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VALUES OF GAUSSIAN HYPERGEOMETRIC SERIES

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ABSTRACT. Let p be prime and let $GF(p)$ be the finite field with p elements. In this note we investigate the arithmetic properties of the Gaussian hypergeometric functions

$$ {}_2F_1(x) = {}_2F_1\left(\begin{matrix}\phi, & \phi \\ & \epsilon\end{matrix}\mid x\right) \quad \text{and} \quad {}_3F_2(x) = {}_3F_2\left(\begin{matrix}\phi, & \phi, & \phi \\ & \epsilon, & \epsilon\end{matrix}\mid x\right), $$

where ϕ and ϵ respectively are the quadratic and trivial characters of $GF(p)$. For all but finitely many rational numbers $x = \lambda$, there exist two elliptic curves ${}_2E_1(\lambda)$ and ${}_3E_2(\lambda)$ for which these values are expressed in terms of the trace of the Frobenius endomorphism. We obtain bounds and congruence properties for these values. We also show, using a theorem of Elkies, that there are infinitely many primes p for which ${}_2F_1(\lambda)$ is zero; however if $\lambda \neq -1, 0, \frac{1}{2}$ or 2, then the set of such primes has density zero. In contrast, if $\lambda \neq 0$ or 1, then there are only finitely many primes p for which ${}_3F_2(\lambda) = 0$. Greene and Stanton proved a conjecture of Evans on the value of a certain character sum which from this point of view follows from the fact that ${}_3E_2(8)$ is an elliptic curve with complex multiplication. We completely classify all such CM curves and give their corresponding character sums in the sense of Evans using special Jacobsthal sums. As a consequence of this classification, we obtain new proofs of congruences for generalized Apéry numbers, as well as a few new ones, and we answer a question of Koike by evaluating ${}_3F_2(4)$ over every $GF(p)$.

1. Introduction

In [12] Greene initiated a study of *Gaussian* hypergeometric series over finite fields. He found that these series possess many properties that are analogous to their ordinary counterparts. In this paper we investigate the values of certain special Gaussian hypergeometric series and explore their number theoretic consequences.

Throughout this paper p is an odd prime. If n is an integer, then $\mathrm{ord}_p(n)$ is the power of p dividing n, and if $\alpha = \frac{a}{b} \in \mathbb{Q}$, then $\mathrm{ord}_p(\alpha) := \mathrm{ord}_p(a) - \mathrm{ord}_p(b)$. As usual, we let $GF(p)$ denote the finite field with p elements, and we extend all characters χ of $GF(p)^\times$ to $GF(p)$ by setting $\chi(0) := 0$. Following Greene we let the appropriate analog of the binomial coefficient be a Jacobi sum. Specifically, if A and B are two characters of $GF(p)$, then $\binom{A}{B}$ is defined by

$$ \binom{A}{B} := \frac{B(-1)}{p} J(A, \bar{B}) = \frac{B(-1)}{p} \sum_{x \in GF(p)} A(x)\bar{B}(1-x). \tag{1} $$

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If A and B are characters of $GF(p)$, then the following identity is known:

$$\binom{A}{B} = \binom{B\bar{A}}{B} B(-1). \tag{2}$$

In this notation, we recall Greene's definition of a 'so-called' Gaussian hypergeometric series.

Definition 1. If $A_0, A_1, \dots A_n$, and $B_1, B_2, \dots B_n$ are characters of $GF(p)$, then the Gaussian hypergeometric series ${}_{n+1}F_n\left(\begin{smallmatrix} A_0, & A_1, & \dots A_n \\ & B_1, & \dots B_n \end{smallmatrix} \mid x\right)$ over $GF(p)$ is defined by

$${}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \dots A_n \\ & B_1, & \dots B_n \end{matrix} \mid x\right) := \frac{p}{p-1}\sum_{\chi}\binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi}\cdots\binom{A_n\chi}{B_n\chi}\chi(x).$$

Throughout this paper the prime p will always be clear from context, and we let $\sum_\chi$ denote a summation over all characters χ of $GF(p)$.

We restrict our attention to the functions ${}_2F_1\left(\begin{smallmatrix} \phi, & \phi \\ & \epsilon \end{smallmatrix} \mid \lambda\right)$ and ${}_3F_2\left(\begin{smallmatrix} \phi, & \phi, & \phi \\ & \epsilon, & \epsilon \end{smallmatrix} \mid \lambda\right)$, where ϕ is the quadratic character and ϵ is the trivial one. For convenience we shall denote these values by ${}_2F_1(\lambda)$ and ${}_3F_2(\lambda)$.

In [13], [17], [18] these special values were investigated in connection with congruence properties of generalized Apéry numbers, the arithmetic of certain special elliptic curves, and conjectured character sums. If $I(t;p)$ denotes the character sum

$$I(t;p) := \sum_{x,y\in GF(p)} \phi(1+x)\phi(1+y)\phi(x+ty)\phi(x)\phi(y),$$

then Evans, Pulham and Sheehan (see [11]) conjectured that

$$I(1;p) = \phi(2)(3x^2 - 2y^2) = \phi(2)(4x^2 - p)$$

when $p \equiv 1, 3 \mod 8$ and x and y are integers for which $p = x^2 + 2y^2$. In [13] Greene and Stanton proved this conjecture by evaluating ${}_3F_2(-1)$ for every prime p.

In section 3 we investigate the arithmetic properties of ${}_2F_1(\lambda)$. Most of our results are deduced by expressing this value in terms of the trace of the Frobenius endomorphism on an elliptic curve in Legendre normal form. In section 4 we explore the arithmetic of ${}_3F_2(\lambda)$, which we express in terms of the trace of the Frobenius endomorphism of another explicit elliptic curve. By finding all λ for which this curve has complex multiplication, we obtain analogous character sum evaluations for $I\left(\frac{4}{\lambda-4};p\right)$. In the case where $\lambda = 8$, we obtain the character sum in the Evans, Pulham, and Sheehan conjecture. These sums are given in section 5, where we also obtain congruences for generalized Apéry numbers.

2. Preliminaries

Let $E = E/\mathbb{Q}$ be the set of $\mathbb{Q}-$rational points (x, y) satisfying the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where $a_i \in \mathbb{Q}$. The discriminant $\Delta(E)$ of the curve E is defined by the auxiliary constants

$$b_2 = a_1^2 + 4a_2, \quad b_4 = a_1a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6$$

and

$$b_8 = a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2.$$

Using this notation, the discriminant $\Delta(E)$ and the $j-$invariant $j(E)$ of E are given by

$$\Delta(E) := -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6, \tag{3}$$

and

$$j(E) := \frac{(b_2^2 - 24b_4)^3}{\Delta(E)}. \tag{4}$$

If $\Delta(E) \neq 0$, then E is an elliptic curve. Throughout E will denote an elliptic curve over $\mathbb{Q}$.

By Mordell's theorem, the points of E, including the point at infinity, form a finitely generated abelian group. Specifically, E is isomorphic to a group of the form $E \simeq \mathrm{Tor}(E) \times \mathbb{Z}^r$, where $\mathrm{Tor}(E)$, the torsion subgroup of E, is a finite abelian group and r is a non-negative integer.

The Hasse-Weil $L-$function of E, denoted by $L(E,s)$, is defined by examining the reductions $\bar{E}$ of E. If p is a prime of good reduction (i.e. $p \nmid \Delta(E)$), then define the integer $a(p)$ by

$$a(p) = 1 + p - N_p, \tag{5}$$

where N_p is the number of points of $\bar{E}$ rational over $GF(p)$ (including the point at ∞). If E is given by $y^2 = x^3 + Ax^2 + Bx + C$, where $A, B, C \in \mathbb{Z}$, then for such p the integer $a(p)$ is given by the character sum

$$a(p) = -\sum_{x \in GF(p)} \phi(x^3 + Ax^2 + Bx + C). \tag{6}$$

If $p|\Delta(E)$, then p is a prime of bad reduction, and $a(p) = 0, \pm 1$ depending on the nature of the singularity. The Hasse-Weil $L-$function for the elliptic curve is defined by

$$L(E,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} := \prod_{p|\Delta(E)} \frac{1}{1 - a(p)p^{-s}} \prod_{p \nmid \Delta(E)} \frac{1}{1 - a(p)p^{-s} + p^{1-2s}}.$$

If p is a prime for which E has good reduction, then the integer $a(p)$ can be interpreted as the trace of the Frobenius endomorphism on E (see [15], [22]). For our purposes, we will be interested in the arithmetic nature of these integers $a(p)$, since it turns out that ${}_2F_1(\lambda)$ and ${}_3F_2(\lambda)$ are functions in $a(p)$. Hasse proved that for every prime p

$$| a(p) | < 2\sqrt{p}. \tag{7}$$

This is the 'so-called' Riemann hypothesis for elliptic curves.

These integers possess some interesting congruence properties. Since the reduction map $(x,y) \to (x \mod p, y \mod p)$ is an injective map on $\mathrm{Tor}(E)$ when p is a prime of good reduction [22, p. 176], it follows that if $| \mathrm{Tor}(E) | = M$, then $M \mid N_p$. Therefore by (5) it is easy to see that

$$a(p) \equiv p + 1 \mod M. \tag{8}$$

The curves with *complex multiplication* are the only such E for which there are simple formulas for the integers $a(p)$. Moreover, the only values of $j(E)$ for which E has complex multiplication are [6, p. 83]

$$\begin{aligned} j(E) \in \{&1728, 66^3, 20^3,\ 0, 2\cdot 30^3, -3\cdot 160^3, -15^3, 255^3, \\ &-32^3, -96^3, -960^3, -5280^3, -640320^3\}. \end{aligned} \tag{9}$$

We shall also be interested in those primes p for which $a(p) = 0$. These primes are the supersingular primes, and if E has complex multiplication, then the set of primes p for which $a(p) = 0$ has density $\frac{1}{2}$. In fact, if E has complex multiplication by the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ and p is a prime of good reduction, then $a(p) = 0$ for every prime p where $\left(\frac{-d}{p}\right) = -1$. However for an elliptic curve without complex multiplication, Elkies [7] proved that there are infinitely many such primes but the number of such primes $< x$ is $\ll x^{\frac{3}{4}}$. Hence the set of supersingular primes for an elliptic curve over $\mathbb{Q}$ without complex multiplication has density zero.

One last idea we need regarding elliptic curves is the notion of a *quadratic twist.* Let E be an elliptic curve given by

$$E: \quad y^2 = x^3 + ax^2 + bx + c,$$

where $a, b, c \in \mathbb{Q}$. If D is a square-free integer, then the $D-$quadratic twist of E, denoted E_D, is given by the equation

$$E_D: \quad y^2 = x^3 + aDx^2 + bD^2x + cD^3. \tag{10}$$

If $L(E,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ and $L(E_D, s) = \sum_{n=1}^{\infty} \frac{a_D(n)}{n^s}$, then it turns out that if p is a prime for which both E and E_D have good reduction and $\gcd(p, 6) = 1$, then

$$a(p) = \left(\frac{D}{p}\right) a_D(p). \tag{11}$$

In particular, for all but finitely many primes p, $a(p)$ and $a_D(p)$ are equal up to a choice of sign.

Proposition 1. *Let E be the elliptic curve with complex multiplication by $\mathbb{Q}(i)$ defined by*

$$E: \quad y^2 = x^3 - x.$$

If $L(E,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, then for odd primes p

$$a(p) = \begin{cases} 0 & \text{if } \quad p \equiv 3 \mod 4, \\ (-1)^{\frac{x+y-1}{2}} 2x & \text{if } p \equiv 1 \mod 4,\ x^2 + y^2 = p,\ \text{and } x \text{ odd.} \end{cases}$$

Moreover, the $q-$series of the Mellin transform of $L(E,s)$ is given by

$$\sum_{n=1}^{\infty} a(n) q^n = q \prod_{n=1}^{\infty} (1 - q^{4n})^2 (1 - q^{8n})^2.$$

Proof. If $\phi_2(-1)$ is the Jacobsthal sum defined by

$$\phi_2(-1) := \sum_{x=0}^{p-1} \phi(x^3 - x),$$

then the formula for $a(p)$ follows by (6) from the well known formula [2, Ch. 6]

$$\phi_2(-1) = \begin{cases} 0 & \text{if } p \equiv 3 \pmod 4, \\ (-1)^{\frac{x+y+1}{2}} \cdot 2x & \text{if } p \equiv 1 \pmod 4, x^2+y^2=p. \text{ and } x \text{ odd.} \end{cases}$$

The fact that $\sum_{n=1}^{\infty} a(n)q^n = q\prod_{n=1}^{\infty}(1-q^{4n})^2(1-q^{8n})^2$ is well known and can be found in [16], [19]. $\square$

Remark 1. The elliptic curve $y^2 = x^3 - x$ and its quadratic twists are the elliptic curves which arise in Tunnell's analysis of the congruent number problem [16].

Proposition 2. *Let E be the elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ defined by*

$$E: \quad y^2 = x^3 + 1.$$

If $L(E,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, then for primes $p \neq 2, 3$

$$a(p) = \begin{cases} 0 & \textit{if } \ p \equiv 2 \mod 3, \\ (-1)^{x+y-1}\left(\frac{x}{3}\right) \cdot 2x & \textit{if } \ p \equiv 1 \mod 3, \textit{and } x^2+3y^2 = p. \end{cases}$$

Moreover, the $q-$series of the Mellin transform of $L(E,s)$ is given by

$$\sum_{n=1}^{\infty} a(n)q^n = q\prod_{n=1}^{\infty}(1-q^{6n})^4.$$

Proof. If $\psi_3(1)$ is the Jacobsthal sum defined by

$$\psi_3(1) := \sum_{x=0}^{p-1} \phi(x^3+1),$$

then the formula for $a(p)$ follows by (6) from the well known result [2, Ch.6]

$$\psi_3(1) = \begin{cases} 0 & \text{if } p \equiv 2 \pmod 3, \\ (-1)^{x+y}\left(\frac{x}{3}\right) \cdot 2x & \text{if } p \equiv 1 \pmod 3, \text{and } x^3+3y^2 = p. \end{cases}$$

The fact that $\sum_{n=0}^{\infty} a(n)q^n = q\prod_{n=1}^{\infty}(1-q^{6n})^4$ is well known (see [19]). $\square$

3. Special values of ${}_2F_1\left(\begin{matrix}\phi & \phi \\ & \epsilon\end{matrix} | \ x\right)$

Before we discuss the general case, we first give the evaluation of ${}_2F_1(1)$ over every $GF(p)$.

Proposition 3. *If p is an odd prime, then the value of ${}_2F_1(1)$ over $GF(p)$ is*

$${}_2F_1(1) = -\frac{\phi(-1)}{p}.$$

Proof. By [12, Th. 4.9], it is known that ${}_2F_1(1) = \phi(-1)\binom{\phi}{\phi}$. However by (1) this may be rewritten as ${}_2F_1(1) = \frac{J(\phi,\phi)}{p}$ which is well known [14, p. 93] to equal ${}_2F_1(1) = -\frac{\phi(-1)}{p}$. $\square$

Now we investigate the values of ${}_2F_1(\lambda)$ when $\lambda \neq 1$. For a rational number λ, let ${}_2E_1(\lambda)$ denote the curve over $\mathbb{Q}$ defined by

$$ {}_2E_1(\lambda): \quad y^2 = x(x-1)(x-\lambda). \tag{12} $$

If $\lambda \neq 0$ or 1, then by (3) and (4) ${}_2E_1(\lambda)$ is an elliptic curve in Legendre normal form, where

$$ \Delta({}_2E_1(\lambda)) := 16\lambda^2(\lambda-1)^2 $$

and

$$ j({}_2E_1(\lambda)) = \frac{256(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}. $$

Let $L({}_2E_1(\lambda), s) = \sum_{n=1}^{\infty} \frac{{}_2a_1(n;\lambda)}{n^s}$ be the Hasse-Weil $L-$function for ${}_2E_1(\lambda)$.

With this notation we recall the following fact which was proved in [18].

Theorem 1. *If $\lambda \in \mathbb{Q} - \{0,1\}$ and p is an odd prime for which* $\mathrm{ord}_p(\lambda(\lambda-1)) = 0$, *then*

$$ {}_2F_1(\lambda) = -\frac{\phi(-1)_2a_1(p;\lambda)}{p}. $$

Proof. By [13, Th. 3.5], ${}_2F_1(\lambda)$ may be rewritten as the character sum

$$ {}_2F_1(\lambda) = \frac{\phi(-1)}{p} \sum_{x \in GF(p)} \phi(x)\phi(1-x)\phi(1-\lambda x). $$

By replacing x by $\frac{x}{\lambda}$ we obtain

$$ {}_2F_1(\lambda) = \frac{\phi(-1)}{p} \sum_{x \in GF(p)} \phi\left(\frac{x}{\lambda}\right)\phi\left(\frac{x}{\lambda}-1\right)\phi(x-1). $$

Since $\phi(\lambda^2) = 1$, it is easy to see that

$$ {}_2F_1(\lambda) = \frac{\phi(-1)}{p} \sum_{x \in GF(p)} \phi(x(x-1)(x-\lambda)). $$

Since p is a prime with good reduction if $\mathrm{ord}_p(\lambda(\lambda-1)) = 0$, the observation in (6) completes the proof. □

The first corollary follows from Hasse's theorem and appears in [18].

Corollary 1. *If $\lambda \in \mathbb{Q} - \{0,1\}$ and p is an odd prime for which* $\mathrm{ord}_p(\lambda(\lambda-1)) = 0$, *then*

$$ |\ {}_2F_1(\lambda)\ | < \frac{2}{\sqrt{p}}. $$

Corollary 2. *Let $\lambda \in \mathbb{Q} - \{0,1\}$ and p an odd prime with* $\mathrm{ord}_p(\lambda(\lambda-1)) = 0$.

(i) If $1-\lambda$ is a perfect rational square, then

$$ {}_2F_1(\lambda) \equiv -\phi(-1)(1+p) \mod 8. $$

(ii) If both λ and $\lambda - 1$ are perfect rational squares, then

$$ {}_2F_1(\lambda) \equiv -\phi(-1)(1+p) \mod 8. $$

(iii) In the remaining cases

$$_2F_1(\lambda) \equiv -\phi(-1) - 1 \mod 4.$$

Proof. The torsion subgroups of $E(M,N) : y^2 = x(x+M)(x+N)$ where $M \neq N \in \mathbb{Q}$ are [20]:

• The torsion subgroup of $E(M,N)$ contains $\mathbb{Z}_2 \times \mathbb{Z}_4$ if M and N are both squares, or $-M$ and $N-M$ are both squares, or $-N$ and $M-N$ are both squares.

• The torsion subgroup of $E(M,N)$ is $\mathbb{Z}_2 \times \mathbb{Z}_8$ if there exists a non-zero integer d such that $M = d^2u^4$ and $N = d^2v^4$, or $M = -d^2v^4$ and $N = d^2(u^4 - v^4)$, or $M = d^2(u^4 - v^4)$ and $N = -d^2v^4$ where (u, v, w) forms a Pythagorean triple (i.e. $u^2 + v^2 = w^2$).

• The torsion subgroup of $E(M,N)$ is $\mathbb{Z}_2 \times \mathbb{Z}_6$ if there exist integers a and b such that $\frac{a}{b} \notin \{-2, -1, -\frac{1}{2}, 0, 1\}$ and $M = a^4 + 2a^3b$ and $N = 2ab^3 + b^4$.

• In all other cases, the torsion subgroup of $E(M,N)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

If $\lambda = \frac{\alpha}{\beta}$, then by multiplying the equation for $_2E_1(\lambda)$ by β^6 and replacing (β^2x, β^3y) by (x, y), we obtain an isomorphic curve

$$y^2 = x(x - \beta^2)(x - \alpha\beta).$$

Now by letting $M = -\beta^2$ and $N = -\alpha\beta$, we find that in cases (i) and (ii) $\mathbb{Z}_2 \times \mathbb{Z}_4$ is contained in the torsion subgroup of $E_{\mathbb{Q}}(M,N)$.

It is not hard to show that in these cases $\mathbb{Z}_2 \times \mathbb{Z}_8$ is not the torsion subgroup of $E_{\mathbb{Q}}(M,N)$. Therefore in (i) and (ii) the torsion subgroup of $E_{\mathbb{Q}}(M,N)$ has order 8.

Moreover, then in (iii) it is clear that the torsion subgroup of $_2E_1(\lambda)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, a group of order 4.

Since the reduction map is injective on the torsion subgroup, it follows that 8 divides the order of $\overline{_2E_1(\lambda)}$ in cases (i) and (ii), and that 4 divides the order of $\overline{_2E_1(\lambda)}$ in case (iii).

Therefore by (8) we find in cases (i) and (ii) that

$$_2F_1(\lambda) \equiv -\frac{\phi(-1)}{p}(p+1) \mod 8,$$

and in case (iii) that

$$_2F_1(\lambda) \equiv -\frac{\phi(-1)}{p}(p+1) \mod 4.$$

The congruences now follow easily from the facts that $p^{-1} \equiv p \mod 8$ and $\phi(-1)p \equiv 1 \mod 4$ for all odd primes p. $\square$

Remark 2. By Corollary 2 it is easy to see that if $\lambda \in \mathbb{Q} - \{0, 1\}$ and p is an odd prime for which $\mathrm{ord}_p(\lambda(\lambda - 1)) = 0$, then $_2F_1(\lambda) \equiv 0 \mod 2$.

Now we evaluate the $_2F_1(\lambda)$ when $_2E_1(\lambda)$ has complex multiplication. Since there are only 13 $j-$invariants for elliptic curves with complex multiplication (9), it is easy to verify that $\lambda = -1, \frac{1}{2}$, and 2 are the only such values; moreover in these cases $_2E_1(\lambda)$ is isomorphic to the *congruent number* elliptic curve $y^2 = x^3 - x$.

Theorem 2. *Let $\lambda \in \{-1, \frac{1}{2}, 2\}$. If p is an odd prime, then*

$$_2F_1(\lambda) = \begin{cases} 0 & \text{if } \ p \equiv 3 \mod 4, \\ \frac{2x(-1)^{\frac{x+y+1}{2}}}{p} & \text{if } \ p \equiv 1 \mod 4, \ \ x^2 + y^2 = p, \ \text{and} \ \ x \ \text{odd.} \end{cases}$$

Proof. By (12) we find that the elliptic curves ${}_2E_1(-1), {}_2E_1\left(\frac{1}{2}\right)$, and ${}_2E_1(2)$ are isomorphic to $y^2 = x^3 - x$. To see this simply replace x by $x+1$ in ${}_2E_1(2)$, and also x by $x = x + \frac{1}{2}$ in ${}_2E_1\left(\frac{1}{2}\right)$. Therefore the theorem follows from Proposition 1 and Theorem 1. $\square$

From the proof of this theorem we obtain the following immediate corollary. Let $c_e(n)$ (resp. $c_o(n)$) denote the number of four colored partitions of n into an even (resp. odd) number of distinct parts where the parts of the latter two colors are even. Note that two parts with the same numerical value are distinct if they have different colors. Then let $c(n)$ be the partition function defined by

$$c(n) := c_e(n) - c_o(n). \tag{13}$$

Corollary 3. *If $\lambda \in \{-1, \frac{1}{2}, 2\}$ and $p \equiv 1 \mod 4$ is prime, then*

$${}_2F_1(\lambda) = -\frac{c\left(\frac{p-1}{4}\right)}{p}.$$

Proof. From the proof of the previous theorem we know that ${}_2E_1(\lambda)$ is isomorphic to the elliptic curve: $y^2 = x^3 - x$. The result now follows by Proposition 1, Theorem 2, and the fact that the generating function for $c(n)$ is

$$\sum_{n=0}^{\infty} c(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})^2.$$

$\square$

In Theorem 2 it was shown that if $\lambda \in \{-1, \frac{1}{2}, 2\}$, then ${}_2F_1(\lambda) = 0$ for all primes $p \equiv 3 \mod 4$. More generally, it is of interest to examine the zeros of ${}_2F_1(\lambda)$.

Theorem 3. *If $\lambda \notin \{-1, 0, \frac{1}{2}, 1, 2\}$, then ${}_2F_1(\lambda) = 0$ for infinitely many primes. However, the set of such primes has density* 0.

Proof. For these λ, the elliptic curve ${}_2E_1(\lambda)$ does not have complex multiplication. In [7] Elkies proved that every elliptic curve over $\mathbb{Q}$ without complex multiplication has infinitely many supersingular primes. Moreover, the number of supersingular primes $\leq x$ is $\ll x^{\frac{3}{4}}$. However, a prime p of good reduction is supersingular if and only if ${}_2a_1(p;\lambda) = 0$. Therefore the result now follows from Theorem 1. $\square$

4. Special values of ${}_3F_2\left(\begin{matrix}\phi & \phi & \phi \\ & \epsilon & \epsilon\end{matrix}\,\middle|\, x\right)$

Before we prove the main theorem regarding the values of ${}_3F_2(\lambda)$ for an arbitrary rational λ, we first evaluate ${}_3F_2(1)$, which is a special case of a result of Evans [10].

Theorem 4. *If p is an odd prime, then ${}_3F_2(1)$ is given by*

$${}_3F_2(1) = \begin{cases} 0 & \text{if } p \equiv 3 \mod 4, \\ \frac{4x^2-2p}{p^2} & \text{if } p \equiv 1 \mod 4, p = x^2 + y^2, \text{ and } x \equiv 1 \mod 2. \end{cases}$$

Proof. By [12, 4.37] it follows that

$${}_3F_2(1) = \begin{cases} 0 & \text{if } \phi \neq \square, \\ \binom{\sqrt{\phi}}{\phi}\binom{\phi\sqrt{\phi}}{\overline{\sqrt{\phi}}} + \binom{\phi\sqrt{\phi}}{\phi}\binom{\overline{\sqrt{\phi}}}{\phi\sqrt{\phi}} & \text{if } \phi = \square. \end{cases}$$

By (2) and the fact that ϕ is a square if and only if $p \equiv 1 \mod 4$, we obtain

$$(14)\qquad {}_3F_2(1) = \begin{cases} 0 & \text{if } p \equiv 3 \mod 4, \\ \binom{\sqrt{\phi}}{\phi}\binom{\phi\overline{\sqrt{\phi}}}{\phi} + \binom{\phi\sqrt{\phi}}{\phi}\binom{\overline{\sqrt{\phi}}}{\phi} & \text{if } p \equiv 1 \mod 4. \end{cases}$$

Hence we may assume that $p \equiv 1 \mod 4$. Note that the two summands for ${}_3F_2(1)$ in (14) are complex conjugates. So it suffices to compute the first summand, which we denote by S. By definition we obtain

$$(15)\qquad S = \binom{\sqrt{\phi}}{\phi}\binom{\phi\overline{\sqrt{\phi}}}{\phi} = \frac{J(\sqrt{\phi}, \phi)J(\phi\overline{\sqrt{\phi}}, \phi)}{p^2}.$$

It is known [14, p. 305] that, for an arbitrary character A,

$$J(A, \phi) = A(4)J(A, A).$$

Therefore we find that

$$J(\sqrt{\phi}, \phi) = \sqrt{\phi}(4)J(\sqrt{\phi}, \sqrt{\phi}),$$

and

$$J(\phi\overline{\sqrt{\phi}}, \phi) = \phi\overline{\sqrt{\phi}}(4)J(\phi\overline{\sqrt{\phi}}, \phi\overline{\sqrt{\phi}}) = \phi\overline{\sqrt{\phi}}J(\sqrt{\phi}, \sqrt{\phi}).$$

The last simplification follows from the fact that $\phi\overline{\sqrt{\phi}} = \sqrt{\phi}$.

By combining these facts we find that

$$S = \frac{J^2(\sqrt{\phi}, \sqrt{\phi})}{p^2}.$$

By [14, 9.9.4] we find that

$$-\sqrt{\phi}(-1)J(\sqrt{\phi}, \sqrt{\phi}) = x + iy,$$

$x^2 + y^2 = p$, and $x + iy \equiv 1 \mod (2 + 2i)$. In particular note that this implies that x is odd. Hence we find that

$$S = \frac{\phi(-1)(x^2 - y^2 + 2xyi)}{p^2}.$$

Since ${}_3F_2(1) = S + \bar{S}$, we obtain

$${}_3F_2(1) = \frac{2x^2 - 2y^2}{p^2} = \frac{4x^2 - 2p}{p^2}.$$

$\square$

Now we evaluate the remaining ${}_3F_2(\lambda)$. For a rational number λ let ${}_3E_2(\lambda)$ denote the curve over $\mathbb{Q}$ defined by

$$(16)\qquad {}_3E_2(\lambda): \quad y^2 = x^3 - \lambda^2 x^2 + (4\lambda^3 - \lambda^4)x + \lambda^6 - 4\lambda^5.$$

If $\lambda \neq 0$ or 4, then by (3) and (4) ${}_3E_2(\lambda)$ is an elliptic curve over $\mathbb{Q}$ with discriminant

$$\Delta({}_3E_2(\lambda)) := 1024\lambda^9(\lambda - 4)$$

and $j-$invariant

$$j({}_3E_2(\lambda)) = \frac{256(\lambda - 3)^3}{\lambda - 4}.$$

Let $L({}_3E_2(\lambda),s)=\sum_{n=1}^{\infty}\frac{{}_3a_2(n;\lambda)}{n^s}$ be the Hasse-Weil $L-$function for ${}_3E_2(\lambda)$. Before we evaluate these Gaussian hypergeometric series, we first present a lemma.

Lemma 1. *Let $\lambda\in\mathbb{Q}$ and let p be an odd prime for which* $\mathrm{ord}_p(\lambda)=0$. *Then*

$$\frac{p}{p-1}\sum_{\chi}\binom{\phi\chi^2}{\chi}\binom{\phi\chi}{\chi}\bar{\chi}(\lambda)$$
$$=\frac{\phi(2)}{p}\sum_{\substack{x\in GF(p)\\ x\neq-\lambda^2}}\phi(x^3-\lambda^2x^2+(4\lambda^3-\lambda^4)x+\lambda^6-4\lambda^5).$$

Proof. Using identity (2), the sum in the lemma may be reduced to

$$\frac{p}{p-1}\sum_{\chi}\binom{\phi\bar{\chi}}{\chi}\binom{\phi\chi}{\chi}\bar{\chi}(\lambda)\chi(-1).$$

However by (1) this reduces to

$$\frac{1}{p-1}\sum_{\chi}\binom{\phi\chi}{\chi}\bar{\chi}(\lambda)\sum_{x\in GF(p)}\phi\bar{\chi}(x)\bar{\chi}(1-x),$$

which after switching the order of summation becomes

$$=\frac{1}{p-1}\sum_{x\in GF(p)}\phi(x)\sum_{\chi}\binom{\phi\chi}{\chi}\bar{\chi}(\lambda)\bar{\chi}(x)\bar{\chi}(1-x)$$
$$=\frac{1}{p-1}\sum_{\substack{x\in GF(p)\\ x\neq1}}\phi(x)\sum_{\chi}\binom{\phi\chi}{\chi}\chi\left(\frac{1}{\lambda x(1-x)}\right).$$

Now by applying (2) again we find that the sum in the lemma is

$$=\frac{1}{p-1}\sum_{\substack{x\in GF(p)\\ x\neq1}}\phi(x)\sum_{\chi}\binom{\phi}{\chi}\chi\left(\frac{-1}{\lambda x(1-x)}\right).\tag{17}$$

Now recall that the binomial theorem (see [12]) for a character A over $GF(p)$ says that

$$A(1+x)=\delta(x)+\frac{p}{p-1}\sum_{\chi}\binom{A}{\chi}\chi(x),\tag{18}$$

where $\delta(x)=1$ (resp. 0) if $x=0$ (resp. $x\neq0$). Since $\frac{1}{\lambda x(1-x)}\neq0$, this implies that

$$\frac{1}{p-1}\sum_{\chi}\binom{\phi}{\chi}\chi\left(\frac{-1}{\lambda x(1-x)}\right)=\frac{1}{p}\phi\left(1-\frac{1}{\lambda x(1-x)}\right).$$

Therefore we may rewrite (17) as

$$=\frac{1}{p}\sum_{\substack{x\in GF(p)\\ x\neq1}}\phi(x)\phi\left(1-\frac{1}{\lambda x(1-x)}\right)$$
$$=\frac{1}{p}\sum_{x\in GF(p)}\phi(x)\phi(\lambda x(1-x)-1)\phi(\lambda x(1-x)).$$

Note that the latter sum includes $x = 1$ without loss of generality since $\phi(0) = 0$. Replacing x by $\frac{x+1}{2}$, we see that the sum becomes

$$\begin{aligned}
&= \frac{\phi(2)}{p} \sum_{x \in GF(p)} \phi(x+1)\phi(\lambda(1-x^2)-4)\phi(\lambda(1-x^2)) \\
&= \frac{\phi(2)}{p} \sum_{\substack{x \in GF(p) \\ x \neq -1}} \phi(\lambda(1-x^2)-4))\phi(\lambda(1-x)) \\
&= \frac{\phi(2)}{p} \sum_{\substack{x \in GF(p) \\ x \neq -\lambda^2}} \phi(x^3 - \lambda^2 x^2 + (4\lambda^3 - \lambda^4)x + \lambda^6 - 4\lambda^5).
\end{aligned}$$

The last simplification is made by multiplying through the cubic polynomial in x by λ^6, a perfect square, and then replacing x by $\frac{x}{\lambda^2}$. □

Theorem 5. *If $\lambda \in \mathbb{Q} - \{0, 4\}$ and p is an odd prime for which $\operatorname{ord}_p(\lambda(\lambda-4)) = 0$, then*

$${}_3F_2\left(\frac{4}{4-\lambda}\right) = \frac{\phi(\lambda^2 - 4\lambda)({}_3a_2(p;\lambda)^2 - p)}{p^2}.$$

Proof. Following Greene and Stanton [13, 3.5], we define the function $f(x)$ by

$$f(x) := \frac{p}{p-1} \sum_{\chi} \binom{\phi\chi^2}{\chi} \binom{\phi\chi}{\chi} \chi\left(\frac{x}{4}\right).$$

If p is an odd prime and $\operatorname{ord}_p(\lambda) = 0$, then

$$f\left(\frac{4}{\lambda}\right) = \frac{p}{p-1} \sum_{\chi} \binom{\phi\chi^2}{\chi} \binom{\phi\chi}{\chi} \chi\left(\frac{1}{\lambda}\right) = \frac{p}{p-1} \sum_{\chi} \binom{\phi\chi^2}{\chi} \binom{\phi\chi}{\chi} \bar{\chi}(\lambda).$$

Therefore by Lemma 1 we find that

$$f\left(\frac{4}{\lambda}\right) = \frac{\phi(2)}{p} \sum_{\substack{x \in GF(p) \\ x \neq -\lambda^2}} \phi(x^3 - \lambda^2 x^2 + (4\lambda^3 - \lambda^4)x + \lambda^6 - 4\lambda^5).$$

However, it is now easy to see by (6) that

$$f\left(\frac{4}{\lambda}\right) = \frac{\phi(2)}{p} \left(-{}_3a_2(p;\lambda) - \phi(-2\lambda)\right). \tag{19}$$

The key identity [12, 4.5] is

$$\begin{aligned}
&\phi\left(\frac{1-u}{u}\right) {}_3F_2\left(\frac{u}{u-1}\right) \\
&\qquad = \phi(u) f^2(u) + \frac{2\phi(-1)}{p} f(u) - \frac{p-1}{p^2}\phi(u) + \frac{(p-1)}{p^2}\delta(1-u).
\end{aligned}$$

By setting $u = \frac{4}{\lambda}$ (when $\lambda \neq 0, 4$), we obtain

$${}_3F_2\left(\frac{4}{4-\lambda}\right) = \phi(\lambda-4)\left(\phi(\lambda) f^2\left(\frac{4}{\lambda}\right) + \frac{2\phi(-1)}{p} f\left(\frac{4}{\lambda}\right) - \frac{(p-1)}{p^2}\phi(\lambda)\right).$$

By making the substitution for $f\left(\frac{4}{\lambda}\right)$ as in (19), we obtain the result. □

By Hasse's theorem we obtain

Corollary 4. *If* $\lambda \in \mathbb{Q} - \{0, 4\}$ *and* p *is an odd prime for which* $\mathrm{ord}_p(\lambda(\lambda-4)) = 0$, *then*

$$|\ {}_3F_2\left(\frac{4}{4-\lambda}\right)| < \frac{3}{p}.$$

Corollary 5. *If* $\lambda \in \mathbb{Q} - \{0, 4\}$, *then let* M *be the order of the torsion subgroup of* ${}_3E_2(\lambda)$. *If* p *is an odd prime for which* $\mathrm{ord}_p(\lambda(\lambda - 4)) = 0$ *and* $\gcd(p, M) = 1$, *then*

$${}_3F_2\left(\frac{4}{4-\lambda}\right) \equiv \phi(\lambda^2 - 4\lambda)(1 + p^{-1} + p^{-2}) \mod M.$$

Moreover, for all odd primes p *for which* $\mathrm{ord}_p(\lambda(\lambda - 4)) = 0$

$${}_3F_2\left(\frac{4}{4-\lambda}\right) \equiv 1 \mod 2.$$

Proof. By (8), if p is an odd prime for which ${}_3E_2(\lambda)$ has good reduction and $\gcd(p, M) = 1$, then ${}_3a_2(p; \lambda) \equiv 1 + p \mod M$. Therefore by Theorem 5 it follows that

$${}_3F_2\left(\frac{4}{4-\lambda}\right) \equiv \frac{\phi(\lambda^2 - 4\lambda)(p^2 + p + 1)}{p^2} \mod M.$$

This completes the proof of the first assertion. To obtain the second claim we simply need to show that M is even. This is easy to see since the point $(\lambda^2, 0)$ is a point of order 2 on ${}_3E_2(\lambda)$. ☐

For these Gaussian hypergeometric functions, we find that ${}_3F_2(\lambda) = 0$ for at most a finite number of primes. In particular, if p is an odd prime for which $\mathrm{ord}_p(\lambda(\lambda - 4)) = 0$, then ${}_3F_2\left(\frac{4}{4-\lambda}\right) = 0$ implies that ${}_3a_2(p; \lambda)^2 = p$, which is absurd since ${}_3a_2(p; \lambda)$ is an integer.

Corollary 6. *If* $\lambda \in \mathbb{Q} - \{0, 4\}$ *and* p *is an odd prime for which* $\mathrm{ord}_p(\lambda(\lambda-4)) = 0$, *then*

$${}_3F_2\left(\frac{4}{4-\lambda}\right) \neq 0.$$

Now we give all the explicit evaluations for those cases where ${}_3E_2(\lambda)$ has complex multiplication. Since the $j-$invariant for ${}_3E_2(\lambda)$ is $j({}_3E_2(\lambda)) = \dfrac{256(\lambda - 3)^3}{\lambda - 4}$, and the only $j-$invariants for elliptic curves over $\mathbb{Q}$ with complex multiplication are given in (9), it is easy to verify that the only λ for which ${}_3E_2(\lambda)$ has complex multiplication are $\lambda = \frac{9}{2}, 36, 8, 3, -12, \frac{63}{16}$, and -252.

Theorem 6. (Complex multiplication evaluations) *If*

$$\lambda \in \{\frac{9}{2}, 36, 8, 3, -12, \frac{63}{16}, -252\},$$

then for every odd prime p *for which* $\mathrm{ord}_p(\lambda(\lambda - 4)) = 0$ *the value* ${}_3F_2\left(\frac{4}{4-\lambda}\right)$ *is given by:*

$$(i) \qquad {}_3F_2(-8) = \begin{cases} -\frac{1}{p} & \text{if} \quad p \equiv 3 \mod 4, \\ \frac{4x^2-p}{p^2} & \text{if} \quad p \equiv 1 \mod 4,\ x^2 + y^2 = p,\ \text{and } x \text{ odd}. \end{cases}$$

$$(ii) \qquad {}_3F_2\left(\frac{-1}{8}\right) = \begin{cases} -\frac{\phi(2)}{p} & \text{if} \quad p \equiv 3 \mod 4, \\ \frac{\phi(2)(4x^2-p)}{p^2} & \text{if} \quad p \equiv 1 \mod 4,\ x^2 + y^2 = p,\ \text{and } x \text{ odd}. \end{cases}$$

$$(iii)\qquad {}_3F_2(-1) = \begin{cases} -\frac{\phi(2)}{p} & \textit{if } \ p \equiv 5,7 \mod 8, \\ \frac{\phi(2)(4x^2-p)}{p^2} & \textit{if } \ p \equiv 1,3 \mod 8, \ \textit{and } x^2+2y^2=p. \end{cases}$$

$$(iv)\qquad {}_3F_2(4) = \begin{cases} -\frac{\phi(-3)}{p} & \textit{if } \ p \equiv 2 \mod 3, \\ \frac{\phi(-3)(4x^2-p)}{p^2} & \textit{if } \ p \equiv 1 \mod 3, \ \textit{and } x^2+3y^2=p. \end{cases}$$

$$(v)\qquad {}_3F_2\left(\tfrac{1}{4}\right) = \begin{cases} -\frac{\phi(3)}{p} & \textit{if } p \equiv 2 \mod 3, \\ \frac{\phi(3)(4x^2-p)}{p^2} & \textit{if } p \equiv 1 \mod 3, \ \textit{and } x^2+3y^2=p. \end{cases}$$

$$(vi)\qquad {}_3F_2(64) = \begin{cases} -\frac{\phi(-7)}{p} & \textit{if } \ p \equiv 3,5,6 \mod 7, \\ \frac{\phi(-7)(4x^2-p)}{p^2} & \textit{if } \ p \equiv 1,2,4 \mod 7, \ \textit{and } x^2+7y^2=p. \end{cases}$$

$$(vii)\qquad {}_3F_2\left(\tfrac{1}{64}\right) = \begin{cases} -\frac{\phi(7)}{p} & \textit{if } p \equiv 3,5,6 \mod 7, \\ \frac{\phi(7)(4x^2-p)}{p^2} & \textit{if } p \equiv 1,2,4 \mod 7, \ \textit{and } x^2+7y^2=p. \end{cases}$$

Proof. To prove this theorem it suffices to determine the explicit values of ${}_3a_2(p;\lambda)$. From these values the formulas in the theorem are easily deduced from Theorem 5.

(i) In this case $\lambda = \frac{9}{2}$ and $j\left({}_3E_2\left(\frac{9}{2}\right)\right) = 1728$. The equation for ${}_3E_2\left(\frac{9}{2}\right)$ is

$$y^2 = x^3 - 81x^2 - 729x + 59049$$

which after replacing x by $x+27$ becomes

$$y^2 = x^3 - 36x.$$

Hence ${}_3E_2(\lambda)$ by (10) is the 6−quadratic twist of $y^2 = x^3 - x$. In particular, for every prime for which $\gcd(p,6)=1$ we find by (11) that

$$ {}_3a_2\left(p;\frac{9}{2}\right) = \left(\frac{6}{p}\right) a(p).$$

Therefore by Proposition 1 we obtain

$$a(p) = \begin{cases} 0 & \text{if } p \equiv 3 \mod 4, \\ (-1)^{\frac{x+y-1}{2}} 2x & \text{if } p \equiv 1 \mod 4, \ x^2+y^2=p, \text{ and } x \equiv 1 \mod 2. \end{cases}$$

(ii) By [12, Th. 4.2], it is well known that

$$ {}_3F_2\left(\frac{1}{t}\right) = \phi(-t)\,{}_3F_2(t). \tag{20}$$

Therefore by (i) we find that

$$ {}_3F_2\left(-\frac{1}{8}\right) = \phi(2)\,{}_3F_2(-8).$$

(iii) In this case $\lambda = 8$, $j({}_3E_2(8)) = 20^3$, and so ${}_3E_2(8)$ has complex multiplication by $\mathbb{Q}(\sqrt{-2})$. This case was evaluated by Greene and Stanton in [13, 4.13] using the evaluation of a certain Brewer sum given by Berndt and Evans [1, 5.17].

(iv) In this case $\lambda = 3$, $j({}_3E_2(3)) = 0$, and so ${}_3E_2(3)$ is an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-3})$. The equation for ${}_3E_2(3)$ is

$$y^2 = x^3 - 9x^2 + 27x - 243,$$

which after replacing x by $x+3$ becomes

$$y^2 = x^3 - 6^3,$$

which by (10) is the -6-quadratic twist of $y^2 = x^3 + 1$. Therefore from Proposition 2 and (11) it follows that

$$ {}_3a_2(p;3) = \begin{cases} 0 & \text{if } p \equiv 2 \pmod 3, \\ (-1)^{x+y-1}\left(\frac{x}{3}\right)\left(\frac{-6}{p}\right)\cdot 2x & \text{if } p \equiv 1 \pmod 3 \text{ and } x^2+3y^2=p. \end{cases} $$

(v) Using (20), we find that ${}_3F_2\left(\frac{1}{4}\right) = \phi(-1){}_3F_2(4)$, and so the result now follows from the proof of (iv).

(vi) In this case $\lambda = \frac{63}{16}, j\left({}_3E_2\left(\frac{63}{16}\right)\right) = -15^3$, and so ${}_3E_2\left(\frac{63}{16}\right)$ has complex multiplication by $\mathbb{Q}(\sqrt{-7})$. In this case the equation for ${}_3E_2\left(\frac{63}{16}\right)$ is

$$ y^2 = x^3 - 3969x^2 + 250047x - 992436543. $$

By (10) this is the 42-quadratic twist of an elliptic curve with conductor 49, and by the work of Rajwade [21] we find that

$$ {}_3a_2\left(p;\frac{63}{16}\right) = \begin{cases} 0 & \text{if } p \equiv 3,4,5 \pmod 7, \\ (-1)^{\frac{p^2-1}{8}}\left(\frac{x}{7}\right)\left(\frac{21}{p}\right)\cdot 2x & \text{if } p \equiv 1,2,4 \pmod 7. \end{cases} $$

(vii) As in proving (ii) and (v), we find using (20) that ${}_3F_2\left(\frac{1}{64}\right) = \phi(-1){}_3F_2(64)$, and so the result follows from the proof of (vi). $\square$

Remark 3. In [18] Koike asks for an explicit evaluation of ${}_3F_2(4)$. Part (iv) of Theorem 6 provides the complete solution to this question.

Remark 4. In the proof of Theorem 6 we did not have to explicitly compute the various ${}_3a_2(p;\lambda)$; it was only necessary to compute ${}_3a_2(p;\lambda)$ up to a choice of sign. However, since determining the explicit value was not too difficult, we chose to do so.

It turns out that some of these special values also have a combinatorial interpretation in terms of colored partition functions. If we let $d_e(n)$ (resp. $d_o(n)$) denote the number of four colored partitions of n into an even (resp. odd) number of distinct parts, then the partition function $d(n) := d_e(n) - d_o(n)$ has the generating function

$$ \sum_{n=0}^{\infty} d(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^4. $$

Corollary 7. *If $p \equiv 1 \pmod 4$ is prime, then*

$$ {}_3F_2\left(-\frac{1}{8}\right) = \frac{\phi(2)(c\left(\frac{p-1}{4}\right)^2 - p)}{p^2} \quad \textit{and} \quad {}_3F_2(-8) = \frac{(c\left(\frac{p-1}{4}\right)^2 - p)}{p^2}. $$

If $p \equiv 1 \pmod 6$ is prime, then

$$ {}_3F_2\left(\frac{1}{4}\right) = \frac{\phi(3)(d\left(\frac{p-1}{6}\right)^2 - p)}{p^2} \quad \textit{and} \quad {}_3F_2(4) = \frac{\phi(-3)(d\left(\frac{p-1}{6}\right)^2 - p)}{p^2}. $$

D. Stanton has pointed out, in unpublished notes, that some of the evaluations in Theorem 6 have *nice* classical analogs. For instance ${}_3F_2\begin{pmatrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}; x \\ & 1, & 1 \end{pmatrix}$, a reasonable analog of the Gaussian ${}_3F_2(x)$, can for special x be explicitly evaluated using the following special case of Clausen's theorem [12, 4.2] and known evaluations which are listed in [8]. As examples, we list

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ & 1, & 1\end{matrix};x\right) = \frac{1}{\sqrt{1-x}}\cdot {}_2F_1\left(\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ & 1\end{matrix};\frac{x}{x-1}\right)^2 $$

and known values of ${}_2F_1\left(\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ & 1\end{matrix};\frac{x}{x-1}\right)^2$. In this way it turns out that

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ & 1, & 1\end{matrix};-1\right) = \frac{1}{\sqrt{2}}\left(\frac{\Gamma(1)\Gamma(1/2)}{\Gamma(5/8)\Gamma(7/8)}\right)^2, $$

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ & 1, & 1\end{matrix};\frac{1}{4}\right) = \frac{3}{4\sqrt{3}}\left(\frac{\Gamma(4/3)\Gamma(1)}{\Gamma(3/2)\Gamma(5/6)}\right)^2. $$

It is interesting to note that $\sqrt{2}$ and $\sqrt{3}$ occur in these evaluations, which nicely corresponds to the fact that the associated Gaussian evaluations follow from the ideal structure in $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-3})$. If $x = -\frac{1}{8}$ or $\frac{1}{64}$, then obvious explicit analogs are unknown to the author, although D. Stanton has suggested methods for deriving them, again using Clausen's theorem. However, it is prudent to note that it is not truly clear what the proper notion of an *analog* should be. In general there will be problems with convergence, and perhaps some care must be taken when associating characters with rational numbers.

5. Number theoretic applications

First we investigate the character sums $I(t;p)$ defined in the introduction. In [13] Greene and Stanton proved a conjecture of Evans, Pulham, and Sheehan by evaluating $I(1;p)$ for every prime p. We solve the analogous problem for all t for which ${}_3E_2\left(\frac{4+4t}{t}\right)$ is an elliptic curve with complex multiplication. It seems extremely unlikely that there are any other values of t for which $I(t;p)$ will be easily evaluated for all primes p.

Corollary 8. *If* $t \in \{-1, 8, \frac{1}{8}, 1, -4, -\frac{1}{4}, -64, -\frac{1}{64}\}$, *then for every odd prime* p *for which* $\mathrm{ord}_p\left(\frac{4+4t}{t}\right) = 0$, *the character sum* $I(t;p)$ *is given by:*

(i) $$I(-1,p) = \begin{cases} 0 & \textit{if } p \equiv 3 \mod 4,\\ 4x^2 - 2p & \textit{if } p \equiv 1 \mod 4,\ x^2+y^2 = p,\ \textit{and } x \textit{ odd.}\end{cases}$$

(ii) $$I(8;p) = \begin{cases} -p & \textit{if } p \equiv 3 \mod 4,\\ 4x^2 - p & \textit{if } p \equiv 1 \mod 4,\ x^2+y^2 = p,\ \textit{and } x \textit{ odd.}\end{cases}$$

(iii) $$I\left(\tfrac{1}{8};p\right) = \begin{cases} -\phi(2)p & \textit{if } p \equiv 3 \mod 4,\\ \phi(2)(4x^2 - p) & \textit{if } p \equiv 1 \mod 4,\ x^2+y^2 = p,\ \textit{and } x \textit{ odd.}\end{cases}$$

(iv) $$I(1;p) = \begin{cases} -\phi(2)p & \textit{if } p \equiv 5,7 \mod 8,\\ \phi(2)(4x^2 - p) & \textit{if } p \equiv 1,3 \mod 8,\ \textit{and } x^2+2y^2 = p.\end{cases}$$

(v) $$I(-4;p) = \begin{cases} -\phi(-3)p & \textit{if } p \equiv 2 \mod 3,\\ \phi(-3)(4x^2 - p) & \textit{if } p \equiv 1 \mod 3,\ \textit{and } x^2+3y^2 = p.\end{cases}$$

(vi) $$I\left(\tfrac{-1}{4};p\right) = \begin{cases} -\phi(3)p & \textit{if } p \equiv 2 \mod 3,\\ \phi(3)(4x^2 - p) & \textit{if } p \equiv 1 \mod 3,\ \textit{and } x^2+3y^2 = p.\end{cases}$$

$$(vii) \qquad I(-64;p) = \begin{cases} -\phi(-7)p & \text{if } p \equiv 3,5,6 \mod 7, \\ \phi(-7)(4x^2-p) & \text{if } p \equiv 1,2,4 \mod 7, \text{ and } x^2+7y^2=p. \end{cases}$$

$$(viii) \qquad I\left(-\tfrac{1}{64};p\right) = \begin{cases} -\phi(7)p & \text{if } p \equiv 3,4,5 \mod 7, \\ \phi(7)(4x^2-p) & \text{if } p \equiv 1,2,4 \mod 7, \text{ and } x^2+7y^2=p. \end{cases}$$

Proof. By [13, 2.10], it is known that

$$I(t;p) = p^2 {}_3F_2(-t).$$

The result now follows as an immediate corollary to Theorem 4 and Theorem 6. □

Now we show how some of these evaluations imply congruences for generalized Apéry numbers.

Definition 2. Given a pair of non-negative integers m and ℓ, the generalized Apéry number $A(n;m,\ell)$ is defined by

$$A(n;m,\ell) := \sum_{k=0}^{n} \binom{n+k}{k}^m \binom{n}{k}^\ell.$$

The generalized Apéry numbers $C(n)$ are defined by

$$C(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

We first recall a proposition proved by Koike [17]:

Proposition 4. *If $p = 2f+1$ is prime and $w = m+\ell$, then*

$$A(f;m,\ell) \equiv \left(\frac{p}{p-1}\right)^{w-1} {}_wF_{w-1}\left(\begin{matrix} \phi, & \phi, & \ldots, \phi \\ & \epsilon, & \ldots, \epsilon \end{matrix} \mid (-1)^\ell \right) \mod p.$$

Using the evaluations in this paper, we obtain the following congruences for generalized Apéry numbers. These congruences were first proved by Beukers and Stienstra [3], [4], [5]. Koike proved these in [17], and the only difference in our proofs is that here we have all the explicit evaluations of the relevant hypergeometric functions.

Corollary 9. *In the above notation:*

(*i*) *The generalized Apéry numbers $A(n;1,1)$ satisfy the congruence*

$$A(f;1,1) \equiv \begin{cases} 0 \mod p & \text{if } p \equiv 3 \mod 4, \\ 2x(-1)^{\frac{x+y-1}{2}} \mod p & \text{if } p \equiv 1 \mod 4,\ x^2+y^2=p, \text{ and } x \text{ odd.} \end{cases}$$

(*ii*) *The generalized Apéry numbers $A(n;1,2)$ satisfy the congruence*

$$A(f;1,2) \equiv \begin{cases} 0 \mod p & \text{if } p \equiv 3 \mod 4, \\ 4x^2 \mod p & \text{if } p \equiv 1 \mod 4,\ x^2+y^2=p, \text{ and } x \text{ odd.} \end{cases}$$

(*iii*) *The generalized Apéry numbers $A(n;2,1)$ satisfy the congruence*

$$A(f;2,1) \equiv \begin{cases} 0 \mod p & \text{if } p \equiv 5,7 \mod 8, \\ \phi(2)4x^2 \mod p & \text{if } p \equiv 1,3 \mod 8, \text{ and } x^2+2y^2=p. \end{cases}$$

Proof. These congruences follow immediately from Proposition 4, Theorems 2, 4, and 6(*iii*). □

Following Koike [17], if $p = 2f + 1$ is prime, then

$$C(f) \equiv \left(\frac{p}{p-1}\right)^2 {}_3F_2(4) \mod p.$$

Beukers and Stienstra proved congruences for these $C(f)$ which follow immediately from Theorem 6 (iv).

Corollary 10. *Using the notation above, the generalized Apéry numbers $C(n)$ satisfy*

$$C(f) \equiv \begin{cases} 0 \mod p & \text{if } p \equiv 2 \mod 3, \\ \phi(-3)4x^2 \mod p & \text{if } p \equiv 1 \mod 3. \end{cases}$$

Now we give some variations of such congruences which follow from the explicit evaluations given in Theorem 6. First we define another type of generalized Apéry number.

Definition 3. Given a pair of non-negative integers m and l, and a rational number r, the generalized Apéry number $D(n;m,l,r)$ is defined by

$$D(n;m,l,r) := \sum_{k=0}^{n} \binom{n+k}{k}^m \binom{n}{k}^l r^{lk}.$$

Proposition 5. *If $p = 2f + 1$ is prime and $w = m + l$, then*

$$D(f;m,l,r) \equiv \left(\frac{p}{p-1}\right)^{w-1} {}_wF_{w-1}\left(\begin{matrix}\phi, & \phi, & \dots, \phi \\ & \epsilon, & \dots, \epsilon\end{matrix} \mid (-r)^l\right) \mod p.$$

Proof. Let ω denote the Teichmüller character which is defined by $\omega(x) := x \mod p$, for integers x. By [17, Lemma 1] we find that

$$\begin{aligned} D(f;m,l,r) &= \sum_{k=0}^{f} \binom{f+k}{k}^m \binom{f}{k}^l r^{kl} \\ &\equiv \left(\frac{p}{p-1}\right)^w \sum_{k=0}^{f} \binom{\phi\omega^k}{\omega^k}^m \binom{\phi}{\omega^k}^l \omega^k\left(r^l\right) \mod p \\ &\equiv \left(\frac{p}{p-1}\right)^w \sum_{\chi} \binom{\phi\chi}{\chi}^m \binom{\phi}{\chi}^l \chi\left(r^l\right) \mod p. \end{aligned}$$

However, since

$$\binom{\phi\chi}{\chi} = \chi(-1)\binom{\phi}{\chi},$$

it follows that

$$\begin{aligned} D(f;m,l,r) &\equiv \left(\frac{p}{p-1}\right)^w \sum_{\chi} \binom{\phi\chi}{\chi}^m \binom{\phi\chi}{\chi}^l \chi\left(-r^l\right) \mod p \\ &= \left(\frac{p}{p-1}\right)^{w-1} {}_wF_{w-1}\left(\begin{matrix}\phi, & \phi, & \dots\phi \\ & \epsilon, & \dots & \epsilon\end{matrix} \mid (-r)^l\right) \mod p. \end{aligned}$$

□

Therefore by Theorem 6, we obtain the following immediate corollary:

Corollary 11. *Using the notation above, if r is a rational number and p is an odd prime for which $\mathrm{ord}_p(r((-r)^l - 1)) = 0$, then the generalized Apéry numbers $D(f; m, l, r)$ satisfy:*

$$(i) \qquad D(f;2,1,8) \equiv D(f,0,3,2) \equiv \begin{cases} 0 \mod p & \text{if } \phi(-1) = -1, \\ 4x^2 \mod p & \text{if } \phi(-1) = 1,\ x^2+y^2 = p,\ x \text{ odd.} \end{cases}$$

$$(ii) \qquad D\left(f;2,1,\frac{1}{8}\right) \equiv D\left(f,0,3,\frac{1}{2}\right) \equiv \begin{cases} 0 \mod p & \text{if } \phi(-1) = -1, \\ \phi(2)4x^2 \mod p & \text{if } \phi(-1) = 1,\ x^2+y^2 = p,\ x \text{ odd.} \end{cases}$$

$$(iii) \qquad D(f;2,1,1) \equiv D(f,0,3,1) \equiv \begin{cases} 0 \mod p & \text{if } \phi(-2) = -1, \\ \phi(2)4x^2 \mod p & \text{if } \phi(-2) = 1,\ x^2+2y^2 = p. \end{cases}$$

$$(iv) \qquad D(f;1,2,\pm 2) \equiv D(f,2,1,-4) \equiv \begin{cases} 0 \mod p & \text{if } \phi(-3) = -1, \\ \phi(-3)4x^2 \mod p & \text{if } \phi(-3) = 1,\ x^2+3y^2 = p. \end{cases}$$

$$(v) \qquad D\left(f;1,2,\pm\frac{1}{2}\right) \equiv D\left(f,2,1,-\frac{1}{4}\right) \equiv \begin{cases} 0 \mod p & \text{if } \phi(-3) = -1, \\ \phi(3)4x^2 \mod p & \text{if } \phi(-3) = 1,\ x^2+3y^2 = p. \end{cases}$$

$$(vi) \qquad D(f,1,2,\pm 8) \equiv D(f,0,3,-4) \equiv D(f,2,1,-64) \equiv \begin{cases} 0 \mod p & \text{if } \phi(-7) = -1, \\ \phi(-7)4x^2 \mod p & \text{if } \phi(-7) = 1,\ x^2+7y^2 = p. \end{cases}$$

$$(vii) \qquad D\left(f;1,2,\pm\frac{1}{8}\right) \equiv D\left(f,0,3,-\frac{1}{4}\right) \equiv D\left(f,2,1,-\frac{1}{64}\right) \equiv \begin{cases} 0 \mod p & \text{if } \phi(-7) = -1, \\ \phi(7)4x^2 \mod p & \text{if } \phi(-7) = 1,\ x^2+7y^2 = p. \end{cases}$$

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