

SOME RAMIFICATIONS OF A THEOREM
OF BOAS AND POLLARD CONCERNING
THE COMPLETION OF A SET OF FUNCTIONS IN L^2

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ABSTRACT. About fifty years ago, R. P. Boas and Harry Pollard proved that an orthonormal system that is completable by the adjunction of a finite number of functions also can be completed by multiplying the elements of the given system by a fixed, bounded, nonnegative measurable function. In subsequent years, several variations and extensions of this theorem have been given by a number of other investigators, and this program is continued here. A mildly surprising corollary of one of the results is that the trigonometric and Walsh systems can be multiplicatively transformed into quasibases for $L^1[0, 1]$.

1. A curious connection between disparate mathematical ideas occurs in the theory of complete systems of real-valued measurable functions. The first of these notions is due to Boas and Pollard [3], who showed that certain incomplete systems in L^2 may be transformed into complete systems by multiplying the elements of the system by a (fixed) bounded, nonnegative measurable function.

Subsequently, Talalyan [17], [18] proved that a system Φ is complete in measure on a set E if and only if, for every positive ε , Φ is complete in $L^2(E_\varepsilon)$, where E_ε is a measurable subset of E that has measure greater than $|E| - \varepsilon$. In addition, Talalyan showed that if Φ has these properties, so also does any family obtained from Φ by deleting a finite number of its members. Later, Goffman and Waterman [6] gave a new proof of this theorem and observed that it is always possible to make certain infinite deletions from a system that is complete in measure so as to leave a residual system that also satisfies this condition.

The somewhat surprising coincidence of these ideas was noted in [15], where the following result was established.

Theorem A. *Let E be a measurable set of finite, positive measure, and let $\Phi = \{\varphi_n : n = 1, 2, \dots\}$ be a subset of $L^2(E)$. Then, the following conditions are equivalent:*

- (BP) *There exists a bounded, measurable function, m , such that $\{m\varphi_n : n = 1, 2, \dots\}$ is complete in $L^2(E)$;*
- (M) *Φ is complete in measure on E ;*

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(T) For every positive ε , there exists $E_\varepsilon \subset E$ such that $|E_\varepsilon| > |E| - \varepsilon$, and Φ is complete in $L^2(E_\varepsilon)$.

Although the details are not given in [15], adaptations of the arguments given there can be used to establish the corresponding theorem in which the role of L^2 is played by any space L^p , with $1 \leq p < +\infty$. It follows, in particular, that all cofinite, and some coinfinite, subsets of a system that is complete in $L^p(E)$ are multiplicatively completable.

In the ensuing years, many investigations of problems related to the foregoing have been undertaken, and the theory continues to be developed along various lines. It is to these matters that the present article is devoted.

2. The following notation and terminology are employed in the sequel. Let B be a Banach space, and let B^* be the conjugate space associated therewith. A system $X = \{x_n : n = 1, 2, \dots\}$, of elements of B , is *complete* in B iff the closure of the set of all finite linear combinations of elements of X coincides with B . If, for every natural number j , $x_j \notin \text{cl}(X \setminus \{x_j\})$, then X is termed *minimal*. This condition is equivalent to the following [7, 264]: there exists a *conjugate system*

$$X^* = \{x_n^* : n = 1, 2, \dots\} \subset B^*$$

such that

$$x_i^*(x_j) = \delta_{ij}, \quad \forall i \in \mathbb{N}, \quad \forall j \in \mathbb{N}.$$

A set $X^* \subset B^*$ is *total* on B if the condition $x^*(x) = 0, \forall x^* \in X^*$, implies that $x = 0$.

If $X = \{x_n : n = 1, 2, \dots\}$ is a minimal, complete subset of B , and if its associated conjugate system is total on B , then X is a *basis in the sense of Markusiewicz*, or, simply, an M -basis for B [16, 219]. A system $X = \{x_n : n = 1, 2, \dots\}$ is a *quasi-basis* for B iff there exists a system $\{y_n^* : n = 1, 2, \dots\} \subset B^*$ such that, for all x in B , $\sum_{n=1}^{\infty} y_n^*(x)x_n$ converges to x in the norm of B . This latter generalization of a Schauder basis, introduced by Gelbaum [5], is a weaker notion, since the associated coefficient functionals need not be uniquely determined [16, 278, 766]. Nevertheless, one can employ the Banach–Steinhaus theorem to show that the operators

$$S_n = \sum_{k=1}^n y_k^*(\cdot)x_k, \quad n \in \mathbb{N},$$

are uniformly bounded in this setting as well.

The Haar functions may be defined in the following manner: For all $t \in [0, 1]$, let

$$h_1(t) = 1,$$

and for $k = 0, 1, 2, \dots; j = 1, 2, \dots, 2^k$, let

$$h_k^{(j)}(t) = \begin{cases} 2^{\frac{k}{2}}, & \text{if } \frac{2j-2}{2^{k+1}} < t < \frac{2j-1}{2^{k+1}}; \\ -2^{\frac{k}{2}}, & \text{if } \frac{2j-1}{2^{k+1}} < t < \frac{2j}{2^{k+1}}; \\ 0, & \text{otherwise;} \end{cases}$$

and, for $n = 2^k + j$, let $h_n = h_k^{(j)}$.

For each $a \in (0, 1)$, let I_a be the real function of period 1 defined by the conditions

$$I_a(t) = \begin{cases} \frac{a-1}{a}, & \text{if } t \in [0, a]; \\ 1, & \text{if } t \in (a, 1]. \end{cases}$$

3. In [2], Braun strengthened the theorem of Boas–Pollard and generalized that theorem, so as to include other L^p -spaces, in the following manner.

Theorem B. *Let E be a measurable set of finite positive measure, and let $\Phi = \{\varphi_n : n = 1, 2, \dots\}$ be a Schauder basis for some space $L^p(E)$, with $1 \leq p < +\infty$. Then, to every natural number N there corresponds a bounded measurable function m such that every element f of $L^p(E)$ can be represented by a series $\sum_{k=N+1}^{\infty} a_k m \varphi_k$ that converges to f in the L^p -norm.*

In other terminology, Theorem B asserts that $\{m\varphi_k : k > N\}$ is a system of representation for $L^p(E)$. Neither does the theorem claim, nor are the arguments originally used to establish this result sufficient to show, that $\{m\varphi_k : k > N\}$ is, again, a Schauder basis for $L^p(E)$. Thus, one is led to inquire whether it is always possible to obtain a Schauder basis, from the residual system, by means of this sort of multiplicative transformation.

This question was addressed in the series of monographs [8]–[12], where it was shown that one of three situations may obtain. In the first of these, one may remove any finite number of elements from the basis and the residual system can be multiplicatively transformed into a basis. In [10], it was shown that the Haar system has this property; a characterization of the usable multiplier functions was given; and it was observed that certain coinfinite subsystems of the Haar system could be transformed in this way. At the other end of the spectrum, there are Schauder bases for $L^p(E)$ for which no cofinite subsystem can be multiplicatively transformed into a basis for $L^p(E)$. The trigonometric and Walsh systems are familiar examples of this type. (See [9], [11], [12].) Finally, there are Schauder bases for which some, but not all, of its cofinite subsets can be multiplicatively transformed into bases.

Taking now a different tack, suppose that the requirement that Φ be a Schauder basis is replaced by some weaker condition. Then, a variety of interesting propositions of Boas–Pollard type obtain.

Theorem 1. *Let $\Phi = \{\varphi_n : n = 1, 2, \dots\}$ be a quasibasis for some $L^p(E)$, $1 \leq p < \infty$, with E a measurable set of finite, positive measure. Then, to every natural number, N , there corresponds a bounded, measurable function, m , such that $\{m\varphi_n : n > N\}$ is a system of representation for $L^p(E)$.*

For the proof of this modest assertion, it suffices to repeat, mutatis mutandis, the arguments employed by Braun in his demonstration of Theorem B, but a somewhat more extensive modification of those arguments yields a much stronger result.

Theorem 2. *Let $\Phi = \{\varphi_n : n = 1, 2, \dots\}$ be a quasibasis for some $L^p(E)$, $1 \leq p < \infty$, with E a measurable set of finite, positive measure. Then, to every nonnegative integer, N , there corresponds a bounded, measurable function, m , such that $\{m\varphi_n : n > N\}$ is a quasibasis for each $L^r(E)$, $1 \leq r \leq p$.*

Corollary 3. *Let $\Phi = \{\varphi_n : n = 1, 2, \dots\}$ be a complete, orthonormal system on E , a measurable set of finite, positive measure. Then, there exists a bounded,*

measurable function, m , such that $\{m\varphi_n : n = 1, 2, \dots\}$ is a quasibasis in each space $L^r(E)$, $1 \leq r \leq 2$.

In particular, the corollary shows that the trigonometric and Walsh systems can be multiplicatively transformed into quasibases for $L^1[0, 1]$, a result hitherto unknown.

The following proof of Theorem 2 depends upon a proposition of Menshov–Talalyan type. (See, for example, [18].)

Lemma 4. *Let $\Phi = \{\varphi_n : n = 1, 2, \dots\}$ be a quasibasis for some space $L^p[0, 1]$, with $1 \leq p < \infty$, and let $f \in L^p[0, 1]$. Then, to each positive real number, ε , and each natural number, N , there correspond a measurable set e and a Φ -polynomial, $P = \sum_{k=N}^M a_k \varphi_k$, such that:*

$$e \subset [0, 1], \quad \text{and} \quad |e| < \varepsilon;$$

$$\|f - P\|_{L^p(G)} < \varepsilon, \quad \text{where } G = [0, 1] \setminus e;$$

and

$$\sup \left\{ \left\| \sum_{k=N}^s a_k \varphi_k \right\|_{L^p(E)} : N \leq s \leq m \right\} < \varepsilon + \|f\|_{L^p(E)},$$

for every measurable subset, E , of G .

Proof. The argument employs the following result, due to Fejér (see [19, 49, 376].)

Lemma C. *Let $\beta : \mathbb{R} \rightarrow \mathbb{R}$ be periodic, of period 1, and let $\beta|_{[0,1]} \in L^p[0, 1]$, for some $p \in [1, +\infty]$. Then, for every $\alpha \in L^q[0, 1]$, where p and q are conjugates, one has*

$$\lim_n \int_0^1 \alpha(t)\beta(nt)dt = \int_0^1 \alpha(t)dt \int_0^1 \beta(t)dt.$$

Certainly, one may assume, without loss of generality, that $0 < \varepsilon < 1$.

Let $\Psi = \{\psi_n^* : n = 1, 2, \dots\}$ be a system of coefficient functionals associated with Φ , and, for each n , let $\psi_n \in L^q[0, 1]$ be a representative of ψ_n^* . Then, the n^{th} partial-sum operator is given by

$$S_n(\cdot) = \sum_{k=1}^n \psi_k^*(\cdot)\varphi_k = \sum_{k=1}^n a_k(\cdot)\varphi_k,$$

where

$$a_k(\cdot) = \int_0^1 (\cdot)\psi_k dt.$$

Since

$$\lim_n \|\alpha - S_n(\alpha)\|_p = 0, \quad \forall \alpha \in L^p[0, 1],$$

an application of the Banach–Steinhaus theorem yields a positive constant, C_p , such that

$$\|S_n\| \leq C_p, \quad \forall n \in \mathbb{N}.$$

There exist a natural number n and a corresponding step function

$$g = \sum_{i=1}^{2^n} \gamma_i \chi_{\Delta_n^i}, \quad \text{where } \Delta_n^i = \left(\frac{i-1}{2^n}, \frac{i}{2^n} \right), \quad i = 1, \dots, 2^n,$$

such that

$$\|f - g\|_p < \frac{\varepsilon}{4},$$

and

$$\|g\|_{L^p(\Delta_n^i)} < \frac{\varepsilon}{8C_p}, \quad i = 1, \dots, 2^n.$$

(For example, one could use an integral mean of f associated with a dyadic partition, of $[0, 1]$, of sufficiently high order.)

Let $n_1 = N$, let

$$B_1 = \max\{\|\varphi_k\|_p : 1 \leq k \leq n_1\},$$

and let

$$\eta_1 = \frac{\varepsilon}{2^{n+3}n_1B_1}.$$

By virtue of Fejér's lemma, there exists a natural number, s_1 , such that $s_1 > n$, and

$$|b_k^1| = \left| \int_{\Delta_n^1} g(t)\psi_k(t)I_{\frac{\varepsilon}{2}}(2^{s_1}t)dt \right| < \eta_1, \quad \forall k \in [1, n_1].$$

Let g_1 be the function defined on $[0, 1]$ by the equation

$$g_1(t) = \gamma_1\chi_{\Delta_n^1}(t)I_{\frac{\varepsilon}{2}}(2^{s_1}t),$$

let n_2 be a natural number, greater than n_1 , such that

$$\left\| g_1 - \sum_{k=1}^{n_2-1} a_k(g_1)\varphi_k \right\|_p < \varepsilon 2^{-n-3},$$

and let

$$P_1 = \sum_{k=n_1}^{n_2-1} a_k(g_1)\varphi_k.$$

Then,

$$\begin{aligned} \|g_1 - P_1\|_p &\leq \left\| g_1 - \sum_{k=1}^{n_2-1} a_k(g_1)\varphi_k \right\|_p + \left\| \sum_{k=1}^{n_1-1} a_k(g_1)\varphi_k \right\|_p \\ &< \varepsilon 2^{-n-3} + \left\| \sum_{k=1}^{n_1-1} b_k^1\varphi_k \right\|_p \\ &\leq \varepsilon 2^{-n-3} + n_1\eta_1B_1 < \varepsilon 2^{-n-2}. \end{aligned}$$

Writing, for the moment, $a = \frac{\varepsilon}{2}$, one finds that

$$\begin{aligned} \int_{\Delta_n^1} |g_1|^p dt &= ((1-a)/a)^p |\gamma_1|^p a |\Delta_n^1| + |\gamma_1|^p (1-a) |\Delta_n^1| \\ &= [(1-a)^p + a^{p-1}(1-a)] |\gamma_1|^p |\Delta_n^1| / a^{p-1} \leq (2/a^{p-1}) \int_{\Delta_n^1} |g|^p dt; \end{aligned}$$

thus,

$$\begin{aligned} \sup \left\{ \left\| \sum_{k=n_1}^m a_k(g_1)\varphi_k \right\|_p^p : n_1 \leq m < n_2 \right\} &\leq 2^p C_p^p \|g_1\|_p^p \\ &\leq (2^{p+1} C_p^p / a^{p-1}) \int_{\Delta_n^1} |g|^p dt < (2^{2p} C_p^p / \varepsilon^{p-1}) (\varepsilon / 8 C_p)^p \\ &= \varepsilon 2^{-p} \leq \varepsilon / 2. \end{aligned}$$

Proceeding inductively, suppose that the polynomials associated with the intervals Δ_n^j , $1 \leq j < i$, have been determined. Let

$$B_i = \max\{\|\varphi_k\|_p : 1 \leq k \leq n_i\},$$

and let

$$\eta_i = \frac{\varepsilon}{2^{n+3} n_i B_i}.$$

Another application of Fejér’s lemma yields an integer, s_i , such that

$$s_i > n,$$

and

$$|b_k^i| = \left| \int_{\Delta_n^i} g(t)\psi_k(t) I_{\frac{\varepsilon}{2}}(2^{s_i}t) dt \right| < \eta_i, \quad \forall k \in [1, n_i].$$

Following the same course traversed above, one determines a natural number n_{i+1} , greater than n_i , such that

$$\left\| g_i - \sum_{k=1}^{n_{i+1}-1} a_k(g_i)\varphi_k \right\| < 2^{-n-3}\varepsilon.$$

Then, for

$$P_i = \sum_{k=n_i}^{n_{i+1}-1} a_k(g_i)\varphi_k,$$

one has

$$\|g_i - P_i\|_p < 2^{-n-3}\varepsilon + \eta_i n_i B_i < 2^{-n-2}\varepsilon,$$

and

$$\sup \left\{ \left\| \sum_{k=n_i}^m a_k(g_i)\varphi_k \right\|_p^p : n_i \leq k < n_{i+1} \right\} \leq C_p^p \|g_i\|_p^p < \frac{\varepsilon}{2}.$$

Finally, let

$$P = \sum_{i=1}^{2^n} P_i,$$

let

$$e = \left\{ t \in [0, 1] : \sum_{i=1}^{2^n} I_{\frac{\varepsilon}{2}}(2^{s_i}t) \chi_{\Delta_n^i}(t) \neq 1 \right\},$$

and let

$$G = [0, 1] \setminus e.$$

From the definition of $I_{\frac{\varepsilon}{2}}$, one has

$$|\{t \in \Delta_n^i : I_{\frac{\varepsilon}{2}}(2^{s_i}t) \neq 1\}| = \left(\frac{\varepsilon}{2}\right) |\Delta_n^i|,$$

so that

$$|e| = \sum_{i=1}^{2^n} \left(\frac{\varepsilon}{2}\right) |\Delta_n^i| < \varepsilon.$$

Since g_i agrees with g on G ,

$$\|f - P\|_{L^p(G)} = \|(f - g) + (g - P)\|_{L^p(G)} \leq \frac{\varepsilon}{2} + \sum_{i=1}^{2^n} \|g_i - P_i\| < \varepsilon.$$

As for the third condition, one notes that, for every measurable subset, E , of G ,

$$\begin{aligned} \max \left\{ \left\| \sum_{i=1}^j P_i \right\|_{L^p(E)} : 1 \leq j \leq 2^n \right\} &\leq \max \left\{ \left\| \sum_{i=1}^j (P_i - g_i) \right\|_{L^p(E)} : 1 \leq j \leq 2^n \right\} \\ &+ \left\| \sum_{i=1}^j g_i - f \chi_{\cup_{i=1}^j \Delta_n^i} \right\|_{L^p(E)} + \|f\|_{L^p(E)} \\ &< \sum_{i=1}^j 2^{-n-2} \varepsilon + \left\| \sum_{i=1}^j (g_i - f) \chi_{\Delta_n^i} \right\|_{L^p(E)} + \|f\|_{L^p(E)} \\ &\leq \frac{\varepsilon}{4} + \|g - f\|_p + \|f\|_{L^p(E)} < \frac{\varepsilon}{2} + \|f\|_{L^p(E)}; \end{aligned}$$

thus, for $m \geq N$, say $n_j \leq m < n_{j+1}$, one has

$$\left\| \sum_{k=N}^m a_k \varphi_k \right\|_{L^p(E)} \leq \left\| \sum_{i=1}^j P_i \right\|_{L^p(E)} + \left\| \sum_{k=n_j}^m a_k (g_j) \varphi_k \right\|_{L^p(E)} < \varepsilon + \|f\|_{L^p(E)}. \quad \square$$

Proof of Theorem 2. Without loss of generality, one may assume that $E = [0, 1]$.

If $\{f_k : k = 1, 2, \dots\}$ is any denumerable subset of $L^p[0, 1]$, then, by virtue of Lemma 3, one may follow the trail blazed by Braun in order to construct a

measurable function, m , with $0 \leq m(t) \leq 1$, for all $t \in [0, 1]$, a double sequence of Φ -polynomials $\{P_{kj}\}_{k=1, j=k}^\infty$,

$$P_{kj} = \sum_{i=n_{k-1}(j)+1}^{n_k(j)} a_i \varphi_i,$$

with

$$N = n_0(1) < n_1(1) = n_0(2) < n_1(2) < n_2(2) = n_0(3) < \dots < n_0(j) < n_1(j) < \dots < n_j(j) = n_0(j+1) < \dots,$$

and a sequence $\{D_n\}_{n=1}^\infty$ of measurable subsets of $[0, 1]$ such that:

- (i) $|[0, 1] \setminus D_n| < \delta_n, \{\delta_n\}_{n=1}^\infty \downarrow 0$;
- (ii) $\|f_k - m \sum_{j=k}^\ell P_{kj}\|_p \leq 2^{-\ell}, \forall k, \forall \ell \geq k$;
- (iii) $\sup \left\{ \left\| m \sum_{i=n_{k-1}(\ell)+1}^s a_i \varphi_i \right\|_p : s \leq n_k(\ell) \right\} \leq 2^{-\ell+2}, \text{ if } \ell > k$;

and, for every measurable set $F \subset D_\ell$,

- (iv) $\sup \left\{ \left\| m \sum_{i=n_{\ell-1}(\ell)+1}^s a_i \varphi_i \right\|_{L^p(F)} : s \leq n_\ell(\ell) \right\} \leq 2^{-\ell+1} + \|f_\ell\|_{L^p(F)}$;
- (v) $\sup \left\{ \left\| m \sum_{i=n_{\ell-1}(\ell)+1}^s a_i \varphi_i \right\|_{L^p([0,1] \setminus D_\ell)} : s \leq n_\ell(\ell) \right\} \leq 2^{-\ell+2}$.

Now let $\{f_k : k = 1, 2, \dots\}$ be the Haar system, normalized with respect to the L^p -norm, and let $\{g_k : k = 1, 2, \dots\}$ be the corresponding conjugate system. One associates with $\{m\varphi_k : k > N\}$ the system Ψ whose members are determined in the following manner:

$$\psi_i = a_i g_k, \text{ if } n_{k-1}(\ell) < i \leq n_k(\ell), \text{ for some } k \text{ and } \ell \geq k.$$

Then, with the functionals defined on $L^r[0, 1]$ by setting

$$b_i(\cdot) = \int_0^1 (\cdot) \psi_i dt, \text{ for } i = N + 1, N + 2, \dots,$$

$\{m\varphi_i : i > N\}$ proves to be a quasibasis for $L^r[0, 1]$, for each $r \in [1, p]$.

The following estimates suffice for the demonstration.

Let f be an arbitrary element of $L^r[0, 1]$, and let $\sum_{k=1}^\infty c_k(f) f_k$ be the expansion of f in the p -normalized Haar system. Since

$$|c_k(f)| = \left| \int_0^1 f g_k dt \right| = \left| \int_0^1 \left[f - \sum_{j=1}^{k-1} c_j f_j \right] g_k dt \right| \leq \left\| f - \sum_{j=1}^{k-1} c_j f_j \right\|_r \cdot \|g_k\|_{r'},$$

and since $\{f_k : k = 1, 2, \dots\}$ is a Schauder basis for $L^r[0, 1]$, it follows that

$$\lim_k |c_k(f)| / \|g_k\|_{r'} = 0.$$

The partial sums $S_n(f)$ of the series

$$\sum_{i=N+1}^\infty b_i(f) m\varphi_i$$

fall into two disjoint classes according as

$$n = n_k(\ell), \text{ or } n_{k-1}(\ell) < n < n_k(\ell),$$

for some natural numbers k and ℓ (with $\ell \geq k$). One estimates the error made in approximating f by $S_n(f)$, for each variety of n .

In the first case, if $k < \ell$, then one has

$$\begin{aligned} \left\| f - \sum_{i=N+1}^n b_i(f)m\varphi_i \right\|_r &= \left\| f - \sum_{j=1}^k \sum_{i=j}^{\ell} c_j mP_{ji} - \sum_{j=k+1}^{\ell-1} \sum_{i=j}^{\ell-1} c_j mP_{ji} \right\|_r \\ &\leq \left\| f - \sum_{j=1}^{\ell-1} c_j f_j \right\|_r + \left\| \sum_{j=1}^{\ell-1} c_j f_j - \sum_{j=1}^k \sum_{i=j}^{\ell} c_j mP_{ji} - \sum_{j=k+1}^{\ell-1} \sum_{i=j}^{\ell-1} c_j mP_{ji} \right\|_r \\ &= \left\| f - \sum_{j=1}^{\ell-1} c_j f_j \right\|_r + \left\| \sum_{j=1}^k c_j \left(f_j - \sum_{i=j}^{\ell} mP_{ji} \right) + \sum_{j=k+1}^{\ell-1} c_j \left(f_j - \sum_{i=j}^{\ell-1} mP_{ji} \right) \right\|_r \\ &\leq \left\| f - \sum_{j=1}^{\ell-1} c_j f_j \right\|_r + \sum_{j=1}^k |c_j| \left\| f_j - \sum_{i=j}^{\ell} mP_{ji} \right\|_r + \sum_{j=k+1}^{\ell-1} |c_j| \left\| f_j - \sum_{i=j}^{\ell-1} mP_{ji} \right\|_r \\ &\leq \left\| f - \sum_{j=1}^{\ell-1} c_j f_j \right\|_r + 2^{-\ell+1} \sum_{j=1}^{\ell-1} |c_j|, \end{aligned}$$

while, if $k = \ell$, then a similar analysis yields

$$\left\| f - \sum_{i=N+1}^n b_i(f)m\varphi_i \right\|_r \leq \left\| f - \sum_{j=1}^{\ell} c_j f_j \right\|_r + 2^{-\ell} \sum_{j=1}^{\ell} |c_j|.$$

Now,

$$|c_j| = |c_j(f)| = \left| \int_0^1 f g_j dt \right| \leq \|f\|_r \|g_j\|_{r'}, \quad \frac{1}{r} + \frac{1}{r'} = 1,$$

and

$$\begin{aligned} \|g_j\|_{r'} &= \|h_j\|_p \|h_j\|_{r'} = \|h_j\|_{\infty}^2 |\Delta_j|^{(\frac{1}{p} + \frac{1}{r'})}, \text{ where } \Delta_j \text{ is the support of } h_j, \\ &= |\Delta_j|^{(\frac{1}{p} + \frac{1}{r'} - 1)} = |\Delta_j|^{1 - (\frac{1}{q} + \frac{1}{r})} \leq j^{(\frac{1}{q} + \frac{1}{r} - 1)} \leq j; \end{aligned}$$

thus,

$$2^{-\ell+1} \sum_{j=1}^{\ell} |c_j| \leq 2^{-\ell} \ell(\ell + 1) \|f\|_r.$$

Finally, if $n_{k-1}(\ell) < n < n_k(\ell)$, with $k \in \mathbb{N}$, $\ell \in \mathbb{N}$ and $k \leq \ell$, then, for $k < \ell$,

$$\begin{aligned} \|f - S_n(f)\|_r &\leq \left\| f - \sum_{j=1}^{\ell-1} c_j f_j \right\|_r + 2^{-\ell} \ell(\ell + 1) \|f\|_r + \left\| c_k \sum_{i=n_{k-1}(\ell)+1}^n a_i m\varphi_i \right\|_p \\ &\leq \left\| f - \sum_{j=1}^{\ell-1} c_j f_j \right\|_r + 2^{-\ell} [\ell(\ell + 1) + 4k] \|f\|_r. \end{aligned}$$

On the other hand, if $k = \ell$, then

$$\|f - S_n(f)\|_r \leq \left\| f - \sum_{j=1}^{\ell-1} c_j f_j \right\|_r + 2^{-\ell} \ell(\ell + 1) \|f\|_r + \left\| c_\ell \sum_{i=n_{\ell-1}(\ell)+1}^n a_i m \varphi_i \right\|_r.$$

Setting

$$\sigma_{\ell n} = \sum_{i=n_{\ell-1}(\ell)+1}^n a_i m \varphi_i,$$

one has, by virtue of condition (v),

$$\begin{aligned} \|\sigma_{\ell n}\|_r^r &= \|\sigma_{\ell n}\|_{L^r([0,1] \setminus D_\ell)}^r + \|\sigma_{\ell n}\|_{L^r(D_\ell)}^r \\ &\leq 2^{(-\ell+2)r} + \|\sigma_{\ell n}\|_{L^r(D_\ell \cap \Delta_\ell)}^r + \|\sigma_{\ell n}\|_{L^r(D_\ell \setminus \Delta_\ell)}^r. \end{aligned}$$

Now, from condition (iv), one finds that

$$\begin{aligned} \|\sigma_{\ell n}\|_{L^r(D_\ell \setminus \Delta_\ell)}^r &\leq \|\sigma_{\ell n}\|_{L^p(D_\ell \setminus \Delta_\ell)}^r \\ &\leq (2^{-\ell+1} + \|f_\ell\|_{L^p(D_\ell \setminus \Delta_\ell)})^r = 2^{(-\ell+1)r}, \end{aligned}$$

while

$$\begin{aligned} \|\sigma_{\ell n}\|_{L^r(D_\ell \cap \Delta_\ell)}^r &\leq \|\sigma_{\ell n}\|_{L^p(D_\ell \cap \Delta_\ell)}^r \cdot |\Delta_\ell|^{1-\frac{r}{p}} \\ &\leq 2^r |\Delta_\ell|^{1-\frac{r}{p}} (2^{(-\ell+1)r} + \|f_\ell\|_p^r) = 2^r |\Delta_\ell|^{1-\frac{r}{p}} (2^{(-\ell+1)r} + 1)^p. \end{aligned}$$

Thus,

$$\begin{aligned} \left\| c_\ell \sum_{i=n_{\ell-1}(\ell)+1}^n a_i m \varphi_i \right\|_r &= |c_\ell| \|\sigma_{\ell n}\|_r \\ &\leq \left(2^{(-\ell+2)r+1} + 2^{r+p} |\Delta_\ell|^{1-\frac{r}{p}} \right)^{\frac{1}{r}} \cdot o(|\Delta_\ell|^{\frac{1}{p}-\frac{1}{r}}), \text{ as } \ell \rightarrow +\infty, \end{aligned}$$

and it follows that, for all $r \in [1, p]$,

$$\lim_n \|S_n(f) - f\|_r = 0.$$

□

4. Continuing along this avenue of investigation, corresponding results are obtainable if a completeness condition is substituted for the basis hypothesis.

Theorem 5. *Let $\Phi = \{\varphi_n : n = 1, 2, \dots\}$ be a minimal, complete system in some $L^p(E)$, $1 \leq p < \infty$, with E a measurable set of finite, positive measure. Then, to every natural number, N , there corresponds a bounded, measurable function, m , such that $\{m\varphi_n : n > N\}$ is a minimal, complete system in $L^p(E)$.*

The demonstration given below depends upon the following result from [11].

Theorem D. *Let E be a measurable set of finite, positive measure, let $\{f_n : n = 1, 2, \dots\}$ be a minimal, complete system in some $L^p(E)$, $1 \leq p < +\infty$, and let $\{f_n^* : n = 1, 2, \dots\}$ be the corresponding conjugate system. If N is a natural number, and if m is a bounded, measurable function, then the system $\{mf_n : n > N\}$ will be complete and/or minimal in $L^p(E)$ iff the following conditions (1) and/or (2), are satisfied:*

(1) A function of the form

$$(1/m) \sum_{k=1}^N a_k f_k^*$$

will be an element of $L^q(E)$ ($\frac{1}{p} + \frac{1}{q} = 1$; $q = +\infty$, if $p = 1$) iff every a_k is 0.

(2) To every $n > N$ there corresponds a unique numerical sequence $\{a_k^{(n)}\}_{k=1}^N$ such that

$$g_n = (1/m) \left[\sum_{k=1}^N a_k^{(n)} f_k^* + f_n^* \right]$$

belongs to $L^q(E)$.

Proof of Theorem 5. Let the elements of the system conjugate to Φ be denoted by φ_n^* , $n \in \mathbb{N}$, and let F be the (measurable) subset of E which contains those points that are simultaneously points of approximate continuity for each φ_n^* . Since each φ_n^* is approximately continuous almost everywhere in E , one has $|F| = |E|$. Let the natural number N be specified. One defines, inductively, a set $T = \{t_j : 1 \leq j \leq N\} \subset F$, on which $\{\varphi_1^*, \dots, \varphi_N^*\}$ is linearly independent, in the following manner:

- (i) Let t_1 be any point of F such that $\varphi_1^*(t_1) \neq 0$;
- (ii) Suppose elements t_1, \dots, t_k of F have been chosen so that $\sum_{j=1}^k a_j \varphi_j^*(t_i) = 0$, $\forall i = 1, \dots, k$, only if each $a_j = 0$. Let the numbers b_j be chosen (uniquely) so that $\theta_{k+1}^* = \varphi_{k+1}^* + \sum_{j=1}^k b_j \varphi_j^*$ satisfies $\theta_{k+1}^*(t_j) = 0$, for $1 \leq j \leq k$. Since $\int_F \varphi_{k+1} \theta_{k+1}^* dt = \int_E \varphi_{k+1} \varphi_{k+1}^* dt = 1$, θ_{k+1}^* is nonzero on a subset of F that has positive measure. Let t_{k+1} be any point of the latter set. It follows, at once, that $\{(\varphi_j^*(t_1), \dots, \varphi_j^*(t_{k+1})) : 1 \leq j \leq k+1\}$ is linearly independent.

For each $n > N$, let

$$\psi_n^* = \varphi_n^* + \sum_{k=1}^N a_k^{(n)} \varphi_k^*$$

where the numbers $a_k^{(n)}$ have been determined (uniquely) by the conditions

$$\psi_n^*(t_j) = 0, \quad 1 \leq j \leq N,$$

and, for each natural number n , let m_n be the function given by

$$m_n(t) = \max\{|\psi_k^*(t)| : N+1 \leq k \leq N+n\}.$$

Then, $m_n(t_j) = 0$, $\forall n \in \mathbb{N}$, $\forall t_j \in T$, and every m_n is approximately continuous at each t_j .

Since $\int_E \varphi_{N+1}^* \varphi_{N+1} dt = \int_E \psi_{N+1}^* \varphi_{N+1} dt$, and since

$$1 = \int_E \varphi_{N+1}^* \varphi_{N+1} dt \leq \|\varphi_{N+1}^*\|_q \|\varphi_{N+1}\|_p,$$

it follows that $|\psi_{N+1}^*|$, and thus also m_1 , are positive on a subset of E that has positive measure. Let λ_1 be a positive number such that

$$F_1 = F \cap \{t : m_1(t) > \lambda_1\}$$

has positive measure, and let

$$E_1 = (F \setminus F_1) \cap \{t : m_2(t) \neq 0\}.$$

If $|E_1| > 0$, then one chooses a $\lambda_2 \in (0, \lambda_1/2)$ such that

$$F_2 = E_1 \cap \{t : m_2(t) > \lambda_2\}$$

satisfies $|F_2| > |E_1|/2$. If $|E_1| = 0$, one sets $F_2 = \phi$ and takes $\lambda_2 = \lambda_1/2$.

Proceeding inductively, suppose that the triples $(\lambda_1, E_1, F_1), \dots, (\lambda_n, E_n, F_n)$ have been defined so that

$$E_k = (F \setminus \bigcup_{i=1}^k F_i) \cap \{t : m_{k+1}(t) \neq 0\}, \quad \text{for all } 1 \leq k \leq n,$$

and, if $|E_k| = 0$, then

$$F_{k+1} = \phi, \text{ and } \lambda_{k+1} = \lambda_k/2,$$

whereas, if $|E_k| > 0$, then $0 < \lambda_{k+1} < \lambda_k/2$,

$$F_{k+1} = E_k \cap \{t : m_{k+1}(t) > \lambda_{k+1}\}, \text{ and } |F_{k+1}| > |E_k|/2.$$

If $|E_n| = 0$, take $\lambda_{n+1} = \lambda_n/2$, and let $F_{n+1} = \phi$. If $|E_n| > 0$, choose $\lambda_{n+1} \in (0, \lambda_n/2)$ such that

$$F_{n+1} = E_n \cap \{t : m_{n+1}(t) > \lambda_{n+1}\}$$

satisfies $|F_{n+1}| > |E_n|/2$.

Let the function m_0 be defined, on E , by setting

$$m_0(t) = \lambda_n, \text{ if } t \in F_n, \quad \forall n \in \mathbb{N},$$

and, for those t that lie in $E \setminus \bigcup_{n=1}^\infty F_n$, let $m_0(t) = \lambda_1$. A function of the type guaranteed by the theorem will be obtained, in the sequel, by modifying the structure of m_0 in the neighborhood of the set T .

Since each m_n is approximately continuous on T , and since $m_n(t) > \lambda_n, \forall t \in \bigcup_{r=1}^n F_r$, it follows that each t_j is a point of density of $F \setminus \bigcup_{r=1}^n F_r, \forall n \in \mathbb{N}$.

Let $T' = \{t_j : t_j \text{ is a point of density of } F_0 = \bigcup_{n=1}^\infty F_n\}$, and, if T' is nonempty, relabel the elements of T so that $T' = \{t_j : 1 \leq j \leq r\}$, and, if $T \setminus T'$ is nonempty, let $T \setminus T' = \{\tau_j : 1 \leq j \leq s\}$.

Let $2\delta_0 = \min\{|t_i - t_j| : 1 \leq i < j \leq N\}$, and let $I = (t_1 - \delta_0, t_1 + \delta_0)$. Since each $\varphi_j^*, j = 1, \dots, N$, is approximately continuous at t_1 , there is, for each natural number n , a δ_n^* in $(0, \delta_0)$ such that

$$\begin{aligned} & \left| \left\{ t \in F : |\varphi_j^*(t) - \varphi_j^*(t_1)| < \frac{1}{n} \right\} \cap (t_1 - \delta, t_1 + \delta) \right| \\ & > \left(2 - \frac{1}{2N} \right) \delta, \quad \forall \delta \in (0, \delta_n^*), \quad j = 1, \dots, N. \end{aligned}$$

Let

$$\begin{aligned} J_n(\delta) &= \bigcap_{j=1}^N \left\{ t \in F : |\varphi_j^*(t) - \varphi_j^*(t_1)| < \frac{1}{n} \right\} \\ & \quad \cap \left((t_1 - \delta, t_1 + \delta) \setminus \left(t_1 - \frac{\delta}{2}, t_1 + \frac{\delta}{2} \right) \right). \end{aligned}$$

Since $t_1 \in T'$, there is an $\eta_1^* > 0$ such that

$$|F_0 \cap (t_1 - \eta, t_1 + \eta)| > \frac{3}{2}\eta, \quad \forall \eta \in (0, \eta_1^*),$$

Let δ_1 be an element of $(0, \min\{\delta_1^*, \eta_1^*\})$, let $J_1 = J_1(\delta_1)$, and let

$$S = \{n : |F_n \cap J_1| > 0\}.$$

Because $F_0 \cap J_1$ has positive measure, it follows that S is nonempty. Let j_1 be any element of S , and let

$$H_1 = F_{j_1} \cap J_1.$$

Since $F_0 \setminus \bigcup_{i=1}^{j_1} F_i$ has density 1 at t_1 , there is an $\eta_2^* > 0$ such that

$$\left| \left(F_0 \setminus \bigcup_{i=1}^{j_1} F_i \right) \cap (t_1 - \eta, t_1 + \eta) \right| > \frac{3}{2}\eta, \quad \forall \eta \in (0, \eta_2^*).$$

Let $\delta_2 \in (0, \min\{\frac{\delta_1}{2}, \delta_2^*, \eta_2^*\})$, and let $J_2 = J_2(\delta_2)$. Then, $|(F_0 \setminus \bigcup_{i=1}^{j_1} F_i) \cap J_2| > 0$, from which follows the existence of a natural number, $j_2 > j_1$, such that

$$H_2 = F_{j_2} \cap J_2$$

has positive measure.

Proceeding inductively, one determines a decreasing sequence $\{\varepsilon_n\}_{n=1}^\infty$, of positive numbers, an increasing sequence $\{j_n\}_{n=1}^\infty$, of natural numbers, and a sequence $\{H_n\}_{n=1}^\infty$, of measurable sets, such that $\lim_n \delta_n = 0$; $H_i \cap H_j = \phi$, $\forall i, \forall j, i \neq j$; $|H_n| > 0$ and $H_n \subset (t_1 - \delta_n, t_1 + \delta_n)$, $\forall n$; and each of $\varphi_1^*, \dots, \varphi_N^*$ is continuous at t_1 relative to $G_1 = \bigcup_{n=1}^\infty H_n$.

If $p > 1$, one defines the function h_1 , on E , in the following manner:

$$h_1(t) = \begin{cases} 1, & \text{if } t \in E \setminus G_1; \\ \mu_n |H_n|^{-\frac{1}{q}}, & \text{if } t \in H_n, n = 1, 2, \dots; \end{cases}$$

where $\mu_n = \lambda_{j_n}$. Then $h_1 \in L^q(E)$, since

$$\int_E h_1^q dt = \sum_{n=1}^\infty \mu_n^q + |E \setminus G_1| \leq |E| + \sum_{n=1}^\infty \lambda_n^q < +\infty,$$

but $h_1/m_0 \notin L^q(E \cap (t_1 - \varepsilon, t_1 + \varepsilon))$, $\forall \varepsilon > 0$, since, if $\delta_{n_0} < \varepsilon$, then

$$\int_{G \cap (t_1 - \varepsilon, t_1 + \varepsilon)} (h_1/m_0)^q dt \geq \sum_{n=n_0}^\infty \frac{|H_n|}{|H_n|} = +\infty.$$

If $p = 1$, one takes $h_1 = 1$, so that

$$h_1 \in L^\infty(E), \text{ while } h_1/m_0 \notin L^\infty(G_1 \cap (t_1 - \varepsilon, t_1 + \varepsilon)), \quad \forall \varepsilon > 0.$$

Similarly, there exist, for each $j = 1, \dots, r$, a measurable set $G_j \subset (t_j - \delta_0, t_j + \delta_0)$, and a measurable function h_j , such that each of $\varphi_1^*, \dots, \varphi_N^*$ is continuous at t_j relative to G_j ; $h_j(t) = 1$, if $t \in E \setminus G_j$; $(m_0/h_j)\chi_{G_j}$ is bounded; $h_j \in L^q(E)$; and $h_j/m_0 \notin L^q(G_j \cap (t_j - \varepsilon, t_j + \varepsilon))$, $\forall \varepsilon > 0$.

If $T \setminus T' \neq \emptyset$, one notes that, for every $j \in \{1, \dots, s\}$, and for every $\theta > 0$, one must have

$$\limsup_{\theta \rightarrow 0^+} \frac{|(\tau_j - \theta, \tau_j + \theta) \cap (F \setminus F_0)|}{2\theta} = \beta_j > 0,$$

since, in the contrary case, one would have

$$\begin{aligned} 1 &= \lim_{\theta \rightarrow 0^+} \frac{|(\tau_j - \theta, \tau_j + \theta) \cap F|}{2\theta} \\ &= \lim_{\theta \rightarrow 0^+} \left\{ \frac{|(\tau_j - \theta, \tau_j + \theta) \cap F_0|}{2\theta} + \frac{|(\tau_j - \theta, \tau_j + \theta) \cap (F \setminus F_0)|}{2\theta} \right\} \\ &= \lim_{\theta \rightarrow 0^+} \frac{|(\tau_j - \theta, \tau_j + \theta) \cap F_0|}{2\theta}, \end{aligned}$$

an impossibility, since no τ_j is a point of density of F_0 . Thus, if α_j is a fixed element of $(0, \beta_j)$, there will be a decreasing sequence, $\{\theta_n\}_{n=1}^\infty$, such that $\theta_1 < \delta_0$, $\lim_n \theta_n = 0$, and

$$\frac{|K_n|}{2\theta_n} > \alpha_j,$$

for all n , where

$$K_n = (F \setminus F_0) \cap \left((\tau_j - \theta_n, \tau_j + \theta_n) \setminus (\tau_j - \theta_{n+1}, \tau_j + \theta_{n+1}) \right), \quad \forall n \in \mathbb{N}.$$

(In general, these sequences will depend upon j , of course.)

If $p > 1$, one defines the function h_{r+j} , on E , by setting

$$h_{r+j}(t) = \begin{cases} |K_n|^{-\frac{1}{q}}, & \text{if } t \in K_n, \quad n = 1, 2, \dots; \\ 1, & \text{otherwise.} \end{cases}$$

If $p = 1$, one lets

$$h_{r+j}(t) = \begin{cases} \frac{1}{\theta_n}, & \text{if } t \in K_n, \quad n = 1, 2, \dots, \\ 1, & \text{otherwise.} \end{cases}$$

Finally, one sets

$$m = m_0 \left/ \prod_{j=1}^N h_j \right.$$

Certainly m is both bounded and measurable. In order to show that $\{m\varphi_n : n = N + 1, \dots\}$ is a minimal, complete system for $L^p(E)$, it suffices to show that conditions (1) and (2) of Theorem D are satisfied.

For the first of these, let $g = \sum_{k=1}^N a_k \varphi_k^*$, where a_1, \dots, a_N are real numbers, and suppose that $g/m \in L^q(E)$. Such a g must vanish on T' , for otherwise, there would exist a j in $\{1, \dots, r\}$ and a positive number ε such that

$$|g(t)| > |g(t_j)|/2 > 0, \quad \forall t \in G_{j\varepsilon} = G_j \cap (t_j - \varepsilon, t_j + \varepsilon),$$

from which would follow

$$\|g/m\|_{E,q} \geq \|g/m\|_{G_{j\varepsilon},q} \geq (|g(t_j)|/2) \|h_j/m_0\|_{G_{j\varepsilon},q} = +\infty.$$

As for the points of $T \setminus T'$, if such there be, suppose that, for some j , $2\gamma = |g(\tau_j)| > 0$. Let $\{\theta_n\}_{n=1}^\infty$ and $\{K_n\}_{n=1}^\infty$ be the sequences associated with τ_j , in the construction given above. By virtue of the approximate continuity of g at τ_j , there is a positive number η such that

$$|\{t : |g(t)| > \gamma\} \cap (\tau_j - \theta, \tau_j + \theta)| > \left(1 - \frac{\alpha_j}{2}\right)(2\theta), \quad \forall \theta \in (0, \eta);$$

thus, if n_0 be chosen so that $\theta_{n_0} < \eta$, then

$$|\{t : |g(t)| > \gamma\} \cap K_n| > \frac{1}{2}|K_n|, \quad \forall n \geq n_0.$$

It follows that, in the case $p > 1$,

$$\int_E |g/m|^q dt \geq \gamma^q \sum_{k=n_0}^{\infty} |K_n|^{-1} \left(\frac{1}{2}|K_n|\right) = +\infty,$$

while

$$\begin{aligned} \|g/m\|_{E,q} &\geq \|g/m\|_{K_n,q}, & \forall n \geq n_0, \\ &\geq \gamma/\theta_n \rightarrow +\infty, \end{aligned}$$

if $p = 1$. Hence, g vanishes at each point of T .

Because the set $\{(\varphi_j^*(t_1), \dots, \varphi_j^*(t_N)) : 1 \leq j \leq N\}$ is linearly independent, one must have $a_k = 0, \forall k$, so that condition (1) is satisfied.

A parallel argument shows that if, for some $n > N$ and real numbers a_1, \dots, a_N ,

$$(1/m) \left[\varphi_n^* + \sum_{k=1}^N a_k \varphi_k^* \right] \in L^q(E),$$

then

$$\varphi_n^*(t_j) + \sum_{k=1}^N a_k \varphi_k^*(t_j) = 0, \quad \forall j = 1, \dots, N,$$

which conditions ensure the satisfaction of condition (2). □

In [13] an analogous result was proved for complete, orthonormal systems in L^2 , and similar questions concerning some specific *CONS* were investigated in articles [11], [4], and [14]. For the classical orthonormal systems examined in [11], [14], it follows automatically, from the minimality and completeness, in L^p , of $\{m\varphi_n : n = N + 1, \dots\}$, that the system thus multiplicatively completed is, in fact, an M -basis. This is not generally the case, however, as the following proposition shows.

Theorem 6. *There exist a system $\{\varphi_n : n = 1, 2, \dots\}$ that is an M -basis for every space $L^p[0, 1], 1 \leq p < \infty$, and a bounded, measurable function, m , such that $\Psi = \{m\varphi_n : n = 2, \dots\}$ is minimal, complete in $L^p[0, 1]$, for all p in $[1, \infty)$, while Ψ is an M -basis for none of these spaces.*

Proof. The demonstration makes use of a construction suggested by Gelbaum [5]. (See also [16, 236].) Let $\{\alpha_j\}_{j=0}^{\infty}$ be an increasing sequence of real numbers, such that $\alpha_0 = 1$, and

$$\beta_j = 1/\alpha_{j-1} - 1/\alpha_j = 1/(j + 1)^3, \quad \forall j \in \mathbb{N},$$

and let

$$\varphi_n = \sum_{j=1}^n 2\alpha_{j-1} \cos \pi(j - 1)(\cdot), \quad \forall n \in \mathbb{N}.$$

Since $\Theta = \{\cos \pi j(\cdot) : j = 0, 1, \dots\}$ is complete in $L^p[0, 1]$, for every p in $[1, \infty)$, it follows that $\Phi = \{\varphi_n : n = 1, 2, \dots\}$ is complete in each of those spaces. If $\Phi^* = \{\varphi_n^* : n = 1, 2, \dots\}$, where

$$\varphi_n^* = (1/\alpha_{n-1}) \cos \pi(n-1)(\cdot) - (1/\alpha_n) \cos \pi n(\cdot), \quad \forall n \in \mathbb{N},$$

then $\{\Phi, \Phi^*\}$ is a biorthogonal system; thus, Φ is minimal in each space $L^p[0, 1]$, $1 \leq p < \infty$.

Now suppose that f is an integrable function such that $\varphi_n^*(f) = 0, \forall n \in \mathbb{N}$. Then,

$$(1/\alpha_{n-1}) \int_0^1 f(t) \cos \pi(n-1)t dt - (1/\alpha_n) \int_0^1 f(t) \cos \pi nt dt = 0, \quad n = 0, 1, \dots,$$

so that, for each nonnegative integer k ,

$$(1/\alpha_k) \int_0^1 f(t) \cos k\pi t dt = (1/\alpha_n) \int_0^1 f(t) \cos \pi nt dt, \quad \forall n > k.$$

By virtue of the Riemann-Lebesgue lemma, it follows that

$$\int_0^1 f(t) \cos k\pi t dt = 0, \quad \forall k = 0, 1, \dots;$$

thus, from the totality of Θ on $L[0, 1]$, it follows that $f(t) = 0$, a.e. Hence, Φ^* is total in $L^1[0, 1]$ and, thus, also in each $L^p[0, 1]$ with $1 < p < \infty$.

Let m be the identity function on $[0, 1]$, and let $\Psi = \{m\varphi_n : n = 2, \dots\}$. Since $a\varphi_1^*/m \in L^p[0, 1]$ iff $a = 0$, and, since, for each $n > 1$, $(1/m)[a^{(n)}\varphi_1^* + \varphi_n^*] \in L[0, 1]$ iff

$$a^{(n)}(1 - 1/\alpha_1) + 1/\alpha_{n-1} - 1/\alpha_n = 0,$$

iff

$$a^{(n)} = -\beta_n(1 - 1/\alpha_1)^{-1} = -8\beta_n,$$

an application of Theorem D shows that Ψ is minimal complete in $L^p[0, 1]$, for every p in $[1, \infty)$. Moreover, for $i \geq 2, j \geq 2$,

$$\int_0^1 [t\varphi_i(t)][(1/t)(\varphi_j^*(t) - 8\beta_j\varphi_1^*(t))]dt = \delta_{ij};$$

thus, the system conjugate to Ψ is $\Psi^* = \{\psi_n^* : n = 2, \dots\} = \{(1/m)(\varphi_n^* - 8\beta_n\varphi_1^*) : n = 2, \dots\}$.

Since $\alpha_n < 2$, for every n , it follows that $\sum_{n=2}^\infty \beta_n\varphi_n$ converges in each space $L^p[0, 1]$, $1 \leq p < \infty$. Denoting the sum of this series by f , one finds that

$$\int_0^1 [8f(t) - \varphi_1(t)]t\psi_n^*(t)dt = 0, \quad \forall n \geq 2;$$

thus, if Ψ^* were total in $L^p[0, 1]$, it would follow that

$$8f(t) - \varphi_1(t) = 0, \text{ a.e.}$$

Since this is impossible, by virtue of the minimality of the system Φ , in each space $L^p[0, 1]$, $1 \leq p < \infty$, it cannot be the case that Ψ is an M -basis for any of the L^p -spaces. □

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