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COHEN-MACAULAY SECTION RINGS

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Abstract. In this paper, we study the section rings of sheaves of Cohen-Macaulay algebras (over a field F) on a ranked poset. A necessary and sufficient condition for these rings to be Cohen-Macaulay will be given. This is a further generalization of a result of Yuzvinsky, which generalizes Reisner's theorem concerning Stanley-Reisner rings.

0. Introduction

Throughout this paper, every ring is associative, commutative, Noetherian and has an identity element 1. Every ring homomorphism maps 1 to 1. If J is an ideal of a ring R, we use the convention

$$\operatorname{bight}(J) = \max\{\operatorname{ht}(P) \mid P \text{ is a minimal prime of } J\}.$$

Let X be a finite poset with its natural increasing order topology and $\mathfrak{A}$ a sheaf of rings on X. The algebraic structure of the section ring $\Gamma(\mathfrak{A})$ is sometimes related to the topological structure of X. One interesting thing is that a Stanley-Reisner ring can be viewed as the section ring of a sharp flasque sheaf of regular algebras on an atomic prelattice. Under this observation, Yuzvinsky studied the section rings of sharp flasque sheaves of regular algebras (over a field F) on an atomic prelattice, and obtained a necessary and sufficient condition for such rings to be Cohen-Macaulay (abbr. CM) [8, Theorem 6.4]. This generalizes Reisner's theorem, first proved in [6], concerning the CM property of Stanley-Reisner rings.

Now, before we state the main result of the paper, let us recall some notions and notations which will be used throughout without further explanations. All other unexplained notions and notations are standard and can be found in [4], [5].

Let X be a finite poset. We denote by $\operatorname{rk}(X)$ the number of elements of a longest chain in X. A ranked poset is a finite poset in which every maximal chain has the same length. For $z \in X$, we always put $X_z = \{x \in X \mid x > z\}$, $\overline{X}_z = \{x \in X \mid x \geq z\}$, $X^z = \{x \in X \mid x < z\}$, $\overline{X}^z = \{x \in X \mid x \leq z\}$. We shall use the convention that $\operatorname{rk}(z) = \operatorname{rk}(\overline{X}^z)$, and $X_{\min} = \{x \in X \mid x \text{ is a minimal element}\}$. Moreover, the collection of all the chains forms an abstract simplicial complex. It is clear that the dimension of this complex is $\operatorname{rk}(X) - 1$. We denote by F a fixed arbitrary field and by $\widetilde{H}^i(X, F)$ the usual ith reduced cohomology group of the above abstract complex. A poset X is said to be F-spherical if for every $z \in X$, the reduced cohomology groups $\widetilde{H}^i(X^z, F) = 0$ for $0 \leq i \leq \operatorname{rk}(z) - 2$. The poset X is said to be F-acyclic if $\widetilde{H}^i(X, F) = 0$ for $i \geq 0$.

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Let X be a poset. The natural increasing order topology on X is defined as the topology in which a subset Y is open if and only if $x \geq z \in Y$ implies $x \in Y$. The general definition of a sheaf on a topological space (see, e.g., [4]) can be easily specified for the order topology. For example, a presheaf $\mathfrak{A} = (A_x, \rho_{yx})$ of rings on a basis of the increasing order topology of X consists of rings $(\mathfrak{A})_x = A_x$ for $x \in X$ which are called stalks of $\mathfrak{A}$, and ring homomorphisms $\rho_{yx} : A_x \to A_y$ for every $x, y \in X$ such that $x \leq y$. The homomorphisms should satisfy the following conditions:

(i) $\rho_{xx} = 1_{A_x}$ for every $x \in X$, and

(ii) $\rho_{zy}\rho_{yx} = \rho_{zy}$ for every $x, y, z \in X$ with $x \leq y \leq z$.

We will use the same notation $\mathfrak{A} = (A_x, \rho_{yx})$ to denote the sheaf yielded by the unique sheafification of the presheaf $\mathfrak{A} = (A_x, \rho_{yx})$.

Let $\mathfrak{A} = (A_x, \rho_{yx})$ be a sheaf of rings on a finite poset X. It is easy to see that the section ring of $\mathfrak{A}$ is $\Gamma(\mathfrak{A}) = \{s \in \prod_{x\in X} A_x \mid \rho_{yx}(s(x)) = s(y), x, y \in X, x \leq y\}$, where $s(x)$ is the image of s under the natural projection of $\prod_{y\in X} A_y$ to A_x. Clearly, $R = \Gamma(\mathfrak{A})$ is a subring of $\prod_{x\in X} A_x$, and $\Gamma(\mathfrak{A} \mid \overline{X}_x) = A_x$. One can prove easily that R is Noetherian. For every nonempty subset $Y \subseteq X$, we denote by ρ_Y the restriction homomorphism $R \to \Gamma(\mathfrak{A} \mid Y)$, and by convention we set $\rho_x = \rho_{\overline{X}_x}$ for $x \in X$. Also we always put $J_x = \mathrm{Ker}\rho_x = \{s \in \Gamma(\mathfrak{A}) \mid s(x) = 0\}$. Clearly, J_x is an ideal of R, $J_x \subset J_y$ for $x \leq y$, and $\bigcap_{x\in X_{\min}} J_x = 0$. We say that $\mathfrak{A}$ is sharp if $\mathrm{ht}(\mathrm{Ker}(\rho_{yx})) \neq 0$ in A_x for every $x < y$. When A_x is a domain for each $x \in X$, this definition agrees with the definition in [8]. The sheaf $\mathfrak{A}$ is called flasque if ρ_U is an epimorphism for every nonempty open subset U. In particular, if $\mathfrak{A}$ is flasque all homomorphisms ρ_x, ρ_{yx} are epimorphisms. In this case, we have $A_x \simeq R/J_x$. If $\mathfrak{A}$ is sharp flasque, it is easy to see that no minimal prime ideal of J_x can be a minimal prime ideal of J_y for $x < y$. This fact will play an important role in the proof of our main result when we localize the sheaf $\mathfrak{A}$ at a prime P of R.

In this paper, our main goal is to prove the following result, which generalizes the main theorem of [8], in which the algebras A_x are assumed to be regular domains and the poset is assumed to be an atomic prelattice.

Theorem 2.4. *Let X be a ranked poset and $\mathfrak{A} = (A_x, \rho_{yx})$ be a sharp flasque sheaf of CM F-algebras on X. Put $R = \Gamma(\mathfrak{A})$. Then the following conditions are equivalent:*

(i) R is CM;

(ii) X is F-spherical, and X^z is F-acyclic for every $z \in X$ with $\mathrm{bight}(J_z) > \mathrm{rk}(z) - 1$.

1. Section Rings of Flasque Sheaves of CM Local Rings

In [8], Yuzvinsky studied the basic properties of sheaves of integral domains on a finite poset. In this section, we obtain similar results for the sheaves of CM local rings on a finite poset. So in the propositions below $\mathfrak{A} = (A_x, \rho_{yx})$ is a fixed sharp flasque sheaf of CM local rings on a finite poset X, and $R = \Gamma(\mathfrak{A})$.

Proposition 1.1. *Let x, y be two elements of X. If J_x is contained in some minimal prime ideal of J_y, then $x \leq y$.*

Proof. Put $Y = \overline{X}_x \cup \overline{X}_y$. Let $z_1, \ldots, z_k$ be the successors of y and set $K_i = \mathrm{Ker}\ \rho_{z_iy}$ $(1 \leqslant i \leqslant k)$. We have $K_i \simeq J_{z_i}/J_y$. Since $\mathfrak{A}$ is sharp, we get that $\mathrm{ht}\ (J_{z_i}/J_y) > 0$ in the ring R/J_y, which means that $\mathrm{ht}\ (\bigcap_{i=1}^{k} K_i) > 0$. As A_y is

CM, there exists a section $t \in \Gamma(\mathfrak{A} \mid Y)$ such that $t(y) \in \bigcap_{i=1}^{k} K_i$ is an A_y-regular element and $t(u) = 0$ for $u \in Y \setminus \{y\}$. Extend t to a section $\overline{t} \in R$. Clearly, $\overline{t} \in J_x$ and $\overline{t}$ cannot be an element of each minimal prime ideal over J_y. □

Proposition 1.2. *The set of minimal prime ideals of R is equal to $\bigcup_{x \in X_{min}} \{P \mid P$ is a minimal prime of $J_x\}$.*

Proof. Clearly, $\bigcap_{\mathrm{rk}\ (x)=1} J_x = 0$. So for an arbitrary minimal prime ideal P of R, we have $P \supset \bigcap_{\mathrm{rk}\ (x)=1} J_x$. This shows that $P \supseteq J_{x_0}$ for some $x_0 \in X_{\min}$, and P is a minimal prime ideal of J_{x_0}.

Conversely, for $x \in X_{\min}$, let P be a minimal prime ideal of J_x. If P is not a minimal prime ideal of R, then there exists a minimal prime ideal P' such that $P \supset P'$ and $P' \not\supseteq J_x$. On the other hand, $P' \supseteq J_y$ for some $y \in X_{\min}$ and $y \neq x$. Hence $P \supseteq P' \supseteq J_y$, by Proposition 1.1, which is a contradiction. So P is a minimal prime ideal of R. □

Proposition 1.3. *If X is a ranked poset with a unique maximal element and R is a local CM ring, then there exists an R-sequence $r_1, \dots, r_{n-1}$ such that $r_1(x) = \cdots = r_{i-1}(x) = 0$ for every element $x \in X$ with $\mathrm{rk}\ (x) = i \geqslant 2$. Furthermore, if $\mathrm{ht}\ J_x = i-1$ for some $x \in X$ $(\mathrm{rk}\ (x) = i)$, then $r_i(x), \dots, r_{n-1}(x)$ is an A_x-sequence.*

Proof. First, we construct elements r_i $(1 \leqslant i \leqslant n-1)$. For every i $(1 \leqslant i \leqslant n)$, we put $J(i) = \bigcap_{\mathrm{rk}\ (x)=i} J_x$. Since R and A_x (for every $x \in X$) are CM local rings, we have $\mathrm{ht} J_x + \dim A_x = \dim R$ for every $x \in X$. Now we claim

$$0 = \mathrm{ht}\ J(1) < \mathrm{ht}\ J(2) < \cdots < \mathrm{ht}\ J(n).$$

In fact, if there exists i_0 $(0 \leqslant i_0 < n)$ such that $\mathrm{ht}\ J(i_0) = \mathrm{ht}\ J(i_0+1)$, then there exists a minimal prime ideal P over $J(i_0+1)$ which is also a minimal prime ideal over $J(i)$. Put $Y_{i_0} = \{x \in X \mid \mathrm{rk}\ (x) \geqslant i_0\}$ and $Y_{i_0+1} = \{x \in X \mid \mathrm{rk}\ (x) \geqslant i_0+1\}$. Since $\mathfrak{A}$ is flasque, $R/J(i)$ and $R/J(i+1)$ are the rings of sections of the flasque sheaves $\mathfrak{A} \mid Y_{i_0}$ and $\mathfrak{A} \mid Y_{i_0+1}$, respectively. Hence according to Proposition 1.2, P must be a minimal prime ideal for some J_x and J_y, where $\mathrm{rk}\ (x) = i_0$, $\mathrm{rk}\ (y) = i_0+1$. It follows from Proposition 1.1 that $x < y$, so the sharp property implies $\mathrm{ht} J_x < \mathrm{ht}\ J_y$. On the other hand, R, A_x, A_y are all CM local rings, so J_x and J_y must be unmixed ideals. This shows that $\mathrm{ht}\ J_x = \mathrm{ht}\ J_y$, which is a contradiction.

Hence we can choose an R-sequence $(r_1, \dots, r_{n-1})$ such that $r_i \in J(i+1) \setminus J(i)$. This completes the construction of the elements r_i.

The second assertion follows easily from CM properties. □

2. Main Result

In this section, we discuss our main result. First of all, we quote some important facts from [3], [8] and [9], which will be used several times in the proof of the main result.

Lemma 2.1 (cf. [8, Section 4,(4.1)])**.** *Let X be a ranked poset and $\mathfrak{A} = (A_x, \rho_{yx})$ be a sheaf of F-algebras on X. If X is F-spherical, then the sheaf cohomology groups $H^i(X, \mathfrak{A})$ can be computed as homology groups of the following complex of $\Gamma(\mathfrak{A})$-modules:*

$$0 \longrightarrow C^0 \xrightarrow{\partial_0} \dots \xrightarrow{\partial_{r-2}} C^{n-1} \longrightarrow 0$$

where $C^i = \bigoplus_{\mathrm{rk}(x)=i+1} \widetilde{H}^{i-1}(X^x, A_x),\ i \geqslant 1,\ n = \mathrm{rk}(X)$, and $C^0 = \bigoplus_{\mathrm{rk}(x)=1} A_x$.

Remark. In Lemma 2.1, since every A_x $(x \in X)$ is an algebra over the field F, it follows from the Universal Coefficient Theorem that each C^i is the direct sum of A_x with $\mathrm{rk}(x) = i+1$, taken each with some multiplicity. In particular, if X^x is F-acyclic, then A_x is not a summand of C^i.

Let X be a poset and $\mathfrak{A} = (A_x, \rho_{yx})$ a sheaf of rings on X. Set $R = \Gamma(\mathfrak{A})$ and let P be a prime ideal of R. Put $Y = \{x \in X \mid J_x \subseteq P\}$. Note that $Y \neq \emptyset$. The localization of A_x and ρ_{yx} at P forms a sheaf $\mathfrak{A}_P$ of local rings on Y.

Lemma 2.2 ([9, Proposition 3.1])**.** *$R_P \simeq \Gamma(\mathfrak{A}_P)$. $\mathfrak{A}$ is flasque if and only if $\mathfrak{A}_P$ is flasque for every prime ideal P of R.*

Due to Proposition 1.1 and Lemma 2.2, we can prove the following proposition word for word as Proposition 6.1 in [8].

Proposition 2.3. *Let X be a poset and $\mathfrak{A}$ a sharp flasque sheaf of CM local rings on X. If $z \in X$ is such that X^z is not connected, then* $\mathrm{depth}_{J_z}(R) \leq 1$.

Now we are ready for the main result.

Theorem 2.4. *Let X be a ranked poset and $\mathfrak{A} = (A_x, \rho_{yx})$ a sharp flasque sheaf of CM F-algebras on X. Put $R = \Gamma(\mathfrak{A})$. Then the following conditions are equivalent:*

(i) R is CM;

(ii) X is F-spherical, and X^z is F-acyclic for every $z \in X$ with $\mathrm{bight} J_z > \mathrm{rk}(z) - 1$.

Proof. First we make some reductions. Since $\mathfrak{A}$ is sharp and flasque, the localization $\mathfrak{A}_P$ of $\mathfrak{A}$ at P is also a sharp sheaf on $Y = \{z \in X \mid J_z \subseteq P\}$. Thus, by Lemma 2.2, $\mathfrak{A}_P$ is a sharp flasque sheaf of CM local F-algebras on Y with the section ring R_P. It is clear that Y is a closed subposet of X. We observe that Y has a unique maximal element. Suppose on the contrary that we can choose two elements z_1, z_2 that are maximal in Y. Put $U = \{z_1, z_2\}$. It is clear that U is open in Y. Take an arbitrary element $s \in R_P$. Since $\mathfrak{A}_P$ is flasque, there exists $t \in R_P$ such that $t(z_1) = 0$ and $t(z_2) = s(z_2)$. As $t \in J_{z_1} R_P$ and $s - t \in J_{z_2} R_P$, we have $s \in J_{z_1} R_P + J_{z_2} R_P$. This implies $R_P = J_{z_1} R_P + J_{z_2} R_P$, a contradiction because J_{z_1}, J_{z_2} are contained in P. Hence Y has a unique maximal element.

We have proved that $\mathfrak{A}_P$ is a sharp flasque sheaf of CM local F-algebras on a ranked closed subposet Y with a unique maximal element. To prove (i)$\Rightarrow$(ii) in the theorem, it suffices to prove that, for every $z \in X$, $\overline{X}^z$ is F-spherical and X^z is F-acyclic if $\mathrm{bight}(J_z) > \mathrm{rk}\,(z) - 1$. Let us choose a minimal prime ideal P of J_z such that $\mathrm{ht}(P)$=$\mathrm{bight}(J_z)$. Localizing $\mathfrak{A}$ at P, we obtain a sharp flasque sheaf $\mathfrak{A}_P$ of CM local F-algebras on a ranked closed subposet Y with a unique maximal element. The section ring is the CM local ring R_P. Note that the sharp assumption implies that P cannot contain J_x for every $x \in X$ and $x > z$. Thus z is the unique maximal element of Y. So $Y = \overline{X}^z$. Since $\mathrm{ht}(J_z R_P)$=$\mathrm{bight}(J_z)$, we deduce that $\mathrm{bight}(J_z) > \mathrm{rk}\,(z) - 1$ if and only if $\mathrm{ht}(J_z R_P) > \mathrm{rk}\,(z) - 1$. Hence we can reduce the proof of (i)$\Rightarrow$(ii) to the case that $\mathfrak{A}$ is a sharp flasque sheaf of CM local F-algebras on a ranked poset X with a unique maximal element, and the section ring is a CM local ring.

Moreover, to prove (ii) $\Rightarrow$ (i) in the theorem, it suffices to prove that R_M is CM for every maximal ideal M of R. Let M be a maximal ideal of R. Localizing $\mathfrak{A}$ at M, we obtain a sharp flasque sheaf $\mathfrak{A}_M$ of CM local F-algebras on a ranked closed subposet Y, and Y has a unique maximal element. Note that if X is F-spherical, then Y is also F-spherical. Furthermore, $\mathrm{bight}(J_zR_M) \leq \mathrm{bight}(J_z)$ for $z \in Y$. This implies that $\mathrm{bight}(J_z) > \mathrm{rk}\ (z)-1$ for z in Y if $\mathrm{bight}(J_zR_M) > \mathrm{rk}\ (z)-1$. Therefore the condition (ii) still holds for $\mathfrak{A}_P$ on Y. Thus we can also reduce the proof of (ii)⇒(i) to the case that $\mathfrak{A}$ is a sharp flasque sheaf of CM local F-algebras on a ranked poset X with a unique maximal element.

So, in the following proof we assume that $\mathfrak{A} = (A_x, \rho_{yx})$ is a sharp flasque sheaf of CM local F-algebras on a ranked poset X with a unique maximal element m, and the section ring of $\mathfrak{A}$ is a local ring R with the unique maximal ideal M. Put $n = \mathrm{rk}\ (X)$ and $d = \dim\ R$.

(i)⇒(ii). Clearly, $\dim A_x + \mathrm{ht} J_x = d$ and $\mathrm{bight}(J_x) = \mathrm{ht}\ J_x$, for every $x \in X$, because A_x and R are CM. We use induction on n to prove the conclusion.

For $n = 1$, the result is trivial. For $n = 2$, it is clear that X is F-spherical. It remains to prove X^x is F-acyclic if $\mathrm{ht}(J_x) > \mathrm{rk}\ (x) - 1$. According to Proposition 1.2, $\mathrm{ht}(J_x) = 0$ for every $x \neq m$. It suffices to prove that X^m is F-acyclic if $\mathrm{ht}(J_m) > 1$. Suppose on the contrary that X^m is not connected. So by Proposition 2.3, $\mathrm{depth} J_m \leq 1$. Thus $\mathrm{ht} J_m \leq 1$ because R is CM, a contradiction.

Now, suppose the conclusion holds for those posets with rank less than n ($n \geq 3$). Let us consider the case of X with $\mathrm{rk}(X) = n$. For every $x \in X$ ($x \neq m$), as we have seen in the above, the localization $\mathfrak{A}_{P_x}$ of $\mathfrak{A}$ at P_x is a sharp flasque sheaf of CM local F-algebras on $\overline{X}^x$ with the section ring R_{P_x}. Since $\mathrm{rk}(\overline{X}^x) \leq n-1$, by the induction hypothesis, we conclude that $\overline{X}^x$ is F-spherical and X^x is F-acyclic if $\mathrm{ht} J_x > \mathrm{rk}\ (x) - 1$. This implies that X^m is F-spherical and X^x is F-acyclic for every $x \in X^m$ with $\mathrm{ht}(J_x) > \mathrm{rk}\ (x) - 1$.

By Lemma 2.1, the sheaf cohomology groups $\mathrm{H}^i(X^m, \mathfrak{A} \mid X^m)$ can be computed as homology of the complex

$$0 \to C^0 \to C^1 \to C^2 \to \cdots \to C^{n-2} \to 0. \tag{2.5}$$

It follows from the remark in Section 2 that each C^i ($0 \leq i \leq n-2$) is the direct sum of some A_x with $\mathrm{rk}(x) = i+1$ and $\mathrm{ht}(J_x) = \mathrm{rk}\ (x)-1$. According to Proposition 1.3, there exists an R-sequence $r_1, \cdots, r_{n-1}$ such that $r_1(x) = r_2(x) = \cdots = r_i(x) = 0$ and $r_{i+1}(x), \cdots, r_{n-1}(x)$ is an A_x-sequence for $x \in X$ with $\mathrm{rk}(x) = i+1$ and $\mathrm{ht}(J_x)$=$\mathrm{rk}(x) - 1$. Hence the sequence $r_i, r_{i+1}, \cdots, r_{n-1}$ is a C^i-sequence.

By Proposition 4.1 in [8], we have isomorphisms

$$\mathrm{H}^i(X^m, \mathfrak{A} \mid X^m) \simeq \widetilde{\mathrm{H}}^i(X^m, A_m) \quad \text{for} \quad i > 0 \tag{2.6}$$

and the short exact sequence

$$0 \to R \to \mathrm{H}^0(X^m, \mathfrak{A} \mid X^m) \to \widetilde{\mathrm{H}}^0(X^m, A_m) \to 0.$$

Since $\mathrm{rk}(m) \geq 3$, it follows from the sharp assumption on $\mathfrak{A}$ that $\mathrm{ht}(J_m) \geq 2$. Thus $\mathrm{depth}(J_m) \geq 2$ because R is CM. By Proposition 2.3, X^m is connected. This implies that $\widetilde{\mathrm{H}}^0(X^m, A_m) = 0$. So $\mathrm{H}^0(X^m, \mathfrak{A} \mid X^m) = R$, that is, we have the following augmented complex of (2.5):

$$0 \to R \to C^0 \to C^1 \to \cdots \to C^{n-2} \to 0. \tag{2.7}$$

To prove that X is F-spherical, it sufficies to prove $\widetilde{\mathrm{H}}^i(X^m, F) = 0$ for $0 \leq i \leq n-3$. Suppose $\widetilde{\mathrm{H}}^i(X^m, F) \neq 0$ for some i ($0 \leq i \leq n-3$). As we have proved $\widetilde{\mathrm{H}}^0(X^m, F) = 0$, we have $n \geq 4$. By the Universal Coefficient Theorem, we have that not all groups $\widetilde{\mathrm{H}}^i(X^m, A_m)$ $(1 \leq i \leq n-3)$ are trivial. Thus, by (2.5), not all groups $\mathrm{H}^i(X^m, \mathfrak{A} \mid X^m) = 0$ $(1 \leq i \leq n-3)$. We will prove that this is impossible by showing that $r_1, r_2, \cdots, r_{k+2}$ is not R-regular. The method is similar to a discussion of Yuzvinsky in the proof of Theorem 5.1 in [8].

Let us consider the sequence (2.7). Put $Z^i = \mathrm{Ker}\,(C^i \to C^{i+1})$ and $B^i = \mathrm{Im}\,(C^{i-1} \to C^i)$ $(0 \leq i \leq n-2)$; here we use the convention $C^{-1} = R$. We will use another convention in the following by setting the boundary homomorphisms of (2.6) equal to ∂. Let $c \in Z^k \backslash B^k$. We define recursively a triangular matrix with entries $a_{pq} \in C^{P-1}$, $q = p+1, \cdots, k+2$, $p = 0, 1, \cdots, k+1$, subject to the relations

$$\sum_{q=p+1}^{k+2} r_q^2 \partial a_{pq} = 0, \qquad p = 0, 1, \cdots, k+1, \tag{2.8}$$

$$\partial a_{pq} = -r_{p+1}^2 a_{p+1,q}, \qquad q = p+2, \cdots, k+2, p = 0, 1, \cdots, k+1. \tag{2.9}$$

For $p = k+1$, we put $a_{k+1,k+2} = c$. It is clear that $a_{k+1,k+2}$ satisfies (2.8). Suppose that $q < k+1$ and a system $a_{p+1,p+2}, \cdots, a_{p+1,k+2}$ is already defined satisfying (2.9). Since $r_{p+1} \in \bigcap_{\mathrm{rk}(x)=p+2} J_x$, we have $r_{p+1}C^p \subseteq Z^p$. Moreover,

$$r_{p+1}^2 C^p \subseteq B^p.$$

If $p < k$, this follows from the exactness of (2.7) at C^p. For $p = k$, $Z^k/B^k \simeq \widetilde{\mathrm{H}}^k(X^m, A_m)$ and $r_{p+1} \in J_m$. So $r_{p+1}Z^k \subseteq B^k$. This implies that $r_{p+1}^2 C^p \subseteq B^p$. Hence, we can choose $a_{pq} \in C^{p-1}$ $(q = p+2, \cdots, k+2)$ satisfying (2.9).

In order to define $a_{p,p+1}$ we observe that $b = \sum_{q=p+2}^{k+2} r_q^2 a_{p+1,q} \in Z^p$. Furthermore, we observe that $b \in B^p$. If $r < k$, this follows from the exactness of (2.7) at C^p; If $p = k$, we have $b = r_{k+2}^2 c$. Since $c \in Z^k$, we have $r_{k+2}^2 c \in B^p$. Hence we can can choose $a_{p,p+1} \in C^{p-1}$ satisfying $\partial a_{p,p+1} = b$. One easily checks that

$$a_{pq}\ (p = 0, 1, \cdots, k+1,\ q = p+1, \cdots, k+2)\ \text{ satisfy (2.8).}$$

Put $I_t B^i = (r_t^2, \cdots, r_{k+2}^2)B^i$ $(i \leq t \leq k+1)$, $I_t C^i = (r_t^2, \cdots, r_{k+1}^2)C^i$ and $b_i = \partial a_{0i}$ $(1 \leq i \leq k+2)$. For $p = 0$ (2.8) reduces to $\sum_{i=1}^{k+2} r_i^2 b_i = 0$ or $r_{k+2}^2 b_{k+2} \in (r_1^2, \cdots, r_{k+1}^2)R = I_1 B^0$ (identifying B^0 with R). Now, we claim that $b_{k+2} \notin I_1 B_0$. Suppose on the contrary $b_{k+2} = \partial a_{0,k+2} \in I_1 B_0$. Using this as the base of induction on p, we assume that for some p, $0 < p < k-1$, we have $\partial a_{p,k+2} \in I_{p+1}B^p$. Due to (2.9), this can be rewritten as $r_{p+1}^2 a_{p+1,k+2} \in I_{p+1}B^p$, or, in other words, there exists $e \in B^p$ such that $r_{p+1}^2(a_{p+1,k+2} - e) \in I_{p+2}B^p$. By the fact that $r_{p+1}^2, \cdots, r_{k+1}^2$ is C^p-regular, we have that $(a_{p+1,k+2} - e) \in I_{p+2}C^p$. This implies that $\partial(a_{p+1,k+2}) \in I_{p+2}C^p$. In particular, for $p = k-1$ we have $r_{k+1}^2 c = -\partial a_{k,k+2} \in r_{k+1}^2 B^k$. Since r_{k+1}^2 is C^k-regular, it follows that $c \in B^k$, which is a contradiction because $c \notin B^k$. Thus $b_{k+2} \notin (r_1^2, \cdots, r_{k+1}^2)R$. Since $r_1^2, \cdots, r_{k+2}^2$ is an R-sequence and $r_{k+2}^2 b_{k+2} \in (r_1^2, \cdots, r_{k+1}^2)R$, we have $b_{k+2} \in (r_1^2, \cdots, r_{k+1}^2)R$, a contradiction. Therefore, X must be F-spherical.

Now, it remains to prove that X^m is acyclic if $\mathrm{ht}J_m > n-1$. Suppose this conclusion is false, that is, $\mathrm{ht}P_m > n-1$ and X^m is not acyclic. Since X is F-spherical and $\mathfrak{A}$ is flasque, the complex

$$0 \longrightarrow R \longrightarrow C^0 \xrightarrow{\partial_0} C^1 \longrightarrow \cdots \longrightarrow C^{n-2} \xrightarrow{\partial_{n-2}} C^{n-1} \longrightarrow 0$$

from Lemma 2.1 is acyclic. Observe that $\mathrm{ht}(J_x) \geq \mathrm{rk}(x) - 1$ for every $x \in X$, and X^x is F-acyclic if $\mathrm{ht}J_x > \mathrm{rk}(x) - 1$ $(x \neq m)$. Thus each C^i is either a CM R-module of dimension $d - i$ or a zero R-module for each i $(0 \leqslant i \leqslant n-2)$ by the induction hypothesis. The above hypothesis means that $C^{n-1} \neq 0$. Put $Z^i = \mathrm{Ker}\ \partial_i$ $(1 \leqslant i \leqslant n-2)$. Let us consider the following short exact sequences:

$$\begin{aligned} &0 \to R \to C^0 \to Z^1 \to 0, \\ &0 \to Z^1 \to C^1 \to Z^2 \to 0, \\ &\quad \cdots \qquad\qquad\quad \cdots \\ &0 \to Z^{n-2} \to C^{n-2} \to C^{n-1} \to 0. \end{aligned} \tag{2.10}$$

We assert that $C^i \neq 0$ and $\dim\ Z^i = d - i$ $(1 \leqslant i \leqslant n-2)$. In fact, from the sharp assumption on $\mathfrak{A}$, we have $\dim\ C^{n-2} > \dim\ C^{n-1}$. This implies $C^{n-2} \neq 0$ and $\dim C^{n-2} = d - n + 2$. Consider the last short exact sequence in (2.10); we have $\dim Z^{n-2} = d - n + 2$ because $\dim C^{n-2} = \max\{\dim Z^{n-2}, \dim C^{n-1}\}$. Similarly, we can prove $C^{n-3}, \cdots, C^1$ are all nonzero modules and $\dim\ Z^i = d - i$ $(1 \leqslant i \leqslant n-3)$. Since R, C^0 are CM, by applying the local cohomology functors $\{\mathrm{H}^i_M(\_)\}_{i \geqslant 0}$ to the first short exact sequence in (2.8), we obtain the following long exact sequence:

$$\cdots \to \mathrm{H}^{i-1}_M(Z^1) \to \mathrm{H}^i_M(R) \to \mathrm{H}^i_M(C^0) \to \mathrm{H}^i_M(Z^1) \to \cdots .$$

As R, C^0 are CM modules of dimension d, we have $\mathrm{H}^i_M(R) = 0$, $\mathrm{H}^i_M(C^0) = 0$ for $i < d$. So $\mathrm{H}^i_M(Z^1) = 0$ for $i < d - 1$. This implies that Z^1 is CM because $\dim Z^1 = d - 1$. Continue the processes, we can prove that $Z^2, \cdots, Z^{n-2}$ are CM. Now, let us consider the long exact sequence of local cohomology derived from the last short exact sequence in (2.10):

$$\cdots \to\ \mathrm{H}^i_M(Z^{n-2}) \to \mathrm{H}^i_M(C^{n-2}) \to \mathrm{H}^i_M(C^{n-1}) \to \cdots .$$

Since $\mathrm{H}^i_M(Z^{n-2}) = 0$ and $\mathrm{H}^i_M(C^{n-2}) = 0$ for $i < d-n+2$, we have $\mathrm{H}^i_M(C^{n-1}) = 0$ for $i < d-n+1$. Note that C^{n-1} is a nonzero CM module and $\dim C^{n-1} < d-n+2$. So $\dim\ C^{n-1} = d-n+1$, i.e. $\mathrm{ht}\ P_m = n-1$, a contradiction. Hence X^m is acyclic for $\mathrm{ht}P_m > n-1$.

(ii) $\Rightarrow$ (i). First, we claim that $\dim A_x = d$ for every $x \in X_{\min}$. We use induction on n. If $n = 1$, there is nothing to prove. For $n = 2$, suppose there exist $x, y \in X_{\min}$ such that $\dim A_x < \dim A_y$. This implies $\dim A_m < \dim A_x < \dim A_y$, by the sharpness hypothesis on $\mathfrak{A}$. Hence $\mathrm{ht}(J_m/J_y) = \dim A_y - \dim A_m \geq 2$. It follows that $\mathrm{ht}J_m \geq 2$. By the assumption (ii), X^m is F-acyclic, which is a contradiction. So our claim is true for $\mathrm{rk}(X) = 2$.

For $n > 2$, put $X_1 = \{x \in X_{\min} \mid \dim A_x = d\}$. It follows from Proposition 1.2 that $X_1 \neq \emptyset$. Suppose our claim is false, that is, $X_1 \neq X_{\min}$. Since X is F-spherical and $\mathrm{rk}(X^m) \geq 2$, we have $\widetilde{\mathrm{H}}^0(X^m, F) = 0$. This implies that X^m is connected. Hence we can choose an element $z \in X$ with $\mathrm{rk}(z) = n-1$ such that $z > x$ and $z > y$ for some $x \in X_1$ and $y \in X_{\min} \setminus X_1$. Let P be a minimal prime ideal of J_z. Localizing the sheaf $\mathfrak{A}$ at P, we obtain a sharp flasque sheaf of CM local F-algebras $\mathfrak{A}_P$ on $\overline{X}^z$ which still satisfies the condition (ii). So by induction $\dim R_P/J_xR_P = \dim R_P/J_yR_P$. As A_x and A_y are CM local rings, we assert that $\dim A_x = \dim A_y$. This contradicts the choices of x and y. So $X_1 = X_{\min}$, and that ends the proof of our claim.

By the claim and Proposition 1.2, one can see easily that every maximal chain of primes of R has length d, because each A_x $(x \in X_{\min})$ is CM. Hence we conclude that for every $x \in X$, bight(J_x)=ht(J_x) and dim$A_x = d-$ ht (J_x). As X is F-spherical and $\mathfrak{A}$ is flasque, by Lemma (2.1), we have the following long exact sequence:

$$0 \to R \to C^0 \xrightarrow{\partial_0} \cdots \xrightarrow{\partial_{n-2}} C^{n-1} \to 0.$$

By the F-acyclicity of X^x for every $x \in X$ with ht$J_x > \mathrm{rk}(x) - 1$ and the obvious fact that ht$J_z \geq \mathrm{rk}(z) - 1$ for every $z \in X$, we assert that each C^i $(0 \leq i \leq n-1)$ is either a CM R-module of dimension $d - i$ or a zero R-module.

Put $Z^i = \mathrm{Ker}\ \partial_i$ $(0 \leqslant i \leqslant n-2)$ and $Z^{n-1} = C^{n-1}$. It is clear that $Z^0 \simeq R$. Let us consider the following short exact sequences:

$$\begin{aligned} &0 \to Z^0 \to C^0 \to Z^1 \to 0,\\ &0 \to Z^1 \to C^1 \to Z^2 \to 0,\\ &\quad\cdots \qquad\qquad \cdots\\ &0 \to Z^{n-2} \to C^{n-2} \to C^{n-1} \to 0. \end{aligned} \tag{2.11}$$

If for some i $(1 \leq i \leq n-1)$, $C^i = 0$, we have $C^j = 0$ for $j > i$ because either dim$C^j = d - j$ or $C^j = 0$. So without loss of generality, we can assume $C^i \neq 0$ for all i $(0 \leqslant i \leqslant n-1)$. For each i $(0 \leqslant i \leqslant n-2)$, we have dim $C^i = \max\{\dim Z^i, \dim Z^{i+1}\}$. Thus dim$C^i$ = dim$Z^i = d - i$ because dim$C^i >$ dimC^{i+1} and dim$Z^{i+1} \leq$ dimC^{i+1}. Applying the local cohomology functors $\{\mathrm{H}^i_M(\_)\}_{i\geqslant 0}$ to the last short exact sequence in (2.11), we derive the long exact sequence

$$\cdots \to \mathrm{H}^i_M(Z^{n-2}) \to \mathrm{H}^i_M(C^{n-2}) \to \mathrm{H}^i_M(C^{n-1}) \to \mathrm{H}^{i+1}_M(Z^{n-2}) \to \cdots.$$

Observe that $\mathrm{H}^i_M(C^{n-2}) = 0$ for $i < d-n+2$ and $\mathrm{H}^i_M(C^{n-1}) = 0$ for $i < d-n+1$. Hence $\mathrm{H}^i_M(Z^{n-2}) = 0$ for $i < d-n+2$. So Z^{n-2} is CM. Continue the processes along with other short exact sequences in (2.11), we can prove $Z^{n-2}, \cdots, Z^1, Z^0$ are all CM. Therefore R is CM. □

Remark. There is a gap in the proof of the Theorem 3.3 of the paper [8], since $J(i) = \bigcap_{\mathrm{rk}\ (x)=i} J_x$ may not be a unmixed ideal of R, i.e. the heights of the minimal prime ideals of $J(i)$ may not be the same. So a maximal R-sequence in $J(i)$ may not be a maximal R-sequence in every minimal prime ideal of $J(i)$. Thus $r_1, \ldots, r_{n+d-1}$ in the proof of Theorem 3.3 of [8] may not be a standard system. Hence even in the case that A_x $(x \in X)$ are all CM local domains, the F-spherical properties in (i) ⇒(ii) of our Theorem 2.4 cannot be deduced directly from the results in [8].

There are two interesting corollaries derived directly from Theorem 2.4.

Corollary 2.12. *Let $\mathfrak{A}$, X and R be as in Theorem 2.4, and let R be CM. If X^y is not F-acyclic, then* bight $(J_x) = \mathrm{rk}\ (x) - 1$ *for all $x \in X$ with $x < y$.*

Corollary 2.13. *Let $\mathfrak{A}$, X and R be as in Theorem 2.4, and let R be CM. If* bight $J_x > \mathrm{rk}\ (x) - 1$ *for some $x \in X$, then X^y is F-acyclic for $y > x$.*

Before we end the paper, we give a simple application of the main result, Theorem 2.4. Let $P = \{x, y, u, v\}$ be a poset with the relations

$$u < x, \quad v < x, \quad u < y, \quad v < y.$$

Let X, Y be two indeterminants and $\mathfrak{A}$ be a sheaf on P with stalks:

$$A_u = F[X], \quad A_v = F[Y], \quad A_x = F[X]/(X^2 - 1), \quad A_y = F$$

and with homomorphisms

$$\rho_{xu}(f(X)) = f(X) \bmod (X^2 - 1), \quad \rho_{yu}(f(X)) = f(0),$$

$$\rho_{xv}(f(Y)) = f(X) \bmod (X^2 - 1), \quad \rho_{yv}(f(Y)) = f(0).$$

It is easy to check that $\mathfrak{A}$ is a sharp flasqe sheaf of CM F-algebras on P. Thus by Theorem 2.4 the section ring $R = \Gamma(\mathfrak{A})$ is CM.

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