in ED$, one obtains a complete set of representatives of the N-orbits in $Fix_{ET}(H)$.

Now two vertices $[v_1, g_1]$ and $[v_2, g_2]$ belonging to edges in this set represent the same N-orbit in $Fix_{VT}(H)$ if and only if $v_1 = v_2$ and some element of $g_1^{-1}G(v_1)g_2$ normalizes H. Since $G(v_1)$ is central-by-finite, checking this involves a finite search. Thus, the underlying graph D_N may be constructed. It is then a finite problem to re-select representatives of the N-orbits in $Fix_{ET}(H)$ so that these edges and the adjacent vertices form a subtree of $Fix_T(H)$ containing a fundamental N-transversal. Note that if $y \in VT \cup ET$ and $g \in G$ such that $[y, g^{-1}] \in Fix_T(H)$ (so $H^g \leq G(y)$), then $N_{[y,g^{-1}]} = N \cap G(y)^{g^{-1}} = (N_{G(y)}(H^g))^{g^{-1}}$, so all stabilizers in N are computable. From this, we may construct the graph of groups $(N(-), D_N)$ (see [2, 8.4], [3, I.4], or [13, I.5.4]). □

Actually, for the purposes of this paper, we shall need only to be able to recognize whether or not $N_G(H)$ is finite.

4. Finiteness of Out(G): A necessary condition

Lemma 4.1. *If G is an amalgamated free product $H \underset{U}{*} K$, then $N_G(U) = N_H(U) \underset{U}{*} N_K(U)$. If G is an HNN extension $H \underset{U}{*} t$, then either $N_G(U) = N_H(U) \underset{U}{*} s$ where $s \in tH \cap N_G(U)$ or (if $tH \cap N_G(H) = \varnothing$ so U^t is not H-conjugate to U) $N_G(U) = N_H(U) \underset{U}{*} N_L(U)$ where $L = H^{t^{-1}}$.*

Proof. If $G = H \underset{U}{*} K$, G is the fundamental group of a graph of groups $(G(-), D)$ in which D is a segment, with vertex groups H and K and edge group U. G acts transitively on the edges (but not the vertices) of the universal cover T and so, since U is the stabilizer of an edge of T, $N_G(U)$ acts transitively on the edges (but not the vertices) of the subtree $Fix_T(U)$. Therefore, $N_G(U) = N_H(U) \underset{U}{*} N_K(U)$.

If $G = H \underset{U}{*} t$, then the corresponding graph of groups is a loop with vertex group H and edge group U. Again, $N_G(U)$ is edge transitive on $Fix_T(U)$ and a simple computation reveals that it is vertex-transitive precisely when U and U^t are conjugate in H. Accordingly, there are two possibilities for the structure of $N_G(U)$ as described in the statement of the lemma. □

Lemma 4.2. *Suppose that $G = H \underset{U}{*} K$ where H is finitely generated and virtually free, U is finite, and $K \neq U$. If* $\mathrm{Out}(G)$ *is finite, then $N_{\mathrm{Aut}(H)}(U)$ is finite. In particular, either $N_H(U)$ is finite or H is finite-by-(infinite cyclic).*

Proof. We may assume that $H \neq U$, for otherwise the conclusion is trivial. For any $\alpha \in C_{\mathrm{Aut}(H)}(U)$, there is an element $\beta \in C_{\mathrm{Aut}(G)}(U)$ such that $h^\beta = h^\alpha$ for all $h \in H$ and $k^\beta = k$ for all $k \in K$. Since $\mathrm{Out}(G)$ is finite, there are a positive integer n and an element g of G such that $h^g = h^{\beta^n} = h^{\alpha^n}$ and $k^g = k^{\beta^n} = k$ for all $h \in H$ and $k \in K$. Then $g \in N_G(H) \cap C_G(K) \leq H \cap K = U$ and so, $g^m = 1$ for some positive integer m. It follows that $\alpha^{nm} = 1$ and hence we have proved that $C_{\mathrm{Aut}(H)}(U)$ is periodic. By Lemma 2.4, $C_{\mathrm{Aut}(H)}(U)$ is finite and so $N_{\mathrm{Aut}(H)}(U)$ is finite. Also, $N_H(U)/Z(H) \cong N_{\mathrm{Inn}(H)}(U)$ and so, if $N_H(U)$ is not finite, neither is $Z(H)$. In this event, the fact that H is virtually free implies that H is a finite extension of a central infinite cyclic group and hence, is finite-by-(infinite cyclic). □

Lemma 4.3. *Suppose that $G = H \underset{U}{*} t$ where H is virtually free, U is finite and $H \neq U$. If* $\mathrm{Out}(G)$ *is finite, then $N_H(U)$ and $N_H(U^t)$ are both finite.*

Proof. Since $G = H \underset{U}{*} t = H \underset{V}{*} t^{-1}$ where $V = U^t$, it suffices to prove that $N_H(U)$ is finite and hence, that $C_H(U)$ is finite. In fact, because H is virtually free, it is enough to show that $C_H(U)$ is periodic. Let $c \in C_H(U)$. There is an automorphism α of G such that $h^\alpha = h$ for all $h \in H$ and $t^\alpha = ct$ and so, if $\alpha^n = i_g \in \mathrm{Inn}(G)$ where $n > 0$, then $g \in C_G(H)$ and $t^g = c^n t$. But since $H \neq U$, $C_G(H) \leq H$. Therefore, $g \in H$ and so the fact that $t^g t^{-1} = c^n \in H$ implies that $g \in U$, whence α has finite order. Since $t^{\alpha^k} = c^k t$ for any integer k, we conclude that c has finite order. □

If H is a finite-by-cyclic group, we shall continue to denote by H^0 the unique maximal finite subgroup of H. Recall that our concern is with graphs of finite-by-cyclic groups $(G(-), D)$ having finite edge groups. As mentioned in the Introduction, it will be convenient to assume that two types of degeneracies in these graphs of groups have been eliminated by adding, deleting or contracting certain edges of D.

First, if $e \in ED$ such that $e^\sigma = e^\tau$ and $G(e) = G(e^\sigma)$ (so $G(e^\sigma)$ is finite and normalized by e), let D^* be the graph obtained from D by deleting the edge e and define $G^*(e^\sigma) = \langle G(e^\sigma), e\rangle$ (a finite-by-cyclic group) and $G^*(y) = G(y)$ for all other elements of $VD^* \cup ED^*$. It is clear that the resulting graph of finite-by-cyclic groups $(G^*(-), D^*)$ has fundamental group isomorphic to that of $(G(-), D)$.

A second type of reducible configuration occurs if for some $e \in ED$, $e^\sigma \neq e^\tau$ and either $G(e) = G(e^\sigma)^0$ or $G(e)^e = G(e^\tau)^0$. Suppose, for example, that $e^\sigma \neq e^\tau$ and $G(e) = G(e^\sigma)^0$. Let $\overline{D}$ be the graph obtained from D by contracting the edge e. If $f \in ED$ such that $f^\sigma = e^\sigma$, then since $G(f) \leq G(e^\sigma)^0 = G(e)$, $G(f)^e \leq G(e^\tau)$. On the other hand, if $f^\tau = e^\sigma$, then since $G(f)^f \leq G(e^\sigma)^0 = G(e)$, the composition fe defines a monomorphism from $G(f)$ into $G(e^\tau)$. If $G(e^\sigma)$ is finite, then using these embeddings, one may define in an obvious way a graph of groups on the underlying graph $\overline{D}$ whose fundamental group is isomorphic to $\pi(G(-), D)$. If $G(e^\sigma)$ is infinite, say $G(e^\sigma) = [G(e)]\langle t\rangle$, the same conclusion applies except that $\overline{D}$ must be modified by adding a loop labelled $e^{-1}t$ at the vertex corresponding to e^τ, this to be interpreted as a monomorphism from $G(e)^e$ to itself. An analogous construction applies in the case that $G(e)^e = G(e^\tau)^0$.

Both of these types of reduction diminish the total number of edges and vertices in the underlying graph by at least one. (Note that the second type of reduction was encountered in the conclusion of the proof of Proposition 2.5, when a certain generalized free product of two finite-by-cyclic groups was regarded as an HNN extension of a finite-by-cyclic group.)

A graph of finite-by-cyclic groups for which there are no possible reductions of the above two types will be called *irreducible*. Thus, in an irreducible graph of finite-by-cyclic groups, whenever $G(e) = G(e^\sigma)^0$ or $G(e)^e = G(e^\tau)^0$ for some edge e of the underlying graph, we may conclude that $e^\sigma = e^\tau$ and $G(e^\sigma)$ is infinite.

By a *terminal* vertex of a directed graph D, we shall mean one of total valence 1 (that is, a vertex v for which $|v^{\sigma^{-1}} \cup v^{\tau^{-1}}| = 1$). We denote the set of all terminal vertices of D by ∂D. The following theorem records several consequences of the assumption that the outer automorphism group of $\pi(G(-), D)$ is finite. It is the last two conclusions which are of principal interest here since these (or rather (iii) and a weaker version of (iv)) will reappear as hypotheses in Section 5 when we consider the converse.

Theorem 4.4. *Let $(G(-), D)$ be an irreducible finite graph of finite-by-cyclic groups with finite edge groups and fundamental group G. If $\mathrm{Out}(G)$ is finite, then the following statements hold:*

(i) *For every $v \in VD$, $N_G(G(v)^0) = G(v)$.*

(ii) *Let $e \in ED$ and let $G(e) \leq H \leq G$. If $G = H \underset{G(e)}{*} t$ for some $t \in G$, then $N_H(G(e))$ and $N_H(G(e)^t)$ are finite. If $G = H \underset{G(e)}{*} K$ for some $K \leq G, K \neq G(e)$ and $(G(u)^0)^x \leq H$ for some $u \in VD, x \in G$, then $N_H(G(e))$ is finite except possibly if $H = G(u)^x$.*

(iii) *Suppose that $e \in ED$, D_0 is a connected component of $D\backslash\{e\}$, and $H_0 = \pi(G(-)|_{D_0}, D_0)$. If $e^\sigma \in VD_0\backslash\partial D$, then $N_{H_0}(G(e))$ is finite. If $e^\tau \in VD_0\backslash\partial D$, then $N_{H_0}(G(e)^e)$ is finite.*

(iv) *If $f \in ED$ and $G(f)$ contains a conjugate of $G(e)$ for some $e \in ED$, $e \neq f$, then $N_G(G(f))$ is finite.*

Proof. We will regard $\pi(G(-), D)$ as $\pi(G(-), D, M)$, where M is a maximal subtree of D. Recall the observation made in Section 3 that if M_1 and M_2 are maximal subtrees of D, then there is an isomorphism from $\pi(G(-), D, M_1)$ to $\pi(G(-), D, M_2)$ which, for each $v \in VD$, maps the natural copy of $G(v)$ in the first group to a conjugate of the natural copy of $G(v)$ in the second. It is because of this that the hypotheses and conclusions of each of the statements in the theorem make sense, despite the fact that no M is specified in the hypotheses. This fact will also allow us in the proof to use any convenient choice of M.

For (i), suppose that $v \in VD$ and $g \in N_G(G(v)^0)\backslash G(v)$ so by Lemma 3.1, there is an edge e of D such that either $e^\sigma = v$ and $G(v)^0 = G(e)$ or $e^\tau = v$ and $G(v)^0 = G(e)^e$. The hypothesis that $(G(-), D)$ is irreducible then implies that $e^\sigma = e^\tau = v$, $G(v)$ is infinite, and $G(v)^0 = G(e) = G(e)^e$. Since e is a loop at v, we can write $G = G_0 \underset{G(e)}{*} e$ where $G_0 = \pi(G(-)|_{D_0}, D_0)$ and $D_0 = D\backslash\{e\}$. By Lemma 4.3, $N_{G_0}(G(e))$ is finite, which is absurd since this group contains $G(v)$.

Conclusion (ii) essentially just summarizes Lemmas 4.2 and 4.3. In fact, by these results, we may assume that $G = H \underset{G(e)}{*} K$ where H is finite-by(infinite cyclic) and $(G(u)^0)^x \leq H$ for some $u \in VD$, $x \in G$. We know that $H^0 \leq G(v)^y$ for some $v \in$

VD, $y \in G$ [3, I.4.9] and so $G(u)^0 \le G(v)^{yx^{-1}}$. By Lemma 3.1 and the irreducibility of $(G(-), D)$, $u = v$ and so $(G(u)^0)^x = H^0$. By (i), $H \le N_G(G(u)^0)^x = G(u)^x$ and so, since $G(u)$ is finite-by-cyclic, $H \triangleleft G(u)^x$. Thus, $G(u)^x \le N_G(H) = H$, whence $H = G(u)^x$.

For (iii), suppose first that $D\backslash\{e\} = D_0$ is connected and M is a maximal subtree of D_0 (and thus, of D). Then $G = H_0 \underset{G(e)}{*} e$, where $H_0 = \pi(G(-)|_{D_0}, D_0, M)$ and by (ii), $N_{H_0}(G(e))$ and $N_{H_0}(G(e)^e)$ are finite. If $D\backslash\{e\}$ consists of two components, say D_0 and D_1, then e belongs to every maximal tree in D (so $e = 1$ as an element of G). If M_0 and M_1 are maximal trees in D_0 and D_1 respectively, then $M = M_0 \cup M_1 \cup \{e\}$ is a maximal tree in D and $G = H_0 \underset{G(e)}{*} H_1$ where $H_i = \pi(G(-)|_{D_i}, D_i, M_i)$. If $N_{H_0}(G(e))$ is not finite, (ii) yields that $H_0 = G(u)^x$ for some $u \in VD$, $x \in G$. By Lemma 3.1, if $v \in VD_0$ and either $v \ne u$ or $x \notin G(u)$ then $G(v) \le G(f)^y$ for some $f \in ED$ with $v \in \{f^\sigma, f^\tau\}$, $y \in G(v) \cup fG(v)$. Thus, the irreducibility of $(G(-), D)$ implies that $VD_0 = \{u\}$ and $H_0 = G(u)$, whence $ED_0 = \varnothing$. Thus, $u \in \{e^\sigma, e^\tau\}$ and is terminal in D, so (iii) is proved.

Finally we prove (iv). This part seems to entail consideration of a number of cases. $D\backslash\{e, f\}$ has at most three connected components which we denote by D_0 and (if they exist) D_1 and D_2. One of these, say D_0, contains at least one vertex incident with e and one incident with f. In fact, reversing the orientation of e if necessary (and defining $G(-)$ on the new edge e^{-1} by $G(e^{-1}) = G(e)^e$), we may assume that $e^\sigma \in VD_0$ and similarly, $f^\sigma \in VD_0$. Let M_i be a maximal subtree of D_i. For the presentation of G, we choose a maximal subtree M of D such that M contains each M_i. If $H_i = \pi(G(-)|_{D_i}, D_i, M_i)$, we have a natural embedding of H_i in $G = \pi(G(-), D, M)$.

By the remark following Proposition 3.4, we may assume that $G(e)^g \le G(f)$, where g corresponds to a path in D_0 from e^σ to f^σ and hence, $g \in H_0$. The possibilities are:

1. $D\backslash\{e, f\}$ has 3 components D_0, D_1 and D_2, whence e and f both lie in EM.
2. $D\backslash\{e, f\}$ has 2 components D_0 and D_1. Here, there are three subcases:
   (a) $D\backslash\{e\}$ is disconnected (so $e \in EM$) but $D\backslash\{f\}$ is connected. In this case, $f^\tau \in VD_0$ and so $f \notin EM$ (since f^σ and f^τ lie in VM_0).
   (b) $D\backslash\{f\}$ is disconnected (so $f \in EM$) but $D\backslash\{e\}$ is connected (so $e \notin EM$).
   (c) $D\backslash\{e\}$ and $D\backslash\{f\}$ are both connected. Here, e^τ and f^τ both lie in VD_1 and exactly one of e or f belongs to EM. We may assume here that $f \in EM$.
3. $D\backslash\{e, f\}$ is connected, whence $M = M_0$ and neither e nor f belongs to EM.

It is convenient now to think of G as the fundamental group of the graph of groups $(H(-), X)$, where VX is the set of connected components D_i of $D\backslash\{e, f\}$, $EX = \{e, f\}$ with the obvious incidence functions, $H(D_i) = H_i$ for all i, $H(e) = G(e)$, and $H(f) = G(f)$. The five cases described above then correspond to the possibilities for the graph X that are shown in Figure 1.

Correspondingly, we have the following possible structures for G :

1. $G = H_1 \underset{G(e)}{*} H_0 \underset{G(f)}{*} H_2$,

2(a). $G = (H_0 \underset{G(f)}{*} f) \underset{G(e)}{*} H_1 = (H_0 \underset{G(e)}{*} H_1) \underset{G(f)}{*} f$,

(b). $G = (H_0 \underset{G(e)}{*} e) \underset{G(f)}{*} H_1 = (H_0 \underset{G(f)}{*} H_1) \underset{G(e)}{*} e$, (c). $G = (H_0 \underset{G(f)}{*} H_1) \underset{G(e)}{*} e$,

3. $G = (H_0 \underset{G(e)}{*} e) \underset{G(f)}{*} f = (H_0 \underset{G(f)}{*} f) \underset{G(e)}{*} e$.

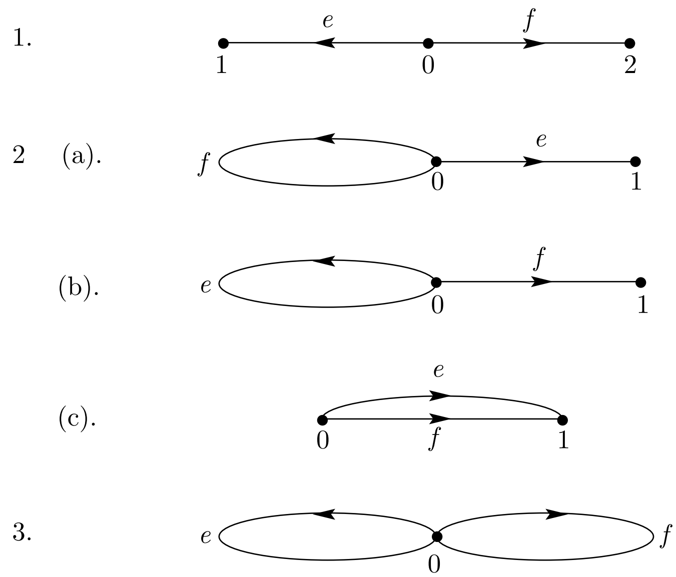

FIGURE 1

First, we claim that G may be realized as the fundamental group of a graph of groups in which the edge groups are $G(e)^g$ and $G(f)$. This is essentially a matter of applying appropriate Tietze transformations to the presentations corresponding to each of the cases above (using the fact that $g \in H_0$). The five possibilities may be dealt with as follows, the final factorization yielding in each case the desired presentation:

1. $G = (H_1 \underset{G(e)}{*} H_0)^g \underset{G(f)}{*} H_2 = H_1^g \underset{G(e)^g}{*} H_0 \underset{G(f)}{*} H_2$,

2(a). $G = G^g = (H_0 \underset{G(f)}{*} f)^g \underset{G(e)^g}{*} H_1^g = (H_0 \underset{G(f)}{*} f) \underset{G(e)^g}{*} H_1^g$,

(b). $G = (H_0 \underset{G(e)^g}{*} g^{-1}e) \underset{G(f)}{*} H_1$,

(c). $G = (H_0 \underset{G(f)}{*} H_1) \underset{G(e)^g}{*} g^{-1}e$,

3. $G = (H_0 \underset{G(f)}{*} f) \underset{G(e)^g}{*} g^{-1}e$.

Let $U = G(e)^g$ and $V = G(f)$ so $U \leq V$ and, after relabeling the subgroups and HNN indeterminates, the five cases above may be written as follows:

1. $G = F_1 \underset{U}{*} F_0 \underset{V}{*} F_2$,

2(a). $G = (F_0 \underset{V}{*} t) \underset{U}{*} F_1$,

(b). $G = (F_0 \underset{U}{*} t) \underset{V}{*} F_1$,

(c). $G = (F_0 \underset{V}{*} F_1) \underset{U}{*} t$,

3. $G = (F_0 \underset{V}{*} t) \underset{U}{*} s$.

Suppose now that $U \neq V$. Then in each of the five cases, Corollary 3.2 together with conclusion (ii) of this theorem (already proved) yields that $N_G(V)$ is finite.

For example, in case 1, $G = F_1 \underset{U}{*} F$ where $F = F_0 \underset{V}{*} F_2$, and so Corollary 3.2 implies that $N_G(V) \le F$. But by conclusion (ii) (with $H = F, K = F_1$), $N_F(U)$ is finite and so since $C_F(V) \le C_F(U)$, $N_G(V) = N_F(V)$ is finite. The other cases are similar and so, we may assume from now on that $U = V$. Cases 2(a) and (b) are then identical. In case 2(c) $U = V \le F_1$ and $U^t = G(e)^e \le F_1$ and so $G = (F_0 \underset{U}{*} F_1) \underset{U}{*} t = F_0 \underset{U}{*} (F_1 \underset{U}{*} t)$. Thus, up to an exchange of the subscripts 0 and 1, case 2(c) is also equivalent to 2(a). So now there are only three cases.

1. $G = F_1 \underset{U}{*} F_0 \underset{U}{*} F_2$,
2. $G = (F_0 \underset{U}{*} t) \underset{U}{*} F_1$,
3. $G = (F_0 \underset{U}{*} t) \underset{U}{*} s$.

The arguments now seem to be somewhat different for each case, though they all depend heavily on Lemma 4.1 and on conclusion (ii) of the theorem.

In the first case, if $F = F_0 \underset{U}{*} F_2$ then $N_G(U) = N_{F_1}(U) \underset{U}{*} N_F(U)$ and $N_F(U) = N_{F_0}(U) \underset{U}{*} N_{F_2}(U)$ by Lemma 4.1. But by conclusion (ii) of this theorem, $N_F(U)$ is finite and so $N_F(U) = N_{F_0}(U)$ or $N_{F_2}(U)$. Accordingly, $N_G(U) = N_{F_1}(U) \underset{U}{*} N_{F_0}(U)$ or $N_{F_1}(U) \underset{U}{*} N_{F_2}(U)$. But Lemma 4.1 yields that $N_{F_1}(U) \underset{U}{*} N_{F_0}(U) = N_{F_1 \underset{U}{*} F_0}(U)$ and, because $\langle F_1, F_2 \rangle = F_1 \underset{U}{*} F_2, N_{F_1}(U) \underset{U}{*} N_{F_2}(U) = N_{F_1 \underset{U}{*} F_2}(U)$. By (iii) (and because G may be written $F_0 \underset{U}{*} (F_1 \underset{U}{*} F_2))$, both of these are finite and thus, $N_G(U)$ is finite, as required.

In case 2, Lemma 4.1 implies that $N_G(U) = N_{F_0 \underset{U}{*} t}(U) \underset{U}{*} N_{F_1}(U)$. By conclusion (iii) of this theorem, the first factor is finite and so, again by Lemma 4.1, $N_{F_0 \underset{U}{*} t}(U) = N_{F_0}(U)$ or $N_{F_0^{t^{-1}}}(U)$. In the former instance, we have $N_G(U) = N_{F_0}(U) \underset{U}{*} N_{F_1}(U) = N_{F_0 \underset{U}{*} F_1}(U)$ and by (iii), this is finite. In the latter instance, $N_G(U) = N_{F_0^{t^{-1}}}(U) \underset{U}{*} N_{F_1}(U)$. But $G = (F_0 \underset{U}{*} t)^{t^{-1}} \underset{U}{*} F_1 = (F_0^{t^{-1}} \underset{U^{t^{-1}}}{*} t) \underset{U}{*} F_1 = (F_0^{t^{-1}} \underset{U}{*} t^{-1}) \underset{U}{*} F_1 = (F_0^{t^{-1}} \underset{U}{*} F_1) \underset{U}{*} t^{-1}$, an HNN extension of $F_0^{t^{-1}} \underset{U}{*} F_1$ and so, by (iii), $N_G(U) = N_{F_0^{t^{-1}}}(U) \underset{U}{*} N_{F_1}(U) = N_{F_0^{t^{-1}} \underset{U}{*} F_1}(U)$ is finite. Thus, case 2 is resolved.

Finally, assume that case 3 applies so $G = (F_0 \underset{U}{*} t) \underset{U}{*} s = (F_0 \underset{U}{*} s) \underset{U}{*} t$. Suppose that $U^{tx} = U$ for some $x \in F_0 \underset{U}{*} s$. Then $G = (F_0 \underset{U}{*} s) \underset{U}{*} tx = (F_0 \underset{U}{*} tx) \underset{U}{*} s$ and (ii) implies that $N_{F_0 \underset{U}{*} tx}(U)$ is finite, which is absurd since tx has infinite order. Thus, we may assume that U^t and U are not conjugate in $F_0 \underset{U}{*} s$. By Lemma 4.1, $N_G(U) = N_{F_0 \underset{U}{*} s}(U) \underset{U}{*} N_{(F_0 \underset{U}{*} s)^{t^{-1}}}(U)$ and also, by (ii), $N_{F_0 \underset{U}{*} s}(U)$ is finite (so it must be equal to either $N_{F_0}(U)$ or $N_{F_0^{s^{-1}}}(U)$. In addition, by Corollary 3.2, $N_{F_0 \underset{U}{*} s}(U^t) \le F_0$ and so $N_{(F_0 \underset{U}{*} s)^{t^{-1}}}(U) = N_{F_0^{t^{-1}}}(U)$. Therefore, $N_G(U) = N_{F_0}(U) \underset{U}{*} N_{F_0^{t^{-1}}}(U)$ or $N_{F_0^{s^{-1}}}(U) \underset{U}{*} N_{F_0^{t^{-1}}}(U)$. But (ii) implies that $N_{F_0 \underset{U}{*} t}(U)$ is finite and, by Lemma 4.1, this group is $N_{F_0}(U) \underset{U}{*} N_{F_0^{t^{-1}}}(U)$. Thus, we may assume that $N_G(U) = N_{F_0^{s^{-1}}}(U) \underset{U}{*} N_{F_0^{t^{-1}}}(U)$.

Now $G = (F_0 \underset{U}{*} s) \underset{U}{*} t = (F_0 \underset{U^s}{*} s^{-1}) \underset{U^s}{*} s^{-1}t = (F_0 \underset{U^s}{*} s^{-1}t) \underset{U^s}{*} s^{-1}$ and so $N_{F_0 \underset{U^s}{*} s^{-1}t}(U^s)$ is finite (by (ii)). But $N_{F_0 \underset{U^s}{*} s^{-1}t}(U^s) = N_{F_0}(U^s) \underset{U^s}{*} N_{F_0^{t^{-1}s}}(U^s) =$

$(N_{F_0^{s^{-1}}}(U) \underset{U}{*} N_{F_0^{t^{-1}}}(U))^s = N_G(U)^s$ by Lemma 4.1, and so we conclude that $N_G(U)$ is finite. This completes the proof of case 3 and of the theorem. □

5. The main theorem

The objective now is to prove that if G is the fundamental group of an irreducible finite graph of finite-by-cyclic groups with finite edge groups, then the last two conclusions of Theorem 4.4 are actually sufficient to ensure the finiteness of $\mathrm{Out}(G)$. (In fact, we shall replace the second condition with a somewhat weaker one.) We consider the following two hypotheses on a graph of groups $(G(-), D)$:

Hypothesis 5.1. For every $e \in ED$, if D_0 is a connected component of $D\backslash\{e\}$ and $H_0 = \pi(G(-)|_{D_0}, D_0)$, then $N_{H_0}(G(e))$ is finite if $e^\sigma \in VD_0\backslash\partial D$ and $N_{H_0}(G(e)^e)$ is finite if $e^\tau \in VD_0\backslash\partial D$.

Hypothesis 5.2. For every $e \in ED$, if $G(e)$ is G-conjugate to $G(f)$ for some $f \in ED$, $f \neq e$, then $N_G(G(e))$ is finite.

Note that by Propositions 3.4 and 3.5, each of these hypotheses is effectively verifiable from the graph of groups.

Lemma 5.3. *Let $(G(-), D)$ be an irreducible graph of finite-by-cyclic groups satisfying Hypothesis 5.1. Then*

(i) *for every non-terminal vertex u of D, $G(u)$ is finite, and*
(ii) *for every vertex v of D, $N_G(G(v)^0) = G(v)$.*

Proof. If u is non-terminal and $u = e^\sigma$ where $e \in ED$, then by Hypothesis 5.1, $N_{G(u)}(G(e))$ is finite and similarly, if $u = e^\tau$, then $N_{G(u)}(G(e)^e)$ is finite. In either case, $Z(G(u))$ is finite whence, because it is finite-by-cyclic, $G(u)$ is finite. This proves (i).

If v is any vertex of D, let $g \in N_G(G(v)^0)$ and suppose that $g \notin G(v)$. Then by Lemma 3.1, $G(v)^0 \leq G(v) \cap G(v)^g \leq G(e)^x$ for some $e \in ED$ with $v \in \{e^\sigma, e^\tau\}$, $x \in G(v) \cup eG(v)$. It follows that $G(v)^0 = G(e)$ if $v = e^\sigma$ and $G(v)^0 = G(e)^e$ if $v = e^\tau$. If $G(v)$ were finite or if e^σ and e^τ were distinct, this would contradict the irreducibility of $(G(-), D)$. But if $e^\sigma = e^\tau = v$, then v is certainly not a terminal vertex of D and so by (i), $G(v)$ cannot be infinite. This contradiction completes the proof of (ii). □

Lemma 5.4. *Let H be a finite-by-(torsion-free) subgroup of a group G, $\alpha \in \mathrm{Aut}(G)$, and F be a finite α-invariant subgroup of H. If $H^\alpha = H^x$ for some $x \in G$, then for some positive integer m, if $g = xx^\alpha x^{\alpha^2} \cdots x^{\alpha^{m-1}}$, then $g \in C_G(F)$ and $H^{\alpha^m} = H^g$.*

Proof. Since $F = F^\alpha \leq H^\alpha = H^x$, it follows by successive applications of α that for each positive integer n, if $g_n = xx^\alpha \cdots x^{\alpha^{n-1}}$, then $F \leq H^{\alpha^n} = H^{g_n}$ and so, since H is finite-by-(torsion-free), $F^{g_n^{-1}} \leq H^0$ (where H^0 is the unique maximal finite subgroup of H). Since H^0 contains only finitely many distinct G-conjugates of F, there exist integers $k > l$ with $F^{g_k^{-1}} = F^{g_l^{-1}}$. If $j = k - l$, it follows that $g_j \in N_G(F)$. Let $y = g_j$ and $\beta = \alpha^j$. Since $N_G(F)$ is α-invariant and $N_G(F)/C_G(F)$ is finite, there exist integers $r > s > 0$ such that $yy^\beta \cdots y^{\beta^r} \equiv yy^\beta \cdots y^{\beta^s}$ modulo $C_G(F)$, whence $yy^\beta \cdots y^{\beta^{r-s-1}} \in C_G(F)$. But $yy^\beta \cdots y^{\beta^{r-s-1}} = g_m$ where $m = j(r-s)$ and so, since $H^{\alpha^m} = H^{g_m}$, the proof is complete. □

At this point, the universal cover T of the graph of groups $(G(-), D)$ begins to play at least as significant a role in the argument as the underlying graph D. To minimize notational clutter, we will denote edges or vertices of T by bold-face letters such as $\mathbf{e}, \mathbf{v}$ etc., the corresponding italic letters such as e, v etc. representing the projections of these edges or vertices in D. (Of course, as before, edges of D will also be identified with elements of the fundamental group of $(G(-), D)$.)

Lemma 5.5. *Let $(G(-), D)$ be an irreducible finite graph of finite-by-cyclic groups with finite edge groups. Let G and T be, respectively, the fundamental group and the universal cover of $(G(-), D)$. Let $\mathbf{e} \in ET$, $N = N_G(G_{\mathbf{e}})$, and assume that N is finite. If $\mathbf{u}, \mathbf{v} \in Fix_{VT}(G_{\mathbf{e}})$ and $\alpha \in \mathrm{Aut}(G)$ such that α centralizes $G_{\mathbf{u}}$ then some non-trivial power of α also centralizes $G_{\mathbf{v}}$.*

Proof. Since $G_{\mathbf{e}} \le G_{\mathbf{u}} \le C_G(\alpha)$, N is α-invariant and so, because N is finite, some positive power of α centralizes it. Thus, with no loss of generality, we may assume that $N \le C_G(\alpha)$. Because $G_{\mathbf{e}} \le G_{\mathbf{v}}$, $Z(G_{\mathbf{v}}) \le N$ and so, because $G_{\mathbf{v}}$ is finite-by-cyclic, it is actually finite. By Lemma 2.1, some positive power of α sends $G_{\mathbf{v}}$ to a conjugate of itself and so again, we may assume that α has this property. By Lemma 5.4 (with $H = G_v$ and $F = G_e$), $G_{\mathbf{v}}^{\alpha^m} = G_{\mathbf{v}}^g$ for some positive integer m and some $g \in C_G(G_{\mathbf{e}}) \le N$. If $g^k = 1$, then $G_{\mathbf{v}}$ is invariant under α^{mk}. But since $G_{\mathbf{v}}$ is finite, α^{mk} has finite order on $G_{\mathbf{v}}$ and so a non-trivial power of α centralizes $G_{\mathbf{v}}$. ☐

If X is a G-set and $Y \subseteq X$, $G_{\{Y\}}$ denotes the setwise-stabilizer of Y.

Lemma 5.6. *Let $(G(-), D)$ be a graph of groups with fundamental group G and universal cover T. Let $e \in ED$, $u \in \{e^\sigma, e^\tau\}$, D_0 be the connected component of $D\backslash\{e\}$ containing u, and $H_0 = \pi(G(-)|_{D_0}, D_0)$. Let $\mathbf{e} \in ET$ such that $\mathbf{e}$ projects onto e and is incident with $[u, 1]$ and let L be the connected component of $T\backslash\mathbf{e}^G$ containing $[u, 1]$. Then $L = \{[y, h] \colon y \in VD_0 \cup ED_0, h \in H_0\}$ and moreover, if $N = N_G(G_{\mathbf{e}})$ then $K = Fix_L(G_{\mathbf{e}})$ is the connected component of $Fix_T(G_{\mathbf{e}})\backslash\mathbf{e}^N$ containing $[u, 1]$. In particular, $G_{\{L\}} = H_0$ and $N_{\{K\}} = N \cap H_0$.*

Proof. Let $T_0 = \{[y, h] \colon y \in VD_0 \cup ED_0, h \in H_0\}$. T_0 is the universal cover of the graph of groups $(G(-)|_{D_0}, D_0)$ and thus, is an H_0-subtree of T. Since $e \notin ED_0$, $T_0 \subseteq T\backslash\mathbf{e}^G$ and since $[u, 1] \in VT_0$, it follows that $T_0 \subseteq L$. If $T_0 \ne L$, then there exists an edge $[f, x]$ of L which is not in ET_0 but which is incident with a vertex of T_0. The projection f of this edge in D is then incident with a vertex of D_0 (since T_0 projects onto D_0) and so, since D_0 is a connected component of $D\backslash\{e\}$, $f \in ED_0$. But if $[f, x]^\sigma \in VT_0$, then $[f^\sigma, x] = [f^\sigma, h]$ for some $h \in H_0$. Therefore, $xh^{-1} \in G(f^\sigma) \le H_0$, whence $x \in H_0$ and $[f, x] \in ET_0$, a contradiction. Similarly, if $[f, x]^\tau \in VT_0$, then $[f^\tau, f^{-1}x] = [f^\tau, h]$ for some $h \in H_0$. Therefore, $f^{-1}xh^{-1} \in G(f^\tau) \le H_0$, whence again $x \in H_0$ and we have the contradiction $[f, x] \in ET_0$. We conclude that $T_0 = L$.

If $K = Fix_L(G_{\mathbf{e}})$, then K is a subtree of L and so is contained in the connected component of $Fix_T(G_{\mathbf{e}})\backslash\mathbf{e}^N$ containing $[u, 1]$, say C. But as a subtree of T containing $[u, 1]$, C must be contained in L, and so we conclude that $C \subseteq L \cap Fix_T(G_{\mathbf{e}}) = K$. Hence, $K = C$ as claimed.

From the representation of L as $\{[y, h] \colon y \in VD_0 \cup ED_0, h \in H_0\}$, it is clear that $H_0 \le G_{\{L\}}$. On the other hand, if $g \in G_{\{L\}}$, then $[u, g] = [u, 1]^g \in VL$ and so $[u, g] = [u, h]$ for some $h \in H_0$. Since $G(u) \le H_0$, it follows that $g \in H_0$. Thus, $H_0 = G_{\{L\}}$.

Clearly, $N_{\{K\}} \leq N_{\{L\}}$. On the other hand, since $Fix_T(G_{\mathbf{e}})$ is N-invariant, $K = L \cap Fix_T(G_{\mathbf{e}})$ is invariant under $N_{\{L\}}$. Thus, $N_{\{K\}} = N_{\{L\}} = N \cap G_{\{L\}} = N \cap H_0$. □

Lemma 5.7. *Suppose that $(G(-), D)$ is a graph of groups satisfying Hypothesis 5.1 and with fundamental group G and universal cover T. Suppose that $\mathbf{e} \in ET$ and let $N = N_G(G_{\mathbf{e}})$. If K is a connected component of $Fix_T(G_{\mathbf{e}})\backslash \mathbf{e}^N$, then K is finite and $N_{\{K\}} = N_{\mathbf{w}}$ for some $\mathbf{w} \in VK$.*

Proof. K is N-conjugate to a connected component of $Fix_T(G_{\mathbf{e}})\backslash \mathbf{e}^N$ which contains a vertex incident with $\mathbf{e}$, so we may assume that K contains $\mathbf{u} \in \{\mathbf{e}^\sigma, \mathbf{e}^\tau\}$. If u is the projection of $\mathbf{u}$ in D, then $\mathbf{u} = [u, x] = [u, 1]^x$ for some $x \in G$ and so, after conjugation if necessary, we may assume also that $\mathbf{u} = [u, 1]$. It follows now by the previous lemma that if D_0 is the component of $D\backslash\{e\}$ containing u and $H_0 = \pi(G(-)|_{D_0}, D_0)$, then $N_{\{K\}} = N_{H_0}(G_{\mathbf{e}})$. If $N_{H_0}(G_{\mathbf{e}})$ is infinite, then Hypothesis 5.1 implies that D_0 consists of the single vertex u, which is terminal in D. It follows that in this case, $K = \{\mathbf{u}\}$ and $N_{\{K\}} = N_{\mathbf{u}}$. If $N_{H_0}(G_{\mathbf{e}})$ is finite, it fixes some $\mathbf{w} \in VK$ by [3, I.4.9] and, since $N_{\mathbf{w}}$ must leave K invariant, $N_{\{K\}} = N_{\mathbf{w}}$. Also, because each orbit of $N_{\{K\}}$ in K is finite and because $Fix_T(G_{\mathbf{e}})$ is N-finite (by Lemma 3.3), K must be finite. □

We show now that statement (iv) of Theorem 4.4 is also a consequence of Hypotheses 5.1 and 5.2.

Lemma 5.8. *Let $(G(-), D)$ be a graph of groups with finite edge stabilizers and fundamental group G which satisfies Hypotheses 5.1 and 5.2. If e and f are distinct edges of D such that $G(e)^g \leq G(f)$ for some $g \in G$, then $N_G(G(f))$ is finite.*

Proof. If T is the universal cover of $(G(-), D)$, it suffices to show that if $\mathbf{e}$ and $\mathbf{f}$ are non-conjugate edges of T and $G_{\mathbf{e}} \leq G_{\mathbf{f}}$, then $N_G(G_{\mathbf{f}})$ is finite. Because of Hypothesis 5.2, we may assume that $G_{\mathbf{e}} \neq G_{\mathbf{f}}$ and so, if $N = N_G(G_{\mathbf{e}})$, $Fix_T(G_{\mathbf{f}}) \subseteq Fix_T(G_{\mathbf{e}})\backslash \mathbf{e}^N$. Since $Fix_T(G_{\mathbf{f}})$ is connected, it follows from Lemma 5.7 that $Fix_T(G_{\mathbf{f}})$ is finite. In particular, $\mathbf{f}$ has only finitely many $N_G(G_{\mathbf{f}})$-conjugates and so, $|N_G(G_{\mathbf{f}})\colon\ G_{\mathbf{f}}|$ is finite. But by hypothesis, all edge stabilizers are also finite and so $N_G(G_{\mathbf{f}})$ is finite. □

The next lemma is the key to the proof of the main theorem of this paper. The conclusion is similar to that of Lemma 5.5 (which is used in the proof).

Lemma 5.9. *Let $(G(-), D)$ be an irreducible finite graph of finite-by-cyclic groups with finite edge groups which satisfies Hypotheses 5.1 and 5.2. Let T be the universal cover of $(G(-), D)$ and $G = \pi(G(-), D)$. Let $\mathbf{e} \in ET, \mathbf{u}, \mathbf{v} \in Fix_{VT}(G_{\mathbf{e}})$, and assume that $G_{\mathbf{u}}$ is finite. If $\alpha \in \mathrm{Aut}(G)$ centralizes $G_{\mathbf{u}}$, then some non-trivial power of α also centralizes $G_{\mathbf{v}}$.*

Proof. Let $N = N_G(G_{\mathbf{e}})$. If $\mathbf{y} \in Fix_{VT}(G_{\mathbf{e}})$, let $K(\mathbf{y})$ denote the connected component of $Fix_T(G_{\mathbf{e}})\backslash \mathbf{e}^N$ containing $\mathbf{y}$ so, by Lemma 5.7, $N_{\{K(\mathbf{u})\}} = N_{\mathbf{w}}$ for some $\mathbf{w} \in VK(\mathbf{u})$ and $N_{\{K(\mathbf{v})\}} = N_{\mathbf{z}}$ for some $\mathbf{z} \in VK(\mathbf{v})$. Let $T(\mathbf{e})$ be the N-tree whose vertices are the $K(\mathbf{y})$, $\mathbf{y} \in Fix_{VT}(G_{\mathbf{e}})$, and whose edges are the N-conjugates of $\mathbf{e}$ with the obvious incidence functions $\overline{\sigma}$ and $\overline{\tau}$ (that is, $\mathbf{f}^{\overline{\sigma}} = K(\mathbf{f}^\sigma)$ and $\mathbf{f}^{\overline{\tau}} = K(\mathbf{f}^\tau)$ for any $\mathbf{f} \in \mathbf{e}^N$). Of course, by Lemma 5.7, the stabilizer in N of each vertex of $T(\mathbf{e})$ is finite-by-cyclic.

If K is a connected component of $Fix_T(G_{\mathbf{e}})\backslash \mathbf{e}^N$ then for any $\mathbf{f} \in EK$, $N_G(G_{\mathbf{f}})$ is finite by the preceding lemma. If $\mathbf{p}$ and $\mathbf{q}$ are two vertices of K, then applying Lemma 5.5 to the ends of each edge in the geodesic from $\mathbf{p}$ to $\mathbf{q}$ in K, we see that if an automorphism of G centralizes $G_{\mathbf{p}}$, then some non-trivial power of that automorphism must centralize $G_{\mathbf{q}}$. The upshot of this is that we may assume that $G_{\mathbf{w}} \leq C_G(\alpha)$ (since $\mathbf{w} \in VK(\mathbf{u})$) and moreover, it will suffice to prove that a non-trivial power of α centralizes $G_{\mathbf{y}}$ for *some* $\mathbf{y} \in VK(\mathbf{v})$. By induction on the length of the $T(\mathbf{e})$-geodesic between $K(\mathbf{u})$ and $K(\mathbf{v})$, we may assume that $K(\mathbf{u})$ and $K(\mathbf{v})$ are the distinct ends of an edge $\mathbf{f} \in \mathbf{e}^N = ET(\mathbf{e})$ and, in fact, since $G_{\mathbf{f}} = G_{\mathbf{e}}$, it is no loss to assume that $\mathbf{f} = \mathbf{e}$. Note also that if a component K of $Fix_T(G_{\mathbf{e}})\backslash \mathbf{e}^N$ has at least one edge, then $G_{\mathbf{p}}$ is finite for every $\mathbf{p} \in VK$. (This is so because if $\mathbf{p} \in VK$ and $\mathbf{f} \in EK \subseteq Fix_{ET}(G_{\mathbf{e}})\backslash \mathbf{e}^N$ such that $\mathbf{f}$ is incident with $\mathbf{p}$, then $\mathbf{f} \notin \mathbf{e}^G$ and so, again by Lemma 5.8, $N_G(G_{\mathbf{f}})$ is finite. Since $Z(G_{\mathbf{p}}) \leq N_G(G_{\mathbf{f}})$ and $G_{\mathbf{p}}$ is finite-by-cyclic, it follows that $G_{\mathbf{p}}$ is finite.) In particular, because $G_{\mathbf{u}}$ is finite by hypothesis, $G_{\mathbf{w}}$ must also be finite.

Because N is edge-transitive on $T(\mathbf{e})$ and $\{e^{\overline{\sigma}}, e^{\overline{\tau}}\} = \{K(\mathbf{u}), K(\mathbf{v})\}$, N is either a free product $N_{\{K(\mathbf{u})\}} \underset{G_{\mathbf{e}}}{*} N_{\{K(\mathbf{v})\}} = N_{\mathbf{w}} \underset{G_{\mathbf{e}}}{*} N_{\mathbf{z}}$ or an HNN extension $N_{\{K(\mathbf{u})\}} \underset{G_{\mathbf{e}}}{*} t = N_{\mathbf{w}} \underset{G_{\mathbf{e}}}{*} t$ (which may also be regarded as a free product $N_{\mathbf{w}} \underset{G_{\mathbf{e}}}{*} [G_{\mathbf{e}}]\langle t\rangle)$. In either case, Proposition 2.5 implies that $C_{\mathrm{Aut}(N)}(N_{\mathbf{w}})$ is finite. Since N is α-invariant, it follows that some power of α centralizes N. Thus, we may assume that $N \leq C_G(\alpha)$.

Suppose that $G_{\mathbf{z}}$ is infinite whence, by the remarks above, $\mathbf{z} = \mathbf{v}$. Then $Z(G_{\mathbf{v}})$ is infinite so $G_{\mathbf{v}} = N_G(Z(G_{\mathbf{v}}))$ by Corollary 3.2. But because $G_{\mathbf{e}} \leq G_{\mathbf{v}}$, $Z(G_{\mathbf{v}}) \leq N \leq C_G(\alpha)$. It follows that $G_{\mathbf{v}}$ is α-invariant and so, since it is finite-by-cyclic, $G_{\mathbf{v}}$ is centralized by some non-trivial power of α. Hence, we shall assume that $G_{\mathbf{z}}$ is finite.

Now if N is transitive on $VT(\mathbf{e})$, then $\mathbf{u}^g \in VK(\mathbf{v})$ for some $g \in N$ and we have $G_{\mathbf{u}^g} = (G_{\mathbf{u}})^g \leq C_G(\alpha)$. Thus, we may also assume that $K(\mathbf{u})$ and $K(\mathbf{v})$ represent distinct N-orbits and so $N = N_{\mathbf{w}} \underset{G_{\mathbf{e}}}{*} N_{\mathbf{z}}$.

By Lemma 2.1, some power of α leaves invariant each conjugacy class of finite subgroups of G and hence, it is no loss of generality to assume that α itself has this property. Therefore, by Lemma 5.4, we may, in fact, assume that $(G_{\mathbf{z}})^\alpha = (G_{\mathbf{z}})^x$ for some $x \in C_G(N_{\mathbf{z}})$. But all edges of $T(\mathbf{e})$ are N-conjugate and so by Corollary 3.2, if $N_{\mathbf{z}} \neq G_{\mathbf{e}}$, then $C_G(N_{\mathbf{z}}) = C_N(N_{\mathbf{z}}) \leq N_{\mathbf{z}} \leq G_{\mathbf{z}}$. In this event, $G_{\mathbf{z}}$ is α-invariant and since $\mathrm{Aut}(G_{\mathbf{z}})$ is finite, some non-trivial power of α centralizes $G_{\mathbf{z}}$. Since $\mathbf{z} \in VK(\mathbf{v})$, the proof is complete in this case. Finally, if $N_{\mathbf{z}} = G_{\mathbf{e}}$, then $N = N_{\mathbf{w}}$ is finite and in this situation, Lemma 5.5 applies. □

Finally, we have reached our main objective.

Theorem 5.10. *Let $(G(-), D)$ be an irreducible finite graph of finite-by-cyclic groups with finite edge groups, and let $G = \pi(G(-), D)$. Then* $\mathrm{Out}(G)$ *is finite if and only if $(G(-), D)$ satisfies Hypotheses 5.1 and 5.2.*

Proof. One direction has already been settled, Hypothesis 5.1 being conclusion (iii) of Theorem 4.4 and Hypothesis 5.2 being a consequence of (iv). So we assume now that Hypotheses 5.1 and 5.2 hold and proceed to prove that $\mathrm{Out}(G)$ must be finite.

If all vertices of D are terminal, then either D consists of a single vertex (so $\mathrm{Aut}(G)$ is finite) or D is a segment, in which case Proposition 2.5 implies that

$\mathrm{Out}(G)$ is finite. Thus, we may assume that some vertex v of D is non-terminal whence, by Lemma 5.3 (i), $G(v)$ is finite.

Let M be a maximal subtree of D and take G to be $\pi(G(-), D, M)$. Let T be the corresponding universal cover of $(G(-), D)$ and let Y be the fundamental G-transversal consisting of all vertices of T of the form $[u, 1], u \in VD$, and all edges of the form $[e, 1], e \in ED$. Let $\widetilde{Y}$ be the subtree of T spanned by Y (so $E\widetilde{Y} = EY$ and $V\widetilde{Y}$ consists of the vertices of Y together with all vertices $[e, 1]^\tau = [e^\tau, e^{-1}]$, $e \in ED \backslash EM$.

By Lemma 2.1, there is a subgroup A of finite index in $\mathrm{Aut}(G)$ which contains $\mathrm{Inn}(G)$ and which maps each finite subgroup of G (and in particular, $G(v)$) to a conjugate of itself. Also, to prove that $\mathrm{Out}(G)$ is finite, it suffices to show that $C_A(G(v))$ is finite and in fact, by Lemma 2.4, it is enough to show that it is periodic. Thus, we let $\alpha \in C_A(G(v)) = C_A(G_{\mathbf{v}})$ where $\mathbf{v} = [v, 1]$, and proceed to prove that α has finite order.

We claim first that some power of α centralizes the stabilizer $G_{\mathbf{u}}$ of each vertex $\mathbf{u}$ of $\widetilde{Y}$. Since the edges of Y represent distinct G-orbits, it is clear that any non-terminal vertex of $\widetilde{Y}$ projects onto a non-terminal vertex of D and thus, all intermediate vertices on the geodesic from $\mathbf{v}$ to any other vertex of $\widetilde{Y}$ have finite stabilizers in G. Therefore, to prove the claim it suffices to show that if $\mathbf{w} \in VT$ such that $G_{\mathbf{w}}$ is finite, $\alpha \in C_A(G_{\mathbf{w}})$, and $\mathbf{e}$ is an edge of $\widetilde{Y}$ connecting $\mathbf{w}$ to some other vertex $\mathbf{u}$ of $\widetilde{Y}$, then some power of α centralizes $G_{\mathbf{u}}$. But this is precisely the upshot of Lemma 5.9, so the claim is proven.

Replacing α by an appropriate power of itself, we may assume now that α centralizes the stabilizer of each vertex $\mathbf{u}$ of $\widetilde{Y}$. If $e \in ED \backslash EM$ and $\mathbf{e} = [e, 1]$, let $\mathbf{w} = \mathbf{e}^\tau \in V\widetilde{Y} \backslash VY$. Considering e as an element of G, $\mathbf{w}^e = ([e, 1]^\tau)^e = [e^\tau, 1] \in VY$. If $x \in G_{\mathbf{w}} \leq C_G(\alpha)$, then $x^e \in G_{\mathbf{w}^e} \leq C_G(\alpha)$ and so $x^e = (x^e)^\alpha = x^{e^\alpha}$. Therefore, $e^\alpha = ce$ for some element c of $C_G(G_{\mathbf{w}})$. Note that the projection w of $\mathbf{w}$ is not terminal in D (because $\mathbf{w}^e \in VY$ and so either D is a loop or w is incident with both e and an edge of M). Thus, $G_{\mathbf{w}}$ is finite and so, by Lemma 5.3 (ii), $C_G(G_{\mathbf{w}}) = Z(G_{\mathbf{w}})$ is finite. It follows that $c \in C_G(\alpha)$ and c has finite order, say k, whence $e^{\alpha^k} = c^k e = e$.

It has now been shown that some positive power of α fixes each of the subgroups $G_{\mathbf{u}}$, $\mathbf{u} \in VY$, and also all $e \in ED \backslash EM$ (regarded as elements of G). But these elements generate G and hence, α has finite order. □

It is perhaps worth confirming that Hypotheses 5.1 and 5.2 are actually independent. For example, if $G = \langle x, u, v, t \colon x^2 = u^3 = v^3 = t^2 = 1, u^t = u^{-1}, v^t = v^{-1} \rangle$ (a free product of a cyclic group of order 2 with the free product of two copies of the symmetric group S_3 amalgamated over a subgroup of order 2), then Hypothesis 5.2 holds (since $N_G(\langle t \rangle) = \langle t \rangle$) but 5.1 does not. There is an automorphism of this group which maps x to x^{uv} and fixes u, v and t, and this induces an element of infinite order in $\mathrm{Out}(G)$. On the other hand, if $G = \langle x, y, u, v \colon x^6 = y^6 = u^3 = v^2 = 1, x^2 = u = y^2, u^v = u^{-1} \rangle = \langle x \rangle \underset{\langle u \rangle}{*} \langle u, v \rangle \underset{\langle u \rangle}{*} \langle y \rangle$, then Hypothesis 5.1 holds but 5.2 does not. Here, the automorphism which fixes x, y and u and maps v to v^{xy} corresponds to an element of infinite order in $\mathrm{Out}(G)$.

Although it can, of course, be proved much more directly, one obvious corollary of Theorem 5.10 is the following:

Corollary 5.11. *If G is an HNN extension of a finite group, then* $\mathrm{Out}(G)$ *is finite.*

To simplify the statements of Corollaries 5.12 and 5.14, we adopt the following *ad hoc* terminology: If $v \in VD$, the subgroups of $G(v)$ of the form $G(e)$ (where e is an outgoing edge) and $G(e)^e$ (where e is an incoming edge) will both be called v-incident edge groups corresponding to e.

Corollary 5.12. *Let $(G(-), D)$ be an irreducible finite graph of finite-by-cyclic groups in which all edge groups and non-terminal vertex groups are finite, and let $G = \pi(G(-), D)$. Suppose that for any vertex $u \in VD$, no u-incident edge group is conjugate in $G(u)$ to a subgroup of another u-incident edge group unless the two correspond to the same edge of D. Then* $\mathrm{Out}(G)$ *is finite.*

Proof. The argument used in Proposition 3.4 shows that under these hypotheses, no G-conjugate of one edge group of D is a subgroup of any other edge group. Thus, Hypothesis 5.2 holds vacuously. Also, if T is the universal cover of $(G(-), D)$, then for any edge $\mathbf{e} \in ET$, $Fix_{ET}(G_{\mathbf{e}})$ must consist only of conjugates of $\mathbf{e}$ and so, in fact, $Fix_{ET}(G_{\mathbf{e}}) = \mathbf{e}^N$ where $N = N_G(G_{\mathbf{e}})$. Therefore, each connected component of $Fix_{ET}(G_{\mathbf{e}})\backslash \mathbf{e}^N$ consists only of a single vertex. If $\mathbf{u} \in \{\mathbf{e}^\sigma, \mathbf{e}^\tau\}$, D_0 is the connected component of $D\backslash\{e\}$ containing u, and $H_0 = \pi(G(-)|_{D_0}, D_0)$, then by Lemma 5.6, $N_{H_0}(G_{\mathbf{e}}) \leq G_{\mathbf{u}}$. If $N_{H_0}(G_{\mathbf{e}})$ is not finite, then by hypothesis, u is terminal in D. Thus, Hypothesis 5.1 holds and the desired conclusion follows from Theorem 5.10. □

Corollary 5.13. *Any finite connected directed graph is the underlying graph of a graph of finite abelian groups whose fundamental group has finite outer automorphism group.*

Proof. Assign edge groups so that the groups corresponding to any two distinct edges are non-trivial abelian of coprime order. Let each vertex group be the direct product of the edge groups corresponding to edges with which the vertex is incident. Corollary 5.12 then applies. □

If $G = \langle x, y, z, t \colon x^3 = y^3 = z^3 = t^2 = 1, x^t = x^{-1}, y^t = y^{-1}, z^t = z^{-1}\rangle$, a generalized free product of three copies of the symmetric group S_3 with a subgroup $\langle t\rangle$ of order two amalgamated, then $N_G(\langle t\rangle) = \langle t\rangle$ and so, by Theorem 5.10, $\mathrm{Out}(G)$ is finite. Thus, the hypotheses of Corollary 5.12 are sufficient but not necessary for the finiteness of $\mathrm{Out}(G)$. However, for graphs of groups whose vertex groups are either finite nilpotent or infinite cyclic extensions of finite nilpotent groups, the converse of this corollary valid. Somewhat more generally, we have

Corollary 5.14. *Let $(G(-), D)$ be an irreducible finite graph of finite-by-cyclic groups with finite edge groups, and let $G = \pi(G(-), D)$. In addition, assume that for every $e \in ED$, $G(e)$ is subnormal in $G(e^\sigma)$ and $G(e)^e$ is subnormal in $G(e^\tau)$. Then* $\mathrm{Out}(G)$ *is finite if and only if the following conditions hold:*

(a) *All non-terminal vertex groups are finite and*
(b) *For any vertex $u \in VD$, no u-incident edge group is conjugate in $G(u)$ to a subgroup of another u-incident edge group unless the two correspond to the same edge of D.*

Proof. Of course, that the conditions are sufficient for the finiteness of $\mathrm{Out}(G)$ is just a special case of Corollary 5.12. Assume now that $\mathrm{Out}(G)$ is finite. By Theorem 4.4 and Lemma 5.3 (i), conclusion (a) holds, so it remains to prove (b).

Suppose that e and f are two (not necessarily adjacent) edges of D chosen such that $G(e)^g \leq G(f)$ for some $g \in G$ and, subject to this, $|G(f)\colon G(e)^g|$ is as large as possible. By Theorem 4.4, $N_G(G(f))$ is finite. In particular, if $R = G(f^\sigma)$ and $S = (G(f^\tau))^{f^{-1}}$ then $U = \langle N_R(G(f)), N_S(G(f))\rangle$ is finite. Therefore, $U \leq G(v)^x$ for some $v \in VD, x \in G$. Now $N_R(G(f)) \neq G(f)$ since otherwise, the subnormality hypothesis implies that $R = G(f)$, contradicting the irreducibility of $(G(-), D)$. Similarly, $N_S(G(f)) \neq G(f)$. The choice of the pair e and f then assures that neither $N_R(G(f))$ nor $N_S(G(f))$ is contained in a conjugate of any edge group. Since $N_R(G(f)) \leq G(f^\sigma) \cap G(v)^x$, it follows from Lemma 3.1 that $f^\sigma = v$ and $x \in G(f^\sigma)$. Similarly, since $N_S(G(f)) \leq G(f^\tau)^{f^{-1}} \cap G(v)^x$, Lemma 3.1 implies that $f^\tau = v$ and (regarding f as an element of G) $xf \in G(f^\tau)$. Therefore, $f^\sigma = f^\tau$ and $f = x^{-1}(xf) \in G(f^\sigma)$. The fact that $f^\sigma = f^\tau$ certainly implies that f^σ is not terminal in D, whence $G(f^\sigma)$ is finite. But it also means that f does not belong to any maximal subtree of D and so, as an element of G, f has infinite order. Thus, we have a contradiction and so, no such pair of edges e and f exists. Therefore, (b) holds and the corollary is proved. □

We note that the property of having only a finite number of outer automorphisms is inherited by certain subgroups of a virtually free group.

Corollary 5.15. *Let $(G(-), D)$ be an irreducible finite graph of finite-by-cyclic groups with finite edge groups and with fundamental group G. Let B be a connected subgraph of D and $H = \pi(G(-)|_B, B)$. If $\mathrm{Out}(G)$ is finite, then $\mathrm{Out}(H)$ is finite.*

Proof. If $(G(-), D)$ is irreducible, so is $(G(-)|_B, B)$, If $(G(-), D)$ satisfies Hypotheses 5.1 and 5.2, so does $(G(-)|_B, B)$. □

To conclude, we record the more geometric formulation of the main theorem described in the Introduction.

Theorem 5.16. *Let G be a group and suppose that T is a G-finite G-tree such that all edge stabilizers are finite and all vertex stabilizers are finite-by-cyclic. Assume that for any edge $\mathbf{e}$ of T with incident vertices $\mathbf{u}$ and $\mathbf{v}$, if $G_\mathbf{e} = G^0_\mathbf{u}$, then $\mathbf{u}$ and $\mathbf{v}$ are G-conjugate and $G_\mathbf{u}$ is infinite. Then $\mathrm{Out}(G)$ is finite if and only if the following three conditions hold:*

- (i) *If $\mathbf{v}$ is a vertex of T whose projection in T/G is non-terminal, then the stabilizer in G of $\mathbf{v}$ is finite.*
- (ii) *If $\mathbf{e}$ is any edge of T and $N = N_G(G_\mathbf{e})$, then each connected component of $Fix_T(G_\mathbf{e})\backslash\mathbf{e}^N$ is finite.*
- (iii) *If $\mathbf{e}$ is any edge of T such that $G_\mathbf{e} = G_\mathbf{f}$ for some edge $\mathbf{f}$ which is not G-conjugate to $\mathbf{e}$, then $Fix_T(G_\mathbf{e})$ is finite.*

Proof. The second sentence of the theorem simply assures the irreducibility of the associated graph of groups as defined in Section 4. We have seen in Lemmas 5.3 (i), 5.7 and 5.8 that Hypotheses 5.1 and 5.2 imply statements (i), (ii) and (iii) of Theorem 5.16, and so it only remains to prove the converse. We show that (i) and (ii) imply Hypothesis 5.1 and that (iii) implies Hypothesis 5.2.

Let $\mathbf{e}$ be any edge in the universal cover T with $N = N_G(G_\mathbf{e})$ and suppose that $\mathbf{u}$ is a vertex of T incident with $\mathbf{e}$ so $\mathbf{u} \in Fix_T(G_\mathbf{e})$. By Lemma 5.6, if K is the component of $Fix_T(G_\mathbf{e})\backslash e^N$ containing $\mathbf{u}$, then $N_{H_0}(G_\mathbf{e}) = N_{\{K\}}$. Also, $N_u \leq N_{\{K\}}$. By statement (ii) above, K is finite and, in particular, it contains only finitely many N-conjugates of $\mathbf{u}$. Thus, $|N_{\{K\}}\colon N_{\mathbf{u}}| < \infty$. But if $\mathbf{u}$ is non-terminal in T, statement (i) says that $G_{\mathbf{u}}$ (and hence, $N_{\mathbf{u}}$) is finite. Therefore, $N_{H_0}(G_{\mathbf{e}}) = N_{\{K\}}$ is finite and it follows that Hypothesis 5.1 holds.

Suppose that $\mathbf{e}$ and $\mathbf{f}$ are non-conjugate vertices of T such that $G_{\mathbf{e}} = G_{\mathbf{f}}$. Assuming statement (iii) of Theorem 5.16, $Fix_T(G_{\mathbf{e}})$ is finite and so $|N_G(G_{\mathbf{e}})\colon G_{\mathbf{e}}|$, the number of $N_G(G_{\mathbf{e}})$-conjugates of $\mathbf{e}$, is finite. It follows that $N_G(G_{\mathbf{e}})$ is finite, whence Hypothesis 5.2 holds. $\square$

To confirm a remark made in the Introduction, consider the group $G = (H_1 \times Z_1) \underset{U}{*} (H_2 \times Z_2)$, where Z_1 and Z_2 are infinite cyclic, H_1 and H_2 are finite, and the amalgamated subgroup U is equal to its own normalizer in both H_1 and H_2. Then $\mathrm{Out}(G)$ is finite but, because $N_G(U) = (U \times Z_1) \underset{U}{*} (U \times Z_2) = U \times (Z_1 * Z_2)$, $\mathrm{Out}(N_G(U))$ is not.

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