THE TRANSFER AND SYMPLECTIC COBORDISM

MALKHAZ BAKURADZE

Abstract. The main result of this paper is the nilpotency formula $\phi_i^4 = 0$, $\forall i \geq 1$ for N. Ray classes $\phi_i$ in the torsion of the symplectic bordism ring $\text{MSp}_\ast$.

INTRODUCTION

This paper is organised as follows. Section 1 is devoted to calculation of the transfer homomorphism in the symplectic cobordism theory [D], [BG]. In particular, using the results of [BM], [Fe], [Sn] we calculate the transfer homomorphism for projective bundles associated with universal $\text{Spin}(m)$ bundles, $m = 3, 4, 5$. This section includes the following corollary in the case $m = 3$:

Let $N$ be the normalizer of the torus $U(1)$ in $\text{Sp}(1)$; $\zeta \rightarrow B\text{Sp}(1)$ be the universal $\text{Sp}(1)$ bundle and $\Lambda$ be the universal $\text{Spin}(3)$ bundle over $B\text{Spin}(3) = B\text{Sp}(1)$. Then the bundle $p : BN \rightarrow B\text{Sp}(1)$ is the projective bundle associated with $\Lambda$. Let

$x = pf_1(\zeta)$;

$y = pf_1(p^*(\zeta))$;

$e = pf_3(p^*(\Lambda \otimes R H))$

be the Conner-Floyd symplectic Pontryagin classes and

$\tau^*_p : \text{MSp}^*(BN) \rightarrow \text{MSp}^*(B\text{Sp}(1))$

be the transfer homomorphism. Then $\tau^*_p$ satisfies the relations

(1) $\tau^*_p(1) = 1$;

(2) $\tau^*_p(e) = 0$.

In Section 2 we establish a connection of the Euler class $e$ with the classes $\phi_i$ defined as follows:

Recall from [R] the classes $\theta_i$ arising from the expansion

$pf_1((\eta - R) \otimes_R (\zeta - H)) = s \sum_{i \geq 1} \theta_i pf_1(\zeta) = s \sum_{i \geq 1} \theta_i x^i$

in $\text{MSp}^i(S^1 \wedge B\text{Sp}(1))$, where $s$ is the generator of $\text{MSp}^i(S^1)$, $\eta \rightarrow S^1$ is the non-trivial real line bundle and $\zeta$ is as above. Also recall the relabelling $\theta_{2i} = \phi_i$ in...
MSp_{8n-3}, and from [Ro] that $\theta_{2i-1} = 0$ for $i > 1$. As proved in [R], each $\phi_i$ is an indecomposable torsion element of order 2.

It is shown in [Na] that the homomorphism $\pi^*$ induced by $\pi : BU(1) \to BSp(1)$ is not a monomorphism in the symplectic cobordism theory. In particular (see Section 2)

$$\pi^*(\theta_1 x + \sum_{i \geq 1} \phi_i x^{2i}) = 0.$$ 

Using this observation and the results of [G], [GR], we state that in $MSp^*(BN)$

$$e = \sum_{i \geq 1} \phi_i^4 y^{8i} (1 + \sum_{j \geq 1} \alpha_j y^j)$$

for some coefficients $\alpha_j \in MSp_*$.

Applying (1), (2), (3) we have

$$\tau^*_p(e) = 0$$

by (2),

$$= \tau^*_p(\sum_{i \geq 1} \phi_i^4 y^{8i} (1 + \sum_{j \geq 1} \alpha_j y^j))$$

by (3),

$$= \sum_{i \geq 1} \phi_i^4 x^{8i} (1 + \sum_{j \geq 1} \alpha_j x^j) \tau^*_p(1)$$

by the transfer property,

$$= \sum_{i \geq 1} \phi_i^4 x^{8i} (1 + \sum_{j \geq 1} \alpha_j x^j)$$

by (1).

Thus we obtain

$$\sum_{i \geq 1} \phi_i^4 x^{8i} = 0$$

in $MSp^*(BSp(1)) = MSp_*[[x]]$.

This proves

**Theorem.** $\phi_i^4 = 0, \forall i \geq 1$.

We cannot use a reasoning similar to that of Section 2 for the self-conjugate cobordism, since in this theory it is impossible to construct characteristic classes with the required properties. Namely, as proved in [BaNa], for arbitrary natural classes

$$P_i(\xi^n) \in SC^{2i}(X)$$

in the self-conjugate cobordism theory

$$P(\xi^n) = 1 + P_1(\xi^n) + ... + P_n(\xi^n),$$

where $\xi^n \to X$ is the SC-vector bundle, the following conditions are contradictory:

1. $P_1(\xi^n)$ is the Euler class (normalization);
2. $P(\xi^n + \xi^m) = P(\xi^n) P(\xi^m)$ (the Whitney formula).

That is why in Section 3 we calculate the transfer homomorphism for the bundle of flags of the bundle $\Lambda$. As a corollary we obtain a new proof of the nilpotency...
formula for the N. Ray classes in the self-conjugate cobordism, which was proved for the first time in [Na].

As is known from [Mo] and [V], various three-fold products of N. Ray’s family are nontrivial. In Section 4 we shall prove

**Proposition 4.1.** All four-fold products of the N. Ray classes are zero, and the images of double products of these classes in self-conjugate cobordism are zero.

I would especially like to thank Professor Mark Mahowald for his encouragement and advice. My thanks are also due to the referee for many useful suggestions.

1. Calculation with Transfer

The result of this section is

**Proposition 1.** Let \( G_m = \text{Spin}(m) \) and \( \xi^m \to BG_m \) be the universal \( \text{Spin}(m) \) bundle, \( m = 3, 4, 5 \). Let \( p_m : P(\xi^m) \to BG_m \) be the associated projective bundle with fibre \( \mathbb{R}P^{m-1} \), and \( \lambda_m \to P(\xi^m) \) be the canonical real line bundle. Then the transfer homomorphism \( \tau_m^*: MSp^*(P(\xi^m)) \to MSp^*(BG_m) \) satisfies the relations

1. \( \tau_m^*(c_m^n) = 0 \), for all \( n \geq 1 \), where \( c_m = pf_1(\lambda_m \otimes_R H) \) is the first Conner-Floyd symplectic Pontryagin class;
2. \( \tau_m^*(1) = \chi(\mathbb{R}P^{m-1}) \), where \( \chi(\mathbb{R}P^{m-1}) \) is the Euler characteristic of \( \mathbb{R}P^{m-1} \) and hence is equal to 1 if \( m = 3, 5 \), and to 0 if \( m = 4 \);
3. \( \tau_m^*(e_m) = 0 \),

where \( e_m = e(p_m^*(\xi^m \otimes_R H)) \) is the Euler class.

For the proof we need the following facts.

1.4. \( \text{Spin}(m) \) bundles. It is well known that the groups

\( \text{Spin}(2), \text{Spin}(3), \text{Spin}(4), \text{Spin}(5), \text{Spin}(6) \)

are isomorphic to

\( S^1 = U(1), S^3 = Sp(1) = SU(2), Sp(1)^2, Sp(2), SU(4) \).

The inclusions \( \text{Spin}(i) \to \text{Spin}(i + 1) \) up to an isomorphism are described as follows:

- \( \text{Spin}(2) \to \text{Spin}(3) \) is the standard \( U(1) \to Sp(1) \);
- \( \text{Spin}(3) \to \text{Spin}(4) \) is the diagonal homomorphism \( Sp(1) \to Sp(1)^2 \);
- \( \text{Spin}(4) \to \text{Spin}(5) \) is the embedding \( Sp(1)^2 \to Sp(2) \) of diagonal matrices.
- \( \text{Spin}(5) \to \text{Spin}(6) \) is the embedding of matrices \( A \) for which \( A^TJA = J \), where

\[
J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.
\]

Denote \( \text{Spin}(m) \) by \( G_m \) and consider \( N_m \), the normalizer of \( G_m \) in \( G_{m+1} \). Then
$N_2$ consists of $U(1)$ and $jU(1)$, where $j$ is the quaternionic unit;
$N_3$ consists of matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, $a$ is the quaternion, $aa = 1$;
$N_4$ consists of matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$, where $a$ and $b$ are quaternions, $aa = b\bar{b} = 1$.

The universal $Spin(m)$ bundles $\xi^m$ have the following description.

Case $m = 5$. $G_5 = Spin(5) = Sp(2)$ acts by conjugation on the 5-dimensional vector space of $2 \times 2$ quaternionic Hermitian matrices of zero trace. These matrices are of the form $\begin{pmatrix} a_0 & b \\ b & -a_0 \end{pmatrix}$, where $a_0$ is real, $b$ is a quaternion and $(b, a_0) \in R^5$.

Let $E \to BG_5$ be the principal $Spin(5)$ bundle. Then the above action of $G_5$ on $R^5$ defines the sphere bundle of $\xi^5$,

$$BG_4 = E \times_{G_5} S^4 \to BG_5;$$
and the projective bundle of $\xi^5$,

$$BN_4 = E \times_{G_5} RP^4 \to BG_5.$$

Case $m = 4$. The universal $Spin(4)$ bundle $\xi^4$ is

$$\zeta_1 \otimes_H \zeta_2^* \to BS\text{p}(1)^2,$$
where $\zeta_1, \zeta_2$ are the canonical symplectic line bundles, $\zeta_2^*$ is the symplectic conjugate of $\zeta_2$ and $(q_1, q_2) \in S\text{p}(1)^2 = G_4$ acts on $R^4 \cong H$ by $v \to q_1 v q_2^{-1}$.

This defines the sphere bundle and the projective bundle of $\xi^4$:

$$BG_3 = E \times_{G_4} S^3 \to BG_4,$$

$$BN_3 = E \times_{G_4} RP^3 \to BG_4.$$

Case $m = 3$. The universal $Spin(3)$ bundle $\xi^3$ is

$$\Lambda \to BS\text{p}(1),$$
where $1 + \Lambda = \zeta \otimes_H \zeta^*$. $G_3 = Sp(1)$ acts on $R^3$ as conjugation on the pure quaternion.

This defines the sphere bundle and the projective bundle of $\xi^3$:

$$BG_2 = E \times_{G_3} S^2 \to BG_3,$$

$$BN_2 = E \times_{G_3} RP^2 \to BG_3.$$

Consider now the standard inclusion $RP^3 \to RP^4$. This is $G_4$ equivariant, where $G_4$ acts on $RP^3$ as above and on $RP^4$ as a subgroup of $G_5$.

This defines the inclusion of the projective bundle $P(\xi^4)$ in $P(\xi^4 + 1)$:

$$l : BN_3 = P(\xi^4) = E \times_{G_4} RP^3 \to E \times_{G_4} RP^4 = P(\xi^4 + 1).$$

The inclusion $RP^2 \to RP^3$, induced by the embedding of the pure quaternions into $H \cong R^4$ is $G_3$-equivariant. Here $G_3$ acts on $RP^2$ as above and on $RP^3$ as a subgroup of $G_4$.

This defines the inclusion

$$m : BN_2 = P(\xi^3) = E \times_{G_3} RP^2 \to E \times_{G_3} RP^3 = P(1 + \xi^3).$$
Let
\[ \lambda_3 \to P(\xi^3), \quad \lambda_4 \to P(\xi^4), \]
\[ \tilde{\lambda}_4 \to P(1 + \xi^3), \quad \tilde{\lambda}_5 \to P(\xi^4 + 1) \]
be the canonical real line bundles. Then it is easy to see

**Lemma 1.5.** \( l^i(\lambda_4) = \lambda_3, \quad m^i(\tilde{\lambda}_5) = \lambda_4. \)

**1.6. Double coset formula.** Let \( G \) be a compact Lie group and \( H \) and \( K \) closed subgroups.

Recall that the bundle \( \rho(H, G) : BH \to BG \) has the fibre \( G/H \) and structure group \( G \). Consider the pullback of \( BH \) to \( BK \),

\[
\begin{array}{ccc}
\Gamma = \times_K (G/K) & \to & BH \\
\downarrow \gamma & & \downarrow \rho(G,H) \\
BK & \to & BG
\end{array}
\]

\( \gamma : \Gamma \to BK \) has the fibre \( G/H \) and structure group \( K \).

Let \( \tau^*(H, G) \) be the transfer homomorphism associated to \( \rho(H, G) \). Then there is a formula for calculation of \( \rho^*(K, G)\tau^*(H, G) \).

**Theorem [Fe].** Let \( \gamma : \Gamma \to BK \) be the fibre bundle with fibre \( F = G/H \) and structure group \( K \) acting on the left on \( F \). Let \( \{M\} \) be the set of orbit-type manifold components of the orbit space \( K|F \), and let \( q \) be any \( K \)-orbit in \( M \). Let \( \tilde{q} \) be the subbundle of \( \gamma \) corresponding to \( q \). Let \( k : \tilde{q} \to \gamma \) be the inclusion and \( \chi^*(M) = \chi(M - M) \). Then

\[ \tau^*_\gamma = \sum \chi^*(M)\tau^*_\tilde{q} k^*, \]

where the sum is over all the orbit-type manifold components \( \{M\} \).

**1.7. Calculations with transfer for sphere bundles.** For the proof of Proposition 1 we need the following

**Lemma 1.8.** Let \( G_m = \text{Spin}(m) \) and \( \rho(G_{m-1}, G_m) : BG_{m-1} \to BG_m \) be the sphere bundle of the universal \( \text{Spin}(m) \) bundle \( \xi^m \). Then

\[ \tau^*(G_{m-1}, G_m)(1) = \chi(S^{m-1}), \quad m = 3, 4, 5, \]

in symplectic cobordism. Here \( \chi(S^{m-1}) \) is the Euler characteristic and because of this is equal to 2 if \( m = 3, 5 \), and 0 if \( m = 4 \).

**Proof.** Case \( m = 4 \). For the diagonal map \( \rho = \rho(\text{Sp}(1), \text{Sp}(1)^2) \) we have \( \rho(x_1) = \rho(x_2) \), where \( x_i = c_{f_1}(\zeta_i); \zeta_1, \zeta_2 \) are the canonical line symplectic bundles.

By the transfer property for \( \tau^* = \tau^*(\text{Sp}(1), \text{Sp}(1)^2) \) we have

\[ \tau^*(a)(x_1 - x_2) = \tau^*(\rho^*(x_1 - x_2)a) = 0 \]

\( \forall a \in MS^*(B\text{Sp}(1)) \). Since \( MS^*(B\text{Sp}(1)^2) = MS^*[[x_1, x_2]] \), this proves that \( \tau^* \) is the trivial homomorphism.

Case \( m = 3 \). Using the double coset formula for \( \rho^*\tau^* \), we see that the double coset space \( \text{Sp}(1)|\text{Sp}(1)^2|\text{Sp}(1) \) is the line segment, with isotropy group \( \text{Sp}(1) \) at
the endpoints and conjugate group of \( U(1) \) in \( Sp(1) \) in the interior. Taking into account the case \( m = 4 \), we have

\[
0 = \tau^*(G_3, G_4)(1) = 2\tau^*(G_3, G_3)(1) - \tau^*(U(1), G_3)(1) = 2 - \tau^*(U(1), G_3)(1).
\]

Since \( \rho(U(1), G_3) \) is the sphere bundle of \( \xi^3 \), this proves the case \( m = 3 \).

Case \( m = 5 \). The sphere bundle of \( \xi^5 \) agrees with \( \rho(Sp(1)^2, Sp(2)) \). On the other hand this bundle is the quaternionic projective bundle associated to the universal symplectic plane bundle, and the statement is known from [D, p.235]. One may prove this case by the method we will use in the following section.

**Proof of Propositions 1.1 and 1.2. Case** \( m = 5 \). It is shown in [Sn, ch.1] that the following diagram of the stable maps is commutative (see also Remark 1.11):

\[
BG_4 = BSp(1)^2 \quad \longrightarrow \quad BSp(2) = BG_5
\]

\[
\downarrow \quad \nearrow \tau
\]

\[
BZ_2 \wr Sp(1) = BN_4
\]

where \( BSp(1)^2 \to BSp(2) \) is induced by \( \rho(Sp(1)^2, Sp(2)) \) and \( BSp(1)^2 \to BZ_2 \wr Sp(1) \) by \( \rho(Sp(1)^2, Z_2 \wr Sp(1)) \).

Since, as it is well known, \( \rho^*(Sp(1)^2, Sp(2)) \) is a monomorphism, this proves the case \( m = 5 \).

**Proof of Propositions 1.1 and 1.2. Case** \( m = 4 \). The following lemma immediately follows from the definitions of 1.4.

**Lemma 1.9.** The double coset space \( G_4|G_5|N_4 \) is a line segment. One endpoint corresponds to an orbit consisting of one point \( (0, \pm 1) \in RP^4 \), where

\[
G_4|N_4 = RP^4 = \{ (v, w) | v \text{ is a quaternion}, w \text{ is a real}, vv^* + w^2 = 1 \}.
\]

The point \( (0, \pm 1) \) is a fixed point. The other endpoint corresponds to \( RP^5 \), consisting of points \( (\pm v, 0) \in RP^4 \). The isotropy groups for these points are conjugate groups of \( N_3 \) in \( G_4 \). The open interval corresponds to orbits \( S^3 \) consisting of points \( \pm(v, w), 0 < vv^* < 1 \). The isotropy groups for these points are conjugate groups of \( G_3 \) in \( G_4 \).

**Proof.** For the point \( (0, \pm 1) \in RP^4 \) the isotropy group is obviously \( Sp(1)^2 \). For the points \( (\pm v, 0) \in RP^4 \), the isotropy group \( K_v \) for the given \( (\pm v, 0) \) consists of elements \( (vqv^{-1}, q) \) and \( (-vqv^{-1}, q) \) from the group \( Sp(1)^2 \). Hence

\[
K_v = gN_3g^{-1}, \quad g = (v, 1).
\]

For the points \( (\pm v, \pm w) \in RP^4, 0 < vv^* < 1 \), we have

\[
(q_1, q_2)(\pm v, \pm w) = (\pm q_1vq_2^{-1}, \pm w);
\]

\[
v = q_1vq_2^{-1};
\]

\[
q_1 = vq_2v^{-1}.
\]

So for the given \( (\pm v, \pm w) \) the isotropy group is the conjugate group of \( Sp(1) \) in \( Sp(1)^2 \).
Combining Lemma 1.9 and the double coset formula for $\rho^*(G_4, G_5)T(N_4, G_5)$, we have

$$\rho^*(G_4, G_5)\tau^*(N_4, G_5)(1) = 1 - \tau^*(G_3, G_4)(1) + \tau^*(N_3, G_4)(1).$$

Since $\tau^*(N_4, G_5)(1) = 1$ and $\tau^*(G_3, G_4)(1) = 0$, this proves $\tau^*(N_3, G_4)(1) = 0$.

Consider now $\rho^*(G_4, G_5)\tau^*(N_4, G_5)(c_5^5)$. Again using the double coset formula above, this is decomposed into three summands. Of these, the two summands corresponding to the subbundles identity $BG_4 \rightarrow BG_4$ and $BG_3 \rightarrow BG_4$ are zero since there are no nontrivial real line bundles over $BG_3$ and $BG_4$. As for the third summand, it coincides with $\tau^*(N_3, G_4)(c_4^a)$ by Lemma 1.5.

Hence we have

$$\rho^*(G_4, G_5)\tau^*(N_4, G_5)(c_5^5) = 0$$

by the case $m = 5$,

$$= 0 - 0 + \tau^*(N_3, G_4)(c_4^a).$$

This proves the case $m = 4$.

**Proof of Propositions 1.1, 1.2.** Case $m = 3$. Consider now the double coset formula for $\rho^*(G_3, G_4)\tau^*(N_3, G_4)$.

Recall, from 1.4, that the homogeneous space $G_4/N_3$ is the projective space $RP^3 = \{\pm h, hh^* = 1, h \in H\}$.

It is easy to see the following.

**Lemma 1.10.** The double coset space $G_4|G_4\backslash N_3$ is a line segment. One endpoint corresponds to an orbit consisting of one point $(\pm 1) \in RP^3$. This point is fixed. The other endpoint corresponds to $RP^2$, consisting of points $\{\pm h, h$ pure quaternion, $hh^* = 1\}$. The isotropy group for the given $(\pm h)$ is the conjugate group of $N_2$ in $Sp(1)$. The open interval corresponds to orbits $S^2$, consisting of points $(\pm h)$, whose real parts differ from 0 and $\pm 1$. The isotropy groups for these points are the conjugate groups of $U(1)$ in $Sp(1)$.

Using now the double coset formula, we obtain

$$0 = \rho^*(G_3, G_4)\tau^*(N_3, G_4)(1)$$

by the case $m = 4$,

$$= 1 - \tau^*(U(1), Sp(1))(1) + \tau^*(N_2, Sp(1))(1)$$

by Lemma 1.10,

$$= 1 - 2 + \tau^*(N_2, Sp(1))(1)$$

by Lemma 1.8.

This proves that $\tau^*(N_2, Sp(1))(1) = 1$.

In the same spirit we obtain

$$0 = \rho^*(G_3, G_4)\tau^*(N_3, G_4)(c_4^a)$$

by the case $m = 4$,

$$= 0 - 0 + \tau^*(N_2, Sp(1))(c_3^a)$$

by Lemma 1.10 and Lemma 1.5.

This proves the case $m = 3$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Proof of Proposition 1.3. For \( m = 3 \) formula (1.3) coincides with (2) from the Introduction, which is the case we need to prove.

The projectivisation \( p : BN \rightarrow BSp(1) \) of the bundle \( \Lambda = \xi^3 \) defines the canonical splitting over \( BN \)

\[
p^*(\Lambda) = \mu + \lambda,
\]

where \( \mu \) and \( \lambda \) are a plane and a linear real bundle respectively.

Then we have the splitting

\[
p^*(\Lambda \otimes_R H) = \mu \otimes_R H + \lambda \otimes_R H.
\]

Apply now the Whitney formula to express the symplectic characteristic classes of the bundle \( p^*(\Lambda \otimes_R H) \) in terms of the classes \( \mu \otimes_R H \) and \( \lambda \otimes_R H \). We obtain the equations

\[
p f_1(p^*(\Lambda \otimes_R H)) = pf_1(\mu \otimes_R H) + pf_1(\lambda \otimes_R H);
\]

\[
p f_2(p^*(\Lambda \otimes_R H)) = pf_2(\mu \otimes_R H) + pf_1(\mu \otimes_R H)pf_1(\lambda \otimes_R H);
\]

\[
e = pf_3(p^*(\Lambda \otimes_R H)) = pf_2(\mu \otimes_R H)pf_1(\lambda \otimes_R H).
\]

Let \( c = pf_1(\lambda \otimes_R H) \). Then the above equations give an exposition of \( e \) in terms of \( c \) and \( pf_i(p^*(\Lambda \otimes_R H)) \), \( i = 1, 2 \):

\[
e = pf_2(p^*(\mu \otimes_R H))c = [pf_2(p^*(\Lambda \otimes_R H)) - pf_1(\mu \otimes_R H)c]c
\]

\[
e = pf_2(p^*(\Lambda \otimes_R H))c - [pf_1(p^*(\Lambda \otimes_R H)) - c]c^2
\]

\[
e = pf_2(p^*(\Lambda \otimes_R H))c - pf_1(p^*(\Lambda \otimes_R H))c^2 + c^2.
\]

Now apply the transfer homomorphism \( \tau^*_p \) to this equation:

\[
\tau^*_p(e) = \tau^*_p[p^*(pf_2(\Lambda \otimes_R H))]c - \tau^*_p[p^*(pf_1(\Lambda \otimes_R H))c^2] + \tau^*_p(c^3).
\]

Taking into account the transfer property \( \tau^*_p(p^*(t)) = tr^*_p(1) \), we obtain

\[
\tau^*_p(e) = pf_2(\Lambda \otimes_R H)\tau^*_p(c) - pf_1(\Lambda \otimes_R H)\tau^*_p(c^2) + \tau^*_p(c^3).
\]

But by virtue of Proposition 1.2 we have \( \tau^*_p(c) = \tau^*_p(c^2) = \tau^*_p(c^3) = 0 \). Therefore \( \tau^*_p(e) = 0 \).

The proofs of the cases \( m = 4, 5 \) are quite analogous. However the case \( m = 4 \) also follows from Proposition 1.1, namely, from the equality \( \tau^*_p(1) = 0 \):

\[
\tau^*_p(e_4) = \tau^*_p(p^*(pf_4(\xi^4 \otimes_R H))) = pf_4(\xi^4 \otimes_R H)\tau^*_p(1) = pf_4(\xi^4 \otimes_R H) = 0.
\]

Then, as proved in [GR], every \( Spin(5) \) bundle and, in particular, \( \xi^4 \), is \( MSp \)-orientable and has zero Euler class. Thus \( pf_5(\xi^5 \otimes_R H) = 0 \), so we have nothing to prove in the case \( m = 5 \).
1.11. Remark on Propositions 1.1 and 1.2. Case $m = 5$. The commutativity of above diagram is stated by the method of equivariant vector fields on the homogeneous spaces [BM]. Namely there is [Sn, Example 1.13] an $Sp(1)^2$ equivariant vector field on $Sp(2)/Z_2 \wr Sp(1)$ with one singular point. Using this field we shall see here that in the case of the projective bundle $P(\xi^4 + 1)$ the transfer map is stably homotopic to the section of this bundle defined by the direct summand $1$.

We need a simple particular case of [BM, Corollary 2.11]. Namely let $\pi : E \to B$ be the fiber bundle with fiber $F$. Suppose that $F$ admits a $G$ equivariant vector field with one singular point (fixed under the action $\chi$) and the Euler characteristic $\chi(F) = 1$. This fixed point obviously defines a section $i : B \to E$. Then $i$ suspends to the transfer map $\tau(\pi)$, that is, $i^+ = \tau(\pi)$ in the track group $\{B^+, E^+\}$.

Taking into account Lemma 1.9, we see that the projective bundle $P(\xi^4 + 1)$, that is, the pullback of $BN_4 \to BG_5$ to $BG_4$, has section defined by the fixed point $(0, \pm 1) \in RP^4$ under the action of $G_4$. This section agrees with the section of $P(\xi^4 + 1)$ defined by the direct summand $1$.

Lemma 1.12. The above section of the projective bundle $P(\xi^4 + 1)$ suspends to the transfer map.

Proof. Following [BM] we construct a $G_4 = Sp(1)^2$ equivariant vector field on $RP^4 = G_5/N_4$ with one zero point. It is easy to see that

$$G_5/N_4 = GL_2(H)/Z_2 \wr B(H)$$

where $H$ is the quaternions, $GL_2(H)$ is the full linear group of $2 \times 2$ matrices, $B(H)$ are the all upper triangular matrices and the generator of $Z_2$ is \(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\).

This follows from the fact, that $GL_2(H)$ acts on $G_5/N_4 = S^4$, that is, on the manifold of flags $F_1 \subset F_2 = H^2$, with the isotropy group $B(H)$.

Now let $v$ be a vector from the Lie algebra of $GL_2(H)$, for which

$$\omega = exp(v) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

where $x, y$ are real numbers and $x \neq y$.

Consider now the field $\varphi_v$ on $GL_2(H)$ defined by the right translations:

$$\varphi_v(g) = dR_g(v), \quad g \in GL_2(H).$$

This field induces the field $g_v$ on $GL_2(H)/Z_2 \wr B(H)$. The field $g_v$ is $Sp(1)^2$ equivariant, since $Sp(1)$ is a subgroup of the centralizer of $\omega$. For the zero points of $g_v$ note that coset of $g$ is the zero point if and only if $g^{-1}\omega g \in Z_2 \wr B(H)$, that is, $g \in Z_2 \wr B(H)$. Thus $g_v$ has one zero point. This proves Lemma 1.12.

The above lemma proves the analog of Proposition 1 for the projective bundle $P(\xi^4 + 1) \to BSp(1)^2$. But since this bundle is the pullback of $P(\xi^5) \to BG_5$ to $BG_4$ and the homomorphism induced by $BSp(1)^2 = BG_4 \to BG_5 = BSp(2)$ is a monomorphism, this proves the case $m = 5$.

2. Proof of (3) from the introduction

We need the following fact.
Proposition [Na]. In $\text{MSp}^*(\text{BU}(1))$
\[ \theta_1 z + \sum_{k \geq 1} \phi_k z^{2k} = 0, \]
where $z = pf_1(\xi + \bar{\xi})$; $\xi$ is the canonical complex line bundle; $\theta_1, \phi_i$ are the Ray classes.

This follows immediately from the bundle relation
\[ \eta \otimes R (\xi + \bar{\xi}) = \xi + \bar{\xi} \]
in $KSp^0(S^1 \times \text{BU}(1))$ and from the definition of Ray classes.

Then, as it is known, any Spin(4) bundle is $\text{MSp}^*$ orientable. This follows from the isomorphism $K\Omega^4 = KSp^0$. For the given $KO$ orientation class of Spin(4) bundle this isomorphism determines the symplectic bundle over the corresponding Thom space, and the first Conner-Floyd symplectic Pontryagin class of this symplectic bundle will be taken as the symplectic orientation class. So the Spin(4) bundle $\zeta \otimes H \zeta^* = 1 + \Lambda$, and because of this $\Lambda$ is $\text{MSp}^*$ orientable [RS].

By using these results and the fact that the bundle $\text{BU}(1) \to \text{BSp}(1)$ is the sphere bundle of $\Lambda$ it is proved in [G] that the Thom class of the bundle $\Lambda$ can be chosen in such a way that its restriction to the zero section $\tilde{c}(\Lambda)$ has the form
\[ \tilde{c}(\Lambda) = \theta_1 x + \sum_{i \geq 1} \phi_i x^{2i}, \]
where $x = pf_1(\zeta)$. For another proof, see [GR].

Since $2\theta_1 = 2\phi_1 = 0$ [R] and $\theta_1^2 = 0$ [G], we obtain
\[ \sum_{i \geq 1} \phi_i^4 x^{8i} = (\tilde{c}(\Lambda))^4 = \tilde{c}(\Lambda \otimes R H) \]
But $\tilde{c}(\Lambda \otimes R H)$ agrees with the ordinary Euler class $e(\Lambda \otimes R H)$ up to multiplication by a unit of $\text{MSp}^*(\text{BSp}(1)_+)$, and we obtain
\[ \sum_{i \geq 1} \phi_i^4 x^{8i} = e(\Lambda \otimes R H)(1 + \sum_{j \geq 1} \alpha_j x^j)^{-1} \]
for some coefficients $\alpha_j \in \text{MSp}^*$. This proves (3).

3. Nilpotency formula in self-conjugate cobordism

Let $Q = \{ \pm 1, \pm i, \pm j, \pm k \}$ be the quaternion group, $N$ the normalizer of $S^1$ in $S^3$ as above and $Z_4$ a cyclic group generated by $j$.

Recall that $\rho(N, S^3)$ is the projective bundle of the universal Spin(3) bundle $\Lambda \to BS^3$, and we have the canonical splitting
\[ \rho^*(N, S^3)(\Lambda) = \mu + \lambda. \]
Here $\mu$ is a plane and $\lambda$ is a line real bundle.

It is easy to see that the bundle $\rho(Q, N)$ is the projective bundle of $\mu$ and the bundle $\rho(Q, S^3)$ is the bundle of flags of the bundle $\Lambda$.

This defines the splittings
\[ \rho^*(Q, S^3)(\Lambda) = \chi_1 + \chi_2 + \chi_3; \]
\[ \rho^*(Q, N)(\mu) = \chi_2 + \chi_3; \]
\[ \rho^*(Q, N)(\lambda) = \chi_1. \]

Here \( \chi_3 = \chi_1 \otimes_R \chi_2. \)

**Proposition.** The transfer homomorphism \( \tau^*(Q, S^3) \) satisfies the following relations:

\[(3.1) \quad e(\Lambda \otimes_R C) = -\tau^*(Q, S^3)(e^2(\chi_i \otimes_R C)e(\chi_j \otimes_R C)) = 0 \]

in the self-conjugate cobordism theory and

\[(3.2) \quad e(\Lambda \otimes_R H) = -\tau^*(Q, S^3)(e^2(\chi_i \otimes_R H)e(\chi_j \otimes_R H)) = 0 \]

in the symplectic cobordism theory, where \( i, j = 1, 2, 3; i \neq j. \)

**Proof of (3.1).** The double coset space \( N[3^3]N \) is a line segment. The isotropy groups are \( N \) and \( Q \) at the endpoints and \( Z_4 \) (generated by \( j \)) in the interior.

By the double coset theorem and Proposition 1, case \( m = 3 \), we have

\[ 0 = e(\Lambda \otimes_R C) + \tau^*(Q, N)(e(\chi_2 \otimes_R C)) - \tau^*(Z_4, N)(\rho^*(Z_4, N)(e(\lambda \otimes_R C))). \]

But

\[ e(\rho^*(N, S^3)(\Lambda \otimes_R C)) = e(\lambda \otimes_R C)e(\mu \otimes_R C) \]

and \( \rho^*(Z_4, N)(\mu) \) has the section. Hence by the above splittings and transfer properties we obtain

\[ e(\rho^*(N, S^3)(\Lambda \otimes_R C)) = -\tau^*(Q, N)(e(\chi_2 \otimes_R C))e(\mu \otimes_R C) \]

\[ = -\tau^*(Q, N)(e^2(\chi_2 \otimes_R C)e(\chi_3 \otimes_R C)). \]

Since \( \tau^*(N, S^3)(1) = 1 \) by the analogue of Proposition 1 for the self-conjugate cobordism, this proves

\[ e(\Lambda \otimes_R C) = -\tau^*(Q, S^3)(e^2(\chi_2 \otimes_R C)e(\chi_3 \otimes_R C)). \]

We may prove relations analogous to (3.1) by changing \( N \) to its conjugate subgroup in \( S^3 \), but this follows also by symmetry.

Now

\[ \tau^*(Q, S^3)(e^2(\chi_1 \otimes_R C)e(\chi_2 \otimes_R C)) \]

\[ = \tau^*(N, S^3)(\tau^*(Q, N)(e^2(\chi_1 \otimes_R C)e(\chi_2 \otimes_R C))) \]

\[ = \tau^*(N, S^3)(e^2(\lambda \otimes_R C)(-e(\lambda \otimes_R C) + \tau^*(Z_4, N)(\rho^*(Z_4, N)(e(\lambda \otimes_R C)))) \]

\[ = \tau^*(N, S^3)(-e^3(\lambda \otimes_R C) + \tau^*(Z_4, S^3)(\rho^*(Z_4, N)(e^3(\lambda \otimes_R C)))). \]

The first summand is zero by Proposition 1 (by its analogue). The second summand is also zero. This follows immediately from the following theorem

**Theorem [Fe].** Assume \( N_G(H)/H \) is not discrete, where \( N_G(H) \) is the normalizer of \( H \) in \( G \). Then \( \tau^*(H, G) = 0. \)
The proof of (3.2) is analogous.

Now since (see Section 2) the symplectic Euler class of $\Lambda \otimes_R H$ (the Euler class of $\Lambda \otimes_R C$ in $SC^*$) coincides with $\sum_{i \geq 1} \phi_i^4 x^{8i}$ (with the image of $\sum_{i \geq 1} \phi_i^2 x^{4i}$ in $SC^*$ theory) up to multiplication by a unit of $\text{MSp}^0(\text{BS}^3_3 +)$ (by a unit of $\text{SC}^0(\text{BS}^3_3 +)$), this proves

**Corollary 3.3.** $\phi_i^4 = 0$, and the images of $\phi_i^2$ in self-conjugate cobordism are zero.

**Remark 3.4.** It follows from the relation between the transfer and the umkehr map [BG], [BO] that Proposition 1 is true also for $m = 2$ and $m = 6$.

### 4. On four-fold products of Ray classes

Here we improve the above method and obtain

**Proposition 4.1.** All four-fold products of Ray classes are zero, and the images of double products of these classes in self-conjugate cobordism are zero.

The proof is organized as follows:

Let $N$ be the normalizer of the torus $U(1)$ in $Sp(1)$ as above. Consider again the bundle

$$p : BN \to BSp(1)$$

and the map

$$f : BN \to BZ_2$$

induced by projection of $N$ on the Weil group $Z_2$. Let $\tau_p$ be the transfer map for $p$.

We have the following relations.

**Proposition 4.2.** In $\text{MSp}^*(\text{BSp}(1)^4)$ we have

$$\sum_{i,j,k,l \geq 1} \phi_i \phi_j \phi_k \phi_l x_1^{2i} x_2^{2j} x_3^{2k} x_4^{2l} = \sum_{m,n,p,q \geq 0} \tau_p^* f^* (\gamma_{mnpq}) x_1^m x_2^n x_3^p x_4^q,$$

where $\gamma_{mnpq}$ are elements from $\text{MSp}^*(BZ_2)$.

**Proposition 4.3.** In $\text{SC}^*(\text{BSp}(1)^2)$ we have

$$\sum_{i,j \geq 1} \psi_i \psi_j y_1^{2i} y_2^{2j} = \sum_{m,n \geq 0} \tau_p^* f^* (\delta_{mn}) y_1^m y_2^n,$$

where $\psi_i$ is the image of $\phi_i$ in self-conjugate cobordism and the $\delta_{mn}$ are elements from $\text{SC}^*(BZ_2)$.

We shall see later that the map $f \tau_p$ induces trivial homomorphism for any generalized cohomology theory $h^*$.

**Proposition 4.4.**

$$\tau_p^* f^*(a) = 0, \quad \forall a \in \check{h}^*(BZ_2);$$

$$\tau_p^*(1) = 1.$$

Thus the right sides of the relations from 4.2 and 4.3 are zero. This proves Proposition 4.1.
Proofs of 4.2 and 4.3. We need a simple lemma about orientable bundles, whose proof follows from the fact that $KO^4(X) = KSp^0(X)$.

Let $\eta \to BZ_2$ be the universal $O(1)$ bundle and $\zeta$, $\zeta^*$, $\Lambda$ the bundles from the introduction.

**Lemma 4.5.** i) The bundle $\eta \otimes_R \zeta \otimes_H \zeta^* \to BZ_2 \times BSp(1)$ is $MSp$-orientable.

ii) The bundle $\eta \otimes_R \sum_{i=1}^4 \Lambda_i \to BZ_2 \times BSp(1)^4$ is $MSp$-orientable.

iii) The bundle $\eta \otimes_R \sum_{i=1}^2 \Lambda_i \to BZ_2 \times BSp(1)^2$ is $SC$-orientable.

**Proof.**

i) This bundle is a $Spin(4)$ bundle and so is $MSp$-orientable.

ii) Since $\zeta_i \otimes_H \zeta_i^* = \Lambda_i + R^1$, the bundle ii) is $MSp$-orientable as a difference of two $MSp$-orientable bundles

$$\eta \otimes_R \sum_{i=1}^4 \zeta_i \otimes_H \zeta_i^* - \eta \otimes_R H.$$ 

iii) This bundle is a difference of $SC$-orientable bundles

$$\eta \otimes_R \sum_{i=1}^2 \zeta_i \otimes_H \zeta_i^* - \eta \otimes_R C.$$ 

Recall from section 2 that $\tilde{e}(\Lambda) = \theta_1 + \sum_{r \geq 1} \phi_r x^{2i}$, where $x = e(\zeta)$.

Any two orientation classes of the given orientable bundle agrees up to multiplication by an invertible element. So there is

$$\tilde{e} = \tilde{e}(\eta \otimes_R \sum_{i=1}^4 \Lambda_i),$$

which as an element from

$$MSp^*(BZ_2 \times BSp(1)^4) = MSp^*(BZ_2)[[x_1, x_2, x_3, x_4]], \quad x_i = e(\zeta_i),$$

has the form

$$\tilde{e} = \sum_{s=1}^4 (\theta_1 + \sum_{r \geq 1} \phi_r x^{2i}^s) + \sum_{m,n,p,q \geq 0} \gamma_{mnpq} x_1^m x_2^n x_3^p x_4^q$$

$$= \sum_{i,j,k,f \geq 1} \phi_i \phi_j \phi_k \phi_l x_1^{2i} x_2^{2j} x_3^{2k} x_4^l + \sum_{m,n,p,q \geq 0} \gamma_{mnpq} x_1^m x_2^n x_3^p x_4^q.$$ 

Here we take into account the relation $\theta_1 \phi_i \phi_j = 0$ from [G].

Consider now the map

$$g = (f, p) \times 1 : BN \times BSp(1)^3 \to BZ_2 \times BSp(1) \times BSp(1)^3.$$ 

**Lemma 4.6.** $g^*(\tilde{e}) = 0$.

**Proof.** Recall from Section 3 that $p^*(\Lambda) = \mu + \lambda$. But $f^*(\eta) = \lambda$ and $\lambda^2 = 1$. Thus the bundle

$$g^*(\eta \otimes_R \sum_{i=1}^4 \Lambda_i) = \lambda(\mu + \lambda + \Lambda_2 + \Lambda_4 + \Lambda_4)$$

has the section. This proves the lemma.
We now have in $MSp^*(BN \times BSp(1)^3)$ the relation
\[ \sum_{i,j,k,l \geq 1} \phi_i \phi_j \phi_k \phi_l \gamma_{i,j,k,l}(x_1)^{2i} x_2^{2j} x_3^{2k} x_4^{2l} + \sum_{m,n,p,q \geq 0} f^*(\gamma_{mnpq})p^*(x_1)^m x_2^n x_3^p x_4^q = 0. \]

After application of the transfer homomorphism for the bundle
\[ p \times 1 : BN \times BSp(1)^3 \to BSp(1)^4 \]
we get Proposition 4.2.

The proof of 4.3 is analogous.

Proof of 4.4. In fact this is a particular case of Proposition 1, although we should rewrite it as follows:

**Proposition 4.7.** Let $G_m = \text{Spin}(m)$, and let $\xi^m \to BG_m$ be the universal $\text{Spin}(m)$ bundle, $m = 2, 3, 4, 5$. Let $p_m : P(\xi^m) \to BG_m$ be the projective bundle associated to $\xi^m$ and let
\[ f_m : P(\xi^m) \to BZ_2 \]
be the classifying map for the canonical real line bundle $\lambda_m \to P(\xi^m)$. Then $\tau_m(1)$ is equal to 0 if $m = 2, 4$ and equal to 1 if $m = 3, 5$;
\[ \tau_m^*(a) = 0, \quad \forall a \in MSp^*(BZ_2). \]

The case $m = 3$ gives Proposition 4.4.

We also remark that using [Bu] and Proposition 1 one can obtain a new proof of the relation $\theta_1 \theta_2 \theta_3 = 0$ proved in [GR]. Moreover, some relations between the $\theta_i$'s and the generators of the free part of the symplectic cobordism can be also derived. We plan to present the details in a future paper.

**References**


A. Razmadze Mathematics Institute of the Georgian Academy of Sciences, M. Aleksidze st. 1, 380093, Tbilisi, Republic of Georgia

E-mail address: maxo@imath.acnet.ge