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L^p AND OPERATOR NORM ESTIMATES FOR THE COMPLEX TIME HEAT OPERATOR ON HOMOGENEOUS TREES

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ABSTRACT. Let $\mathfrak{X}$ be a homogeneous tree of degree greater than or equal to three. In this paper we study the complex time heat operator $\mathcal{H}_\zeta$ induced by the natural Laplace operator on $\mathfrak{X}$. We prove comparable upper and lower bounds for the L^p norms of its convolution kernel h_ζ and derive precise estimates for the L^p–L^r operator norms of $\mathcal{H}_\zeta$ for ζ belonging to the half plane $\operatorname{Re}\zeta \geq 0$. In particular, when ζ is purely imaginary, our results yield a description of the mapping properties of the Schrödinger semigroup on $\mathfrak{X}$.

Let $\mathfrak{X}$ be a homogeneous tree of degree $q+1$, i.e., a connected graph with no loops in which every vertex is adjacent to $q+1$ other vertices. Unless explicitly stated otherwise, we will assume that $q \geq 2$. We will write $x \sim y$ if x and y are adjacent. $\mathfrak{X}$ carries a natural distance function d, $d(x,y)$ being the number of edges between the vertices x and y, and a natural measure, the counting measure, with respect to which we form the Lebesgue spaces $L^p(\mathfrak{X})$.

On $\mathfrak{X}$ there is also a natural Laplace operator defined by the formula

$$\mathcal{L}f(x) = \frac{1}{q+1} \sum_{x \sim y} [f(x) - f(y)] .$$

$\mathcal{L}$ is easily seen to be bounded on $L^p(\mathfrak{X})$ for every $1 \leq p \leq +\infty$, and self-adjoint on $L^2(\mathfrak{X})$, and therefore the heat operator $\mathcal{H}_t$, which is spectrally defined on $L^2(\mathfrak{X})$ by

$$\mathcal{H}_t f = \int_{\sigma_2(\mathcal{L})} e^{-t\lambda} d\, P_\lambda f \qquad \forall t \in (0, +\infty) \quad \forall f \in L^2(\mathfrak{X}),$$

where $\sigma_2(\mathcal{L})$ denotes the L^2 spectrum of $\mathcal{L}$, and P_λ its spectral resolution, is also given by the series

$$\mathcal{H}_t = \sum_{k=0}^{+\infty} \frac{(-t)^k \mathcal{L}^k}{k!}$$

and the series is absolutely convergent in the $L^p(\mathfrak{X})$ operator norm. Moreover one can replace the positive real parameter t with the complex parameter ζ and obtain a group of bounded operators on $L^p(\mathfrak{X})$.

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A detailed study of the heat operator for $t \geq 0$ was carried out in [CMS2], where comparable upper and lower bounds for the L^p–L^r operator norms of $\mathcal{H}_t$ and for the L^p norms of its convolution kernel h_t were obtained. In this paper we consider the analogous problem for the complex-time heat operator $\mathcal{H}_\zeta$.

We describe the behaviour of the $L^p(\mathfrak{X})$ norms of the heat kernel h_ζ for all p in $[1, +\infty]$, when ζ belongs to the half plane Re $\zeta \geq 0$ (actually our results are easily seen to hold for ζ in the half plane Re $\zeta \geq -\eta$, for fixed $\eta > 0$). Our analysis allows us in fact to obtain estimates for the norm of h_ζ in the Lorentz spaces $L^{p,r}(\mathfrak{X})$ without extra effort, when p and r are such that $1 \leq p < 2$ and $1 \leq r \leq 2$, or $p > 2$ and $1 \leq r \leq \infty$. Using the detailed description of the space of bounded convolution operators from L^p to L^r in terms of Lorentz spaces obtained in [CMS1], we are then able to derive precise bounds for the L^p–L^r operator norms of $\mathcal{H}_\zeta$ for every $1 \leq p \leq r \leq +\infty$, when Re $\zeta \geq 0$. In particular, by taking ζ to be purely imaginary, our estimates describe the behaviour of the L^p and L^p–L^r operator norms of the Schrödinger semigroup.

We note that the results in [CMS2] were obtained by means of techniques of spherical Fourier analysis on $\mathfrak{X}$. In the complex-time setting the analysis becomes considerably more complicated, essentially due to the fact that the convolution kernel h_ζ is no longer positive. Techniques of spherical analysis can still be employed to derive lower bounds, but upper bounds are not so easily obtained in this fashion.

To obviate this difficulty, we use a formula derived in [CMS2], which expresses h_ζ in terms of the heat kernel $h_\zeta^{\mathbb{Z}}$ on the group of integers $\mathbb{Z}$. The latter has an expression in terms of the modified Bessel function $I_n(z)$. Using this, and suitable approximation formulae for $I_n(z)$, we are able to obtain the required upper bounds.

In this respect, our technique is similar to that used to estimate the heat kernel on symmetric spaces of non-compact type, where one realises the heat kernel on the symmetric spaces as the inverse Abel transform of the Euclidean heat kernel (see e.g. [G]). Two differences are worth highlighting. On the one hand, our task is simplified by the fact that in our case the infinitesimal generator of the heat operator is bounded. Consequently we do not have to deal with local analysis, which is a major task in the continuous case, and the heat semigroup is in fact a group. This for example accounts for the L^p–L^r boundedness of the Schrödinger semigroup, a fact that has no counterpart in the continuous setting. On the other hand, the heat kernel on $\mathbb{Z}$ is not an elementary function, and a substantial problem we have to overcome is the control of the error terms introduced when approximating $h_\zeta^{\mathbb{Z}}$ by its asymptotic expansion.

Indeed, these error terms turn out to be uniformly bounded only in fixed sectors $|\arg \zeta| \leq \theta_o < \pi/2$ (see the Appendix below). Nevertheless, by a careful analysis, we are actually able to deduce norm estimates for h_ζ that hold uniformly in fixed half planes strictly contained in Re $\zeta \geq 0$. The extension to the full half plane Re $\zeta \geq 0$ (and indeed to the half planes Re $\zeta \geq -\eta$ with $\eta \geq 0$ fixed) is then achieved using the group property.

The kind of results we obtain is effectively illustrated by our main theorem. To state it we need to introduce a little notation.

For p in $[1, +\infty]$, denote by p' the conjugate index $p/(p-1)$, and let $\delta(p) = 1/p - 1/2$. Let $\gamma : \mathbb{C} \to \mathbb{C}$ be the function defined by

$$\gamma(z) = \frac{q^{1/2}}{q+1}(q^{iz} + q^{-iz}) = \frac{2q^{1/2}}{q+1}\cos(z \log q).$$

Note that the L^p spectrum $\sigma_p(\mathcal{L})$ of $\mathcal{L}$ is the image under $1-\gamma$ of the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq |\delta(p)|\}$, so that, if b_p denotes the infimum of $\inf \operatorname{Re}(\sigma_p(\mathcal{L}))$, we have $b_p = 1 - \gamma(i\delta(p))$ (see, e.g., [FTN], Chapter 2).

Let $\|\mathcal{H}_\zeta\|_{p,r}$ denote the L^p–L^r norm of $\mathcal{H}_\zeta$, and, given two nonnegative functions f and g, both defined on a set $\mathbf{D}$, write $f(\zeta) \sim g(\zeta)$ in $\mathbf{D}$ if there exist positive constants C_1 and C_2 such that $C_1 f(\zeta) \leq g(\zeta) \leq C_2 f(\zeta)$ for all ζ in $\mathbf{D}$. Finally, let $\Phi_p(\zeta)$ be the equivalent norm on $\mathbb{C}$ defined by the formula

$$\Phi_p(\zeta) = \gamma(i\delta(p)) \left((\operatorname{Re}\zeta)^2 + \tanh^2(\delta(p)\log q)(\operatorname{Im}\zeta)^2\right)^{1/2}.$$

With this notation we have the following:

Theorem 1. *Let $(\mathcal{H}_\zeta)$ be the heat operator defined above. Then the following hold:*

(i) *for all* $1 \leq p \leq r \leq +\infty$,

$$\|\mathcal{H}_\zeta\|_{p,r} \sim 1 \qquad \forall \zeta : \ |\zeta| \leq 1;$$

(ii) *if* $p = r = 2$, *then*

$$\|\mathcal{H}_\zeta\|_{p,r} = \exp\left[-b_2 \operatorname{Re} \zeta\right] \qquad \forall \zeta : \ \operatorname{Re} \zeta \geq 0;$$

(iii) *if* $p = r \neq 2$ *is in* $[1, \infty]$, *then*

$$\|\mathcal{H}_\zeta\|_{p,r} \sim \exp\left[-\operatorname{Re} \zeta + \Phi_p(\zeta)\right] \qquad \forall \zeta : \ \operatorname{Re} \zeta \geq 0;$$

(iv) *if p and r are such that either* $1 \leq p < r = 2$ *or* $2 = p < r \leq \infty$, *then*

$$\|\mathcal{H}_\zeta\|_{p;r} \sim \min\left\{1, \operatorname{Re}^{-3/4} \zeta\right\} \exp\left[-b_2 \operatorname{Re} \zeta\right] \qquad \forall \zeta : \ \operatorname{Re} \zeta \geq 0;$$

(v) *if p and r are such that* $1 \leq p < r < 2$, *then*

$$\|\mathcal{H}_\zeta\|_{p;r} \sim |\zeta|^{-1/2r'} \exp\left[-\operatorname{Re}\zeta + \Phi_r(\zeta)\right] \qquad \forall \zeta : \ |\zeta| \geq 1, \operatorname{Re} \zeta \geq 0;$$

(vi) *if p and r are such that* $2 < p < r \leq \infty$, *then*

$$\|\mathcal{H}_\zeta\|_{p;r} \sim |\zeta|^{-1/2p} \exp\left[-\operatorname{Re}\zeta + \Phi_p(\zeta)\right] \qquad \forall \zeta : \ |\zeta| \geq 1, \operatorname{Re} \zeta \geq 0;$$

(vii) *if p and r are such that* $1 \leq p < 2 < r \leq \infty$, *then*

$$\|\mathcal{H}_\zeta\|_{p;r} \sim |\zeta|^{-3/2} \exp\left[-b_2 \operatorname{Re} \zeta\right] \qquad \forall \zeta : \ |\zeta| \geq 1, \operatorname{Re} \zeta \geq 0.$$

1. Preliminaries and notation

Let o be a reference point in $\mathfrak{X}$ and write $|x|$ for $d(x, o)$. We say that a function f on $\mathfrak{X}$ is radial if it depends only on $|x|$. If $E(\mathfrak{X})$ is a function space on $\mathfrak{X}$, we will denote by $E(\mathfrak{X})^\sharp$ the subspace of radial elements in $E(\mathfrak{X})$.

Let G be the group of automorphisms of $(\mathfrak{X}, d)$, and G_o the isotropy subgroup of o. Then $\mathfrak{X}$ may be identified with the coset space G/G_o. By means of this identification functions on $\mathfrak{X}$ can be identified with G_o–right-invariant functions on G, and radial functions correspond to G_o–bi-invariant functions on G.

We endow the totally disconnected unimodular group G with the Haar measure such that the open compact subgroup G_o has unit mass. Then

$$\sum_{x \in \mathfrak{X}} f(x) = \int_G f(g \cdot o)\, dg, \qquad \forall f \in L^1(\mathfrak{X}),$$

and this allows us to define the convolution of two functions on $\mathfrak{X}$ as

$$f_1 * f_2(g \cdot o) = \int_G f_1(h \cdot o)\, f_2(h^{-1} g \cdot o)\, dh \qquad \forall g \in G,$$

whenever the integral makes sense. We observe that, in case f_2 is radial, we can write

$$f_1 * f_2(x) = \sum_{n=0}^{+\infty} f_2(x_n) \sum_{d(x,y)=n} f_1(y),$$

where, for every n, x_n is such that $|x_n| = n$.

It follows that the Laplace operator can be expressed in terms of convolution by the formula

$$\mathcal{L}f = f * (\delta_o - \nu),$$

where δ_o is the Dirac measure at o, and ν is the normalised radial measure concentrated on the set $\mathfrak{S}_1 = \{x \in \mathfrak{X} : |x| = 1\}$. More generally, every G–invariant (in the sense that $\mathcal{K}(f \circ g) = (\mathcal{K}f) \circ g$ for every g in G) continuous operator from $L^p(\mathfrak{X})$ to $L^r(\mathfrak{X})$ (weak-star continuous if $r = +\infty$) is given by right convolution with a G_o–bi-invariant kernel k. We will denote by $Cv_p^r(\mathfrak{X})$ the space of such convolution kernels. The norm of an element k in Cv_p^r is then defined as the norm of the corresponding invariant operator. A simple duality argument shows that $Cv_p^r(\mathfrak{X}) = Cv_{r'}^{p'}(\mathfrak{X})$ with equality of norms, and the easy generalisation of a theorem of Hörmander [H] shows that $Cv_p^r(\mathfrak{X})$ is the trivial space if $p > r$.

Now we summarise the main features of spherical analysis on $\mathfrak{X}$. Let τ denote $2\pi/\log q$, and for every positive real number β let $\mathbf{S}_\beta$ and $\overline{\mathbf{S}}_\beta$ be the strips $\{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$ and $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq \beta\}$, respectively.

The spherical functions are defined to be the radial eigenfunctions of the Laplace operator $\mathcal{L}$ satisfying the normalisation condition $\phi(o) = 1$, and are given by

$$\phi_z(x) = \begin{cases} \left(1 + \dfrac{q-1}{q+1}\,|x|\right) q^{-|x|/2} & \forall z \in \tau\mathbb{Z}, \\ \left(1 + \dfrac{q-1}{q+1}\,|x|\right) q^{-|x|/2}(-1)^{|x|} & \forall z \in \tau/2 + \tau\mathbb{Z}, \\ \mathbf{c}(z)\, q^{(iz-1/2)|x|} + \mathbf{c}(-z)\, q^{(-iz-1/2)|x|} & \forall z \notin (\tau/2)\mathbb{Z}, \end{cases}$$

where $\mathbf{c}$ is the meromorphic function defined by the rule

$$\mathbf{c}(z) = \frac{q^{1/2}}{q+1}\, \frac{q^{1/2+iz} - q^{-1/2-iz}}{q^{iz} - q^{-iz}} \qquad \forall z \notin (\tau/2)\mathbb{Z}.$$

We remark that our parametrisation of spherical functions is different from that in [FTP] and [FTN]; our ϕ_z corresponds to their $\phi_{1/2+iz}$, and $\mathbf{c}$ is reparametrised similarly.

The spherical Fourier transform $\widetilde{f}$ of a function $f \in L^1(\mathfrak{X})^\sharp$ is then defined by the formula

$$\widetilde{f}(z) = \sum_{x \in \mathfrak{X}} f(x)\, \phi_z(x) \qquad \forall z \in \mathbf{S}_{1/2}.$$

By the symmetry properties of the spherical functions, $\widetilde{f}$ is even and τ-periodic in the strip $\mathbf{S}_{1/2}$.

Let $d\mu$ be the Plancherel measure defined on the torus $\mathbb{T} = \mathbb{R}/\tau\mathbb{Z}$, usually identified with $[-\tau/2, \tau/2)$, by the formula

$$d\mu(s) = c_G |\mathbf{c}(s)|^{-2}\, ds,$$

where $c_G = \dfrac{q\log q}{4\pi(q+1)}$. Then the spherical Fourier transformation extends to an isometry of $L^2(\mathfrak{X})^\sharp$ onto $L^2(\mathbb{T}, d\mu(s))$, and corresponding Plancherel and inversion formulae hold:

$$\|f\|_2 = \left(\int_{\tau/2}^{\tau/2} \left|\widetilde{f}(s)\right|^2 d\mu(s)\right)^{1/2} \qquad \forall f \in L^2(\mathfrak{X})^\sharp,$$

and

$$f(x) = \int_{-\tau/2}^{\tau/2} \widetilde{f}(s)\,\phi_s(x)\,d\mu(s) \qquad \forall x \in \mathfrak{X} \quad \forall f \in L^2(\mathfrak{X})^\sharp.$$

See, for instance, [FTN], Chapter 2. We also will need the following version of the classical Hausdorff–Young inequality which was proved in [CMS1].

Theorem 2. *Let $1 \le p < 2$ and $1 \le r \le 2$. If f is in $L^{p,r}(\mathfrak{X})^\sharp$, then $\widetilde{f}$ may be extended to an even, τ-periodic holomorphic function in the strip $\mathbf{S}_{\delta(p))}$, whose boundary values $\widetilde{f}(\cdot + i\delta(p))$ and $\widetilde{f}(\cdot - i\delta(p))$ are the Euclidean Fourier transforms of functions in $L^r(\mathbb{T})$. Moreover, the map $z \mapsto \widetilde{f}(z+\cdot)$ is continuous from $\overline{\mathbf{S}}_{\delta(p)}$ into $L^{r'}(\mathbb{T})$ and*

$$\left(\int_{-\tau/2}^{\tau/2} \left|\widetilde{f}(z+s)\right|^{r'} ds\right)^{1/r'} \le C\,\|f\|_{p,r} \qquad \forall z \in \overline{\mathbf{S}}_{\delta(p)}.$$

We will use the "variable constant convention", denoting by C a constant which may vary from place to place and may depend on any factor quantified before its occurrence, but not on factors quantified afterwards.

2. The complex-time heat kernel on $\mathfrak{X}$

Let $\mathcal{H}_\zeta$ be the complex-time heat operator of $\mathcal{L}$, and h_ζ its convolution kernel, so that

$$\mathcal{H}_\zeta f = f * h_\zeta \qquad \forall f \in \mathfrak{X}.$$

From the power series representation of $\mathcal{H}_\zeta$

$$\mathcal{H}_\zeta = \sum_{k=0}^{+\infty} \frac{(-\zeta)^k}{k!}\mathcal{L}^k,$$

one readily sees that

$$h_\zeta(x) = \sum_{k=0}^{+\infty} \frac{(-\zeta)^k}{k!}(\delta_o - \nu)^{(*k)} = e^{-\zeta}\sum_{k=0}^{+\infty} \frac{\zeta^k}{k!}(\nu)^{(*k)},$$

where $\nu^{(*k)}$ denotes the k-th convolution power of ν. Since $\|\nu\|_1 = 1$ it follows easily that

$$\|h_\zeta\|_1 \le e^{-\operatorname{Re}\zeta + |\zeta|}, \qquad \forall \zeta \in \mathbb{C}.$$

From this one deduces immediately that the series defining $\mathcal{H}_\zeta$ converges in the L^p operator norm, and therefore, by the inclusion properties of L^p spaces, in the L^p–L^r operator norm for every p and r such that $p \le r$. Moreover,

$$|||\mathcal{H}_\zeta|||_{p,r} \le \|h_\zeta\|_1.$$

Note also that the L^2 adjoint of $\mathcal{H}_\zeta$ is $\mathcal{H}_{\overline{\zeta}}$, so that

$$\|\!|\mathcal{H}_\zeta\|\!|_{p,r} = \|\!|\mathcal{H}_{\overline{\zeta}}\|\!|_{r',p'}.$$

As already mentioned, if $\zeta = it$, with $t \in \mathbb{R}$, then the heat operator defined above reduces to the Schrödinger semigroup, $\mathcal{H}_{it}$, which solves the Schrödinger equation

$$i\frac{d}{dt}\mathcal{H}_{it} = -\mathcal{L}\,\mathcal{H}_{it}.$$

Thus the properties listed above, and the norm estimates that we are going to establish, hold in particular for the latter semigroup. The L^p–L^r boundedness of $\mathcal{H}_{it}$ for $1 \le p \le r \le +\infty$ constitutes a significant difference from the case of Schrödinger equation on spaces that are not compact and symmetric. We also note that a direct derivation of the norm estimates for the Schrödinger semigroup would be significantly more difficult.

Since the spherical Fourier transform of $(\delta_o - \nu)$ is $1-\gamma(z)$, the spherical Fourier transform of the heat kernel is

$$\widetilde{h_\zeta}(z) = e^{-\zeta[1-\gamma(z)]},$$

whence, by spherical Fourier inversion,

$$h_\zeta(x) = \int_{-\tau/2}^{\tau/2} \exp\left[-\zeta(1-\gamma(s))\right]\, \phi_s(x)\, d\mu(s) \qquad \forall x \in \mathfrak{X}. \tag{1}$$

As mentioned in the introduction, upper bounds for h_ζ will be derived using a formula that expresses h_ζ in terms of the heat kernel on the group of integers $\mathbb{Z}$. To avoid ambiguity, we will denote with a sub- or superscript $\mathbb{Z}$ objects defined on $\mathbb{Z}$. Thus $\mathcal{L}^{\mathbb{Z}}$ is the Laplacian on $\mathbb{Z}$, which is defined by

$$\mathcal{L}^{\mathbb{Z}} f(d) = f(d) - \frac{f(d+1)+f(d-1)}{2} = f *_{\mathbb{Z}} (\delta_0 - \nu^{\mathbb{Z}}),$$

where δ_d is the Dirac measure at d, and $\nu^{\mathbb{Z}} = (\delta_1 + \delta_{-1})/2$, $h_\zeta^{\mathbb{Z}}$ is the complex-time heat kernel on $\mathbb{Z}$, and so on.

Let $\mathcal{F}$ be the Fourier transformation defined on $\mathbb{Z}$ by the formula

$$\mathcal{F}F(s) = \sum_{d=-\infty}^{+\infty} F(d) q^{i\,dz} \qquad \forall F \in L^1(\mathbb{Z}), \quad \forall s \in \mathbb{T},$$

with corresponding inversion formula

$$F(d) = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \mathcal{F}F(d) q^{-i\,ds}\, ds.$$

The Fourier transform of the convolution kernel associated to $\mathcal{L}^{\mathbb{Z}}$ is $[1-\cos(s\log q)]$; hence that of $h_\zeta^{\mathbb{Z}}$ is $\exp[-\zeta(1-\cos(s\log q))]$ and, by Fourier inversion,

$$\begin{aligned} h_\zeta^{\mathbb{Z}}(d) &= \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \exp\left[-\zeta(1-\cos(s\log q))\right] e^{-isd}\, ds \\ &= \frac{e^{-\zeta}}{2\pi}\int_{-\pi}^{\pi} \exp\left[\zeta\cos s\right] \cos(sd)\, ds = e^{-\zeta} I_{|d|}(\zeta), \end{aligned} \tag{2}$$

where $I_n(\zeta)$ denotes the modified Bessel function of order n, and the last equality follows from one of the integral representations of modified Bessel functions (see, e.g, [W], formula (4), p. 181).

In the following proposition we express h_ζ in terms of $h^{\mathbb{Z}}_\zeta$, and find a contour integral representation of h_ζ which is closely related to one of the representations of modified Bessel functions.

Proposition 3. *The following hold:*
(i)

$$h_\zeta(x) = \frac{2\, e^{-b_2\zeta}}{\gamma(0)\,\zeta}\, q^{-|x|/2} \sum_{k=0}^{\infty} q^{-k}\,(|x|+2k+1)\, h^{\mathbb{Z}}_{\gamma(0)\zeta}(|x|+2k+1).$$

(ii) *For every ζ with* $\operatorname{Re}\zeta > 0$ *and every x in $\mathfrak{X}$*

$$h_\zeta(x) = \frac{1}{2\pi i} q^{-|x|/2} e^{-\zeta} \int_{\infty - i\pi}^{\infty + i\pi} \exp\{\gamma(0)\zeta\cosh w - |x|\, w\}\, \frac{e^{2w}-1}{e^{2w}-q^{-1}}\, dw,$$

where the integral is evaluated over a contour in the complex half plane $\operatorname{Re} w > -\frac{1}{2}\log q$ *that originates from $\infty - i\pi$ and ends at $\infty + i\pi$.*

Proof. (i) is obtained replacing the real parameter t with ζ in the proof of [CMS2], Proposition 2.5, but, for completeness and the convenience of the reader, we outline the argument.

Using the explicit expression of spherical functions, and the identity $\mathbf{c}(-s) = \overline{\mathbf{c}(s)}$, we have

$$\phi_s(x)\, d\mu(s) = c_G \left(\mathbf{c}(-s)^{-1} q^{(is-1/2)|x|} + \mathbf{c}(s)^{-1} q^{(-is-1/2)|x|} \right),$$

for all x in $\mathfrak{X}$ and s in $[-\tau/2, \tau,/2)$. Since $\widetilde{h_\zeta}$ is even we may rewrite the inversion formula (1) as follows:

$$h_\zeta(x) = 2c_G \int_{-\tau/2}^{\tau/2} \exp\left[-\zeta(1-\gamma(s))\right]\, q^{(-is-1/2)|x|} \mathbf{c}(s)^{-1} ds,$$

and then, substituting the expression of $\mathbf{c}(s)$, changing variables, and simplifying, we finally arrive at

$$h_\zeta(x) = \frac{1}{2\pi} q^{-|x|/2} e^{-\zeta} \int_{-\pi}^{\pi} \exp\left\{\gamma(0)\zeta\cos s - is\,|x|\right\} \frac{1-e^{-2is}}{1-q^{-1}e^{-2is}}\, ds. \tag{3}$$

Expanding the denominator in the last integral and rearranging yields

$$h_\zeta(x) = e^{-b_2\zeta}\, q^{-|x|/2} \sum_{k=0}^{\infty} q^{-k} \left[h^{\mathbb{Z}}_{\gamma(0)\zeta}(|x|+2k) - h^{\mathbb{Z}}_{\gamma(0)\zeta}(|x|+2k+2) \right],$$

whence (i) follows using the formula

$$h^{\mathbb{Z}}_\zeta(d) - h^{\mathbb{Z}}_\zeta(d+2) = \frac{2(d+1)}{\zeta}\, h^{\mathbb{Z}}_\zeta(d+1),$$

which in turn is a consequence of (2) and of the recurrence formula $I_{\nu-1}(z) - I_{\nu+1}(z) = (2\nu/z) I_\nu(z)$ satisfied by the modified Bessel functions (see, e.g., [W], formula (1), p. 79).

To prove (ii) we perform the change of variables $u = e^{is}$ in (3) to obtain

$$h_\zeta(x) = \frac{1}{2\pi i} q^{-|x|/2} e^{-\zeta} \int_{\mathcal{C}} e^{\gamma(0)\zeta(u+u^{-1})/2} u^{-(|x|+1)} \frac{u^2-1}{u^2-q^{-1}}\, du,$$

where $\mathcal{C}$ is the unit circle centered at the origin. Since the integrand is singular at $u = 0$, $\pm q^{-1/2}$ and analytic elsewhere, by Cauchy's theorem the contour of integration can be deformed to any curve that goes around the points 0, $\pm q^{-1/2}$ once. Keeping into account the fast decay of the integrand as $\operatorname{Re} u \to -\infty$ in the strip $|\operatorname{Im} u| \leq C$ with C constant, we can in fact deform the contour to a curve that starts from ∞ with argument $-\pi$, encloses the points $0, \pm q^{-1/2}$ and ends at ∞ with argument $+\pi$. Performing the further change of variables $u = e^w$ yields (ii). $\square$

Note that since $\left|h^{\mathbb{Z}}_{\zeta}(d)\right| \leq \min\left\{1, C \operatorname{Re}^{-1/2} \zeta\right\}$ for every ζ with $\operatorname{Re} \zeta \geq 0$, the series in (i) is absolutely convergent in $\operatorname{Re} \zeta \geq 0$ and we have the trivial bound

$$|h_\zeta(x)| \leq C \min\left\{1, |\zeta|^{-1} \operatorname{Re}^{-1/2} \zeta e^{-b_2 \operatorname{Re} \zeta}\right\}.$$

We also remark that by applying the saddle point or the steepest descent method to the contour integral in (ii), one could, at least in principle, obtain the asymptotic expansion of $h_\zeta(x)$ for $|x|$ large, and upper and lower pointwise bounds for $|h_\zeta(x)|$. Either method involves a careful choice of a contour of integration passing through the critical points of the argument of the exponential. This step is fairly easy if $\zeta = t$ is assumed to be real, and indeed in this case one can recover the results of [CMS2], Proposition 2.5, without too much effort. On the other hand, the task of finding suitable integration contours is not entirely trivial when ζ is complex, especially if ζ is not confined to lie in a fixed sector, and the problem of obtaining uniform estimates for the resulting integral might be quite difficult. Alternatively, the asymptotic behavior of modified Bessel functions, and therefore of h_ζ, may also be obtained by means of differential equation techniques, which provide more manageable expressions for the error terms (see [O]). A discussion of the results that can be obtained using this approach is contained in the Appendix, where we also derive explicit upper bounds for the error terms (see Theorem A4). The pointwise upper estimates for $h^{\mathbb{Z}}_{\zeta}(d)$ that can be deduced from Theorem A4 and Proposition 3 (i) above may then be used to obtain norm estimates for h_ζ, as shown in the following corollary, where the norm $\|h_\zeta\|_{p,r}$ of h_ζ in the Lorentz space $L^{p,r}(\mathfrak{X})$ is estimated in terms of suitable weighted $L^r(\mathbb{N})$ norms of $h^{\mathbb{Z}}_{\gamma(0)\zeta}$.

Corollary 4. *For every p and r in $[1, +\infty)$, there exists a constant C such that, for every $\zeta \neq 0$,*

$$\|h_\zeta\|_{p,r} \leq C \frac{e^{-b_2 \operatorname{Re} \zeta}}{|\zeta|} \left(\sum_{d=0}^{+\infty} d^r q^{r\delta(p)d} \left| h^{\mathbb{Z}}_{\gamma(0)\zeta}(d) \right|^r \right)^{1/r}.$$

If $p \leq 2\log q/\log 2$, there exists a constant C' such that the reverse inequality also holds.

Proof. We recall that if $\{o = x_0, x_1, \dots\}$ is a geodesic emanating from o, so that $|x_d| = d$ for every d, then the map

$$f \mapsto \left(\sum_{d=0}^{+\infty} q^{dr/p} |f(x_d)|^r \right)^{1/r}$$

defines an equivalent norm on the space $L^{p,r}(\mathfrak{X})^\sharp$ (see [P]). Accordingly, by (i) above we have

$$\|h_\zeta\|_{p,r} \sim \frac{e^{-b_2 \operatorname{Re} \zeta}}{|\zeta|} \left(\sum_{d=0}^{+\infty} q^{dr/p} \left| \sum_{k=0}^{+\infty} (d+2k+1) q^{-(d+2k+1)/2} h^{\mathbb{Z}}_{\gamma(0)\zeta}(d+2k+1) \right|^r \right)^{1/r}$$
$$= \frac{e^{-b_2 \operatorname{Re} \zeta}}{|\zeta|} \left(\sum_{d=0}^{+\infty} \left| \sum_{k=0}^{+\infty} (d+2k+1) q^{-(2k+1)/p} q^{\delta(p)(d+2k+1)} h^{\mathbb{Z}}_{\gamma(0)\zeta}(d+2k+1) \right|^r \right)^{1/r}.$$

Denote by I the quantity in brackets on the right hand side of the formula above. By Minkowski's inequality

$$I \leq \sum_{k=0}^{+\infty} q^{-(2k+1)/p} \left(\sum_{d=0}^{+\infty} (d+2k+1)^r q^{r\delta(p)(d+2k+1)} \left| h^{\mathbb{Z}}_{\gamma(0)\zeta}(d+2k+1) \right|^r \right)^{1/r}$$
$$\leq \sum_{k=0}^{+\infty} q^{-(2k+1)/p} \left(\sum_{d=0}^{+\infty} d^r q^{r\delta(p)d} \left| h^{\mathbb{Z}}_{\gamma(0)\zeta}(d) \right|^r \right)^{1/r},$$

and the required upper estimate follows.

To prove the lower estimate, we break the inner series in the formula for I in the sum of its first term and the series for $k \geq 1$, we apply Minkowski inequality to estimate the $L^r(\mathbb{N})$ norm of the sum of these two terms by the difference of their norms, and again use the Minkowski inequality to estimate from above the norm of the second term. Thus we arrive at

$$I \geq q^{-1/p} \left(\sum_{d=0}^{+\infty} d^r q^{r\delta(p)(d)} \left| h^{\mathbb{Z}}_{\gamma(0)\zeta}(d) \right|^r \right)^{1/r} \left\{ 1 - \sum_{k=1}^{+\infty} q^{2k/p} \right\}.$$

The lower estimate now follows from the fact that the quantity in braces is equal to $(q^{2/p}-2)/(q^{2/p}-1)$. □

3. Norm estimates for the heat kernel: Lower bounds

In this section we prove lower estimates for the L^p norm of h_ζ, postponing to the following section a discussion of the corresponding upper estimates. For the proof of Theorem 1 we will only use lower bounds for the $L^p(\mathfrak{X})$ norms of the heat kernel, but since without any extra effort our techniques yield bounds for the norm $\|h_\zeta\|_{p,r}$ of h_ζ in the Lorentz space $L^{p,r}(\mathfrak{X})$ for p and r belonging to a large range of values, we state our result in this more general form.

Theorem 5. *Let h_ζ be the complex-time heat kernel. Then the following hold uniformly in* $\operatorname{Re} \zeta \geq 0$*:*

(i) *If $p = 2$,*

$$\|h_\zeta\|_p \sim \min\left\{1, \operatorname{Re}^{-3/4} \zeta\right\} \exp[-b_2 \operatorname{Re} \zeta].$$

(ii) *If p is in $[1,2)$ and r is in $[1,2]$, then*

$$\|h_\zeta\|_{p,r} \geq C \min\left\{1, |\zeta|^{-1/2r'}\right\} \exp\left[-\operatorname{Re} \zeta + \Phi_p(\zeta)\right].$$

(iii) *If* $p=+\infty$, *then,*

$$\|h_\zeta\|_p \geq C \min\left\{1, |\zeta|^{-3/2}\right\} \exp\left[-b_2 \operatorname{Re}\zeta\right].$$

(iv) *If* $2<p<+\infty$, *and* $1\leq r\leq+\infty$ *then*

$$\|h_\zeta\|_{p,r} \geq C \min\left\{1, |\zeta|^{-3/2}\right\} \exp\left[-b_2 \operatorname{Re}\zeta\right].$$

Proof. If $p=2$, by Plancherel theorem

$$\|h_\zeta\|_2^2 = \int_{-\tau/2}^{\tau/2} \exp\left(-2\operatorname{Re}\zeta[1-\gamma(s)]\right) d\mu(s) = \|h_{\operatorname{Re}\zeta}\|_2^2,$$

and (i) follows immediately from the result in the real-time case ([CMS2], Lemma 2.1 (i)).

To prove (ii), note first of all that $\|h_\zeta\|_{p,r}$ is continuous and strictly positive for every ζ. Indeed, the continuity is obvious and the second assertion follows easily from the inequality $\|h_\zeta\|_{p,r} \geq \|h_\zeta\|_\infty$, from the fact that the latter is strictly positive for ζ real positive, and from the group property. Therefore it suffices to prove that (ii) holds for $|\zeta|\geq 1$.

Next use Theorem 2 to get

$$\begin{aligned}\|h_\zeta\|_{p,r} &\geq C\left(\int_{-\tau/2}^{\tau/2}\left|\widetilde{h_\zeta}(s+i\delta(p))\right|^{r'} ds\right)^{1/r'}\\ &= C\, e^{-\operatorname{Re}\zeta b_p}\left(\int_{-\tau/2}^{\tau/2} e^{-r'\operatorname{Re}(\zeta[\gamma(i\delta(p))-\gamma(s+i\delta(p))])}\, ds\right)^{1/r'},\end{aligned} \tag{4}$$

and we are reduced to estimating the last integral. A straightforward computation shows that

$$\begin{aligned}&\gamma(i\delta(p))-\gamma(s+i\delta(p))\\ &\qquad= \frac{2q^{1/2}}{q+1}\cosh(\delta(p)\log q)\big(1-\cos(s\log q)+i\tanh(\delta(p)\log q)\sin(s\log q)\big)\\ &\qquad= \gamma(i\delta(p))\big[1-\cos(s\log q)+i\tanh(\delta(p)\log q)\sin(s\log q)\big],\end{aligned}$$

and therefore

$$\begin{aligned}&\operatorname{Re}\big(\zeta[\gamma(i\delta(p))-\gamma(s+i\delta(p))]\big)\\ &\qquad= \gamma(i\delta(p))\big([1-\cos(s\log q)]\operatorname{Re}\zeta - \tanh(\delta(p)\log q)\sin(s\log q)\operatorname{Im}\zeta\big).\end{aligned}$$

When we insert this and change variables, the integral to be estimated becomes

$$\frac{1}{\log q}\int_{-\pi}^{\pi} e^{-r'\varphi_\zeta(s)}\, ds,$$

where we have set

$$\varphi_\zeta(s) = \gamma(i\delta(p))\big(\operatorname{Re}\zeta[1-\cos s] - \tanh(\delta(p)\log q)\operatorname{Im}\zeta\sin s\big).$$

One readily verifies that φ_ζ attains its minimum at

$$s_\zeta = \tan^{-1}\left(\frac{\operatorname{Im}\zeta}{\operatorname{Re}\zeta}\tanh(\delta(p)\log q)\right),$$

and that

$$\varphi_\zeta(s_\zeta) = \gamma(i\delta(p)) \operatorname{Re} \zeta - \Phi_p(\zeta) \qquad \text{and} \qquad \varphi''_\zeta(s_\zeta) = \Phi_p(\zeta).$$

Moreover, since $\varphi'''_\zeta(s) = -\varphi'_\zeta(s)$ for every s,

$$\varphi''_\zeta(s) \le \Phi_p(\zeta) \qquad \forall s \in [-\pi, +\pi],$$

so that expanding $\phi_\zeta(s)$ around s_ζ yields

$$\begin{aligned} \varphi_\zeta(s) - \varphi_\zeta(s_\zeta) &= \frac{1}{2}\varphi''_\zeta(\overline{s})(s - s_\zeta)^2 \qquad (\overline{s} \ \text{ between } s_\zeta \ \text{ and } s) \\ &\le \frac{1}{2}\Phi_p(\zeta)(s - s_\zeta)^2, \end{aligned}$$

for every s in $[-\pi, \pi]$. Thus,

$$\begin{aligned} \int_{-\pi}^{\pi} e^{-r'\varphi_\zeta(s)}\, ds &\ge e^{-r'(\gamma(i\delta(p)) \operatorname{Re} \zeta - \Phi_p(\zeta))} \int_{-\pi}^{\pi} e^{-r'\Phi_p(\zeta)(s-s_\zeta)^2/2}\, ds \\ &\ge \frac{e^{-r'(\gamma(i\delta(p)) \operatorname{Re} \zeta - \Phi_p(\zeta))}}{\Phi_p^{1/2}(\zeta)} \int_0^{\Phi_p^{1/2}(\zeta)\pi} e^{-r'u^2/2}\, du \\ &\ge C \frac{e^{-r'(\gamma(i\delta(p)) \operatorname{Re} \zeta - \Phi_p(\zeta))}}{\Phi_p^{1/2}(\zeta)}, \end{aligned}$$

for all ζ such that $|\zeta| \ge 1$. (ii) follows on substituting the last inequality in (4) above and using

$$\tanh(\delta(p) \log q)\, |\zeta| \le \Phi_p(\zeta) \le |\zeta|\,.$$

We now prove (iii). We claim that it suffices to prove that

$$\|h_\zeta\|_\infty \ge C\, |\zeta|^{-3/2} \exp\left[-b_2 \operatorname{Re} \zeta\right] \qquad \forall \zeta :\ \operatorname{Re} \zeta \ge a,$$

for some fixed $a > 0$. For then, by the group property and Young's inequality for convolution, it follows that, for $0 \le \operatorname{Re} \zeta \le a$,

$$\|h_\zeta\|_\infty \ge \ \|h_{\,a}\|_1^{-1} \|h_{\,\zeta+a}\|_\infty \ge C \|h_{\,a}\|_1^{-1}\, |\zeta + a|^{-3/2} \exp\left[-b_2(\operatorname{Re} \zeta + a)\right],$$

and the required inequality follows.

Next, since $\|h_\zeta\|_\infty \ge |h_\zeta(o)|$, it suffices to estimate the latter quantity for ζ such that $\operatorname{Re} \zeta \ge a$.

Since $\phi_z(o) = 1$ for every z, and $\mathbf{c}(-s) = \overline{\mathbf{c}(s)}$, by spherical Fourier inversion

$$\begin{aligned} h_\zeta(o) &= c_G \int_{-\tau/2}^{\tau/2} e^{-\zeta[1-\gamma(s)]}\, |\mathbf{c}(s)|^{-2}\ ds \\ &= 2\, c_G e^{-\zeta b_2} \int_0^{\tau/2} e^{-\zeta\gamma(0)[1-\cos(s \log q)]}\, |\mathbf{c}(s)|^{-2}\ ds \end{aligned}$$

Using the identity

$$|\mathbf{c}(s)|^{-2} = \frac{(q+1)^2 \sin^2(s \log q)}{(q+1)^2 \sin^2(s \log q) + (q-1)^2 \cos^2(s \log q)}$$

and performing the change of variable $v = 1 - \cos(s \log q)$, we rewrite

$$h_\zeta(o) = \frac{q(q+1)}{2\pi} e^{-\zeta b_2} \int_0^2 e^{-\zeta\gamma(0) v} v^{1/2} \psi(v)\, dv,$$

where ψ is the function defined by

$$\psi(v) = \frac{(2-v)^{1/2}}{(q+1)^2 v(2-v) + (q-1)^2(1-v)^2}.$$

Since the denominator in the formula above vanishes only for $v = 1 \pm (q+1)/2\sqrt{q}$, the function $v^{1/2}\psi(v)$ is analytic in $\mathbb{C} \setminus \{v : v \leq 0,\ \text{or}\ v \geq 2\}$, the square root taking its principal determination. Therefore, if we denote by $I(\zeta)$ the integral above, by Cauchy's theorem we can write

$$I(\zeta) = I_1(\zeta) + I_2(\zeta),$$

where $I_1(\zeta)$ is the integral over the segment in the complex plane $[0, 2\overline{\zeta}/|\zeta|]$, and $I_2(\zeta)$ is the integral along the arc of circle of radius 2 and center 0 joining $2\overline{\zeta}/|\zeta|$ to 2.

We begin by estimating

$$\begin{aligned} I_1(\zeta) &= \int_0^{2\overline{\zeta}/|\zeta|} e^{-\gamma(0)\zeta v} v^{1/2}\psi(v)\, dv \\ &= \left(\frac{\overline{\zeta}}{|\zeta|}\right)^{3/2} \int_0^2 e^{-\gamma(0)|\zeta| s} s^{1/2} \psi(s\overline{\zeta}/|\zeta|)\, ds. \end{aligned}$$

Since $\psi(v)$ is bounded in $D = \{v : |v| \leq 2, \operatorname{Re} v \geq 0\}$, and tends to $\sqrt{2}/(q-1)^2$ as $v \to 0$, an application of Laplace method (see [O], Chapter 3, §7, pp. 80-84) shows that the integral on the right hand side is equal to

$$\frac{\sqrt{2}\,\Gamma(3/2)}{\gamma(0)^{3/2}(q-1)^2} |\zeta|^{-3/2} \left(1 + o(1)\right)$$

uniformly as $|\zeta|$ tends to $+\infty$ in $\operatorname{Re} \zeta \geq 0$.

It follows that there exists $M_1 > 0$ such that, for every ζ with $|\zeta| \geq M_1$ and $\operatorname{Re} \zeta \geq 0$,

$$|I_1(\zeta)| \geq \frac{1}{2} \frac{\sqrt{2}\,\Gamma(3/2)}{\gamma(0)^{3/2}(q-1)^2} |\zeta|^{-3/2}.$$

We now estimate $I_2(\zeta)$. To fix notation we assume that $\arg\zeta$ belongs to $[0, \pi/2)$, the argument being completely similar if $-\pi/2 < \arg\zeta \leq 0$. Putting $v = 2e^{i\theta}$, we rewrite

$$I_2(\zeta) = 2^{3/2}\, i \int_{-\arg\zeta}^0 e^{-2\gamma(0)\,\zeta\, \exp(i\theta)}\, e^{3i\theta/2} \psi(2e^{i\theta}) d\theta,$$

so that

$$|I_2(\zeta)| \leq 2^{3/2} e^{-2\gamma(0) \operatorname{Re}\zeta} \int_{-\arg\zeta}^0 e^{-2\gamma(0)|\zeta|\,[\cos(\arg\zeta + \theta) - \cos(\arg\zeta)]} \left|\psi(2e^{i\theta})\right| d\theta.$$

Assume first that $\arg\zeta \geq \pi/4$. Since $\cos(\arg\zeta + \theta) - \cos(\arg\zeta) = -\sin(\arg\zeta)\sin\theta + o(\theta)$ and $\left|\psi(2e^{i\theta})\right| = \sqrt{2}|\theta|^{1/2}/(q-1)^2 + o(\sqrt{|\theta|})$ as $\theta \to 0^-$, using the Laplace method one shows that as $|\zeta|$ tends to infinity the integral above is asymptotic to

$$\Gamma(3/2) \frac{\sqrt{2}}{(q-1)^2} \left(2\,\gamma(0) \sin(\arg\zeta)\ |\zeta|\right)^{-3/2}$$

uniformly in $\pi/4 \leq \arg\zeta < \pi/2$. Therefore,

$$|I_2(\zeta)| \leq \frac{\sqrt{2}\,\Gamma(3/2)}{\gamma(0)^{3/2}(q-1)^2}|\zeta|^{-3/2}(\sin(\arg\zeta))^{-3/2}e^{-2\,\gamma(0)\,\mathrm{Re}\,\zeta}\,(1+o(1))\,,$$

where $|o(1)|$ tends to zero uniformly as $\zeta \to \infty$ in the sector $\pi/4 \leq \arg\zeta \leq \pi/2$. It follows that there exist $M_2 > 0$ and $a > 0$ such that

$$|I_2(\zeta)| \leq \frac{1}{4}\frac{\sqrt{2}\,\Gamma(3/2)}{\gamma(0)^{3/2}(q-1)^2}|\zeta|^{-3/2}. \tag{5}$$

On the other hand, it is easily seen that

$$|I_2(\zeta)| \leq 2\pi \sup_D |\psi(v)|\, e^{-\sqrt{2}\gamma(0)|\zeta|}$$

in the sector $0 \leq \arg\zeta \leq \pi/4$. We may therefore conclude that there exist M_3 and a such that (5) holds for every ζ such that $|\zeta| \geq M_3$ and $\mathrm{Re}\;\zeta \geq a$.

Combining the results obtained above shows that there exist M and a such that for every ζ with $|\zeta| \geq M$ and $\mathrm{Re}\;\zeta \geq a$,

$$|I(\zeta)| \geq |I_1(\zeta)| - |I_2(\zeta)| \geq \frac{1}{4}\frac{\sqrt{2}\,\Gamma(3/2)}{\gamma(0)^{3/2}(q-1)^2}|\zeta|^{-3/2},$$

and (iii) follows.

The proof of (iv) is now a simple consequence of (iii) and of the inclusion properties of the Lorentz spaces $L^{p,r}(\mathfrak{X})$. $\square$

4. Norm estimates for the heat kernel: Upper bounds

In the real time case considered in [CMS2], §3, upper estimates for the L^p norms $(1 \leq p \leq 2)$ of the heat kernel were obtained by means of spherical analysis techniques, most notably by an application of the Plancherel formula if $p = 2$, and of [CMS1], Theorem 1.2 if $1 \leq p < 2$.

The extension to the case where the parameter ζ is complex presents no difficulty if $p = 2$ (see Theorem 5 (i)), but fails if $1 \leq p < 2$, essentially because in this case h_ζ is no longer positive.

Instead, we will obtain our upper bounds by a direct method, which in fact allows us to derive sharp upper bounds for the $L^{p,r}(\mathfrak{X})$ norms of h_ζ for $p \neq 2$ and $1 \leq r < +\infty$, and whose key ingredients are Corollary 4 and the pointwise upper estimates for $\left|h^{\mathbb{Z}}_\zeta(d)\right|$ which can be derived from Theorem A4.

We begin by collecting the latter in the following lemma. Fractional powers are assumed throughout to take their principal value when their argument is positive, and to be continuous elsewhere.

Lemma 6. *For every $d \geq 1$ and ζ with $\mathrm{Re}\;\zeta > 0$ we have*

$$\begin{aligned}\left|h^{\mathbb{Z}}_\zeta(d)\right| \leq (1+\tilde{r}_2(d,\zeta))&\frac{\sqrt{2}}{\sqrt{\pi}\,|d^2+\zeta^2|^{1/4}}\\ &\times \exp\left\{-\mathrm{Re}\;\zeta + \mathrm{Re}\left((d^2+\zeta^2)^{1/2}\right) + d\log\left|\frac{\zeta}{d+(d^2+\zeta^2)^{1/2}}\right|\right\},\end{aligned}$$

and the following upper bounds for the error term $\tilde{r}_2(d,\zeta)$ hold: for every θ_o in $(0,\pi/2)$ and $\epsilon > 0$, there exists a constant C such that for every $d \geq 1$ and ζ with

$\mathrm{Re}\ \zeta \geq \eta > 0$,

$$\tilde{r}_2(d,\zeta) \leq \begin{cases} C & \textit{if } |\arg\zeta| \leq \theta_o \textit{ or } |\zeta| \leq (1-\epsilon)d, \\ e^{Cd^{1/2}/\eta^{3/2}} & \textit{if } (1-\epsilon)d \leq |\zeta| \leq (1+\epsilon)d, \\ e^{C/\eta} & \textit{if } (1+\epsilon)d \leq |\zeta|. \end{cases}$$

Proof. Recalling that

$$h_\zeta^{\mathbb{Z}}(d) = e^{-\zeta} I_d(\zeta),$$

the proof is an immediate consequence of Theorem A4 in the Appendix: taking absolute values in (11) we obtain the main term. Since $1/3\sqrt{5} + 1/6 < 1/3$, and $d \geq 1$, the error term $r_1(d)$ in (11) is bounded in absolute value by $1/2$ and therefore $|1 + r_1(d)|^{-1} \leq 2$. It is straightforward to check that the error term $\tilde{r}_2(d,\zeta) = |r_2(d,\zeta)|$ satisfies the stated estimates in $\mathrm{Re}\ \zeta \geq \eta$. □

We may therefore estimate

$$\begin{aligned} \|h_\zeta\|_{p,r} &\leq C \frac{e^{-b_2 \,\mathrm{Re}\ \zeta}}{|\zeta|} \left(\sum_{d=0}^{\infty} d^r q^{rd\delta(p)} \left| h^{\mathbb{Z}}_{\gamma(0)\zeta}(d) \right|^r \right)^{1/r} \\ &\leq C \frac{e^{-\mathrm{Re}\ \zeta}}{|\zeta|} \left(\sum_{d=0}^{\infty} (1 + \tilde{r}_2(d,\zeta))^r \frac{d^r}{|d^2 + \gamma^2(0)\zeta^2|^{r/4}} e^{r\varphi_{p,\zeta}(d)} \right)^{1/r}, \end{aligned} \tag{6}$$

where $\tilde{r}_2(d,\zeta)$ satisfies the upper estimates stated in Lemma 6 and we have set

$$\varphi_{p,\zeta}(s) = s\delta(p)\log q + \mathrm{Re}\left((s^2 + \gamma^2(0)\zeta^2)^{1/2} \right) + s \log \left| \frac{\gamma(0)\zeta}{s + (s^2 + \gamma^2(0)\zeta^2)^{1/2}} \right|.$$

Upper bounds for $\|h_\zeta\|_{p,r}$ can therefore be obtained by estimating the series on the right hand side of (6).

We state in the following lemma a number of properties of the function $\varphi_{p,\zeta}$ that will be used in obtaining the required upper estimates.

Lemma 7. *The following hold:*

(i) *If* $1 \leq p < 2$*, then the maximum value of* $\varphi_{p,\zeta}(s)$ *on* $[0, +\infty)$ *is* $\Phi_p(\zeta)$*, which is attained at*

$$\overline{s} = \frac{q^{1/p} - q^{1/p'}}{q+1} \frac{|\zeta|^2}{[\mathrm{Re}^2\,\zeta + \tanh^2(\delta(p)\log q)\,\mathrm{Im}^2\,\zeta]^{1/2}}.$$

(ii) *If* $p > 2$*, then* $\varphi_{p,\zeta}(s) \leq \delta(p)s + \gamma(0)\,\mathrm{Re}\ \zeta$ *for all* s *in* $[0, +\infty)$*.*

(iii) *If* $|\arg\zeta| \leq \pi/4$*, then*

$$\varphi''_{p,\zeta}(s) \leq -\frac{1}{2\max\{s, \gamma(0)\,|\zeta|\}} \qquad \forall s \geq 0.$$

(iv) *If* $|\arg\zeta| \geq \pi/4$*, then*

$$\varphi''_{p,\zeta}(s) \leq - \begin{cases} \dfrac{\mathrm{Re}\ \zeta}{\gamma(0)\,|\zeta|^2} & \textit{if } 0 \leq s \leq \gamma(0)[\mathrm{Im}^2\,\zeta - \mathrm{Re}^2\,\zeta]^{1/2}, \\ \dfrac{1}{2\max\{s, \gamma(0)\,|\zeta|\}} & \textit{if } \gamma(0)[\mathrm{Im}^2\,\zeta - \mathrm{Re}^2\,\zeta]^{1/2} \leq s. \end{cases}$$

Proof. Note first that straightforward computations yield

$$\varphi'_{p,\zeta}(s) = \delta(p)\log q + \log\left|\frac{\gamma(0)\zeta}{s+(s^2+\gamma^2(0)\zeta^2)^{1/2}}\right|,$$

$$\varphi''_{p,\zeta}(s) = -\operatorname{Re}\Big((s^2+\gamma^2(0)\zeta^2)^{-1/2}\Big) = -\frac{\operatorname{Re}\Big((s^2+\gamma^2(0)\overline{\zeta}^2)^{1/2}\Big)}{|s^2+\gamma^2(0)\zeta^2|}.$$

It follows in particular that $\varphi''_{p,\zeta}(s) < 0$ for every $s \geq 0$.

Assume first that $1 \leq p < 2$. Since $\varphi'_{p,\zeta}(s)$ is equal to $\delta(p)\log q > 0$ for $s = 0$, tends to $-\infty$ as s tends to $+\infty$, and $\varphi''_{p,\zeta}(s) \leq 0$ for every s, it follows that $\varphi_{p,\zeta}$ has a unique maximum in $[0,\infty)$ attained at the point $\overline{s}$ solution of the equation $\varphi'_{p,\zeta}(s) = 0$. Using the formula

$$z^{1/2} = \frac{1}{\sqrt{2}}\left(|z| + \operatorname{Re} z\right)^{1/2} + \frac{i}{\sqrt{2}}\operatorname{sgn}(\operatorname{Im} z)\left(|z| - \operatorname{Re} z\right)^{1/2}, \tag{7}$$

valid if $|\arg z| < \pi$, elementary, although somewhat lengthy, algebraic manipulations show that the equation is satisfied if $\overline{s}$ has the expression given in the statement, and that $\varphi_{p,\zeta}(\overline{s}) = \Phi_p(\zeta)$, therefore proving (i).

Assume now that $p > 2$. Since $\varphi''_{p,\zeta}(s) \leq 0$, we have $\varphi'_{p,\zeta}(s) \leq \varphi'_{p,\zeta}(0) = \delta(p)\log q$, and we obtain (ii) by integrating between 0 and s.

To prove (iii) we use (7) to write

$$\varphi''_{p,\zeta}(s) = -\frac{1}{\sqrt{2}}\frac{[\left|s^2+\gamma^2(0)\zeta^2\right| + \operatorname{Re}(s^2+\gamma^2(0)\zeta^2)]^{1/2}}{|s^2+\gamma^2(0)\zeta^2|}.$$

Since $|\arg\zeta| \leq \pi/4$, $\operatorname{Re}(s^2+\gamma^2(0)\zeta^2) = s^2+\gamma^2(0)[\operatorname{Re}^2\zeta - \operatorname{Im}^2\zeta] > 0$, and we may estimate

$$\varphi''_{p,\zeta}(s) \leq -\frac{1}{\sqrt{2}\,|s^2+\gamma^2(0)\zeta^2|^{1/2}} \leq -\frac{1}{2\max(s,\gamma(0)\,|\zeta|)}.$$

The same estimate clearly holds if $|\arg\zeta| \geq \pi/4$ and $s^2 \geq \gamma^2(0)[\operatorname{Im}^2\zeta - \operatorname{Re}^2\zeta]$. Assume therefore that $s^2 \leq \gamma^2(0)[\operatorname{Im}^2\zeta - \operatorname{Re}^2\zeta]$, so that $\operatorname{Re}(s^2+\gamma^2(0)\zeta^2)$ ranges in the interval $[-\gamma^2(0)(\operatorname{Im}^2\zeta - \operatorname{Re}^2\zeta), 0]$. Since the function

$$\psi_a(t) = \frac{1}{2}\frac{(t^2+a^2)^{1/2}+t}{t^2+a^2}$$

is increasing for $t \leq 0$, and

$$\varphi''_{p,\zeta}(s) = -\left[\psi_a\left(\operatorname{Re}(s^2+\gamma^2(0)\zeta^2)\right)\right]^{1/2},$$

for $a = 2\gamma^2(0)\operatorname{Re}\zeta\,\operatorname{Im}\zeta$, we have

$$\varphi''_{p,\zeta}(s) \leq -\left[\psi_a\left(-\gamma^2(0)[\operatorname{Im}^2\zeta - \operatorname{Re}^2\zeta]\right)\right]^{1/2} = -\frac{\operatorname{Re}\zeta}{\gamma(0)\,|\zeta|^2},$$

as required to conclude the proof of (iv). □

We are now ready to prove the upper bounds for the $L^{p,r}(\mathfrak{X})$ norms of the heat kernel. We remark that for p and r belonging to the range considered in Theorem 5, our upper bounds are comparable with the lower bounds obtained therein. We also recall that comparable upper and lower estimates for $\|h_\zeta\|_p$ when $p = 2$ are already contained in Theorem 5 (i).

Theorem 8. *The following hold:*

(i) *If* $1 \le p < 2$ *and* $1 \le r < +\infty$, *then*

$$\|h_\zeta\|_{p,r} \le C \min\left\{1, |\zeta|^{-1/2r'}\right\} \exp\left(-\operatorname{Re}\zeta + \Phi_p(\zeta)\right) \qquad \forall \zeta : \operatorname{Re}\zeta \ge 0;$$

(ii) *If* $p > 2$ *and* $1 \le r \le +\infty$, *then*

$$\|h_\zeta\|_{p,r} \le C \min\left\{1, |\zeta|^{-3/2}\right\} \exp\left(-b_2 \operatorname{Re}\zeta\right) \qquad \forall \zeta : \operatorname{Re}\zeta \ge 0.$$

Proof. As a preliminary step, we note that it is enough to prove that the stated estimates hold in the half plane $\operatorname{Re}\zeta \ge 1$. The argument is similar to the one used in Proposition 5 (iii).

First of all, interpolating between $\|f * g\|_1 \le \|f\|_1 \|g\|_1$ and $\|f * g\|_p \le \|f\|_p \|g\|_1$, one finds that $\|f * g\|_{p,r} \le \|f\|_{p,r} \|g\|_1$ for all p in $(1, +\infty)$ and r in $[1, +\infty]$.

Since $h_\zeta = h_{\zeta+1} * h_{-1}$ by the group property, and h_{-1} belongs to $L^1(\mathfrak{X})$, we have

$$\|h_\zeta\|_{p,r} \le \|h_{\zeta+1}\|_{p,r} \|h_{-1}\|_1.$$

Thus, assuming for instance that (i) holds in $\operatorname{Re}\zeta \ge 1$, we conclude that the right hand side of the above inequality is bounded above by

$$C \|h_{-1}\|_1 |\zeta + 1|^{-1/2r'} \exp\left[-\operatorname{Re}\zeta + 1 + \Phi_p(\zeta + 1)\right].$$

Since $\Phi_p(\zeta)$ defines an equivalent norm on $\mathbb{C}$, by the triangle inequality

$$|\Phi_p(\zeta + 1) - \Phi_p(\zeta)| \le \Phi_p(1),$$

and our claim follows. The argument is even easier in the case of (ii).

It follows from this and formula (6) that it suffices to estimate the series

$$S_{p,r}(\zeta) = \sum_{d=0}^{\infty} (1 + \tilde{r}_2(d, \zeta))^r \frac{d^r}{|d^2 + \gamma^2(0)\zeta^2|^{r/4}} e^{r\varphi_{p,\zeta}(d)}$$

for ζ in $\operatorname{Re}\zeta \ge 1$.

Assume first that $1 \le p < 2$. We need to show that there exists a constant C such that for every ζ with $\operatorname{Re}\zeta \ge 1$

$$S_{p,r}(\zeta) \le C\, |\zeta|^{r/2+1/2}\, e^{r\Phi_p(\zeta)}. \tag{8}$$

From the expression of $\overline{s}$ in Lemma 7 (i) we deduce that

$$\frac{q^{1/p} - q^{1/p'}}{q+1} |\zeta| \le \overline{s} \le \gamma(i\delta(p))\, |\zeta|,$$

and that there exists $\theta_o \in (\pi/4, \pi/2)$ such that for every z with $\theta_o \le |\arg\zeta| < \pi/2$

$$\frac{\gamma(0) + \gamma(i\delta(p))}{2} |\zeta| \le \overline{s} \le \gamma(i\delta(p))\, |\zeta|.$$

We consider first the case where $|\arg\zeta| \le \theta_o$. Then, by virtue of the error estimates in Lemma 6, $\tilde{r}_2(d, \zeta)$ is uniformly bounded above by a constant which depends only upon θ_o. To estimate $\varphi_{p,\zeta}(s)$ from above we use the identity

$$\varphi_{p,\zeta}(s) - \Phi_p(\zeta) = \int_{\overline{s}}^{s} (s - v) \varphi''_{p,\zeta}(v)\, dv.$$

In the present setting the estimates in Lemma 7 (iii) and (iv) yield

$$\varphi''_{p,\zeta}(v) \leq -\frac{\cos(\theta_o)}{2\max\{v,\gamma(0)\,|\zeta|\}}, \qquad \forall v \geq 0,$$

and we may conclude that there exists a constant $C_0 > 0$, depending only on θ_o, such that

$$\varphi_{p,\zeta}(d) - \Phi_p(\zeta) \leq -C_0 \begin{cases} |\zeta|^{-1}\,(d-\overline{s})^2 & \text{if } d \leq \overline{s}, \\ [d\log(d/\overline{s}) - (d-\overline{s})] & \text{if } d \geq \overline{s}. \end{cases}$$

Moreover,

$$\left|d^2+\gamma^2(0)\zeta^2\right|^2 = (d^2-\gamma^2(0)\,|\zeta|^2)^2 + 4\gamma^2(0)d^2\operatorname{Re}^2\zeta \geq 4\gamma^2(0)d^2\,|\zeta|^2\cos^2\theta_o.$$

It follows that

$$\begin{aligned} S_{p,r}(\zeta) &= e^{r\Phi_p(\zeta)}\sum_{d=1}^{+\infty}\frac{d^r}{|d^2+\gamma^2(0)\zeta^2|^{r/4}}\exp\left(r[\varphi_{p,\zeta}(d)-\Phi_p(\zeta)]\right) \\ &\leq C_1\frac{e^{r\Phi_p(\zeta)}}{|\zeta|^{r/4}}\left(\Sigma_1(\zeta)+\Sigma_2(\zeta)\right), \end{aligned}$$

where

$$\Sigma_1(\zeta) = \sum_{d=1}^{[\overline{s}]} d^{3r/4}e^{-rC_0(d-\overline{s})^2/|\zeta|},$$

and

$$\Sigma_2(\zeta) = \sum_{d=[\overline{s}]+1}^{+\infty} d^{3r/4}e^{-rC_0[d\log(d/\overline{s})-(d-\overline{s})]},$$

$[s]$ denoting the integer part of s.

Since $\overline{s} \leq \gamma(i\delta(p))\,|\zeta|$ and $s \mapsto e^{-rC_o(s-\overline{s})^2/|\zeta|}$ is increasing in $[0,\overline{s}]$, we may write

$$\Sigma_1(\zeta) \leq C_2\,|\zeta|^{3r/4}\left(\int_0^{\overline{s}} e^{-rC_o(s-\overline{s})^2/|\zeta|}\,ds + 1\right),$$

and then the change of variables $v = (\overline{s}-s)/\,|\zeta|^{1/2}$ shows that the integral on the right hand side is bounded above by $C_3\,|\zeta|^{1/2}$, where C_3 does not depend on ζ. Thus we conclude that

$$\Sigma_1(\zeta) \leq C_4\,|\zeta|^{3r/4+1/2}$$

with C_4 independent of ζ in the sector $|\arg\zeta| \leq \theta_o$.

To estimate $\Sigma_2(\zeta)$ we proceed in a similar way. Since by assumption $\operatorname{Re}\zeta \geq 1$, $\overline{s} \geq C_5\,|\zeta| \geq C_5$. By elementary calculus, for every C_0' satisfying $0 < C_0' < C_0$

$$s^{3r/4}e^{-C_0[s\log(s/\overline{s})-(s-\overline{s})]} \leq C\overline{s}^{3r/4}e^{-rC_0'[s\log(s/\overline{s})-(s-\overline{s})]},$$

where

$$\begin{aligned} C &= \sup_{s\geq\overline{s}}\left\{\left(\frac{s}{\overline{s}}\right)^{3r/4}e^{-(C_0-C_0')\overline{s}[(s/\overline{s})\log(s/\overline{s})-(s/\overline{s}-1)]}\right\} \\ &\leq \sup_{u\geq 1}\left\{u^{3r/4}e^{-r(C_0-C_0')[u\log u-u+1]}\right\} < +\infty. \end{aligned}$$

Therefore

$$\Sigma_2(\zeta) \le C_6 \overline{s}^{3r/4} \sum_{d=[|\overline{s}|]+1}^{+\infty} e^{-rC_0'\overline{s}[(d/\overline{s})\log(d/\overline{s})-d/\overline{s}+1]}$$
$$\le C_6 \overline{s}^{3r/4} \left(\int_{\overline{s}}^{+\infty} e^{-rC_0'\overline{s}[(s/\overline{s})\log(s/\overline{s})-s/\overline{s}+1]} ds + 1 \right).$$

Changing variables, we may write the integral on the right hand side as

$$\overline{s} \int_1^{+\infty} e^{-rC_0'\overline{s}[\log u - u + 1]} du \le C_7 \overline{s}^{1/2},$$

where the last inequality follows from an application of the Laplace method. It follows that

$$\Sigma_2(\zeta) \le C_8 |\zeta|^{3r/4+1/2}$$

and therefore that

$$S_{p,r}(\zeta) \le C_9 |\zeta|^{r/2+1/2} e^{r\Phi_p(\zeta)}$$

with C_9 independent of ζ in the sector $|\arg\zeta| \le \theta_o$.

We next consider the case where $\theta_o \le |\arg\zeta| < \pi/2$. In this case $\tilde{r}_2(d,\zeta)$ is no longer uniformly bounded, and we need to proceed with greater care. Since $\gamma(0) < \gamma(i\delta(p))$, our choice of θ_o implies that we now have

$$0 \le \gamma(0)[\mathrm{Im}^2\zeta - \mathrm{Re}^2\zeta]^{1/2} \le \gamma(0)|\zeta| \le \frac{\gamma(0)+\gamma(i\delta(p))}{2}|\zeta| \le \overline{s} \le \gamma(i\delta(p))|\zeta|.$$

To estimate $S_{p,r}(\zeta)$, choose $\epsilon > 0$ and write

$$S_{p,r}(\zeta) = S_1(\zeta) + S_2(\zeta) + S_3(\zeta),$$

where

$$S_1(\zeta) = e^{r\Phi_p(\zeta)} \sum_{d \le (1-\epsilon)\gamma(0)|\zeta|} (1+\tilde{r}_2(d,\zeta))^r \frac{d^r}{|d^2+\gamma^2(0)\zeta^2|^{r/4}} e^{r[\varphi_{p,\zeta}(d)-\Phi_p(\zeta)]},$$

in $S_2(\zeta)$ the sum is over the range $(1-\epsilon)\gamma(0)|\zeta| < d \le [3\gamma(0)+\gamma(i\delta(p))]|\zeta|/4$, and $S_3(\zeta)$ is the series for $d > [3\gamma(0)+\gamma(i\delta(p))]|\zeta|/4$.

Arguing as in the previous case, and using Lemma 7 (iv), we see that

$$\varphi_{p,\zeta}(d) - \Phi_p(\zeta) \le -C_0 \begin{cases} [d\log(d/\overline{s}) - (d-\overline{s})] & \text{if } d \ge \overline{s}, \\ |\zeta|^{-1}(d-\overline{s})^2 & \text{if } \frac{3\gamma(0)+\gamma(i\delta(p))|\zeta|}{4} \le d \le \overline{s}. \end{cases}$$

Since $\varphi_{p,\zeta}'(s) > 0$ in $[0,\overline{s})$, it also follows that

$$\varphi_{p,\zeta}(d) - \Phi_p(\zeta) \le \varphi_{p,\zeta}([3\gamma(0)+\gamma(i\delta(p))]|\zeta|/4) - \Phi_p(\zeta) \le C_{10}|\zeta|$$

for all $d \le (3\gamma(0)+\gamma(i\delta(p))|\zeta|/4$.

We consider first $S_3(\zeta)$. In this case $\frac{\gamma(0)|\zeta|}{d} \le \frac{4\gamma(0)}{3\gamma(0)+\gamma(i\delta(p))} < 1$, and, by the error estimates in Lemma 6, $\tilde{r}_2(d,\zeta)$ is bounded above by a constant. Also,

$$|d^2+\gamma^2(0)\zeta^2| \ge d - \gamma^2(0)|\zeta|^2 \ge \frac{\gamma(i\delta(p))-\gamma(0)}{4} d|\zeta|.$$

Thus we can repeat the argument used in the case where $|\arg\zeta| \leq \theta_o$ to conclude that

$$S_3(\zeta) \leq C_{11} e^{r\Phi_p(\zeta)} |\zeta|^{1/2r+1/2} \qquad \forall\zeta: \ \mathrm{Re}\ \zeta \geq 1.$$

We consider next $S_1(\zeta)$. Here $\frac{\gamma(0)|\zeta|}{d} \geq (1-\epsilon)^{-1} > 1$. Hence, again by Lemma 6, there is a constant C_{12} such that

$$\tilde{r}_2(d,\zeta) \leq C_{12} \qquad \forall\zeta: \ \mathrm{Re}\ \zeta \geq 1, \ \forall d: \ d \leq (1-\epsilon)\gamma(0)\,|\zeta|\,,$$

and

$$\left|d^2 + \gamma^2(0)\zeta^2\right| \geq \gamma^2(0)\,|z|^2 - d^2 \geq \frac{\epsilon}{1-\epsilon}\gamma^2(0)\,|\zeta|^2\,.$$

Thus

$$S_1(\zeta) \leq C_{13} e^{r\Phi_p(\zeta)} \sum_{d \leq (1-\epsilon)\gamma(0)|\zeta|} |\zeta|^{r/2} e^{-C_{10} r|\zeta|} \quad \forall\zeta: \ \mathrm{Re}\ \zeta \geq 1,$$

and the right hand side is exponentially decreasing in $|\zeta|$.

$S_2(\zeta)$ is the most critical term, for here $\gamma(0)\,|\zeta|\,/d$ ranges in an interval containing 1, and therefore, for $\mathrm{Re}\ \zeta \geq 1$, the error term $\tilde{r}_2(d,\zeta)$ is bounded above by $e^{C_{14} d^{1/2}}$. Since the difference $\varphi_{p,\zeta}(s) - \Phi_p(\zeta)$ is bounded above by $-C_{10}\,|\zeta|$ with $C_{10} > 0$ independent of ζ in the range under consideration, and

$$\left|d^2 + \gamma^2(0)\zeta^2\right|^2 = \left(s^2 - \gamma^2(0)\,|\zeta|^2\right)^2 + 4\gamma^2(0)s^2\,\mathrm{Re}^2\,\zeta \geq C_{15}\,|\zeta|^2\,\mathrm{Re}^2\,\zeta,$$

we have

$$S_2(\zeta) \leq C_{16} e^{r\Phi_p(\zeta)} \sum_{\substack{(1-\epsilon)\gamma(0)|\zeta| < d \\ \leq (3\gamma(0)+\gamma(i\delta(p)))|\zeta|/4}} \frac{d^r}{|\zeta|^{r/4}} e^{-r[C_{10}|\zeta| + C_{14} d^{1/2}]}$$

for all ζ in $\mathrm{Re}\ \zeta \geq 1$, and the sum on the right hand side is bounded above by

$$C_{17}\,|\zeta|^{3r/4+1}\, e^{-r|\zeta|[C_{10} - C_{14}|\zeta|^{-1/2}]},$$

which is exponentially decreasing in $|\zeta|$. Combining the above estimates, we conclude that there exists a constant C_{18}, independent of ζ in $\mathrm{Re}\ \zeta \geq 1$, such that when $\theta_o \leq |\arg\zeta| < \pi/2$,

$$S_{p,r}(\zeta) \leq C_{18} \left|\zeta^{r/2+1/2}\right| e^{r\Phi_p(\zeta)},$$

as required to complete the proof of (i).

The proof of (ii) is considerably easier, and we only outline the argument. By the inclusion properties of the Lorentz spaces $L^{p,r}(\mathfrak{X})$ it suffices to show that for every p in $(2,+\infty)$, $\|h_\zeta\|_{p,1}$ satisfies the stated estimate in the half plane $\mathrm{Re}\ \zeta \geq 1$, or, equivalently, that there exists a constant C_{19} such that for every ζ with $\mathrm{Re}\ \zeta \geq 1$,

$$S_{p,1}(\zeta) \leq C_{19}\,|\zeta|^{-1/2}\, e^{\gamma(0)\,\mathrm{Re}\ \zeta}.$$

By Lemma 7 (ii), we may estimate

$$S_{p,1}(\zeta) \leq e^{\gamma(0)\,\mathrm{Re}\ \zeta} \sum_{d \geq 1} (1 + \tilde{r}_2(d,\zeta)) \frac{d}{\left|d^2 + \gamma^2(0)\zeta^2\right|^{1/4}} e^{d\delta(p)} = e^{\gamma(0)\,\mathrm{Re}\ \zeta}\left(\Sigma_1 + \Sigma_2\right),$$

where Σ_1 is the sum over the d's less than or equal to $\gamma(0)\,|\zeta|\,/2$ and Σ_2 is the tail of the series.

Consider first Σ_1. Since Re $\zeta \geq 1$, and $d \leq \gamma(0)\left|\zeta\right|/2$, the error term $\tilde{r}_2(d,\zeta)$ is uniformly bounded. Also, the absolute value at the denominator is bounded from below by a multiple of $\left|\zeta\right|^{1/4}$. Proceeding as in (i), one shows that

$$\Sigma_1 \leq C_{20}\left|\zeta\right|^{-1/2} \qquad \forall\zeta : \text{ Re } \zeta \geq 1.$$

Consider next Σ_2. Since Re $\zeta \geq 1$, in this range of d's the error term grows at most like $e^{C_{14}d^{1/2}}$. On the other hand, $\delta(p) < 0$, so that the factor $e^{d\delta(p)}$ is exponentially decreasing in d, and the term in the denominator is bounded from below by a multiple of $\left|\zeta\right|^{1/4}$. Therefore Σ_2 is exponentially decreasing in $|\zeta|$, and (ii) follows. □

5. Proof of Theorem 1

The proof of Theorem 1 follows the lines of that of [CMS2], Theorem 2.2, and uses the norm estimates of Theorems 5 and 8, and the description of the space of bounded convolution operators given in [CMS1], Theorem 2.4.

Proof of Theorem 1. (i) is an immediate consequence of the chain of inequalities

$$\left\|h_\zeta\right\|_\infty = \left\|\!\left|\mathcal{H}_\zeta\right|\!\right\|_{1,\infty} \leq \left\|\!\left|\mathcal{H}_\zeta\right|\!\right\|_{p,r} \leq \left\|\!\left|\mathcal{H}_\zeta\right|\!\right\|_{p,p} \leq \left\|h_\zeta\right\|_p, \qquad \forall 1 \leq p \leq r \leq \infty,$$

which in turn follows from the fact that the embedding of $L^p(\mathfrak{X})$ into $L^r(\mathfrak{X})$ is norm non-increasing.

To prove (ii) we use the fact that since $\mathcal{H}_\zeta$ is G-invariant, its norm on $L^2(\mathfrak{X})$ coincides with the norm of its restriction to the subspace of $L^2(\mathfrak{X})^\sharp$ of radial functions in $L^2(\mathfrak{X})$. By Plancherel's theorem the latter is equal to

$$\sup_{[-\tau/2,\tau/2]}\left|\widetilde{h_\zeta}(s)\right| = \sup_{[-\tau/2,\tau/2]}\left|e^{-\zeta(1-\gamma(s))}\right| = e^{-b_2 \operatorname{Re} \zeta}.$$

Assume now that $1 \leq p = r < 2$. By Pytlik's theorem (see [P], or [CMS1], Theorem 2.4), and Theorem 8 (i),

$$\left\|\!\left|\mathcal{H}_\zeta\right|\!\right\|_{p,p} \leq \left\|h_\zeta\right\|_{p,1} \leq C\exp\left[-\operatorname{Re}\zeta + \Phi_p(\zeta)\right].$$

On the other hand, by the group property, $\left\|h_{\,2\zeta}\right\|_p \leq \left\|\mathcal{H}_\zeta h_\zeta\right\|_p \leq \left\|\!\left|\mathcal{H}_\zeta\right|\!\right\|_{p,p}\|h_\zeta\|_p$, whence

$$\left\|\!\left|\mathcal{H}_\zeta\right|\!\right\|_{p,p} \geq \left\|h_{\,2\zeta}\right\|_p\|h_\zeta\|_p^{-1}$$

and the corresponding lower estimate follows from Theorem 5 (ii) and Theorem 8(i). The case where $2 < p = r \leq \infty$ follows from duality from the case just considered, and the proof of (iii) is completed.

Consider now the case where $1 \leq p < r = 2$. By the Kunze–Stein property (see [N], or [CMS1], §2), and Theorem 5 (i)

$$\left\|\!\left|\mathcal{H}_\zeta\right|\!\right\|_{p,2} \leq C\|h_\zeta\|_2 \leq C\min\left\{1, \mathrm{Re}^{-3/4}\right\}e^{-b_2 \operatorname{Re}\zeta}.$$

On the other hand,

$$\|h_\zeta\|_2 = \|\mathcal{H}_\zeta\delta_o\|_2 \leq \left\|\!\left|\mathcal{H}_\zeta\right|\!\right\|_{p,2}\|\delta_o\|_p$$

and applying Theorem 5 (i) again yields the reverse inequality. The case where $p = 2 < 2 \leq \infty$ is obtained by duality from the previous case.

To prove (v) we use again the Kunze–Stein property and Theorem 8 (ii) to estimate

$$\|\mathcal{H}_\zeta\|_{p,r} \leq C\|h_\zeta\|_r \leq C\min\{1,|\zeta|^{-1/2r'}\}\exp[-\operatorname{Re}\zeta + \Phi_r(\zeta)],$$

while the reverse inequality follows from $\|h_\zeta\|_r = \|\mathcal{H}_\zeta\delta_o\|_r \leq \|\mathcal{H}_\zeta\|_{p,r}\|\delta_o\|_p$ and Theorem 5 (ii).

The case where $2 \leq p < r \leq +\infty$ follows by duality from (v).

Finally, assume that $1 \leq p < 2 < r \leq +\infty$. According to Theorem 2.4 in [CMS1], if $r \neq p'$, then we have the equality

$$Cv_p^r(\mathfrak{X}) = L^{\min(p',r)}(\mathfrak{X})^\sharp$$

with equivalence of norms, while if $r = p'$ then the we have inclusion

$$L^{p',p'/2}(\mathfrak{X})^\sharp \subseteq Cv_p^{p'}(\mathfrak{X}) \subseteq L^{p'}(\mathfrak{X})^\sharp,$$

and corresponding norm inequalities hold. (vii) then follows at once from Theorem 5 (ii) and Theorem 8 (ii). □

6. Appendix

This Appendix is devoted to a discussion of the uniform approximation of modified Bessel functions and to the proof of the bounds for the error terms used in the proof of Lemma 6.

Let

$$\xi(z) = (1+z^2)^{1/2} + \log\left(\frac{z}{1+(1+z^2)^{1/2}}\right),$$

and set

$$p(z) = (1+z^2)^{-1/2} \qquad \text{and} \qquad U_1(p) = (3p-5p^3)/24.$$

Given a function f and a piecewise C^1 regular path $\omega(t)$ ($t_o \leq t \leq t_1$) contained in the domain of f, denote by $\mathcal{V}_\omega(f)$ the total variation of f along ω defined by $\int_{t_o}^{t_1}|f(\omega(t))\omega'(t)|\,dt$.

An application of the theory of approximation of solutions of differential equations shows that (see [O], Chapter 10, §7), for $\nu > 0$ and $|\arg z| < \pi/2$, we have

$$I_\nu(\nu z) = \frac{1}{1+\eta(\nu,+\infty)}\frac{e^{\nu\xi(z)}}{(2\pi\nu)^{1/2}(1+z^2)^{1/4}}\left(1+\eta(\nu,z)\right), \tag{9}$$

where the fractional powers take their principal values for z positive and are continuous elsewhere, and the error term $\eta(\nu,z)$ is bounded by

$$|\eta(\nu,z)| \leq 2\left\{\frac{2\mathcal{V}_{\omega_z}(U_1)}{\nu}\right\}\frac{\mathcal{V}_{\omega_z}(U_1)}{\nu},$$

where ω_z is any path joining 0 with z in the half plane $\operatorname{Re} z > 0$ satisfying the following conditions: a) ω_z consists of a finite chain of C^2 regular arcs, b) ω_z coincides with the real z-axis in a neighborhood of 0, and c) $\operatorname{Re}\xi(\omega_z(t))$ is nondecreasing. Paths satisfying the above conditions are referred to as ξ-progressive in Olver's terminology.

Explicit upper bounds for the error term depend on the choice of suitable ξ-progressive paths. This requires a preliminary study of the map $z \mapsto \xi(z)$ for $\operatorname{Re} z > 0$. It is readily seen that:

(i) $\xi(\overline{z}) = \overline{\xi(z)}$, so that $\xi(z)$ is real iff z is real and positive, and the points $z = 0, +\infty$ correspond to $\xi = -\infty, +\infty$, respectively;

(ii) $\xi(z) = \log(z/2) + 1 + o(1)$ as $|z| \to 0$, and $\xi(z) = z + o(1)$ as $|z| \to +\infty$;

(iii) $\xi(it) = (1-t^2)^{1/2} + \log(t/[1+(1-t^2)^{1/2}]) + i\pi/2$ if $0 < t < 1$, and $\xi(it) = i\{(t^2-1)^{1/2} + \arctan[t/(t^2-1)^{1/2}]\}$ if $1 < t < +\infty$;

(iv) $\xi(z)$ is asymptotic to $3^{-1}2^{3/2}e^{-i\pi/4}(z-i)^{3/2}$ as $z \to i$;

(v) $d\xi/dz = (1+z^2)^{1/2}/z$ vanishes only for $z = \pm i$;

(vi) For every $\theta \in (-\pi/2, \pi/2)$, Re $\xi(z)$ is non-decreasing along the ray $z = te^{i\theta}$.

It follows that the half plane Re $z > 0$ is mapped conformally onto the region Δ consisting of the union of the half plane Re $\xi > 0$ and the strip $|\mathrm{Im}\ \xi| < \pi/2$ (cf. [O], p. 375).

Suitable variational paths in the z plane can be constructed as the image under the inverse map $\xi \mapsto z$ of the paths in the ξ plane obtained by traveling from $\xi(z)$ parallel to the imaginary axis until the real axis is reached, and then moving along the real axis to $-\infty = \xi(0)$. Alternatively, one could use the path in the z-plane obtained by traveling from z along the ray through z until the circle $|z| = \epsilon$ is reached, and then proceeding to $z = 0$ along a path of the form previously described.

We note that, by the symmetry of the problem, it suffices to estimate $\mathcal{V}_{\omega_z}(U_1)$ when z lies in Im $z \geq 0$.

The following lemmata play a crucial role in obtaining upper estimates for the error terms in formula (9).

Lemma A1. *For ξ in Δ with* Im $\xi > 0$ *denote by l_ξ the vertical line segment joining ξ with the real axis, and let δ_ξ be the distance of l_ξ from $i\pi/2$. Let ω be the image in the z-plane of l_ξ. Then there exists a constant C such that*

$$\mathcal{V}_\omega(U_1) \leq C(1 + \delta_\xi^{-1}).$$

Proof. Dropping for ease of notation the suffix ξ, we parametrise l by $l(s) = \mathrm{Re}\ \xi + is$, $0 \leq s \leq \mathrm{Im}\ \xi$, and write

$$\mathcal{V}_\omega(U_1) = \int_0^{\mathrm{Im}\ \xi} \left|\frac{dU_1}{dp}(p(z(l(s))))\right| \left|\frac{dp}{dz}(z(l(s)))\right| \left|\frac{dz}{d\xi}(l(s))\right| ds, \tag{10}$$

where

$$\frac{dU_1}{dp}(p) = \frac{1-5p^2}{8}, \quad \frac{dp}{dz}(z) = \frac{-z}{(1+z^2)^{3/2}} \quad \text{and} \quad \frac{dz}{d\xi}(\xi) = \frac{z}{(1+z^2)^{1/2}}\bigg|_{z=z(\xi)}.$$

It is not hard to check that $|z(\xi)|$ is bounded if $|\xi|$ is bounded, and that $|z(\xi) - i|$ is bounded away from zero if $|\xi - i\pi/2|$ is bounded away from zero. Moreover, it follows from the asymptotic formulae in (ii) above that $z(\xi) = \xi + o(1)$ as $|\xi|$ tends to $+\infty$ in Re $\xi \geq 0$, and $|z(\xi)| = 2e^{\mathrm{Re}\ \xi - 1}(1 + o(1))$ as Re ξ tends to $-\infty$ in Δ. Therefore $dU_1/dp|_{p=p(z(\xi))}$, $dp/dz|_{z=z(\xi)}$ and $dz/d\xi$ are uniformly bounded if ξ stays away from $i\pi/2$, and

$$\left|\frac{dp}{dz}\right|(z(\xi)), \ \left|\frac{dz}{d\xi}\right|(\xi) \asymp 2e^{\mathrm{Re}\ \xi - 1}$$

as Re $\xi \to -\infty$, and

$$\left|\frac{dp}{dz}\right|(z(\xi)) \asymp 1, \qquad \left|\frac{dz}{d\xi}\right|(\xi) \asymp |\xi|^{-2}$$

as $|\xi| \to +\infty$ in $\operatorname{Re}\xi > 0$. We may conclude from this that $\mathcal{V}_\omega(U_1)$ is uniformly bounded if δ is bounded away from zero, and, in fact, tends to zero as δ tends to $+\infty$.

Assume now that $\delta \le \delta_o$, where δ_o is so small that the asymptotic relation (iv) above implies that the equality $|z-i| = 3^{2/3}2^{-1}|\xi - i\pi/2|^{2/3}(1+o(1))$ holds with $0 \le |o(1)| \le 1/2$ inside the circle $\mathcal{C}_{\delta_o} = \{\xi : |\xi - i\pi/2| = \delta_o\}$.

By what we observed above, the contribution to the integral in (10) of the parts of l lying outside $\mathcal{C}_{\delta_o}$ is uniformly bounded by a constant C_o depending only on δ_o. Hence we need only estimate the contribution to the integral over the part of l lying inside $\mathcal{C}_{\delta_o}$ and since there $|z-i| \ge C_1|\xi - i\pi/2|^{2/3}$ while $1 \le |z+i| \le 2$, the integrand in (10) is bounded above by $C_2|l(s) - i\pi/2|^{-2}$ with C_2 independent of $0 \le \delta \le \delta_o$.

Thus, letting $s_o < s_1$ denote the values of the parameter for which $l(s)$ intersects the circles $\mathcal{C}_{\delta_o}$ and $\mathcal{C}_\delta$, respectively, we may estimate

$$\mathcal{V}_\omega(U_1) \le C_o + C_2 \int_{s_o}^{s_1} |l(s) - i\pi/2|^{-2}\, ds.$$

A simple application of the law of cosines shows that

$$|l(s) - i\pi/2|^2 = |(l(s_1) - i\pi/2) + i(s-s_1)|^2 \ge \delta^2 + (s-s_2)^2,$$

so that the integral on the right hand side is bounded by

$$\int_{s_o}^{s_2} (\delta^2 + (s-s_1)^2)^{-1} ds \le \int_0^{+\infty} (\delta^2 + s^2)^{-1} ds = \pi/2\delta,$$

as required to finish the proof of the lemma. □

Lemma A2. *Maintaining the notation of Lemma* A1, *given ξ in Δ such that $\operatorname{Im}\xi \ge 0$, denote by l_ξ the vertical line segment joining ξ with the real ξ axis, and let δ_ξ be the distance of l_ξ from $i\pi/2$. Then the following hold:*

(i) *For every θ_o in $[0,\pi/2)$ there exists δ_o such that for every z with $0 \le \arg z \le \theta_o$, $\delta_{\xi(z)} \ge \delta_o$.*

(ii) *For every θ_o in $(0,\pi/2)$ and $\epsilon > 0$ there exist positive constants C and δ_1 such that for every z with $\theta_o \le \arg z < \pi/2$*

$$\delta_{\xi(z)} \ge \begin{cases} \delta_1 & \text{if } |z| \le 1-\epsilon, \\ C\,(\operatorname{Re} z)^{3/2} & \text{if } 1-\epsilon \le |z| \le 1+\epsilon, \\ C \operatorname{Re} z & \text{if } 1+\epsilon \le |z|. \end{cases}$$

Proof. To prove (i), note that since $z \mapsto \xi(z)$ maps $\{z : \operatorname{Re} z > 0\}$ conformally onto Δ, the region $\{z : 0 \le \arg z \le \theta_o\}$ is mapped conformally onto the region D_o lying above the real ξ axis and below the image under ξ of the ray $z_o(t) = te^{i\theta_o}$, $0 < t < +\infty$. Since $\operatorname{Re}(\xi(z_o(t)))$ is increasing with t, the segment $l_{\xi(z)}$ is entirely contained in D_o, and therefore its distance from $i\pi/2$ is greater than or equal to the distance δ_o of $\xi(z_o(t))$ from $i\pi/2$.

We now prove (ii). Assume first that $|\zeta| \le 1-\epsilon$. Then $\xi(z)$ belongs to the region D_1 lying to the left of the image under $z \mapsto \xi$ of the circle of radius $1-\epsilon$ centered at

0. If the latter is parametrised by $z_1(t) = (1-\epsilon)e^{i(\pi/2-t)}$, $0 \le t \le \pi/2$, the equality

$$\frac{d}{dt}\xi(z_1(t)) = -i(1+z_1(t)^2)^{1/2}$$

shows that $\mathrm{Re}\,(\xi(z_1(t)))$ is increasing with t, and $\mathrm{Im}\,(\xi(z_1(t)))$ is decreasing. It follows as in (i) that $l_{\xi(z)}$ is entirely contained in D_1, and therefore that its distance from $i\pi/2$ is greater than or equal to the distance δ_1 of $\xi(z_1(t))$ from $i\pi/2$.

Next assume that $1-\epsilon \le |z| \le 1+\epsilon$. Without loss of generality we may assume that θ_o is close enough to $\pi/2$ and $\epsilon > 0$ so small that that for every z satisfying $\theta_o \le \arg z < \pi/2$ and $1-\epsilon \le |z| \le 1+\epsilon$ we have $\xi(z) - i\pi/2 = 3^{-1}2^{3/2}e^{-i\pi/4}(z-i)^{3/2}(1+o(1))$, with $|o(1)| \le 10^{-1}$.

Arguing as in (i), we may infer that the segment $l_\xi(z)$ is entirely contained in the subregion of Δ lying below the image under ξ of the ray $te^{i\arg z}$, $0 < t < +\infty$, and therefore that the distance of $i\pi/2$ from $l_{\xi(z)}$ is greater than or equal to its distance from the curve $\xi(te^{i\arg z})$. On the other hand, the function $t \mapsto \left|te^{i\arg z} - i\right|$ attains its minimum $\cos(\arg z)$ at $t = \sin(\arg z)$, so that the curve $\xi(te^{i\arg z})$ lies outside the image of the circle of radius $\cos(\arg z)$ centered at i. The required conclusion follows on noting that, by the asymptotic relation quoted above,

$$\left|\xi(te^{i\arg z}) - i\pi/2\right| \ge 3^{-1}2^{3/2}\cos^{3/2}(\arg z)(1-10^{-1}) = C\cos^{3/2}(\arg z) \quad \forall t > 0.$$

Finally assume that $|z| \ge 1+\epsilon$. We consider the image under ξ of the circle $t \mapsto |z|\, e^{i(\pi/2-t)}$. Since $\mathrm{Re}\left(\xi(|z|\, e^{i(\pi/2-t)})\right)$ is zero for $t=0$, and is equal to $\mathrm{Re}\,(\xi(z))$ for $t = \pi/2 - \arg z$, we can write

$$\begin{aligned}\mathrm{Re}\,(\xi(z)) &= \int_0^{\pi/2-\arg z} \frac{d}{dt}\,\mathrm{Re}\left(\xi(|z|\, e^{i(\pi/2-t)}\right) dt \\ &= \int_0^{\pi/2-\arg z} \mathrm{Im}\left((1+|z|^2\, e^{2i(\pi/2-t)})^{1/2}\right) dt.\end{aligned}$$

By formula (7) in Section 3, the integrand on the right hand side is equal to

$$2^{-1/2}\left\{\left|1+|z|^2\, e^{2i(\pi/2-t)}\right| - \mathrm{Re}\left(1+|z|^2\, e^{2i(\pi/2-t)})\right)\right\}^{1/2},$$

and a straightforward computation shows that this is bounded from below by

$$(|z|^2\cos(2t)-1)^{1/2} \ge 2^{-1/2}\,|z|\left(1-(1+\epsilon)^{-2}\right)^{1/2}$$

in the range $0 \le t \le \frac{1}{2}\mathrm{arccos}\left(\frac{(1+\epsilon)^2+1}{2(1+\epsilon)^2}\right) = t_\epsilon$.

Thus we can estimate

$$\mathrm{Re}\,(\xi(z)) \ge |z|\left(1-(1+\epsilon)^{-2}\right)^{1/2}\min\{\pi/2-\arg z, \pi/2-t_\epsilon\},$$

and the required conclusion follows. □

After this preparation we are ready to prove the upper bounds for the error terms in formula (9).

Proposition A3. *The following hold:*

(i) $|\eta(\nu,+\infty)| \le (C_o/\nu)e^{C_o/\nu}$, *where* $C_o = 1/3\sqrt{5} + 1/6$.

(ii) *For every* θ_o *in* $[0,\pi/2)$ *and* $\epsilon>0$ *there exist positive constants* C_1, C_2, C_3 *such that*

$$|\eta(\nu,z)| \le \begin{cases} \dfrac{C_1}{\nu}e^{C_1/\nu} & \textit{if } \ |\arg z| \le \theta_o \ \textit{or} \ \ |z| \le 1-\epsilon; \\ \dfrac{C_2(1+\mathrm{Re}^{-3/2}\, z)}{\nu}e^{C_2(1+\mathrm{Re}^{-3/2}\, z)/\nu} & \textit{if } \ 1-\epsilon \le |z| \le 1+\epsilon; \\ \dfrac{C_3(1+\mathrm{Re}^{-1}\, z)}{\nu}e^{C_3(1+\mathrm{Re}^{-1}\, z)/\nu} & \textit{if } \ 1+\epsilon \le |z|\,. \end{cases}$$

Proof. We recall that

$$|\eta(\nu,z)| \le \frac{2\mathcal{V}_{\omega_z}(U_1)}{\nu}e^{2\mathcal{V}_{\omega_z}(U_1)/\nu},$$

where ω_z is any ξ-progressive path joining 0 with z.

To estimate the error term we compute the total variation $\mathcal{V}_{\omega_z}(U_1)$ along the ξ-progressive path ω_z whose image in the ξ plane is given by real line $(-\infty, \mathrm{Re}\,(\xi(z))]$ followed by the vertical segment $l_{\xi(z)} = [\mathrm{Re}\,(\xi(z)) + i0, \xi(z)]$.

Assume first that $z=+\infty$. Then $\omega_{+\infty}(t)$ traces the positive real z axis, and therefore $p(\omega_{+\infty}(t)) = (1+\omega_{+\infty}(t)^2)^{1/2}$ ranges over the interval $[0,1]$. By a change of variables, we can rewrite

$$\mathcal{V}_{\omega_{+\infty}}(U_1) = \int_0^1 \frac{\left|1-5p^2\right|}{8}dp = \frac{1}{6\sqrt{5}} + \frac{1}{12},$$

and (i) follows.

To prove (ii), note that the variation of U_1 along the part of ω_z corresponding to the line $(-\infty, \mathrm{Re}\,(\xi(z))]$ is bounded from above by $\mathcal{V}_{\omega_{+\infty}}(U_1)$. On the other hand, by Lemma A1 and Lemma A2, given $\theta_o>0$ and $\epsilon>0$ there exist constants C_i', $i=1,2,3$, such that the variation of U_1 along the part of ω_z corresponding to $l_{\xi(z)}$ is bounded above by

$$\begin{cases} C_1' & \text{if } \ |\arg z| \le \theta_o \text{ or } \ |z| \le 1-\epsilon, \\ C_2'/\,\mathrm{Re}^{3/2}\, z & \text{if } \theta_o \le |\arg z| < \pi/2 \text{ and } 1-\epsilon \le |z| \le 1+\epsilon, \\ C_3'/\,\mathrm{Re}\; z & \text{if } \theta_o \le |\arg z| < \pi/2 \text{ and } 1+\epsilon \le |z|\,, \end{cases}$$

from which (ii) follows at once. □

Setting $z=\zeta/\nu$ allows us to translate the results so far obtained for $I_\nu(\nu z)$ into the following description of the asymptotic behaviour of $I_\nu(\zeta)$:

Theorem A4. *For every* $\nu \ge 0$ *and every* ζ *with* $\mathrm{Re}\;\zeta>0$ *we have*

$$I_\nu(\zeta) = \frac{1+r_2(\nu,\zeta)}{1+r_1(\nu)}\frac{1}{\sqrt{2\pi}(\nu^2+\zeta^2)^{1/4}}e^{(\nu^2+\zeta^2)^{1/2}+\nu\log(\frac{\zeta}{\nu+(\nu^2+\zeta^2)^{1/2}})}, \tag{11}$$

where

$$|r_1(\nu)| \le C_o/\nu\, e^{C_o/\nu},$$

with $C_o = 1/3\sqrt{5} + 1/6$*, and for every* θ_o *in* $(0, \pi/2)$ *and* $\epsilon > 0$ *there exists a constant* $C > 0$ *such that*

$$|r_2(\nu,\zeta)| \le \begin{cases} C\nu^{-1} & \textit{if } |\arg\zeta| \le \theta_o \textit{ or } \dfrac{|\zeta|}{\nu} \le (1-\epsilon), \\ C(\nu^{-1} + \nu^{1/2}\operatorname{Re}^{-3/2}\zeta)e^{C(\nu^{-1}+\nu^{1/2}\operatorname{Re}^{-3/2}\zeta)} & \textit{if } (1-\epsilon) \le \dfrac{|\zeta|}{\nu} \le (1+\epsilon), \\ C(\nu^{-1} + \operatorname{Re}^{-1}\zeta)e^{C(\nu^{-1}+\operatorname{Re}^{-1}\zeta)} & \textit{if } \dfrac{|\zeta|}{\nu} \ge (1+\epsilon). \end{cases}$$

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