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BILINEAR OPERATORS ON HERZ-TYPE HARDY SPACES

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ABSTRACT. The authors prove that bilinear operators given by finite sums of products of Calderón-Zygmund operators on $\mathbb{R}^n$ are bounded from $H\dot{K}_{q_1}^{\alpha_1,p_1} \times H\dot{K}_{q_2}^{\alpha_2,p_2}$ into $H\dot{K}_q^{\alpha,p}$ if and only if they have vanishing moments up to a certain order dictated by the target space. Here $H\dot{K}_q^{\alpha,p}$ are homogeneous Herz-type Hardy spaces with $1/p = 1/p_1 + 1/p_2$, $0 < p_i \leq \infty$, $1/q = 1/q_1 + 1/q_2$, $1 < q_1, q_2 < \infty$, $1 \leq q < \infty$, $\alpha = \alpha_1 + \alpha_2$ and $-n/q_i < \alpha_i < \infty$. As an application they obtain that the commutator of a Calderón-Zygmund operator with a BMO function maps a Herz space into itself.

1. Introduction and statements of results

Beurling [2] first introduced some primordial form of Herz spaces to study convolution algebras. Later Herz [15] introduced versions of the spaces defined below in a slightly different setting. Since then, the theory of Herz spaces has been significantly developed, and these spaces have turned out to be quite useful in harmonic analysis. For instance, they were used by Baernstein and Sawyer [1] to characterize the multipliers on the standard Hardy spaces.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, $\widetilde{\chi}_k = \chi_k$ if $k \in \mathbb{N}$ and $\widetilde{\chi}_0 = \chi_{B_0}$, where by χ_E we denote the characteristic function of a set E.

Definition 1. Let $\alpha \in \mathbb{R}, 0 < p \leq \infty$ and $0 < q \leq \infty$.

(a) The *homogeneous Herz space* $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \Big[\sum_{k=-\infty}^{\infty} \|2^{k\alpha} f\chi_k\|^p_{L^q(\mathbb{R}^n)}\Big]^{1/p} < \infty;$$

(b) The *non-homogeneous Herz space* $K_q^{\alpha,p}(\mathbb{R}^n)$ is

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \Big[\sum_{k=0}^{\infty} \|2^{k\alpha} f\widetilde{\chi}_k\|^p_{L^q(\mathbb{R}^n)}\Big]^{1/p} < \infty,$$

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and the usual modifications in the definitions above are made when $p = \infty$.

The spaces $K_q^{n(1-1/q),1}(\mathbb{R}^n) \equiv A^q$, with $1 < q < \infty$, are called Beurling algebras and were introduced by Beurling [2] with different, but equivalent norms. The equivalence of the norms is in Feichtinger [8]. The spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ were introduced by Herz [15], also with different norms. Flett [9] gave a characterization of these spaces which is easily seen to be equivalent to Definition 1.

The theory of the Hardy spaces in this setting has been developed considerably. Chen and Lau [3] introduced the Hardy spaces associated with the Beurling algebras A^q on the real line with $1 < q \le 2$. García-Cuerva [10] generalized the theory of [3] to higher dimensions and to all q with $1 < q < \infty$. Lu and Yang [18], [19] established the theory of the corresponding homogeneous spaces. More recently, García-Cuerva and Herrero [11] and Lu and Yang [20]–[23] independently developed the real Hardy space theory for Herz spaces.

Before we introduce these spaces on $\mathbb{R}^n$, we fix some notation. Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with supp $\phi \subseteq B_0$, $\int_{\mathbb{R}^n} \phi(x)\,dx \ne 0$ and $\phi_t(x) = \frac{1}{t^n}\phi(\frac{x}{t})$ for any $t > 0$. $\mathcal{S}'(\mathbb{R}^n)$ denotes the class of tempered distributions on $\mathbb{R}^n$. Let

$$M_\phi(f)(x) = \sup_{t>0} |f * \phi_t(x)|. \tag{1.1}$$

Definition 2. Let $0 < p \le \infty$, $0 < q < \infty$, $\alpha \in \mathbb{R}$ and ϕ be as above.

(a) The *homogeneous Herz-type Hardy space* $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ associated with the Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \ M_\phi(f) \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we set $\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|M_\phi(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$.

(b) The *non-homogeneous Herz-type Hardy space* $HK_q^{\alpha,p}(\mathbb{R}^n)$ associated with $K_q^{\alpha,p}(\mathbb{R}^n)$ is

$$HK_q^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \ M_\phi(f) \in K_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we set $\|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)} = \|M_\phi(f)\|_{K_q^{\alpha,p}(\mathbb{R}^n)}$.

Remark 1. By the real-variable theory established in [14] (see also [23] and [11]), it follows that the norms above do not depend on the choice of ϕ.

Remark 2. When $1 < q < \infty$, $-n/q < \alpha < n(1 - 1/q)$ and $0 < p \le \infty$, we have $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_q^{\alpha,p}(\mathbb{R}^n) = K_q^{\alpha,p}(\mathbb{R}^n)$. See [16], also [14] and [21]. These identities fail when α is not in the above range. It is also easy to see that for $0 < p < \infty$, $H\dot{K}_p^{0,p}(\mathbb{R}^n) = HK_p^{0,p}(\mathbb{R}^n)$ are the usual Hardy spaces $H^p(\mathbb{R}^n)$ discussed in [7]. (We have $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ when $p > 1$).

Herz-type Hardy spaces are good substitutes for the usual Hardy spaces when studying boundedness of non-translation invariant operators (see Lu and Yang [20] and [24] for examples).

The purpose of this paper is to extend the known theory of certain bilinear operators on Hardy spaces to Herz-type Hardy spaces. The "spirit" of our results is as follows:

A bilinear operator $B(f, g)$ maps a product of Herz-type Hardy spaces to another Herz-type Hardy space if and only if it has moments vanishing up to a certain order dictated by the target space. More precisely, let

$$B(f,g)(x)=\sum_{\gamma=1}^{N}(T_\gamma^1 f)(x)\,(T_\gamma^2 g)(x),\quad x\in\mathbb{R}^n, \tag{1.2}$$

where T_γ^1 and T_γ^2 are Calderón-Zygmund operators. Assuming the required vanishing moments condition, Coifman and Grafakos [4] and Grafakos [12] proved that B maps $H^p\times H^q\to H^r$ for a certain range of p's and q's when $1/p+1/q=1/r$. Recently, Grafakos and Li [13] found another proof of the theorem in [4], and they also showed boundedness for the missing pairs of indices p, q, thus establishing $H^p(\mathbb{R}^n)\times H^q(\mathbb{R}^n)\to H^r(\mathbb{R}^n)$ boundedness for B, on the entire range of $0<p,q,r<\infty$ when $1/r=1/p+1/q$. The method developed in [13] avoids the use of the Fourier transform, and it can be adapted in this setting.

We now state our main results. We postpone the definition of an (α,q)–atom until the end of this section. We break up our results in three parts, and we state each part as a separate theorem. Our proofs are inspired by [13], but no prior knowledge of that paper is required for understanding this paper.

Theorem 1. *Let $0<p_1,p_2\le\infty$, $1/p=1/p_1+1/p_2$, $1<q_1,q_2<\infty$, $q\ge1$, $1/q=1/q_1+1/q_2$, $-n/q_i<\alpha_i<n(1-1/q_i)$, $i=1,2$ and $\alpha=\alpha_1+\alpha_2$. Let $s\ge[\alpha+n(1/q-1)]$ be a non-negative integer such that*

$$\int_{\mathbb{R}^n}x^\beta B(f,g)(x)\,dx=0, \tag{1.3}$$

for all multi-indices β with $|\beta|\le s$, and all $f,\ g\in L^2(\mathbb{R}^n)$ with compact support. Then $B(f,g)$ can be extended to a bounded operator from $\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)\times\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$ into $H\dot K_q^{\alpha,p}(\mathbb{R}^n)$.

Theorem 2. *Let $0<p_1,p_2\le\infty$, $1/p=1/p_1+1/p_2$, $1<q_1,q_2<\infty$, $q\ge1$, $1/q=1/q_1+1/q_2$, $\alpha_1\ge n(1-1/q_1)$, $-n/q_2<\alpha_2<n(1-1/q_2)$ and $\alpha=\alpha_1+\alpha_2$. Let $s\ge[\alpha+n(1/q-1)]$ be a non-negative integer such that*

$$\int_{\mathbb{R}^n}x^\beta B(a,g)(x)\,dx=0, \tag{1.4}$$

for all multi-indices β with $|\beta|\le s$, for all $(\alpha_1,q_1)-$atoms a, and all $g\in L^2(\mathbb{R}^n)$ with compact support. Then $B(f,g)$ can be extended to a bounded operator from $H\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)\times\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$ into $H\dot K_q^{\alpha,p}(\mathbb{R}^n)$.

Theorem 3. *Let $0<p_1,p_2\le\infty$, $1/p=1/p_1+1/p_2$, $1<q_1,q_2<\infty$, $q\ge1$, $1/q=1/q_1+1/q_2$, $\alpha_i\ge n(1-1/q_i)$, $i=1,2$, and $\alpha=\alpha_1+\alpha_2$. Let $s\ge[\alpha+n(1/q-1)]$ be a non-negative integer such that*

$$\int_{\mathbb{R}^n}x^\beta B(a,b)(x)\,dx=0, \tag{1.5}$$

for all multi-indices β with $|\beta|\le s$ and all $(\alpha_1,q_1)-$atoms a and $(\alpha_2,q_2)-$atoms b. Then $B(f,g)$ can be extended to a bounded operator from $H\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)\times H\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$ into $H\dot K_q^{\alpha,p}(\mathbb{R}^n)$.

In the theorems above we have assumed that $\alpha\ge n(1-1/q)$. We shall indicate later that the case $\alpha<n(1-1/q)$ follows trivially from Hölder's inequality.

We end this section by reviewing some known facts about Herz-type Hardy spaces that we will use later. We have

Definition 3. Let $1 < q < \infty$, $\alpha \in \mathbb{R}$ and $s \in \mathbb{N} \cup \{0\}$. A function $a(x)$ is said to be a *central* $(\alpha, q)_s$*–atom*, if

(i) $\operatorname{supp} a \subseteq B(r) \equiv \{x \in \mathbb{R}^n : |x| \le r\}$, $r > 0$;
(ii) $\|a\|_{L^q(\mathbb{R}^n)} \le |B(r)|^{-\alpha/n}$; and
(iii) $\int_{\mathbb{R}^n} a(x) x^\beta \, dx = 0, \quad |\beta| \le s.$

Proposition 1 (Atomic decomposition in Herz-type Hardy spaces). *Let* $0 < p \le \infty$, $1 < q < \infty$ *and* $\alpha \in \mathbb{R}$. *For any given* $s \in \mathbb{N} \cup \{0\}$ *and* $f \in H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, *we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \tag{1.6}$$

where the series converges in the sense of distributions, $\lambda_k \ge 0$*, each* a_k *is a central* $(\alpha, q)_s$*−atom with* $\operatorname{supp} a_k \subseteq B_k$*, and*

$$\sum_{k=-\infty}^{\infty} (\lambda_k)^p \le c\|f\|^p_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}. \tag{1.7}$$

Conversely, if $\alpha \ge n(1 - 1/q)$ *and* $s \ge [\alpha + n(1/q - 1)]$*, and if (1.6) holds, then* $f \in H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$*, and*

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \sim \inf \Big\{ \Big(\sum_{k=-\infty}^{\infty} (\lambda_k)^p \Big)^{1/p} \Big\},$$

where the infimum is taken over all the decompositions of f *as above. In this case we call* $(\alpha, q)_s$*−atoms simply* (α, q)*−atoms.*

Remark 3. It is remarkable that atomic decomposition also holds for $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (including $L^q(\mathbb{R}^n)$ as a special case). We shall use this fact in the proof of Theorem 2. The atoms in the decomposition (1.2) can be taken to be supported in dyadic annuli. See [22] (also [10], [11], and [20]) for details regarding the construction of such atoms.

Calderón-Zygmund operators and the Hardy-Littlewood maximal operator are bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ for a certain range of α's and q's. This is stated in the next proposition. In this article, by a C-Z operator, we mean an L^2 bounded singular integral operator with kernel $K(x)$, which is $\mathcal{C}^\infty$ away from the origin, satisfying

(i) $|K(x)| \le c|x|^{-n}$, if $x \ne 0$;
(ii) $\left| \frac{\partial^\beta}{\partial x^\beta} K(x - y) - \frac{\partial^\beta}{\partial x^\beta} K(x - y') \right| \le C_\beta \frac{|y-y'|}{|x-y|^{n+|\beta|+1}}$, if $|x - y| \ge 2|y - y'|$, where $\beta = (\beta_1, \cdots, \beta_n)$ is any multi-index and $|\beta| = \beta_1 + \cdots + \beta_n$.

Proposition 2. *Caderón-Zygmund operators and the Hardy-Littlewood maximal operator are bounded on* $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ *whenever* $-n/q < \alpha < n(1 - 1/q)$, $1 < q < \infty$ *and* $0 < p \le \infty$.

The proof is given in [16] (see also [21] and [14]).

Next we have the following:

Lemma (Hölder's inequality in Herz spaces). *If* $0 < p_i, q_i \le \infty$, $-\infty < \alpha_i < \infty, i = 1, 2,$ $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$ *and* $\alpha = \alpha_1 + \alpha_2$*, then*

$$\|fg\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \le \|f\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)} \|g\|_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.$$

The proof follows immediately by applying the usual Hölder's inequality twice.
Combining this lemma with Proposition 2, we have

Corollary. *The operator $B(f,g)$ defined in (1.2) is bounded from $\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n) \times \dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$ into $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ whenever $-n/q_i < \alpha_i < n(1-1/q_i)$, $i=1,2$, $\alpha = \alpha_1+\alpha_2$ and $-n/q < \alpha < n(1-1/q)$, where $0 < p_1, p_2 \le \infty$, $1/p = 1/p_1 + 1/p_2$, $1 < q, q_1, q_2 < \infty$ and $1/q = 1/q_1 + 1/q_2$.*

It follows from the corollary that it remains to show boundedness from the space $\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n) \times \dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$ into $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ when $\alpha \ge n(1-1/q)$.

We end this section with a few remarks:

Remark 4. It is easy to see that the integrals in (1.3), (1.4) and (1.5) are well-defined for f and g in the corresponding spaces. Also, compactly supported L^2 functions are dense in $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and (α,q)–atoms are dense in $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (see [16]). This means that the vanishing moment hypotheses hold on a dense set of functions.

Remark 5. The order of vanishing moments in the theorems above is assumed to be at least $[\alpha+n(1/q-1)]$. This assumption is natural and optimal from Definition 3 and Proposition 1.

Remark 6. It follows from the proofs that our results are still true if we replace the standard C–Z operators by the central C–Z operators defined in [27]. See also [20].

Remark 7. Throughout this paper, we only discuss the homogeneous Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. Our theorems also hold for the non-homogeneous Herz spaces $K_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_q^{\alpha,p}(\mathbb{R}^n)$. The proofs are similar and are omitted. These spaces are related by the following identities:

$$K_q^{\alpha,p}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$$

whenever $0 < p \le \infty$, $\alpha > 0$, $0 < q \le \infty$; and

$$HK_q^{\alpha,p}(\mathbb{R}^n) = H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$$

whenever $0 < p \le \infty$, $\alpha \ge n(1-1/q)$ and $1 < q < \infty$. See [1], [18] and [21]–[22] for the details.

Remark 8. The discussion of Fourier transforms on Herz or Herz-type Hardy spaces can also be found in the literature. We refer the readers to [8], [9], [26] and [28].

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2. Proof of Theorem 1

Our proof of Theorem 1 uses some standard estimates for maximal functions, the vanishing moment condition (1.3) and the lemma in Section 1.

Let ϕ be as in (1.1). Without loss of generality, we may assume $\phi \geq 0$ (this assumption will remain valid throughout this paper). For $x \in \mathbb{R}^n$, let $\phi_{t,x}(y) = \frac{1}{t^n}\phi(\frac{x-y}{t}),\ y \in \mathbb{R}^n$. We want to show that

$$\Big\|\sup_{t>0}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(f,g)(y)\,dy\Big|\Big\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \leq c\|f\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\|g\|_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}. \tag{2.1}$$

We choose $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $\eta \equiv 1$ on $\{x:\ |x|<2\}$ and supp $\eta \subseteq \{x:\ |x|<4\}$. Define $\eta_0(y) = \eta(\frac{x-y}{t})$ and $\eta_1(y) = 1-\eta_0(y)$. We split $B(f,g)$ as sum

$$B(f,g) = B(\eta_0 f,\eta_0 g) + B(f,\eta_1 g) + B(\eta_1 f, g) - B(\eta_1 f,\eta_1 g). \tag{2.2}$$

Consider $B(f,\eta_1 g)$ first. We have

$$\begin{aligned}
&\sup_{t>0}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(f,\eta_1 g)(y)\,dy\Big| \\
&\quad\leq c\sum_{\gamma=1}^{N}\sup_{t>0}\int_{\mathbb{R}^n}\phi_{t,x}(y)|T_\gamma^1 f(y)||T_\gamma^2(\eta_1 g(y) - T_\gamma^2(\eta_1 g)(x)|\,dy \\
&\qquad\qquad + c\sum_{\gamma=1}^{N}\sup_{t>0}\int_{\mathbb{R}^n}\phi_{t,x}(y)|T_\gamma^1 f(y)||T_\gamma^2(\eta_1 g)(x)|\,dy \\
&\quad\leq c\sum_{\gamma=1}^{N}M(T_\gamma^1 f)(x)M(\eta_1 g)(x) + c\sum_{\gamma=1}^{N}M(T_\gamma^1 f)(x)|T_\gamma^2(\eta_1 g)(x)|,
\end{aligned}$$

where M is the Hardy-Littlewood maximal operator. The last inequality follows easily from standard estimates for C–Z operators (see for instance [4] or [13]). Now we apply the lemma and Proposition 2 in section 1 to this expression and obtain inequality (2.1) with $\eta_1 g$ replacing g on the left hand side. The estimates for $B(\eta_1 f, g)$ and $B(\eta_1 f,\eta_1 g)$ in (2.2) are similar.

We now prove (2.1) for $B(\eta_0 f,\eta_0 g)$. By the vanishing moment condition (1.3), we can subtract the Taylor polynomial P_y^s of $\phi_{t,x}(\cdot)$ at y of degree s and obtain

$$\begin{aligned}
&\sup_{t>0}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(\eta_0 f,\eta_0 g)(y)\,dy\Big| \\
&\quad= \sup_{t>0}\Big|\int_{\mathbb{R}^n}\sum_{\gamma=1}^{N}(\eta_0 f)(y)\Big[(T_\gamma^1)^*[(\phi_{t,x}(\cdot) - P_y^s(\cdot - y))T_\gamma^2(\eta_0 g)(\cdot)](y)\Big]\,dy\Big|,
\end{aligned}$$

where $(T_\gamma^1)^*$ is the adjoint of T_γ^1. This last expression can be estimated by Hölder's inequality and the fractional integral theorem by

$$cM(|f|^{\ell_1})^{1/\ell_1}(x)M(|g|^{\ell_2})^{1/\ell_2}(x),$$

for all $\ell_1,\ \ell_2$ such that $1<\ell_1<q_1$, $1<\ell_2<q_2$ and $1/\ell_1+1/\ell_2 = (n+s+\varepsilon)/n > 1$ for some fixed $0<\varepsilon<1$. By our assumptions on $\alpha_1,\ \alpha_2$, and α, we may choose $\ell_1,\ \ell_2$ as above such that

$$-n/q_1 < \alpha_1 < n(1/\ell_1 - 1/q_1) \ \text{ and } \ -n/q_2 < \alpha_2 < n(1/\ell_2 - 1/q_2). \tag{2.3}$$

Hölder's inequality now gives

$$\Big\|\sup_{t>0}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(\eta_0 f,\eta_0 g)(y)\,dy\Big|\Big\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \le c\Big\|M(|f|^{\ell_1})^{1/\ell_1}\Big\|_{\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\Big\|M(|f|^{\ell_2})^{1/\ell_2}\Big\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.$$

The first term above is

$$\begin{aligned}\Big\|M(|f|^{\ell_1})^{1/\ell_1}\Big\|_{\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)} &= \Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha_1 p_1}\Big\|M(|f|^{\ell_1})^{1/\ell_1}\chi_k\Big\|_{L^{q_1}(\mathbb{R}^n)}^{p_1}\Big\}^{1/p_1}\\ &= \Big\{\sum_{k=-\infty}^{\infty}2^{k\ell_1\alpha_1\frac{p_1}{\ell_1}}\Big\|M(|f|^{\ell_1})\chi_k\Big\|_{L^{q_1/\ell_1}(\mathbb{R}^n)}^{p_1/\ell_1}\Big\}^{1/p_1}\\ &\le c\Big\{\sum_{k=-\infty}^{\infty}2^{k\ell_1\alpha_1\frac{p_1}{\ell_1}}\Big\||f|^{\ell_1}\chi_k\Big\|_{L^{q_1/\ell_1}(\mathbb{R}^n)}^{p_1/\ell_1}\Big\}^{1/p_1}\\ &= c\|f\|_{\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)},\end{aligned}$$

where we used the boundedness of Hardy-Littlewood maximal function on the Herz space $\dot K_{q_1/\ell_1}^{\ell_1\alpha_1,p_1/\ell_1}(\mathbb{R}^n)$ and Proposition 2. Similarly, we obtain that

$$\Big\|M(|g|^{\ell_2})^{1/\ell_2}\Big\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)} \le c\|g\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.$$

We now have estimate (2.1) for $B(\eta_0 f,\eta_0 g)$ as well.

This finishes the proof of Theorem 1.

3. Proof of Theorem 2

The idea of the proof is to break the whole estimate into "dyadic" pieces and give appropriate pointwise estimates on dyadic annuli of C-Z operators acting on functions in some dense subspaces. While the maximal function estimates (in Herz type spaces) are used to treat the "local" parts, the vanishing moment condition (1.4) and the cancellation property of atoms are carefully used to treat the "non-local" parts.

We assume $0 < p_1, p_2 < \infty$ and leave the easy cases $p_1 = \infty$ or $p_2 = \infty$ to the interested reader. We must show that

$$\Big\|\sup_{t>0}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(f,g)(y)\,dy\Big|\Big\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \le c\|f\|_{H\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\|g\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}, \tag{3.1}$$

for all $f \in H\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)$ and $g \in \dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$. Without loss of generality, we may assume that $f = \sum_{i\in\mathbb{Z}}\lambda_i a_i$ and $g = \sum_{j\in\mathbb{Z}}\mu_j b_j$, where $\lambda_j,\ \mu_j \ge 0$, the a_j's are $(\alpha_1, q_1)-$atoms, the b_j's are $(\alpha_2, q_2)_N-$atoms, N is a sufficiently large positive integer, and, by Remark 3, supp $a_i \subseteq \{x \in \mathbb{R}^n:\ 2^{i-2} < |x| \le 2^{i+2}\}$ and supp $b_j \subseteq \{x \in \mathbb{R}^n:\ 2^{j-2} < |x| \le 2^{j+2}\}$. Using the vanishing moment conditions of atoms, together with the standard estimates of C–Z operators, we have, for any non-negative integer s_1,

$$|T_\gamma^1 a_i(x)| \le c\frac{2^{i(n+s_1-\alpha_1-n/q_1)}}{|x|^{n+s_1}},\quad \text{whenever}\quad |x| > 2^{i+3}; \tag{3.2}$$

and

$$|T_\gamma^1 a_i(x)| \le c2^{-i(\alpha_1+n/q_1)},\quad \text{whenever}\quad |x| < 2^{i-3}. \tag{3.2'}$$

(3.2) and (3.2') also hold for b_j when we replace $T_\gamma^1,\ \alpha_1,\ q_1$ and i by $T_\gamma^2,\ \alpha_2,\ q_2$ and j respectively.

Now let

$$S(a_i,b_j)(x)=\sup_{t>0}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(a_i,b_j)(y)\,dy\Big|.$$

Then

$$\begin{aligned}\sup_{t>0}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(f,g)(y)\,dy\Big| &\le \sum_{i,j}\lambda_i\mu_j S(a_i,b_j)(x)\\ &=\sum_{i,j}\lambda_i\mu_j S(a_i,b_j)(x)\chi_{\{|x|\le 2^{i+4}\}}\\ &\qquad+\sum_{i,j}\lambda_i\mu_j S(a_i,b_j)(x)\chi_{\{|x|>2^{i+4}\}}\\ &\equiv \Gamma_1(x)+\Gamma_2(x).\end{aligned}$$

For $\Gamma_1(x)$, we have

$$\begin{aligned}\Gamma_1(x)&=\sum_{i,j}\lambda_i\mu_j S(a_i,b_j)(x)\chi_{\{|x|\le 2^{i+4},\ |x|\le 2^{j-5}\}}\\ &\quad+\sum_{i,j}\lambda_i\mu_j S(a_i,b_j)(x)\chi_{\{|x|\le 2^{i+4},\ 2^{j-5}<|x|\le 2^{j+4}\}}\\ &\quad+\sum_{i,j}\lambda_i\mu_j S(a_i,b_j)(x)\chi_{\{|x|\le 2^{i+4},\ |x|>2^{j+4}\}}\\ &\equiv I_1(x)+I_2(x)+I_3(x).\end{aligned}$$

Let us consider $I_1(x)$ first. In this case we have

$$\begin{aligned}S(a_i,b_j)(x)&\le\sum_{\gamma=1}^N\sup_{0<t\le\frac12|x|}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)T_\gamma^1a_i(y)T_\gamma^2b_j(y)\,dy\Big|\\ &\quad+\sum_{\gamma=1}^N\sup_{t>\frac12|x|}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)\sum_{\gamma=1}^N T_\gamma^1a_i(y)T_\gamma^2b_j(y)\,dy\Big|.\end{aligned}\tag{3.3}$$

The first summand in (3.3) is dominated by

$$\sum_{\gamma=1}^N\sup_{0<t\le\frac12|x|}\Big|\int_{|y|<2^{j-4}}\phi_{t,x}(y)T_\gamma^1a_i(y)T_\gamma^2b_j(y)\,dy\Big|,$$

since $|y|\le|x|+|x-y|\le\frac32|x|<2^{j-4}$. By (3.2') for b_j, this is no more than

$$c\sum_{\gamma=1}^N M(|T_\gamma^1a_i|)(x)2^{-j(\alpha_2+n/q_2)}.\tag{3.4}$$

The second summand in (3.3) is dominated by

$$\begin{aligned}&\sum_{\gamma=1}^N\sup_{t>\frac12|x|}\Big|\int_{|y|<2^{j-4}}\phi_{t,x}(y)T_\gamma^1a_i(y)T_\gamma^2b_j(y)\,dy\Big|\\ &\qquad+\sum_{\gamma=1}^N\sup_{t>\frac12|x|}\Big|\int_{|y|\ge 2^{j-4}}\phi_{t,x}(y)T_\gamma^1a_i(y)T_\gamma^2b_j(y)\,dy\Big|.\end{aligned}\tag{3.5}$$

The first term in (3.5) is estimated as before, and is shown to be bounded by

$$c\sum_{\gamma=1}^{N} M(|T_\gamma^1 a_i|)(x)2^{-j(\alpha_2+n/q_2)},$$

i.e., the term in (3.4). By Hölder's inequality, the second term in (3.5) is dominated by

$$\begin{aligned}(3.6)\quad \sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \Big(\int_{\mathbb{R}^n} \phi_{t,x}(y)|T_\gamma^1 a_i(y)|^{q_2'}\,dy\Big)^{\frac{1}{q_2'}} \Big(\int_{|y|\ge 2^{j-4}} \phi_{t,x}(y)|T_\gamma^2 b_j(y)|^{q_2}\,dy\Big)^{\frac{1}{q_2}} \\ \le c\sum_{\gamma=1}^{N} M(|T_\gamma^1 a_i(y)|^{q_2'})^{\frac{1}{q_2'}}(x)2^{-\frac{jn}{q_2}}\|T_\gamma^2 b_j\|_{L^{q_2}(\mathbb{R}^n)} \\ \le c\sum_{\gamma=1}^{N} M(|T_\gamma^1 a_i(y)|^{q_2'})^{1/q_2'}(x)2^{-j(\alpha_2+n/q_2)},\end{aligned}$$

where we used the fact that $|y|\le|x|+|x-y|\le 3t$ to obtain the range of integration in the second integral.

Combining (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}(3.7)\quad \|I_1\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \le c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{\infty}\lambda_i M(|T_\gamma^1 a_i|)\chi_{\{|x|\le 2^{i+4}\}}\Big\|_{\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)} \\ \times\Big\|\sum_{j=-\infty}^{\infty}\mu_j 2^{-j(\alpha_2+n/q_2)}\chi_{\{|x|\le 2^{j-5}\}}\Big\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)} \\ + c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{\infty}\lambda_i M(|T_\gamma^1 a_i|^{q_2'})^{1/q_2'}\chi_{\{|x|\le 2^{i+4}\}}\Big\|_{\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)} \\ \times\Big\|\sum_{j=-\infty}^{\infty}\mu_j 2^{-j(\alpha_2+n/q_2)}\chi_{\{|x|\le 2^{j-5}\}}\Big\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.\end{aligned}$$

The first norm in the first summand of (3.7) is equal to

$$\begin{aligned} c\sum_{\gamma=1}^{N}\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha_1 p_1}\Big\|\sum_{i=-\infty}^{\infty}\lambda_i M(|T_\gamma^1 a_i|)\chi_{\{|x|\le 2^{i+4}\}}\chi_k\Big\|_{L^{q_1}(\mathbb{R}^n)}^{p_1}\Big\}^{1/p_1} \\ \le c\sum_{\gamma=1}^{N}\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha_1 p_1}\Big(\sum_{i=k-4}^{\infty}\lambda_i\|M(|T_\gamma^1 a_i|)\|_{L^{q_1}(\mathbb{R}^n)}\Big)^{p_1}\Big\}^{1/p_1} \\ \le c\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha_1 p_1}\Big(\sum_{i=k-4}^{\infty}\lambda_i\|a_i\|_{L^{q_1}(\mathbb{R}^n)}\Big)^{p_1}\Big\}^{\frac{1}{p_1}}, \\ \text{by the Hardy-Littlewood maximal theorem} \\ \le c\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{i=k-4}^{\infty}\lambda_i 2^{(k-i)\alpha_1}\Big)^{p_1}\Big\}^{1/p_1}.\end{aligned}$$

When $p_1 \le 1$, since $\alpha_1 > 0$, this last expression is no more than

$$c\Big\{\sum_{i=-\infty}^{\infty}(\lambda_i)^{p_1}\Big(\sum_{k=-\infty}^{i+4}2^{(k-i)\alpha_1 p_1}\Big)\Big\}^{1/p_1} \le c\Big\{\sum_{i=-\infty}^{\infty}(\lambda_i)^{p_1}\Big\}^{1/p_1} \le c\|f\|_{H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}.$$

When $p_1 > 1$, letting $1/p_1 + 1/p_1' = 1$, this expression is no more than

$$c\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{i=k-4}^{\infty}(\lambda_i)^{p_1}2^{(k-i)\alpha_1 p_1/2}\Big)\Big(\sum_{i=k-4}^{\infty}2^{(k-i)\alpha_1 p_1'/2}\Big)^{p_1/p_1'}\Big\}^{1/p_1}$$
$$\le c\Big\{\sum_{i=-\infty}^{\infty}(\lambda_i)^{p_1}\Big(\sum_{k=-\infty}^{i+4}2^{(k-i)\alpha_1 p_1/2}\Big)\Big\}^{1/p_1}$$
$$\le c\Big\{\sum_{i=-\infty}^{\infty}(\lambda_i)^{p_1}\Big\}^{1/p_1} \le c\|f\|_{H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}.$$

The first norm in the second summand of (3.7) can be estimated just as above since $q_1 > q_2'$.

The second norms in both summands of (3.7) are the same, and they are equal to

$$\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p_2}\Big\|\sum_{j=-\infty}^{\infty}\mu_j 2^{-j(\alpha_2+n/q_2)}\chi_{\{|x|\le 2^{j-5}\}}\chi_k\Big\|_{L^{q_2}(\mathbb{R}^n)}^{p_2}\Big\}^{1/p_2}$$
$$\le c\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p_2}\Big(\sum_{j=k+4}^{\infty}\mu_j 2^{-j(\alpha_2+n/q_2)+kn/q_2}\Big)^{p_2}\Big\}^{1/p_2}$$
$$\le c\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k+4}^{\infty}\mu_j 2^{(k-j)(\alpha_2+n/q_2)}\Big)^{p_2}\Big\}^{1/p_2}.$$

Since $\alpha_2 + n/q_2 > 0$, this last expression can be estimated as before by

$$c\Big\{\sum_{j=-\infty}^{\infty}(\mu_j)^{p_2}\Big\}^{1/p_2} \le c\|g\|_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)},$$

for all p_2 such that $0 < p_2 < \infty$.

We therefore obtain

$$\|I_1\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \le c\|f\|_{H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\|g\|_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}. \tag{3.8}$$

The estimation for $I_2(x)$ is easy. We choose q_0 such that $1 < q_0 < q_1$, $1 < q_0' < q_2$ and $1/q_0 + 1/q_0' = 1$, since $1/q_1 + 1/q_2 = 1/q < 1$. We then have

$$\|I_2\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \le c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{\infty}\lambda_i M(|T_\gamma^1 a_i|^{q_0})^{1/q_0}\chi_{\{|x|\le 2^{i+4}\}}\Big\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}$$
$$\times\Big\|\sum_{j=-\infty}^{\infty}\mu_j M(|T_\gamma^2 b_j|^{q_0'})^{1/q_0'}\chi_{\{2^{j-5}<|x|\le 2^{j+4}\}}\Big\|_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.$$

The first norm above is as in (3.7), and it is bounded by $c\|f\|_{H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}$.

The second norm is equal to

$$\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2p_2}\Big(\sum_{j=k-4}^{k+4}\mu_j\Big\|M(|T_\gamma^2b_j|^{q_0'})^{1/q_0'}\chi_{\{2^{j-5}<|x|\le 2^{j+4}\}}\Big\|_{L^{q_2}(\mathbb{R}^n)}\Big)^{p_2}\Big\}^{1/p_2}$$
$$\le c\Big\{\sum_{j=-\infty}^{\infty}(\mu_j)^{p_2}\Big(\sum_{k=j-4}^{j+4}2^{k\alpha_2p_2-j\alpha_2p_2}\Big)\Big\}^{1/p_2}$$
$$\le c\Big\{\sum_{j=-\infty}^{\infty}(\mu_j)^{p_2}\Big\}^{1/p_2}\le c\|g\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.$$

Thus we obtain

$$\|I_2\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}\le c\|f\|_{H\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\|g\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}. \tag{3.9}$$

We now estimate $I_3(x)$. Again, we write $S(a_i,b_j)(x)$ as in (3.3), and we denote by $I_{31}(x)$ the part of $I_3(x)$ where in the majorization of $S(a_i,b_j)(x)$ the supremum is taken over $t\le\frac{1}{2}|x|$. In this case, since $|y|\ge|x|-|x-y|\ge|x|/2>2^{j+3}$, we can use (3.2) with $s_1=1$ for b_j to obtain that each term of the sum is no more than

$$cM(|T_\gamma^1a_i|)(x)\frac{2^{j(n+1-\alpha_2-n/q_2)}}{|x|^{n+1}}. \tag{3.10}$$

Therefore

$$\|I_{31}\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}\le c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{\infty}\lambda_iM(|T_\gamma^1a_i|)\chi_{\{|x|\le 2^{i+3}\}}\Big\|_{\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}$$
$$\times\Big\|\sum_{j=-\infty}^{\infty}\mu_j\frac{2^{j(n+1-\alpha_2-n/q_2)}}{|x|^{n+1}}\chi_{\{|x|>2^{j+4}\}}\Big\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.$$

The first norm above appeared in (3.7), and it is bounded by $c\|f\|_{H\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}$. By a simple calculation, the second norm above is bounded by

$$c\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-5}\mu_j2^{(j-k)(n+1-\alpha_2-n/q_2)}\Big)^{p_2}\Big\}^{1/p_2},$$

which, for all p_2, $0<p_2<\infty$, is bounded above by

$$c\Big\{\sum_{j=-\infty}^{\infty}(\mu_j)^{p_2}\Big\}^{1/p_2}\le c\|g\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)},$$

since $n+1-\alpha_2-n/q_2>0$.

We now denote by I_{32} the part of $I_3(x)$ where in the majorization of $S(a_i,b_j)(x)$ the supremum is taken over $t>\frac{1}{2}|x|$. Let $P_0^s(y)$ be the Taylor polynomial of degree

s of $\phi_{t,x}(y)$ at the origin. By the vanishing moment hypotheses we estimate I_{32} by

$$\begin{aligned}
&\sup_{t>\frac{1}{2}|x|}\Big|\int_{\mathbb{R}^n}(\phi_{t,x}(y)-P_0^s(y)\ )\sum_{\gamma=1}^{N}T_\gamma^1 a_i(y)T_\gamma^2 b_j(y)\,dy\Big|\\
&\quad\le c\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{\mathbb{R}^n}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy\\
&\quad\le c\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{|x-y|>4t}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy\\
&\qquad+c\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{|x-y|\le 4t}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy\\
&\quad\equiv A_1(x)+B_1(x).
\end{aligned}$$

If $|x-y|>4t$, then $|y|\ge|x-y|-|x|\ge 2t>|x|>2^{j+4}$. By (3.2) with $s_1=s+2$ for b_j, we obtain

$$\begin{aligned}
A_1(x)&\le c\sup_{t>\frac{1}{2}|x|}\int_{|x-y|>4t}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)|\frac{2^{j(n+s+2-\alpha_2-n/q_2)}}{|y|^{n+s+2}}\,dy\\
&\le c\frac{2^{j(n+s+2-\alpha_2-n/q_2)}}{|x|^{n+s+2}}\sup_{t>\frac{1}{2}|x|}t\int_{|x-y|>4t}\frac{|T_\gamma^1 a_i(y)|}{|y|^{n+1}}dy\\
&\le c\frac{2^{j(n-\alpha_2-n/q_2)}}{|x|^n}M(|T_\gamma^1 a_i(y)|)(x),
\end{aligned}\tag{3.11}$$

since $2^{j+4}<|x|$. Therefore

$$\begin{aligned}
\|I_{32}^1\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}&\le c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{\infty}\lambda_i M(|T_\gamma^1 a_i(y)|)\chi_{\{|x|\le 2^{i+4}\}}\Big\|_{\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\\
&\qquad\times\Big\|\sum_{j=-\infty}^{\infty}\mu_j\frac{2^{j(n-\alpha_2-n/q_2)}}{|x|^n}\chi_{\{|x|>2^{j+4}\}}\Big\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.
\end{aligned}$$

The second norm above is no more than

$$\begin{aligned}
&c\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-5}\mu_j 2^{(j-k)(n-\alpha_2-n/q_2)}\Big)^{p_2}\Big\}^{1/p_2}\\
&\qquad\le c\Big\{\sum_{j=-\infty}^{\infty}(\mu_j)^{p_2}\Big\}^{1/p_2},\quad\text{since } n-\alpha_2-n/q_2>0.
\end{aligned}$$

The first norm above is estimated as in (3.7). Thus we have the desired estimate for I_{32}^1.

For $B_1(x)$, we have

$$B_1(x) \le c\sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{\substack{|x-y|\le 4t\\ |y|\le 2^{j+3}}} \frac{|y|^{s+1}}{t^{n+s+1}} |T^1_\gamma a_i(y)||T^2_\gamma b_j(y)|\,dy$$
$$+ c\sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{\substack{|x-y|\le 4t\\ |y|> 2^{j+3}}} \frac{|y|^{s+1}}{t^{n+s+1}} |T^1_\gamma a_i(y)||T^2_\gamma b_j(y)|\,dy.$$

The first supremum above is

$$c\frac{2^{j(s+1)}}{|x|^{n/q_2+s+1}} \sup_{t>\frac{1}{2}|x|}\Big(\frac{1}{t^n}\int_{|x-y|\le 4t} |T^1_\gamma a_i(y)|^{q_2'}\,dy\Big)^{1/q_2'} \|T^2_\gamma b_j\|_{L^{q_2}(\mathbb{R}^n)}$$
$$\le c\frac{2^{j(s+1-\alpha_2)}}{|x|^{n/q_2+s+1}} M(|T^1_\gamma a_i|^{q_2'})^{1/q_2'}(x). \tag{3.12}$$

If we can show that

$$\Big\|\sum_{j=-\infty}^{\infty} \mu_j \frac{2^{j(s+1-\alpha_2)}}{|x|^{n/q_2+s+1}}\chi_{\{|x|>2^{j+4}\}}\Big\|_{\dot K^{\alpha_2,p_2}_{q_2}(\mathbb{R}^n)} \le c\Big\{\sum_{j=-\infty}^{\infty}(\mu_j)^{p_2}\Big\}^{1/p_2}, \tag{3.13}$$

then, by (3.7), we have the desired estimate for (3.12). The left hand side of (3.13) is bounded by

$$c\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-5}\mu_j 2^{(j-k)(s+1-\alpha_2)}\Big)^{p_2}\Big\}^{1/p_2},$$

which is dominated by the right hand side of (3.13) if $s+1>\alpha_2$. This is all right since $s \ge [\alpha + n(1/q-1)]$. Using (3.2) with $s_1 = s+1$ for b_j, we obtain that the second supremum in $B_1(x)$ is dominated by

$$c\sup_{t>\frac{1}{2}|x|}\int_{\substack{|x-y|\le 4t\\ |y|>2^{j+3}}} \frac{|y|^{s+1}}{t^{n+s+1}}|T^1_\gamma a_i(y)|\frac{2^{j(n+s+1-\alpha_2-n/q_2)}}{|y|^{n+s+1}}\,dy$$
$$\le c\frac{2^{j(n+s+1-\alpha_2-n/q_2)}}{|x|^{n/q_2+s+1}}\sup_{t>0}\Big(\frac{1}{t^n}\int_{|x-y|\le 4t}|T^1_\gamma a_i(y)|^{q_2'}\,dy\Big)^{1/q_2'}$$
$$\times\Big(\int_{|y|>2^{j+3}}\frac{1}{|y|^{nq_2}}\,dy\Big)^{1/q_2}$$
$$\le cM(|T^1_\gamma a_i|^{q_2'})^{1/q_2'}\frac{2^{j(s+1-\alpha_2)}}{|x|^{n/q_2+s+1}},$$

which is the same as (3.12). From the estimates of $B_1(x)$, we obtain that the contribution of B_1 in I_3, denoted by I^2_{32}, also satisfies

$$\|I^2_{32}\|_{\dot K^{\alpha,p}_q(\mathbb{R}^n)} \le c\|f\|_{H\dot K^{\alpha_1,p_1}_{q_1}(\mathbb{R}^n)}\|g\|_{\dot K^{\alpha_2,p_2}_{q_2}(\mathbb{R}^n)}.$$

Since $I_3(x) \le I_{31}(x) + I^1_{32}(x) + I^2_{32}(x)$, we have

$$\|I_3\|_{\dot K^{\alpha,p}_q(\mathbb{R}^n)} \le c\|f\|_{H\dot K^{\alpha_1,p_1}_{q_1}(\mathbb{R}^n)}\|g\|_{\dot K^{\alpha_2,p_2}_{q_2}(\mathbb{R}^n)}. \tag{3.14}$$

Combining (3.8), (3.9) and (3.14), we obtain the estimate (3.1) for $\Gamma_1(x)$.

Next, we consider $\Gamma_2(x)$. We split this term similarly. We have

$$\begin{aligned}\Gamma_2(x) \le& \sum_{i,j} \lambda_i \mu_j S(a_i, b_j)(x) \chi_{\{2^{i+4}<|x|,\ |x|\le 2^{j-5}\}}\\ &+ \sum_{i,j} \lambda_i \mu_j S(a_i, b_j)(x) \chi_{\{2^{i+4}<|x|,\ 2^{j-5}<|x|\le 2^{j+4}\}}\\ &+ \sum_{i,j} \lambda_i \mu_j S(a_i, b_j)(x) \chi_{\{2^{i+4}<|x|,\ |x|>2^{j+4}\}}\\ \equiv& J_1(x) + J_2(x) + J_3(x).\end{aligned}$$

Consider $J_1(x)$ first. By (3.3) and the vanishing moment condition (1.4), we have

$$\begin{aligned}S(a_i, b_j)(x) \le& \sum_{\gamma=1}^{N} \sup_{0<t\le\frac{1}{2}|x|} \Big| \int_{\mathbb{R}^n} \phi_{t,x}(y) T_\gamma^1 a_i(y) T_\gamma^2 b_j(y)\, dy \Big|\\ &+ \sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{\mathbb{R}^n} \frac{|y|^{s+1}}{t^{n+s+1}} |T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\, dy\\ \equiv& A_2(x) + B_2(x).\end{aligned} \tag{3.15}$$

For $A_2(x)$, we have

$$\begin{aligned}A_2(x) \le& \sum_{\gamma=1}^{N} \sup_{0<t\le\frac{1}{2}|x|} \Big| \int_{|y|<2^{j-4}} \phi_{t,x}(y) T_\gamma^1 a_i(y) T_\gamma^2 b_j(y)\, dy \Big|\\ &+ \sum_{\gamma=1}^{N} \sup_{0<t\le\frac{1}{2}|x|} \Big| \int_{|y|\ge 2^{j-4}} \phi_{t,x}(y) T_\gamma^1 a_i(y) T_\gamma^2 b_j(y)\, dy \Big|.\end{aligned}$$

Notice that in this case $|y| \ge |x| - |x-y| \ge \frac{|x|}{2} > 2^{i+3}$. Using (3.2) and (3.2') respectively, the first term in $A_2(x)$ is bounded by

$$\begin{aligned}&c \sum_{\gamma=1}^{N} \sup_{0<t\le\frac{1}{2}|x|} \int_{|y|<2^{j-4}} \phi_{t,x}(y) \frac{2^{i(n+s_1-\alpha_1-n/q_1)}}{|y|^{n+s_1}} 2^{-j(\alpha_2+n/q_2)}\, dy\\ &\le c \frac{2^{i(n+s_1-\alpha_1-n/q_1)}}{|x|^{n+s_1}} 2^{-j(\alpha_2+n/q_2)},\end{aligned} \tag{3.16}$$

for some non-negative integer s_1 to be determined later. Since

$$\begin{aligned}&\Big\| \sum_{i=-\infty}^{\infty} \lambda_i \frac{2^{i(n+s_1-\alpha_1-n/q_1)}}{|x|^{n+s_1}} \chi_{\{|x|>2^{i+4}\}} \Big\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\\ &= \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_1} \Big(\sum_{i=-\infty}^{k-4} \lambda_i 2^{i(n+s_1-\alpha_1-n/q_1)} \Big\| \frac{1}{|x|^{n+s_1}} \chi_k \Big\|_{L^{q_1}(\mathbb{R}^n)} \Big)^{p_1} \Big\}^{1/p_1}\\ &\le c \Big\{ \sum_{k=-\infty}^{\infty} \Big(\sum_{i=-\infty}^{k-4} \lambda_i 2^{(i-k)(n+s_1-\alpha_1-n/q_1)} \Big)^{p_1} \Big\}^{1/p_1}\\ &\le c \Big\{ \sum_{i=-\infty}^{\infty} (\lambda_i)^{p_1} \Big\}^{1/p_1}, \quad \text{if } s_1 > \alpha_1 - n(1-1/q_1),\end{aligned}$$

by (3.4), the first term in $A_2(x)$ satisfies the desired estimate.

Using (3.2) for a_i, we obtain that the second term in $A_2(x)$ is dominated by

$$\begin{aligned}\sum_{\gamma=1}^{N}\sup_{0<t\le\frac{1}{2}|x|}&\int_{|y|\ge 2^{j-4}}\phi_{t,x}(y)\frac{2^{i(n+s_1-\alpha_1-n/q_1)}}{|y|^{n+s_1}}|T_\gamma^2 b_j(y)|\,dy\\ &\le c\sum_{\gamma=1}^{N}\frac{2^{i(n+s_1-\alpha_1-n/q_1)}}{|x|^{n+s_1}}\sup_{0<t\le\frac{1}{2}|x|}\Big(\int_{\mathbb{R}^n}\phi_{t,x}(y)\,dy\Big)^{1/q_2'}\\ &\qquad\qquad\times\Big(\int_{|y|\ge 2^{j-4}}\phi_{t,x}(y)|T_\gamma^2 b_j(y)|^{q_2}\,dy\Big)^{1/q_2}\\ &\le c\frac{2^{i(n+s_1-\alpha_1-n/q_1)}}{|x|^{n+s_1}}2^{-j(\alpha_2+n/q_2)},\end{aligned}$$

since $\phi_{t,x}(y)\le ct^{-n}$ and $2^{j-4}\le|y|\le 2|x-y|\le 2t$; the latter is due to the facts that $|y|\ge 2^{j-4}$ and $|x|\le 2^{j-5}$. We then obtain the same estimate for the second term in $A_2(x)$ as we did for (3.16).

Now we turn our attention to $B_2(x)$. First we have

$$\begin{aligned}B_2(x)&\le\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{|y|>2^{i+3}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy\\ &\quad+\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{|y|\le 2^{i+3}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy\\ &=B_{21}(x)+B_{22}(x).\end{aligned}$$

Now,

$$\begin{aligned}B_{21}(x)\le&\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{\substack{|y|>2^{i+3}\\|y|\le 2^{j-4}}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy\\ &+\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{\substack{|y|>2^{i+3}\\|y|>2^{j-4}}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy.\end{aligned}$$

Using (3.2) and (3.2') for a_i and b_j respectively, we find that the first term in $B_{21}(x)$ is dominated by

$$\begin{aligned}c\sup_{t>\frac{1}{2}|x|}&\int_{\substack{|y|>2^{i+3}\\|y|\le 2^{j-4}}}\frac{|y|^{s+1}}{t^{n+s+1}}\frac{2^{i(n+s+2-\alpha_1-n/q_1)}}{|y|^{n+s+2}}2^{-j(\alpha_2+n/q_2)}\,dy\\ &\le c\frac{2^{i(n+s+2-\alpha_1-n/q_1)}}{|x|^{n+s+1}}2^{-j(\alpha_2+n/q_2)}\int_{|y|>2^{i+3}}\frac{1}{|y|^{n+1}}\,dy\\ &\le c\frac{2^{i(n+s+1-\alpha_1-n/q_1)}}{|x|^{n+s+1}}2^{-j(\alpha_2+n/q_2)}.\end{aligned}$$

This is the desired estimate from (3.16) when $s+1>\alpha_1+n(1/q_1-1)$. Using (3.2) for a_i, the second term in $B_{21}(x)$ is dominated by

$$\begin{aligned}
&c\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{\substack{|y|>2^{i+3}\\|y|>2^{j-4}}}\frac{|y|^{s+1}}{t^{n+s+1}}\frac{2^{i(n+s+2-\alpha_1-n/q_1)}}{|y|^{n+s+2}}|T_\gamma^2 b_j(y)|\,dy\\
&\qquad\le c\sum_{\gamma=1}^{N}\frac{2^{i(n+s+2-\alpha_1-n/q_1)}}{|x|^{n+s+1}}\Big(\int_{|y|>2^{i+3}}\frac{1}{|y|^{(n/q_2'+1)q_2'}}\,dy\Big)^{1/q_2'}\\
&\qquad\qquad\times\Big(\int_{|y|>2^{j-4}}\frac{|T_\gamma^2 b_j(y)|^{q_2}}{|y|^n}\,dy\Big)^{1/q_2}\\
&\qquad\le c\frac{2^{i(n+s+1-\alpha_1-n/q_1)}}{|x|^{n+s+1}}2^{-j(\alpha_2+n/q_2)}.
\end{aligned}$$

This is the desired estimate. For $B_{22}(x)$, we have

$$\begin{aligned}
B_{22}(x)\le&\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{\substack{|y|\le 2^{i+3}\\|y|\le 2^{j-4}}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy\\
&+\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{\substack{|y|\le 2^{i+3}\\|y|\le 2^{j-4}}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy.
\end{aligned}$$

Using (3.2') for b_j, we have that the first term in $B_{22}(x)$ is dominated by

$$\begin{aligned}
&c\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{|y|\le 2^{i+3}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)|2^{-j(\alpha_2+n/q_2)}\,dy\\
&\qquad\le c\frac{2^{-j(\alpha_2+n/q_2)}}{|x|^{n+s+1}}\sum_{\gamma=1}^{N}\Big(\int_{|y|\le 2^{i+3}}|y|^{(s+1)q_1'}\,dx\Big)^{1/q_1'}\|T_\gamma^1 a_i\|_{L^{q_1}(\mathbb{R}^n)}\\
&\qquad\le c\frac{2^{i(n+s+1-\alpha_1-n/q_1)}}{|x|^{n+s+1}}2^{-j(\alpha_2+n/q_2)}.
\end{aligned}$$

The second term in $B_{22}(x)$ is bounded above by

$$\begin{aligned}
&c\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{2^{j-4}<|y|\le 2^{i+3}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy\\
&\qquad\le c\frac{1}{|x|^{n+s+1}}\sum_{\gamma=1}^{N}\Big[\int_{|y|\le 2^{i+3}}(|y|^{s+1+n/q_2}|T_\gamma^1 a_i(y)|)^{q_2'}\,dy\Big]^{1/q_2'}\\
&\qquad\qquad\times\Big(\int_{2^{j-4}<|y|}\frac{|T_\gamma^2 b_j(y)|^{q_2}}{|y|^n}\,dy\Big)^{1/q_2}\\
&\qquad\le c\frac{2^{i(n+s+1-\alpha_1-n/q_1)}}{|x|^{n+s+1}}2^{-j(\alpha_2+n/q_2)}.
\end{aligned}$$

Thus both terms in $B_{22}(x)$ satisfy the desired estimate. Notice that the above estimates are valid when $s+1>\alpha_1+n(1/q_1-1)$. The minimun value of such s is

$[\alpha + n(1/q - 1)]$. Combining the estimates for A_2 and B_2, we obtain

$$\|J_1\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \leq c\|f\|_{H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\|g\|_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}. \tag{3.17}$$

Next we consider $J_2(x)$. We again have (3.15). The estimation of $B_2(x)$ is the same as before, since we never used the assumption $|x| < 2^{j-5}$ in that case (the assumption that $|x| \leq 2^{j+4}$ suffices for the estimate of $B_2(x)$ in that case.) For the term $A_2(x)$ in the current case, we still have $|y| \geq |x|/2 > 2^{i+3}$, and moreover, we have

$$\begin{aligned} A_2(x) &\leq c\sum_{\gamma=1}^{N} \sup_{0<t\leq\frac{1}{2}|x|} \int_{\mathbb{R}^n} \phi_{t,x}(y) \frac{2^{i(n+s_1-\alpha_1-n/q_1)}}{|y|^{n+s_1}} |T_\gamma^2 b_j(y)|\, dy \\ &\leq c\frac{2^{i(n+s_1-\alpha_1-n/q_1)}}{|x|^{n+s_1}} \sum_{\gamma=1}^{N} \sup_{t>0} \Big(\int_{\mathbb{R}^n} \phi_{t,x}(y)\, dy \Big)^{1/q_0} M(|T_\gamma^2 b_j|^{q_0'})^{1/q_0'}(x), \end{aligned}$$

where q_0 is chosen as in the estimation of I_2. Thus $A_2(x)$ can be estimated as in (3.16) for i and as in I_2 for j. Therefore,

$$\|J_2\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \leq c\|f\|_{H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\|g\|_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}. \tag{3.18}$$

Finally, we consider $J_3(x)$. We still use (3.15), that is, $S(a_i, b_j)(x) \leq A_2(x) + B_2(x)$. For $A_2(x)$, $|y| \geq |x| - |x-y| \geq \frac{|x|}{2} \geq \max(2^{i+3},\ 2^{j+3})$; thus,

$$\begin{aligned} A_2(x) &\leq c\sum_{\gamma=1}^{N} \sup_{0<t\leq\frac{1}{2}|x|} \int_{\mathbb{R}^n} \phi_{t,x}(y) \frac{2^{i(n+\ell+1-\alpha_1-n/q_1)}}{|y|^{n+\ell+1}} \frac{2^{j(n-\alpha_2-n/q_2)}}{|y|^n}\, dy \\ &\leq c\frac{2^{i(n+\ell+1-\alpha_1-n/q_1)}}{|x|^{n+\ell+1}} \frac{2^{j(n-\alpha_2-n/q_2)}}{|x|^n}. \end{aligned} \tag{3.19}$$

It follows from (3.11) and (3.16) that $A_2(x)$ satisfies the required estimate. Now we write

$$\begin{aligned} B_2(x) &\leq c\sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{|y|\geq\max(2^{i+4},\ 2^{j+4})} \frac{|y|^{s+1}}{t^{n+s+1}} |T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\, dy \\ &\quad + c\sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{|y|\leq\min(2^{i+4},\ 2^{j+4})} \frac{|y|^{s+1}}{t^{n+s+1}} |T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\, dy \\ &\quad + c\sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{2^{j+4}<|y|\leq 2^{i+4}} \frac{|y|^{s+1}}{t^{n+s+1}} |T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\, dy \\ &\quad + c\sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{2^{i+4}<|y|\leq 2^{j+4}} \frac{|y|^{s+1}}{t^{n+s+1}} |T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\, dy \\ &\equiv D_1(x) + D_2(x) + D_3(x) + D_4(x). \end{aligned}$$

Let $m,\ \ell$ be non-negative integers to be determined later and $s = m + \ell$.

Using (3.2) for a_i and b_j respectively, we have

$$\begin{aligned}D_1(x) &\le c\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{|y|\ge\max(2^{i+4},\ 2^{j+4})}\frac{|y|^{s+1}}{t^{n+s+1}}\frac{2^{i(n+m+1-\alpha_1-\frac{n}{q_1})}}{|y|^{n+m+1}}\\ &\qquad\qquad\times\frac{2^{j(n+s+1-\alpha_2-\frac{n}{q_2})}}{|y|^{n+s+1}}\,dy\\ &\le c\frac{2^{i(n+m+1-\alpha_1-\frac{n}{q_1})}2^{j(n+s+1-\alpha_2-\frac{n}{q_2})}}{|x|^{n+s+1}}\int_{|y|\ge\max(2^{i+4},\ 2^{j+4})}\frac{1}{|y|^{n+m+1}}\frac{1}{|y|^{n}}\,dy.\end{aligned}$$

If $i\ge j$, the integral above is dominated by

$$\begin{aligned}&\Big(\int_{|y|\ge 2^{i+4}}\frac{1}{|y|^{(n+m+1)q_2'}}\,dy\Big)^{1/q_2'}\Big(\int_{|y|\ge 2^{j+4}}\frac{1}{|y|^{nq_2}}\,dy\Big)^{1/q_2}\\ &\qquad\le c2^{i(-n/q_2-m-1)}2^{j(n/q_2-n)}\le c2^{-j(n+m+1)}2^{-in/q_2}2^{jn/q_2}.\end{aligned}$$

If $i<j$, the integral above is dominated by

$$\begin{aligned}&\Big(\int_{|y|\ge 2^{j+4}}\frac{1}{|y|^{(n+m+1)q_2}}\,dy\Big)^{1/q_2}\Big(\int_{|y|\ge 2^{i+4}}\frac{1}{|y|^{nq_2'}}\,dy\Big)^{1/q_2'}\\ &\qquad\le c2^{j(n/q_2-n-m-1)}2^{-in/q_2}.\end{aligned}$$

In either case, we obtain

$$D_1(x)\le c\frac{2^{i(n+m+1-\alpha_1-n/q)}}{|x|^{n/q_2'+m+1}}\frac{2^{j(\ell-\alpha_2)}}{|x|^{n/q_2+\ell}}. \tag{3.20}$$

In view of (3.12), the second term, when $\ell>\alpha_2$, satisfies the desired estimate. Since

$$\begin{aligned}&\Big\|\sum_{i=-\infty}^{\infty}\lambda_i\frac{2^{i(n+m+1-\alpha_1-n/q)}}{|x|^{n/q_2'+m+1}}\chi_{\{|x|>2^{i+4}\}}\Big\|_{\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\\ &\qquad\le c\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{i=-\infty}^{k-4}\lambda_i2^{(i-k)(n+m+1-\alpha_1-n/q)}\Big)^{p_1}\Big\}^{1/p_1}\\ &\qquad\le c\Big\{\sum_{i=-\infty}^{\infty}(\lambda_i)^{p_1}\Big\}^{1/p_1},\end{aligned}$$

whenever $m+1>\alpha_1+n(1/q-1)$, we obtain the desired estimate for the first term in (3.20) also. Notice that we require $s+1=m+1+\ell>\alpha+n(1/q-1)$.

The estimation for the term $D_2(x)$ is easy. We have

$$\begin{aligned}D_2(x)&\le c\frac{2^{i(m+1)}2^{j\ell}}{|x|^{n+s+1}}\sum_{\gamma=1}^{N}\|T_\gamma^1a_i\|_{L^{q_2'}(\mathbb{R}^n)}\|T_\gamma^2b_j\|_{L^{q_2}(\mathbb{R}^n)}\\ &\le c\frac{2^{i(m+n+1-\alpha_1-n/q)}}{|x|^{n/q_2'+m+1}}\frac{2^{j(\ell-\alpha_2)}}{|x|^{n/q_2+\ell}}.\end{aligned}$$

Both the terms above appeared in (3.20).

Next, we consider $D_3(x)$. By (3.2) for b_j, and the fact that we have $i > j$ in this case, we get

$$\begin{aligned}
D_3(x) &\le c\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{2^{j+4}<|y|\le 2^{i+4}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1 a_i(y)|\frac{2^{j(\ell+m+2+n-\alpha_2-n/q_2)}}{|y|^{\ell+m+2+n}}\,dy\\
&\le c\sum_{\gamma=1}^{N}\frac{2^{i(m+1)}}{|x|^{n/q_2'+m+1}}\frac{2^{j(\ell+1+n-\alpha_2-n/q_2)}}{|x|^{\ell+n/q_2}}\|T_\gamma^1 a_i\|_{L^{q_2'}(\mathbb{R}^n)}\\
&\qquad\qquad\times\Big(\int_{|y|>2^{j+4}}\frac{1}{|y|^{(n+1)q_2}}\,dy\Big)^{1/q_2}\\
&\le c\frac{2^{i(m+n+1-\alpha_1-n/q)}}{|x|^{n/q_2'+m+1}}\frac{2^{j(\ell-\alpha_2)}}{|x|^{n/q_2+\ell}},
\end{aligned}$$

as desired.

Finally, by (3.2) for a_i, and the fact that $i < j$ in the case of $D_4(x)$, we get

$$\begin{aligned}
D_4(x) &\le c\sum_{\gamma=1}^{N}\sup_{t>\frac{1}{2}|x|}\int_{2^{i+4}<|y|\le 2^{j+4}}\frac{|y|^{s+1}}{t^{n+s+1}}\frac{2^{i(n+\ell+m+2-\alpha_1-n/q_1)}}{|y|^{n+\ell+m+2}}|T_\gamma^2 b_j(y)|\,dy\\
&\le c\frac{2^{i(n+m+2-\alpha_1-\frac{n}{q_1})}}{|x|^{\frac{n}{q_2'}+m+1}}\frac{2^{j\ell}}{|x|^{\frac{n}{q_2}+\ell}}\sum_{\gamma=1}^{N}\|T_\gamma^2 b_j\|_{L^{q_2}(\mathbb{R}^n)}\Big(\int_{|y|>2^{i+4}}\frac{1}{|y|^{(n+1)q_2'}}\,dy\Big)^{\frac{1}{q_2'}}\\
&\le c\frac{2^{i(n+m+1-\alpha_1-n/q)}}{|x|^{n/q_2'+m+1}}\frac{2^{j(\ell-\alpha_2)}}{|x|^{n/q_2+\ell}},
\end{aligned}$$

as desired.

Combining (3.19) and the estimates for $D_1(x)$, $D_2(x)$, $D_3(x)$ and $D_4(x)$, we obtain

$$\|J_3\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \le c\|f\|_{H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\|g\|_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}. \tag{3.21}$$

Notice that, from the estimation above, the minimum value of s in (1.4) can be taken as $[\alpha + n(1/q-1)]$.

A combination of (3.8), (3.9), (3.14), (3.17), (3.18) and (3.21) finish the proof ot Theorem 2.

4. Proof of Theorem 3

Our proof of this theorem is also a little technical. The spirit is similar to that of Theorem 2.

We still assume that $0 < p_1,\ p_2 < \infty$ and leave the cases $p_1 = \infty$ or $p_2 = \infty$ to the reader. We have to show that

$$\Big\|\sup_{t>0}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(f,g)(y)\,dy\Big|\Big\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \le c\|f\|_{H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\|g\|_{H\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}, \tag{4.1}$$

for all $f \in H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)$ and $g \in H\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$. Without loss of generality, we may assume that $f = \sum_{i\in\mathbb{Z}}\lambda_i a_i$ and $g = \sum_{j\in\mathbb{Z}}\mu_j b_j$, where $\lambda_j,\ \mu_j \ge 0$, the a_j's are $(\alpha_1, q_1)-$atoms, b_j's are $(\alpha_2, q_2)-$atoms with supp $a_i \subseteq B_i$ and supp $b_j \subseteq B_j$.

Now let $S(a_i, b_j)(x)$ be as in the proof of Theorem 2; that is,

$$S(a_i,b_j)(x) = \sup_{t>0}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(a_i,b_j)(y)\,dy\Big|.$$

Then

$$\sup_{t>0}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(f,g)(y)\,dy\Big| \le \sum_{i,j}\lambda_i\mu_j S(a_i,b_j)(x)$$
$$=\sum_{i\ge j}\lambda_i\mu_j S(a_i,b_j)(x)+\sum_{i<j}\lambda_i\mu_j S(a_i,b_j)(x).$$

We only need to show (4.1) for the part $\sum_{i\ge j}$, since the other part can be estimated in the same way. In fact, the roles of i and j are symmetric. For each x, we have

$$\begin{aligned}\sum_{i\ge j}\lambda_i\mu_j S(a_i,b_j)(x) &= \sum_{i\ge j}\lambda_i\mu_j S(a_i,b_j)(x)\chi_{\{|x|\le 2^{j+5}\}}\\ &\quad+\sum_{i\ge j}\lambda_i\mu_j S(a_i,b_j)(x)\chi_{\{2^{j+5}<|x|\le 2^{i+5}\}}\\ &\quad+\sum_{i\ge j}\lambda_i\mu_j S(a_i,b_j)(x)\chi_{\{|x|>2^{i+5}\}}\\ &\equiv \Omega_1(x)+\Omega_2(x)+\Omega_3(x).\end{aligned} \tag{4.2}$$

Consider Ω_1 first. We can choose q_0 as in the proof of Theorem 2 such that $1<q_0<q_1$ and $1<q_0'<q_2$ with $1/q_0+1/q_0'=1$. We obtain

$$S(a_i,b_j)(x)\le c\sum_{\gamma=1}^{N} M(|T_\gamma^1 a_i|^{q_0})^{1/q_0}(x)M(|T_\gamma^2 b_j|^{q_0'})^{1/q_0'}(x).$$

Thus by Hölder's inequality

$$\begin{aligned}\|\Omega_1\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} &\le c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{\infty}\lambda_i M(|T_\gamma^1 a_i|^{q_0})^{1/q_0}(x)\chi_{\{|x|\le 2^{i+5}\}}\Big\|_{\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\\ &\quad\times\Big\|\sum_{j=-\infty}^{\infty}\mu_j M(|T_\gamma^2 b_j|^{q_0'})^{1/q_0'}(x)\chi_{\{|x|\le 2^{j+5}\}}\Big\|_{\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}\end{aligned} \tag{4.3}$$

Both norms above can be estimated as in (3.7). Therefore, we have

$$\begin{aligned}\|\Omega_1\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} &\le c\Big\{\sum_{i=-\infty}^{\infty}(\lambda_i)^{p_1}\Big\}^{1/p_1}\Big\{\sum_{j=-\infty}^{\infty}(\mu_j)^{p_2}\Big\}^{1/p_2}\\ &\le c\|f\|_{H\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\|g\|_{H\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.\end{aligned} \tag{4.4}$$

Next, we consider Ω_2. First we split

$$\begin{aligned}S(a_i,b_j)(x) &\le \sup_{0<t\le\frac12|x|}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(a_i,b_j)(y)\,dy\Big|\\ &\quad+\sup_{t>\frac12|x|}\Big|\int_{\mathbb{R}^n}\phi_{t,x}(y)B(a_i,b_j)(y)\,dy\Big|\\ &\equiv S_1(a_i,b_j)(x)+S_2(a_i,b_j)(x).\end{aligned} \tag{4.5}$$

For the first term above, since $|y| \ge |x| - |x-y| \ge |x|/2 \ge 2^{j+4}$, we have

(4.6)

$$\begin{aligned}
S_1(a_i, b_j)(x) &\le \sup_{0<t\le \frac{1}{2}|x|} \Big| \int_{|y|>2^{j+3}} \phi_{t,x}(y) B(a_i, b_j)(y)\, dy \Big| \\
&\le c \sum_{\gamma=1}^{N} \sup_{0<t\le\frac{1}{2}|x|} \Big(\int_{|y|>2^{j+3}} \phi_{t,x}(y) |T_\gamma^1 a_i(y)|^{q_2'}\, dy \Big)^{1/q_2'} \\
&\quad \times \sup_{0<t\le\frac{1}{2}|x|} \Big[\int_{|y|>2^{j+3}} \phi_{t,x}(y) \Big(\frac{2^{j(n+1+\ell-\alpha_2-n/q_2)}}{|y|^{n+1+\ell}} \Big)^{q_2} dy \Big]^{1/q_2} \\
&\le c \sum_{\gamma=1}^{N} M(|T_\gamma^1 a_i(y)|^{q_2'})^{1/q_2'}(x) \frac{2^{j(n+1+\ell-\alpha_2-n/q_2)}}{|x|^{n+1+\ell}},
\end{aligned}$$

where we used (3.2) for b_j to obtain the second inequality above. The first term in this last expression appeared in (3.7) and the second term appeared similarly in (3.10). Since $q_1 > q_2'$, and since we can choose ℓ large enough so that $\ell + 1 > \alpha_2 + n(1/q_2 - 1)$, we know that $S_1(a_i, b_j)$ satisfies the required estimate.

Now using the vanishing moment condition (1.5) and Taylor's theorem we obtain

$$\begin{aligned}
S_2(a_i, b_j)(x) &\le c \sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{|x-y|>4t} \frac{|y|^{\ell+1}}{t^{n+\ell+1}} |T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\, dy \\
&\quad + c \sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{|x-y|\le 4t} \frac{|y|^{\ell+1}}{t^{n+\ell+1}} |T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\, dy \\
&\equiv S_{21}(x) + S_{22}(x),
\end{aligned}$$

for some non-negative integer ℓ to be determined later. In $S_{21}(x)$ note that since $|y| \ge |x-y| - |x| \ge 2t > |x|$, we have $|y| > 2^{j+5}$ and $|x-y| \le 2|y|$. Using (3.2) with $s_1 = \ell + 3$ for b_j, we estimate $S_{21}(x)$ as we did $A_1(x)$ in section 3. We obtain

$$\begin{aligned}
S_{21}(x) &\le c \sum_{\gamma=1}^{N} \frac{1}{|x|^{n+\ell+2}} \sup_{t>\frac{1}{2}|x|} \int_{\substack{|x-y|>4t \\ |y|>2^{j+5}}} t |T_\gamma^1 a_i(y)| \frac{2^{j(\ell+3+n-\alpha_2-n/q_2)}}{|y|^{n+2}}\, dy \\
&\le c \frac{2^{j(\ell+3+n-\alpha_2-n/q_2)}}{|x|^{n+\ell+2}} \sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{\substack{|x-y|>4t \\ |y|>2^{j+5}}} \frac{t^{1/q_2'} |T_\gamma^1 a_i(y)|}{|x-y|^{(n+1)/q_2'}} \\
&\qquad \times \frac{1}{|y|^{n/q_2+1}}\, dy, \quad \text{since } 2t \le |y| \\
&\le c \frac{2^{j(\ell+3+n-\alpha_2-n/q_2)}}{|x|^{n+\ell+2}} \sum_{\gamma=1}^{N} \sup_{t>0} \Big(\int_{|x-y|>4t} \frac{t |T_\gamma^1 a_i(y)|^{q_2'}}{|x-y|^{(n+1)}}\, dy \Big)^{1/q_2'} \\
&\qquad \times \Big(\int_{|y|>2^{j+5}} \frac{1}{|y|^{n+q_2}}\, dy \Big)^{1/q_2} \\
&\le c \sum_{\gamma=1}^{N} M(|T_\gamma^1 a_i(y)|^{q_2'})^{1/q_2'}(x) \frac{2^{j(\ell+2+n-\alpha_2-n/q_2)}}{|x|^{n+\ell+2}}.
\end{aligned} \tag{4.7}$$

This is similar to (4.6). For $S_{22}(x)$, we proceed likewise. $S_{22}(x)$ is dominated by $S_{221}(x) + S_{222}(x)$, where

$$S_{221}(x) = c\sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{\substack{|x-y|\le 4t\\ |y|\le 2^{j+3}}} \frac{|y|^{\ell+1}}{t^{n+\ell+1}} |T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy,$$

and

$$S_{222}(x) = c\sum_{\gamma=1}^{N} \sup_{t>\frac{1}{2}|x|} \int_{\substack{|x-y|\le 4t\\ |y|> 2^{j+3}}} \frac{|y|^{\ell+1}}{t^{n+\ell+1}} |T_\gamma^1 a_i(y)||T_\gamma^2 b_j(y)|\,dy.$$

By Hölder's inequality and the fact that $\frac{1}{t} \le \frac{4}{|x-y|} \le \frac{c}{|x|}$, since $|x| > 2^{j+5}$ and $|y| \le 2^{j+3}$ in $S_{221}(x)$, we have

(4.8)

$$\begin{aligned} S_{221}(x) &\le c\frac{1}{|x|^{n/q_2+\ell+1}} \sum_{\gamma=1}^{N} \sup_{t>0} \Big(\frac{1}{t^n}\int_{|x-y|\le 4t} |T_\gamma^1 a_i(y)|^{q_2'}\,dy\Big)^{1/q_2'} \\ &\qquad \times \Big(\int_{|y|\le 2^{j+3}} |y|^{(\ell+1)q_2} |T_\gamma^2 b_j(y)|^{q_2}\,dy\Big)^{1/q_2} \\ &\le c\sum_{\gamma=1}^{N} M(|T_\gamma^1 a_i(y)|^{q_2'})^{1/q_2'}(x) \frac{2^{j(\ell+1-\alpha_2)}}{|x|^{n/q_2+\ell+1}}. \end{aligned}$$

This last expression appeared in (3.12).

For $S_{222}(x)$, we use (3.2) for b_j and obtain

$$\begin{aligned} S_{222}(x) &\le c\frac{2^{j(\ell+2+n-\alpha_2-n/q_2)}}{|x|^{n/q_2+\ell+1}} \sum_{\gamma=1}^{N} \sup_{t>0} \Big(\frac{1}{t^n}\int_{|x-y|\le 4t} |T_\gamma^1 a_i(y)|^{q_2'}\,dy\Big)^{1/q_2'} \\ &\qquad \times \Big(\int_{|y|>2^{j+3}} \frac{1}{|y|^{(n+1)q_2}}\,dy\Big)^{1/q_2} \\ &\le c\sum_{\gamma=1}^{N} M(|T_\gamma^1 a_i(y)|^{q_2'})^{1/q_2'}(x) \frac{2^{j(\ell+1-\alpha_2)}}{|x|^{n/q_2+\ell+1}}. \end{aligned} \tag{4.9}$$

This again appeared in (3.12).

Combining (4.6), (4.7), (4.8) and (4.9), we obtain

$$\|\Omega_2\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \le c\|f\|_{H\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)} \|g\|_{H\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}. \tag{4.10}$$

Finally, we consider Ω_3. As before, we have

$$S(a_i,b_j)(x) \le S_1(a_i,b_j)(x) + S_2(a_i,b_j)(x),$$

where $S_1(a_i,b_j)(x)$ and $S_2(a_i,b_j)(x)$ are defined in (4.5). For $S_1(a_i,b_j)(x)$, we use (3.2) for a_i and b_j respectively and obtain

$$\begin{aligned}
S_1(a_i,b_j)(x) &\le \sup_{0<t\le\frac{|x|}{2}}\Big|\int_{|y|>2^{i+3}}\phi_{t,x}(y)B(a_i,b_j)(y)\,dy\Big| \\
&\le c\sum_{\gamma=1}^{N}\sup_{0<t\le\frac{|x|}{2}}\int_{|y|>2^{i+3}}\phi_{t,x}(y)\frac{2^{i(m+1+n-\alpha_1-n/q_1)}}{|y|^{n+m+1}} \\
&\qquad\qquad\times\frac{2^{j(\ell+1+n-\alpha_2-n/q_2)}}{|y|^{n+\ell+1}}\,dy \\
&\le \frac{2^{i(m+1+n-\alpha_1-n/q_1)}}{|x|^{n+m+1}}\frac{2^{j(\ell+1+n-\alpha_2-n/q_2)}}{|x|^{n+\ell+1}},\quad\text{since } |y|\ge\frac{|x|}{2}.
\end{aligned}\tag{4.11}$$

As long as $m+1>\alpha_1+n/q_2-n$ and $\ell+1>\alpha_2+n/q_2-n$, we know from (4.6) that the above estimates are the required ones.

For $S_2(a_i,b_j)(x)$, using the cancellation property (1.5), we can obtain as before

$$\begin{aligned}
S_2(a_i,b_j)(x) &\le \sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{\mathbb{R}^n}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1a_i(y)||T_\gamma^2b_j(y)|\,dy \\
&\le c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{|y|\le 2^{j+3}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1a_i(y)||T_\gamma^2b_j(y)|\,dy \\
&\quad + c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{2^{j+3}<|y|\le 2^{i+3}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1a_i(y)||T_\gamma^2b_j(y)|\,dy \\
&\quad + c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{|y|>2^{i+3}}\frac{|y|^{s+1}}{t^{n+s+1}}|T_\gamma^1a_i(y)||T_\gamma^2b_j(y)|\,dy \\
&\equiv L_1(x)+L_2(x)+L_3(x).
\end{aligned}$$

Let $s=m+\ell+1$, where $m,\ \ell$ are non-negative integers to be determined later. $L_1(x)$ is similar to $D_2(x)$ in §3. Since $q_1>q_2'$, we obtain

$$\begin{aligned}
L_1(x) &\le c\sum_{\gamma=1}^{N}\frac{2^{i(m+1)}}{|x|^{n/q_2'+m+1}}\frac{2^{j(\ell+1)}}{|x|^{n/q_2+\ell+1}}\|T_\gamma^1a_i\|_{L^{q_2'}(\mathbb{R}^n)}\|T_\gamma^2b_j\|_{L^{q_2}(\mathbb{R}^n)} \\
&\le c\frac{2^{i(m+n+1-\alpha_1-n/q)}}{|x|^{n/q_2'+m+1}}\frac{2^{j(\ell+1-\alpha_2)}}{|x|^{n/q_2+\ell+1}}.
\end{aligned}\tag{4.12}$$

This is similar to (3.20). Whenever $m+1>\alpha_1+n(1/q-1)$ and $\ell+1>\alpha_2$ (thus $s+1>\alpha+n(1/q-1)$), this is the desired estimate.

For $L_2(x)$, we use (3.2) for b_j and obtain

(4.13)

$$\begin{aligned} L_2(x) &\le c \sum_{\gamma=1}^{N} \sup_{t>\frac{|x|}{2}} \int_{2^{j+3}<|y|\le 2^{i+3}} \frac{|y|^{s+1}}{t^{n+s+1}} |T_\gamma^1 a_i(y)| \frac{2^{j(\ell+m+3+n-\alpha_2-n/q_2)}}{|y|^{n+\ell+m+3}}\, dy \\ &\le c \sum_{\gamma=1}^{N} \frac{2^{i(m+1)}}{|x|^{n/q_1+m+1}} \Big(\int_{\mathbb{R}^n} |T_\gamma^1 a_i(y)|^{q_1}\, dy \Big)^{1/q_1} \\ &\qquad \times \frac{2^{j(\ell+2+n-\alpha_2-n/q_2)}}{|x|^{n/q_1'+\ell+1}} \Big(\int_{2^{j+3}<|y|} \frac{1}{|y|^{(n+1)q_1'}}\, dy \Big)^{1/q_1'} \\ &\le c \frac{2^{i(m+1-\alpha_1)}}{|x|^{n/q_1+m+1}} \frac{2^{j(\ell+1+n-\alpha_2-n/q)}}{|x|^{n/q_1'+\ell+1}}. \end{aligned}$$

This is symmetric to (4.12). As long as $m+1>\alpha_1$ and $\ell+1>\alpha_2+n(1/q-1)$ (thus $s+1>\alpha+n(1/q-1)$), this is the desired estimate.

Finally, we consider $L_3(x)$. By using (3.2) for a_i and b_j respectively, we get

(4.14)

$$\begin{aligned} L_3(x) &\le c \sup_{t>\frac{|x|}{2}} \int_{|y|>2^{i+3}} \frac{1}{t^{n/2+m+1}} \frac{2^{i(n+m+1-\alpha_1-n/q_1)}}{|y|^{n+m+1}} \\ &\qquad \times \frac{1}{t^{n/2+\ell+1}} \frac{2^{j(n+s+1-\alpha_2-n/q_2)}}{|y|^{n+s+1}} |y|^{s+1}\, dy \\ &\le c \frac{2^{i(n+m+1-\alpha_1-n/q_1)}}{|x|^{n/2+m+1}} 2^{i(m+1)} \Big(\int_{|y|>2^{i+3}} \frac{1}{|y|^{2(n+m+1)}}\, dy \Big)^{1/2} \\ &\qquad \times \frac{2^{j(n+\ell+1-\alpha_2-n/q_2)}}{|x|^{n/2+\ell+1}} \Big(\int_{|y|>2^{j+3}} \frac{1}{|y|^{2n}}\, dy \Big)^{1/2} \\ &\le c \frac{2^{i(n/2+m+1-\alpha_1-n/q_1)}}{|x|^{n/2+m+1}} \frac{2^{j(n/2+\ell+1-\alpha_2-n/q_2)}}{|x|^{n/2+\ell+1}}. \end{aligned}$$

For the first term above, we have

$$\begin{aligned} &\Big\| \sum_{i=-\infty}^{\infty} \lambda_i \frac{2^{i(n/2+m+1-\alpha_1-n/q_1)}}{|x|^{n/2+m+1}} \chi_{\{|x|>2^{i+5}\}} \Big\|_{\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)} \\ &\qquad \le c \Big\{ \sum_{k=-\infty}^{\infty} \Big(\sum_{i=-\infty}^{k-6} \lambda_i 2^{(i-k)(n/2+m+1-\alpha_1-n/q_1)} \Big)^{p_1} \Big\}^{1/p_1} \\ &\qquad \le c \Big\{ \sum_{i=-\infty}^{\infty} (\lambda_i)^{p_1} \Big\}^{1/p_1}, \quad 0<p_1<\infty, \end{aligned}$$

whenever $m+1>\alpha_1+n/q_1-n/2$. Similar estimates hold for the second term in (4.14) whenever $\ell+1>\alpha_2+n/q_2-n/2$. Therefore $L_3(x)$ also satisfies the desired estimate whenever $s+1>\alpha+n(1/q-1)$.

Combining (4.11), (4.12), (4.13) and (4.14), we obtain

$$\|\Omega_3\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \le c \|f\|_{H\dot K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)} \|g\|_{H\dot K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}. \tag{4.15}$$

Therefore, (4.1) follows from (4.4), (4.10) and (4.15). Notice that the minimum value of s can be taken to be $[\alpha+n(1/q-1)]$.

Theorem 3 is now proved.

5. Applications

The following two corollaries extend the theorem of Coifman, Rochberg and Weiss [6] on the L^p boundedness of the commutator by a C–Z operator and a BMO function to Herz spaces. Note that $H\dot{K}_1^{0,1}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.

Corollary 1. *Let b be in $BMO(\mathbb{R}^n)$ and T be a Calderón-Zygmund operator. Then the commutator*

$$[b,T](f) = bT(f) - T(bf) \tag{5.1}$$

maps $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ into itself when $1 < p, q < \infty$ and $-n/q < \alpha < n(1-1/q)$.

Consider the bilinear operator

$$B(f,g) = (Tf)\,g - f\,(T^*g),$$

where T^* is the adjoint operator of T. One can easily check that $B(f,g)$ has integral zero for all f, g square-integrable and compactly supported functions. By Theorem 1, we obtain

$$\|B(f,g)\|_{H^1} \le C\|f\|_{\dot{K}_q^{\alpha,p}} \|g\|_{\dot{K}_{q'}^{-\alpha,p'}}.$$

Using the duality between H^1 and BMO, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n}[b,T](f)(x)g(x)\,dx\right| &= \left|\int_{\mathbb{R}^n} b(x)\big(g(x)(Tf)(x) - f(x)(T^*g)(x)\big)\,dx\right| \\ &\le \|b\|_{BMO}\|g(Tf) - f(T^*g)\|_{H^1} \\ &\le C\|b\|_{BMO}\|f\|_{\dot{K}_q^{\alpha,p}}\|g\|_{\dot{K}_{q'}^{-\alpha,p'}}.\end{aligned}$$

Now the duality between $\dot{K}_q^{\alpha,p}$ and $\dot{K}_{q'}^{-\alpha,p'}$ gives the required conclusion.

To state the next corollary, we recall the definition of the spaces $C\dot{M}O_q(\mathbb{R}^n)$. For $1 < q < \infty$, $C\dot{M}O_q(\mathbb{R}^n)$ is the set of all measurable functions on $\mathbb{R}^n$ whose q^{th} powers are locally integrable and which satisfy

$$\sup_{r>0}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)}\left|f(x) - \frac{1}{|B(0,r)|}\int_{B(0r} f(y)dy\right|^q dx\right)^{\frac{1}{q}} < \infty. \tag{5.2}$$

The $C\dot{M}O_q$ norm of f is defined to be the expression in (5.2). It can be shown that $C\dot{M}O_q(\mathbb{R}^n)$ is a Banach space, and also that it is the dual of $H\dot{K}_{q'}^{n(1-1/q),1}$. This duality is the homogeneous version of an extension of the duality between H^1 and BMO. See [11] and [19] for details. We now have the following:

Corollary 2. *Let b be in $C\dot{M}O_{q'}(\mathbb{R}^n)$ for some $1 < q < \infty$, and let T be a Calderón-Zygmund operator. Then the commutator $[b,T](f) = bT(f) - T(bf)$ maps $\dot{K}_{q_1}^{\alpha_1,p}(\mathbb{R}^n)$ into $\dot{K}_{q_2'}^{-\alpha_2,p}(\mathbb{R}^n)$ when $1 < p < \infty$, $1/q_1 + 1/q_2 = 1/q$, $-n/q_i < \alpha_i < n(1-1/q_i)$, $i = 1,2$, and $\alpha_1 + \alpha_2 = n(1-1/q)$.*

For the proof we use Theorem 1. As before,

$$\begin{aligned}\left|\int_{\mathbb{R}^n}[b,T](f)(x)g(x)\,dx\right| &\le \|b\|_{C\dot{M}O_{q'}}\|g(Tf) - f(T^*g)\|_{H\dot{K}_q^{n(1-1/q),1}} \\ &\le C\|b\|_{C\dot{M}O_{q'}}\|f\|_{\dot{K}_{q_1}^{\alpha_1,p}}\|g\|_{\dot{K}_{q_2}^{\alpha_2,p'}},\end{aligned}$$

for all g in $\dot{K}_{q_2}^{\alpha_2,p'}$. This implies that $\|[b,T](f)\|_{\dot{K}_{q_2'}^{-\alpha_2,p}} \leq C\|b\|_{C\dot{M}O_{q'}}\|f\|_{\dot{K}_{q_1}^{\alpha_1,p}}$, and Corollary 2 is proved.

By the way, it is still an interesting open problem if the converses of Corollaries 1 and 2 are true. It seems that we do need some different technique from [6] to deal with this, since the Herz spaces are not translation invariant.

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