THE RESIDUAL SPECTRUM OF $U(2,2)$

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Dedicated to Professor H. Shimizu on his sixtieth birthday

Abstract. Let $G$ be the quasi-split unitary group $U(2,2)_{k^2/k}$ of four variables attached to a quadratic extension $k^2/k$ of number fields. An irreducible decomposition of the non-cuspidal discrete spectrum of $L^2$-automorphic forms of $G$ is obtained.

1. Introduction

Let $G$ be a reductive group defined over a number field $k$. For simplicity we assume the center of $G$ is anisotropic over $k$. If we write $\mathbb{A}$ for the ring of adeles of $k$, then the group of $\mathbb{A}$-points $G(\mathbb{A})$ is a locally compact topological group which contains $G(k)$ as a discrete subgroup with finite covolume. The right regular representation $R$ of $G(\mathbb{A})$ on the space of $L^2$-automorphic forms $L^2(G(\mathbb{A})\backslash G(\mathbb{A}))$

$$[R(g)\phi](x) := \phi(xg), \quad (\phi \in L^2(G(k)\backslash G(\mathbb{A})), \ g \in G(\mathbb{A})),$$

is a primary object in the modern theory of automorphic forms. In particular we are interested in the “irreducible decomposition” of this $R$.

We know from the general theory of spectral decomposition that $L^2(G(k)\backslash G(\mathbb{A}))$ decomposes into a direct sum of two $G(\mathbb{A})$-invariant closed subspaces $L^2_{\text{disc}}(G)$ and $L^2_{\text{cont}}(G)$. $L^2_{\text{disc}}(G)$ is a direct sum of irreducible representations of $G(\mathbb{A})$ while $L^2_{\text{cont}}(G)$ is a continuous sum of irreducible $G(\mathbb{A})$-modules. Nevertheless, Langlands’ spectral theory of Eisenstein series ([La], [MW]) reduces the study of $L^2_{\text{cont}}(G)$ to that of $L^2_{\text{disc}}(M)$ for Levi subgroups $M$ of $G$. Thus, the term “irreducible decomposition of $L^2(G(k)\backslash G(\mathbb{A}))$” means that of $L^2_{\text{disc}}(G)$.

For $\phi \in L^2(G(k)\backslash G(\mathbb{A}))$, we define its constant term along a $k$-parabolic subgroup $P$ by

$$\phi_P(g) := \int_{U(k)\backslash U(\mathbb{A})} \phi(ug) \, du.$$

Here $U$ is the unipotent radical of $P$. Then the space $L^2_0(G(k)\backslash G(\mathbb{A}))$ of $L^2$-cusp forms on $G(\mathbb{A})$ is spanned by those $\phi \in L^2(G(k)\backslash G(\mathbb{A}))$ such that $\phi_P$ vanishes almost everywhere for each proper $k$-parabolic subgroup $P$. This is clearly a $G(\mathbb{A})$-invariant closed subspace in $L^2(G(k)\backslash G(\mathbb{A}))$. Since every $\phi \in L^2_0(G(k)\backslash G(\mathbb{A}))$ is rapidly decreasing, $L^2_0(G(k)\backslash G(\mathbb{A}))$ is contained in $L^2_{\text{disc}}(G)$.

Classical Fourier expansions for elliptic modular forms suggest that the Whittaker model for irreducible automorphic representations is a useful tool for the

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irreducible decomposition of $L^2_0(G(k) \backslash G(\mathbb{A}))$. This idea was carried out by Jacquet and Langlands in the case $G = GL(2)$, and later extended to the case $G = GL(n)$ by Jacquet, Shalika and Piatetskii-Shapiro. Their results are quite satisfactory in these cases. But the Whittaker model cannot capture all the irreducible representations appearing in $L^2_0(G(k) \backslash G(\mathbb{A}))$ if $G$ is not $GL(n)$ or $SL(n)$. Thus the irreducible decomposition of $L^2_0(G(k) \backslash G(\mathbb{A}))$ is open except for the cases $G = GL(n)$, $SL(2)$ and $U(2,1)$. (It seems that the only hope in this direction is the theory of twisted endoscopy.)

There is one problem which is accessible with our present knowledge. It is to determine the irreducible decomposition of the residual discrete spectrum, the orthogonal complement in $L^2_{disc}(G) \backslash L^2_0(G(k) \backslash G(\mathbb{A}))$, for rank 2 classical groups $G$. In fact:

1. It was shown by Langlands ([La]) that the residual discrete spectrum is spanned by certain residues of Eisenstein series on $G(k) \backslash G(\mathbb{A})$.
2. Since any Levi factor of $k$-parabolic subgroups of $G$ is rank one, we know from [Sh] that the poles of these Eisenstein series are given by those of certain automorphic $L$-functions.
3. Hence if the poles of these automorphic $L$-functions can be determined, then the irreducible decomposition of the residual discrete spectrum is reduced to those of $L^2_0(M(k)A_M(\mathbb{A}) \backslash M(\mathbb{A}))$ for Levi factors $M$ of $k$-parabolic subgroups of $G$.

Here $A_M$ denotes the split component of the center of $M$.

In the present paper we shall carry this out in the case of a rank 2 quasi-split unitary group $G = U(2,2)_{k'/k}$ attached to a quadratic extension $k'/k$. In this case the automorphic $L$-functions in (2) above are products of Hecke $L$-functions, twisted tensor $L$-functions ([As], [HLR], [G]) and the product $L$-functions for $U(1,1) \times \text{Res}_{k'/k} \mathbb{G}_m$. Their poles are either well-known or easily determined (cf. Appendices A, B). Also the Levi factor in (3) is one of $M_0 := \text{Res}_{k'/k} \mathbb{G}_m \times \text{Res}_{k'/k} \mathbb{G}_m$, $M_1 := \text{Res}_{k'/k} GL(2)$ and $M_2 := \text{Res}_{k'/k} \mathbb{G}_m \times U(1,1)_{k'/k}$. The irreducible decomposition of $L^2_0(M(k)A_M(\mathbb{A}) \backslash M(\mathbb{A}))$ for these $M$ can be deduced from [JL] and [LL]. Now the main result of this paper is stated as follows (Theorem 3.11, Theorem 3.18, Theorem 4.4 and Theorem 5.5).

**Theorem 1.1.** The residual spectrum of $G = U(2,2)_{k'/k}$ is a direct sum of the following irreducible representations of $G(\mathbb{A})$. Each occurs with multiplicity one.

1. The one dimensional representations $\chi \circ \det$. Here $\chi$ runs over all the characters of $U(1,k) \backslash U(1,\mathbb{A})$.
2. The $\theta$-lifts $R(V_k, \chi)$ of the trivial representations of rank 1 unitary groups $U(V, \mathbb{A})$. Here $V_k = V \otimes_k \mathbb{A}$ and $V$ runs over all the 1-dimensional Hermitian spaces over $k'$. $\chi$ runs over all the characters of $k'_{\times}/k''_{\times}$ such that $\chi|_{k_{\times}} = \eta_{k'/k}$. $\eta_{k'/k}$ is the quadratic character corresponding to $k'/k$ by class field theory.
3. The “$\theta$-lifts” of non-trivial 1-dimensional representations of $U(1,1)_{k'/k}(\mathbb{A})$.
4. The global Langlands quotient of $\text{Ind}_{P_1(\mathbb{A})}^G(\mathbb{A})(\mathcal{S}(P_1) \otimes 1_{\nu_1(\mathbb{A})})$. Here $P_1$ is a parabolic subgroup of $G$ whose Levi factor is $M_1$. $\mathcal{S}(P_1) = \pi \otimes | \det |_{A_{k'}}$, where $\pi$ runs over the irreducible cuspidal automorphic representations of $M_1(\mathbb{A})$ such that
   (a) the central character $\omega_\pi$ of $\pi$ restricted to $A_{k''}$ is trivial, and
(b) if we write $H$ for the subgroup $GL(2)_k$ of $M_1 \simeq \text{Res}_{k'/k} GL(2)$, then
\[
\int_{H(k)Z(H,\mathbb{A}) \backslash H(\mathbb{A})} f(h) \, dh \neq 0
\]
for some $f$ in the automorphic realization of $\pi$.

(5) The global Langlands quotients of
\[
\text{Ind}^{G(\mathbb{A})}_{P_2(\mathbb{A})} [\mathcal{S}(P_2, \eta_{k'/k}) \otimes 1_{U_2(\mathbb{A})}] \quad \text{and} \quad \text{Ind}^{G(\mathbb{A})}_{P_2(\mathbb{A})} [\mathcal{S}(P_2, \mathbf{1}) \otimes 1_{U_2(\mathbb{A})}],
\]
where $P_2$ is a parabolic subgroup of $G$ whose Levi factor is $M_2$ and $\mathcal{S}(P_2, \eta_{k'/k})$ runs over all the irreducible cuspidal representations of $M_2(\mathbb{A})$ which satisfy the following conditions. $\mathcal{S}(P_2, \eta_{k'/k})$ is written as $\chi|_{A_{k'}} \otimes \tau$ according to $M_2(\mathbb{A}) \simeq A_{k'}^\times \times U(1,1)_{k'/k}(\mathbb{A})$, where

(a) $\chi|_{A_{k'}} = \eta_{k'/k}$, and
(b) $\tau$ is the $\theta$-lift of its central character $\omega_\tau$ from $U(1)_{k'/k}(\mathbb{A})$ under the Weil representation $\omega_{\psi, \chi^{-1}}$ (see B.1).

Also $\mathcal{S}(P_2, \mathbf{1})$ runs over the irreducible cuspidal representations of $M_2(\mathbb{A})$ of the form $\chi|_{k_{k'}}^{1/2} \otimes \tau$ such that $\chi|_{A_{k'}} = \mathbf{1}$ and $L(s, \tau \times \chi)$ does not vanish at $s = 1/2$.

The residual spectrum of $GL(n)$, which had been conjectured by Jacquet [Ja2], was obtained by Moeglin and Waldspurger [MW2]. Also the same kind of results for $G = Sp(2)$ were obtained independently by H. Kim and the author ([Ki]). Kim also determined the residual spectrum of the split group of type $G_2$. Note that the condition (b) in (4) is equivalent to certain functorial properties of $\pi$ (see [HLR], [Y] and [Fl]).

We now explain the organization of this paper. In §2, we collect some notation and review the basic tools in Langlands’ spectral theory of Eisenstein series from [MW]. In §3 we determine the contributions of cuspidal Eisenstein series from the Borel subgroup $P_0$ to the residual spectrum. This will need a detailed calculation of the $L^2$-inner product of these Eisenstein series (see 3.2). Once the inner product is calculated (Theorem 3.4), we proceed to the irreducible decomposition. In this step the results of Kudla, Sweet, S. T. Lee and C. Zhu on the irreducible decomposition of certain degenerate principal series representations of $U(n, n)$ will be used ([KSw], [Le] and [LeZ]). We review these in 3.3.1. Finally we use the arguments of [KRS], §2, to get statements (1) and (2) in Theorem 1.1. Statement (3) is easily proved by a local Langlands classification argument.

In §§4 and 5, we determine the contributions of cuspidal Eisenstein series from the Siegel parabolic subgroup $P_1$ and from the non-Siegel maximal parabolic subgroup $P_2$ respectively. The main problem in these cases is to determine the poles of certain automorphic $L$-functions. In the Siegel parabolic case, the twisted tensor $L$-function appears, while the product $L$-function of $U(1,1) \times \text{Res}_{k'/k} \mathbb{G}_m$ appears in the non-Siegel parabolic case. The poles of these $L$-functions will be determined in Appendices A, B, respectively. Once the poles are determined, statements (4) and (5) in Theorem 1.1 can be easily deduced from local Langlands classification arguments.

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2. Preliminaries

2.1. Notation and conventions.

2.1.1. Let \( k'/k \) be a quadratic extension of number fields. We write \( \Gamma \) for the Galois group \( \text{Gal}(k'/k) \) and \( \sigma \) for its generator. \( W_k \) denotes the Weil group of \( k \).

We write \( \mathbb{A} \) for the adele ring of \( k \) and write \( \mathbb{A}_\infty \) and \( \mathbb{A}_f \) for its infinite and finite component respectively. \( | \cdot |_k \) denotes the idele norm of \( \mathbb{A}_\times \). A place of \( k \) is conventionally denoted by \( v \). For each place \( v \) of \( k \), we write \( k_v \) for the completion of \( k \) at \( v \) and \( | \cdot |_w \) for the \( v \)-adic norm. In particular, if \( v \) is finite, we let \( \mathcal{O}_v \) be the maximal compact subring of \( k_v \), \( p_v \) the maximal ideal in \( \mathcal{O}_v \) and \( w_v \) a generator of \( p_v \). Also we write \( q_v \) for the cardinality of the residue field of \( k_v \).

We write \( \mathbb{A}_{k'} \) for the adele ring of \( k' \). The idele norm of \( \mathbb{A}_{k'}_\times \) is written \( | \cdot |_{k'} \). We conventionally write a place of \( k' \) by \( w \). The notations \( k'_w \supset \mathcal{O}_w \supset p_w \supset w_w \) are defined similarly as in the case of \( k \). The following conventions on places of \( k \) and \( k' \) will be convenient. If a place \( v \) is inert in \( k' \), then the place of \( k' \) lying over \( v \) will be denoted by \( w \). If \( v \) splits in \( k' \), then the places of \( k' \) lying over \( v \) will be denoted by \( w_1 \) and \( w_2 \). We often write \( k'_v \) for the semisimple algebra \( k' \otimes_k k_v \) over \( k_v \).

2.1.2. We need some notation about the group \( \tilde{G} := \text{Res}_{k'/k} \text{GL}(4) \). The \( k' \)-automorphism on \( \tilde{G} \) determined by the action of \( \sigma \) on \( \text{GL}(4,k') \) is denoted by \( \tilde{\sigma} \). We fix a \( k' \)-group isomorphism

\[
\tilde{G}_{k'} \xrightarrow{\sim} \text{GL}(4)_{k'} \times \text{GL}(4)_{k'},
\]

so that the action of \( \sigma \) on \( \tilde{G} \) is transported to

\[
\sigma : \text{GL}(4)_{k'} \times \text{GL}(4)_{k'} \ni (g_1,g_2) \rightarrow (\sigma(g_2),\sigma(g_1)) \in \text{GL}(4)_{k'} \times \text{GL}(4)_{k'},
\]

and \( \tilde{\sigma} \) is transported to

\[
\tilde{\sigma} : \text{GL}(4)_{k'} \times \text{GL}(4)_{k'} \ni (g_1,g_2) \rightarrow (g_2,g_1) \in \text{GL}(4)_{k'} \times \text{GL}(4)_{k'}.
\]

2.1.3. We define an automorphism \( \theta_2 \) of \( \tilde{G}_{k'} \) by

\[
\theta_2 : \text{GL}(4)_{k'} \times \text{GL}(4)_{k'} \ni (g_1,g_2) \rightarrow (\text{Int} J_2(1g_1^{-1}), \text{Int} J_2(1g_2^{-1})) \in \text{GL}(4)_{k'} \times \text{GL}(4)_{k'},
\]

where

\[
J_2 := \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}.
\]

This clearly commutes with \( \sigma \) and hence is a \( k \)-automorphism of \( \tilde{G} \). Then our group \( G := U(2,2)_{k'/k} \) is defined by \( G := \{ g \in \tilde{G} ; \theta_2 \circ \tilde{\sigma}(g) = g \} \). Note that
$G_{k'} = \{(g, \theta_2(g)) \in GL(4)_{k'} \times GL(4)_{k'} \} \cong GL(4)_{k'}$. Fix a minimal $k$-parabolic subgroup $P_0$ and its Levi factor $M_0$ of $G$ so that

$$P_{0,k'} := \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ * & * & * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ * & * & * & * \end{pmatrix} \right\} \in G_{k'} \right\},$$

$$M_{0,k'} := \left\{ d((x_1, y_1), (x_2, y_2)) \right\} := \left\{ \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ 0 & y_1^{-1} & 0 & 0 \\ 0 & 0 & x_2^{-1} & 0 \\ 0 & 0 & 0 & x_2^{-1} \end{pmatrix}, \begin{pmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & x_2^{-1} & 0 \\ 0 & 0 & 0 & x_2^{-1} \end{pmatrix} \right\} \in G_{k'} \right\}.$$

The unipotent radical of $P_0$ is denoted by $U_0$. We write an element of $M_0$ as $d(x_1, x_2) := \text{diag}(x_1, x_2, \sigma(x_1)^{-1}, \sigma(x_2)^{-1})$ ($x_i \in \text{Res}_{k'/k} \mathbb{G}_m$, $i = 1, 2$). Then the $k$-split component $A_0$ of the center of $M_0$ is given by $A_0 := \{d(x_1, x_2) : x_1, x_2 \in \mathbb{G}_m \}$. We write $G_v$ for the algebraic group over $k_v$ given by the scalar extension $G_v := G \otimes_k k_v$. Then $G_v \cong U(2, 2)_{k_v'/k_v}$ if $v$ is inert in $k'$, and $G_v \cong GL(4)_{k_v}$ if $v$ splits in $k'$. Also we fix a good maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$ as

$$K_v = \left\{ \begin{array}{ll}
G(k_v) \cap GL(4, \mathcal{O}_w), & \text{if } v \not\mid \infty \text{ and } v \text{ is inert in } k', \\
GL(4, \mathcal{O}_v), & \text{if } v \not\mid \infty \text{ and } v \text{ splits in } k', \\
U(4) \cap G(\mathbb{R}) \cong U(2) \times U(2), & \text{if } v = \mathbb{R} \text{ and } v \text{ is inert in } k', \\
O(4), & \text{if } v = \mathbb{R} \text{ and } v \text{ splits in } k', \\
U(4) \cap G(\mathbb{C}) = U(4), & \text{if } v = \mathbb{C}.
\end{array} \right.$$}

Then we have the Iwasawa decomposition $G(\mathbb{A}) = P_0(\mathbb{A})K$. We write $K_\infty$ for $\prod_v K_v \subset G(\mathbb{A}_\infty)$.

We choose a basis $\{e_1, e_2\}$ of $X^*(A_0)$, the character group of $A_0$, so that

$$e_j(d(x_1, x_2)) = x_j \quad (j = 1, 2).$$

Let $\{e_1', e_2'\}$ be its dual basis of $X_e(A_0)$, the group of one-parameter subgroups of $A_0$: $e_1'(x) = d(x, 1)$, $e_2'(x) = d(1, x)$. Name the $P_0$-positive roots of $A_0$ as

$$R^+(P_0, A_0) = \{\alpha_1 := e_1 - e_2, \alpha_2 := 2e_2, \beta_1 := e_1 + e_2, \beta_2 := 2e_1\};$$

then $\Delta(P_0, A_0) := \{\alpha_1, \alpha_2\}$ is the set of simple roots.

2.1.4. Fix a $k$-splitting of $G_{k'}$ as $(P_{0,k'}, M_{0,k'}, \{X_\alpha\})$, where

$$X_{\alpha_1} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$X_{\alpha_2} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
and $X_{\alpha_i'} := \tilde{\sigma}(X_{\alpha_i})$. Here
\[
\alpha_1(d((x_1, y_1), (x_2, y_2))) = x_1x_2^{-1}, \quad \alpha_2(d((x_1, y_1), (x_2, y_2))) = x_2y_2, \\
\alpha_1'(d((x_1, y_1), (x_2, y_2))) = y_1y_2^{-1}
\]
are the simple roots of $M_{0, k'}$ in $P_{0, k'}$. The dual group $\hat{G}$ of $G = U(2, 2)_{k}/k$ equals $GL(4, \mathbb{C})$. We fix a splitting $(\hat{P}_0, \hat{M}_0, \{X_{\alpha}^\vee\}_{\alpha^\vee})$ of $\hat{G}$ to be objects of the form of the first components of $P_{0, k'}, M_{0, k'}$, $\{X_{\alpha}\}_{\alpha}$. This distinguished splitting is stable under the $\Gamma$-action $\sigma(g) := \theta_2(g)$. Now the $L$-group $^L G$ for $G$ is given by $^L G := \hat{G} \times_{\theta_2} W_k$, where $W_k$ acts through $\Gamma$ by $\sigma(g) := \theta_2(g)$.

2.1.5. The $k$-parabolic subgroups of $G$ which contain the minimal parabolic subgroup $P_0$ are called standard parabolic subgroups. They contain a complete system of representatives of $G(k)$-conjugacy classes of $k$-parabolic subgroups. Each standard parabolic subgroup $P$ has a unique Levi component $M$ which contains $M_0$. Then the set of simple roots $\Delta(M \cap P_0, A_0)$ of $A_0$ in $M \cap P_0$ is a subset of $\Delta(P_0, A_0)$. This gives a bijection between the standard parabolic subgroups of $G$ and the subsets of $\Delta(P_0, A_0)$. Thus we have two proper standard parabolic subgroups $P_i = M_i U_i$ ($i = 1, 2$) other than $P_0$, where $P_i$ corresponds to $\{\alpha_i\} \subset \Delta(P_0, A_0)$ under the above bijection. We also adopt analogous notation for parabolic subgroups of $\hat{G}$ which contain $\hat{P}_0$. Then the $L$-group of $P_i = M_i U_i$ ($i = 1, 2$) is given by $\hat{P}_i \times_{\theta_2} W_k$. Here $\hat{P}_i = \hat{M}_i \hat{U}_i$ is such that
\[
\Delta(\hat{P}_0 \cap \hat{M}_i, \hat{M}_0) = \begin{cases} \{\alpha_1^\vee, \alpha_i^\vee\}, & \text{if } i = 1, \\
\{\alpha_2^\vee\}, & \text{if } i = 2. \end{cases}
\]

2.1.6. For a standard parabolic subgroup $P = MU$, we write $A_M$ for the $k$-split component of the center of $M$. We fix a $k$-isomorphism $\Psi_M : \mathbb{G}_m^{\dim A_M} \cong_k A_M$ and write $A_M(\mathbb{R})_+$ for the image of the composit
\[
(\mathbb{R}^\times)^{\dim A_M} \hookrightarrow \mathbb{G}_m^{\dim A_M}(\mathbb{A}_\infty) \xrightarrow{\Psi_M} A_M(\mathbb{A}_\infty).
\]
Here the first injection is the diagonal embedding into
\[
\mathbb{G}_m^{\dim A_M}(\mathbb{A}_\infty) = \prod_{v \mid \infty} \mathbb{G}_m^{\dim A_M}(k_v).
\]

As usual we have real vector spaces $\mathfrak{a}_M := \text{Hom}(X^*(M)_k, \mathbb{R})$ and $\mathfrak{a}_M^* := X^*(M)_k \otimes_{\mathbb{Z}} \mathbb{R}$ dual to each other. Here $X^*(M)_k$ denotes the $k$-rational character group of $M$. We write $\alpha_0$ and $\alpha_0^*$ for $\mathfrak{a}_{M_0}$ and $\mathfrak{a}_{M_0}^*$ respectively. $\mathfrak{a}_M$ is always identified with the subspace $\{\lambda \in \mathfrak{a}_0 : \langle \alpha, \lambda \rangle = 0 \text{ for } \alpha \in \Delta(P_0 \cap M, A_0)\}$ of $\mathfrak{a}_0$. On the other hand, the restriction map $X^*(M)_k \ni \chi \mapsto \chi|_{M_0} \in X^*(M_0)_k$ gives a natural embedding $\mathfrak{a}_M^* \hookrightarrow \mathfrak{a}_0^*$. The Harish-Chandra map $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ is defined by
\[
\langle H_M(m), \chi \rangle = \log |\chi(m)|_k, \quad \text{for any } \chi \in X^*(M)_k.
\]

We write the kernel of this map as $M(\mathbb{A})^1$. Then we have a direct product decomposition $M(\mathbb{A}) = M(\mathbb{A})^1 \times A_M(\mathbb{R})_+$. $H_M$ is often considered as a map from $G(\mathbb{A})$ by the Iwasawa decomposition fixed above:

\[
(2.1) \quad H_M : G(\mathbb{A}) \ni g = umk \mapsto H_M(m) \in \mathfrak{a}_M, \quad (u \in U(\mathbb{A}), m \in M(\mathbb{A}), k \in K).
\]
The Weyl group of $A_0$ in $G$ is denoted by $W$. It acts on $a_0$ and $a_0^\ast$. We write $w_i$ for the simple reflection attached to the simple root $\alpha_i$ ($i = 1, 2$) and $w_-$ for the longest element $w_1 w_2 w_1 w_2 = w_2 w_1 w_2$. We also have the Weyl group $W(A_M)$ of $A_M$ in $G$, which acts on $a_M$. The inner product on $a_0^\ast$ for which $\{e_1, e_2\}$ is an ortho-normal basis is $W$-invariant. This determines a $W(A_M)$-invariant inner product on each $a_M^\ast$. We fix these inner products throughout the paper.

Finally we normalize various measures as in [Ar].

2.1.7. Notation for automorphic forms. In the rest of this paper, we freely use the notations introduced in [MW], II. We briefly review them.

Fix a standard parabolic subgroup $P = MU$. We write the space of automorphic forms on $U(\mathbb{A})M(k)\backslash G(\mathbb{A})$ as $A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))$. $A(M(k)\backslash M(\mathbb{A}))$ denotes the space of automorphic forms on $M(k)\backslash M(\mathbb{A})$. The space of cusp forms on $U(\mathbb{A})M(k)\backslash G(\mathbb{A})$ is written as $A_0(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))$. $A_0(M(k)\backslash M(\mathbb{A}))$ is the space of cusp forms on $M(k)\backslash M(\mathbb{A})$.

Next let $\pi$ be a cuspidal automorphic representation. We write $A(M(k)\backslash M(\mathbb{A}))_{\pi}$ for the $\pi$-isotypic subspace in $A_0(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))$. For each $\phi \in A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))$, we define

$$\phi_k(m) := e^{i\rho_m H_M(m)}\phi(mk) \quad (m \in M(\mathbb{A}))$$

belongs to $A(M(k)\backslash M(\mathbb{A}))_{\pi}$ for any $k \in K$.

Each $\lambda \in a_{M,C}^* := a_M^* \otimes \mathbb{C}$ is identified with the quasi-character

$$M(\mathbb{A}) \ni m \mapsto \exp(\lambda, H_M(m)) \in \mathbb{C}^\times$$

of $M(\mathbb{A})$ trivial on $M(k)$. Recall that two automorphic subrepresentations $\pi$ and $\tau$ of $M(\mathbb{A})$ are equivalent if there exists $\lambda \in a_{M,C}^*$ such that $\tau \simeq \pi \otimes \lambda$. We call a pair $(M, \mathcal{P})$, where $\mathcal{P}$ is an equivalence class of cuspidal automorphic representations of $M(\mathbb{A})$, a cuspidal datum.

Fix a cuspidal datum $(M, \mathcal{P})$ for the rest of this section. Recall the space of Paley-Wiener sections $P_{(M, \mathcal{P})}$. For each $\phi \in P_{(M, \mathcal{P})}$, its Fourier transform $F(\phi)$ is defined.

2.1.8. Notation for Eisenstein series. For each $\pi \in \mathcal{P}$, we write $\omega_\pi$ for its central character, and define $\Re \pi \in a_M^*$. Let $\mathfrak{p}_\pi$ be by

$$\Re \pi : M(\mathbb{A}) \ni m \mapsto |\omega_\pi(m)|_\mathbb{A} \in \mathbb{R}_+^\times, \quad \Im \pi := (\Re \pi)^{-1} \ominus \pi.$$

We also write $-\pi$ for the contragredient of $\pi$ and $\bar{\pi}$ for the complex conjugate of $\pi$. We have a hermitian pairing between $\phi \in A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_{\pi}$ and $\phi' \in A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_{\bar{\pi}}$ given by

$$\langle \phi, \phi' \rangle := \int_{M(\mathbb{A})} \int_{M(\mathbb{A})} \phi(m_1 k) \overline{\phi'(m_1 k)} \ dm_1 dk.$$

Notice that if $\pi \in \mathcal{P}$ then $-\pi \in \mathcal{P}$.

For $\phi : \mathcal{P} \to A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))$ with $\phi(\pi) \in A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_{\pi}$ ($\pi \in \mathcal{P}$), the Eisenstein series attached to $\phi$ is defined by

$$E(\phi, \pi)(g) := \sum_{\gamma \in P(k)\backslash G(k)} \phi(\pi)(\gamma g), \quad g \in G(\mathbb{A}).$$

This converges absolutely on

$$C^+_P(\mathcal{P}) := \{ \pi \in \mathcal{P} : (\Re \pi - \rho_P, \alpha^\vee) > 0, \text{ for all } \alpha \in R^+(A_M, P) \}.$$
Take \( w \in W \) such that \( w(M) \) is again a Levi factor of a standard parabolic subgroup containing \( M_0 \). In the \( U(2,2) \) case this means \( w(M) = M \). We have the intertwining operator
\[
(M(w, \pi)\phi_w)(g) := \int_{U_w(A)} \phi_w(\hat{w}^{-1}ug) \, du,
\]
where \( \phi_w \in A(U(A)M(k)\backslash G(A)) \), \( U_w := U_0 \cap w(U) \) and \( \hat{w} \) is a representative of \( w \) in the normalizer of \( A_0(k) \) in \( G(k) \). This converges absolutely on
\[
C_p^+(\mathfrak{P}, w) := \Big\{ \pi \in \mathfrak{P} \mid \langle \Re \pi - \rho_P, \alpha^\vee \rangle > 0 \quad \text{for all} \quad \alpha \in R^+(A_M, P) \quad \text{such that} \quad \omega \alpha \notin R^+(A_M, P) \Big\}.
\]

2.1.9. Constant terms of Eisenstein series. For a standard parabolic subgroup \( P = MU \) of \( G \), we have the Weyl group \( W^M \subset W \) of \( M \). Let \( P' = M'U' \) be another standard parabolic subgroup. Set
\[
W_M := \left\{ w \in W \mid \begin{array}{l}
i) \ w \text{ is of minimal length in the coset } wW^M, \\
ii) \ w(M) \text{ is again a Levi component of a standard } \\
\text{parabolic subgroup containing } M_0.
\end{array} \right\},
\]
\[
W_{M,M'} := \{ w \in W_M \mid w(M) \subset M' \}.
\]

Now let \( E(\phi, \pi)(g) \) be as in 2.1.8. Then the constant term of \( E(\phi, \pi) \) along the parabolic subgroup \( P' = M'U' \) is given by
\[
E_{P'}(\phi, \pi)(m) = \sum_{w \in W_{M,M'}} E^{M'}(M(w, \pi)\phi(\pi), w\pi)(m) \quad (m \in (A_\mathbb{A})),
\]
where
\[
E^{M'}(M(w, \pi)\phi(\pi), w\pi)(g) := \sum_{\gamma \in \langle wPw^{-1}\gamma(M')\rangle(k) \backslash M'(k)} M(w, \pi)\phi(\pi)(\gamma g).
\]

2.1.10. Eisenstein pseudo-series. For each \( \phi \in P_{(M, \mathfrak{P})} \), we define
\[
\theta_\phi(g) := \sum_{\gamma \in P(k) \backslash G(k)} F(\phi)(\gamma g) \quad (g \in G(A)).
\]

Then the sum on the right hand side converges absolutely and uniformly on any compact subsets of \( G(A) \). \( \theta_\phi \) is in \( L^2(G(k)\backslash G(A)) \), and
\[
\{ \theta_\phi \mid \phi \in P_{(M, \mathfrak{P})}, \quad (M, \mathfrak{P}) \text{ runs over all cuspidal data} \}
\]
spans a dense subspace of \( L^2(G(k)\backslash G(A)) \).

2.1.11. The inner product formula. Let \((M, \mathfrak{P})\) and \((M', \mathfrak{P}')\) be two cuspidal data. Set
\[
W(\mathfrak{P}, \mathfrak{P}') := \{ w \in W_{M,M'} : w(M, \mathfrak{P}) = (M', \mathfrak{P}') \}.
\]
Two data \((M, \mathfrak{P})\) and \((M', \mathfrak{P}')\) are said to be equivalent if the set \( W(\mathfrak{P}, \mathfrak{P}') \) is non-empty. The following theorem is [MW], Théorème II.2.1.
Theorem 2.1. Let \((M, \mathfrak{P})\) and \((M', \mathfrak{P}')\) be as above. Then for \(\phi \in P_{(M, \mathfrak{P})}\) and \(\phi' \in P_{(M', \mathfrak{P}')}\), the hermitian inner product of \(\theta_\phi\) and \(\theta_{\phi'}\) is given by
\[
\langle \theta_\phi, \theta_{\phi'} \rangle_{L^2(G(k)\backslash G(A))} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\dim \mathfrak{a}_M} \int_{\mathfrak{P}, \Re \pi = \lambda_0} \sum_{w \in W(\mathfrak{P}, \mathfrak{P}')} \langle M(w, \pi)\phi(\pi), \phi'(-w\pi) \rangle d\pi.
\]
Here the hermitian pairing in the integrand is that of 2.1.8 and \(\lambda_0 \in \text{Re}(C_P^+(\mathfrak{P}))\).

We write \(\mathfrak{C}\) for the set of equivalence classes of cuspidal data for \(G\). For each equivalence class \(\mathfrak{X} \in \mathfrak{C}\), \(L^2(G(k)\backslash G(A))_\mathfrak{X}\) denotes the closed span of \(\theta_\phi\)'s \((\phi \in P_{(M, \mathfrak{P})}, (M, \mathfrak{P}) \in \mathfrak{X})\) in \(L^2(G(k)\backslash G(A))\). Then as a corollary of Theorem 2.1, we have the direct sum decomposition
\[
L^2(G(k)\backslash G(A)) = \bigoplus_{\mathfrak{X} \in \mathfrak{C}} L^2(G(k)\backslash G(A))_\mathfrak{X}.
\]

3. The contributions of cuspidal data attached to \(P_0\)

To decompose the non-cuspidal spectrum of \(G\), we first decompose the inner product (2.2) in Theorem 2.1. By (2.3), it is enough to analyze \(\langle \theta_\phi, \theta_{\phi'} \rangle_{L^2(G(k)\backslash G(A))}\) for \(\phi\) and \(\phi'\) belonging to the same equivalence class of cuspidal data. We say that a cuspidal datum \((M, \mathfrak{P})\) is attached to a standard parabolic subgroup \(P\) if \(M\) is the Levi factor of \(P\). \(\mathfrak{C}_P\) denotes the set of equivalence classes of cuspidal data attached to \(P\). In this section, we study \(\langle \theta_\phi, \theta_{\phi'} \rangle_{L^2(G(k)\backslash G(A))}\) for \(\phi, \phi'\) belonging to cuspidal data attached to \(P_0\).

As in [MW], V.2.1, we rewrite (2.2) as follows. For \(\mathfrak{X} \in \mathfrak{C}_P\), define
\[
P_\mathfrak{X} := \bigoplus_{(M, \mathfrak{P}) \in \mathfrak{X}} P_{(M, \mathfrak{P})}, \quad \theta_{\phi'} := \sum_{(M, \mathfrak{P}) \in \mathfrak{X}} \theta_{\phi'_{(M, \mathfrak{P})}},
\]
where \(\phi' = (\phi'_{(M, \mathfrak{P})})_{(M, \mathfrak{P}) \in \mathfrak{X}} \in P_\mathfrak{X}\). Take \((M, \mathfrak{P}) \in \mathfrak{X}\) and write
\[
A(\phi, \phi')(\pi) := \sum_{(M, \mathfrak{P}) \in \mathfrak{X}} \sum_{w \in W(\mathfrak{P}, \mathfrak{P}')} \langle M(w, \pi)\phi(\pi), \phi'_{(M, \mathfrak{P})}(-w\pi) \rangle,
\]
where \(\phi \in P_{(M, \mathfrak{P})}\) and \(\phi' \in P_\mathfrak{X}\). From now on we consider
\[
\langle \theta_\phi, \theta_{\phi'} \rangle_{L^2(G(k)\backslash G(A))} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\dim \mathfrak{a}_M} \int_{\mathfrak{P}, \Re \pi = \lambda_0} A(\phi, \phi')(\pi) d\pi
\]
instead of (2.2).

3.1. Singular hyperplanes. Let the notation be as above. For a subset \(\mathfrak{S} \subset \mathfrak{P}\), its vector part \(\mathfrak{S}^0\) is defined to be \(\{ \lambda \in \mathfrak{a}^*_M, \lambda \otimes \mathfrak{S} \subset \mathfrak{S}\}\). A subset \(\mathfrak{S} \subset \mathfrak{P}\) is called an affine subspace with vector part defined over \(\mathbb{R}\) if

1. \(\text{Re}(\mathfrak{S})\) is an affine subspace in \(\mathfrak{a}^*_M\) whose vector part is \(\text{Re}(\mathfrak{S}^0)\),
2. \(\mathfrak{S}^0 := \text{Re}(\mathfrak{S}^0) \otimes \mathbb{R} \subset \mathfrak{a}^*_M\) equals \(\mathfrak{S}^0\).

An affine hyperplane with vector part defined over \(\mathbb{R}\) is an affine subspace \(\mathfrak{S}\) with vector part defined over \(\mathbb{R}\), such that \(\text{codim}_{\mathfrak{a}^*_M} \text{Re}(\mathfrak{S}^0) = 1\). It is known that the singularity set of \(A(\phi, \phi')(\pi)\) is an union of affine hyperplanes with vector part defined over \(\mathbb{R}\). We write \(S_{(M, \mathfrak{P})}^h\) for the set of these singular hyperplanes of \(A(\phi, \phi')\).
In this case ($\phi \in P(M, \mathfrak{a})$, $\phi' \in P_X$). In the following, we give a list of elements in $S^{\mu}_{(M_0, \mathfrak{a})}$ which intersect the positive chamber.

Each $\pi \in \mathfrak{a}$ is decomposed into a restricted tensor product $\otimes_v \pi_v$, where $\pi_v$ is a smooth irreducible representation of $M(k_v)$ if $v$ is finite and of $(\text{Lie} M(k_v) \otimes \mathbb{R} \mathbb{C}, M(k_v) \cap K_v)$ if $v$ is archimedean. We write $\mathfrak{a}_v$ for the set of local representations $\pi_v$, which appear as the $v$-component of some $\pi \in \mathfrak{a}$. By replacing $| \cdot |_v$ with $| |_v$ in 2.1.6, we have the local Harish-Chandra map $H_M: M(k_v) \to \mathfrak{a}_M$. This gives local analogues $\text{Re} \pi_v$, $\text{Im} \pi_v$, $C^{+}_P(\mathfrak{a}_v)$ and $C^{+}_P(\mathfrak{a}_v, w)$ of $\text{Re} \pi$, $\text{Im} \pi$, $C^{+}_P(\mathfrak{a})$ and $C^{+}_P(\mathfrak{a}, w)$, respectively.

We may assume the Eulerian decomposition
\[
(3.2) \quad M(w, \pi)\phi(\pi) = \bigotimes_v M(w, \pi_v)\phi_v(\pi_v) \quad (\pi \in C^{+}_P(\mathfrak{a}, \omega)).
\]

Here $\phi_v(\pi_v)$ is contained in the space $\text{Ind}^{G(k_v)}_{P(k_v)}[\pi_v \otimes 1_{U(k_v)}]$, and
\[
(M(w, \pi_v)\phi_v(\pi_v))(g) = \int_{U_w(k_v)} \phi_v(\pi_v)(\hat{w}^{-1}ng) \, dn \quad (g \in G(k_v))
\]
for $\pi_v \in C^{+}_P(\mathfrak{a}_v, w)$ (cf. [Sh3]).

### 3.1.1. Analytic behavior of local intertwining operators

For section 3.1.1, our cuspidal data $(M, \mathfrak{a})$ are always attached to $P_0$. Since $W_{M_0} = W$, we have to study $M(w, \pi_v)\phi_v(\pi_v)$ for all $w \in W$. We write $\pi \in \mathfrak{a}$ in the form
\[
\pi = \mu_1 \otimes \mu_2 : M_0(\hat{k}) \ni d(x_1, x_2) \mapsto \mu_1(x_1)\mu_2(x_2) \in \mathbb{C}^\times,
\]
where $\mu_i = \otimes_w \mu_{i,w} (i = 1, 2)$ are quasi-characters of $\hat{k}_v^\times / k_v^\times$. We fix a non-trivial additive character $\psi_v$ of $k_v$ at each $v$. In particular we set
\[
\psi_v(x) = \begin{cases} 
\psi_R(x) := \exp(2\pi \sqrt{-1}x) & \text{if } v = \mathbb{R}, \\
\psi_C(x) := \psi_R \circ \text{Tr}_{\mathbb{C}/\mathbb{R}}(x) & \text{if } v = \mathbb{C}.
\end{cases}
\]

Now the normalization factor $r(w, \pi_v)$ for $M(w, \pi_v)$ is given as follows (cf. [KS], §§2–3; [Ar2], §3).

1. **The case of the simple reflection $w_1$.** In this case $(U_0)_{w_1} = \hat{U}_0 \cap \hat{M}_1$ and $\hat{G}_{\alpha_1, D} = \text{Res}_{k^\times / k} SL(2)$, in the notation of [KS], §2. Thus
\[
\begin{align*}
\epsilon(s, \pi_v, r_{w_1}) := & \frac{L(0, \pi_v, r_{w_1})}{L(1, \pi_v, r_{w_1})} \varepsilon(0, \pi_v, r_{w_1}, \psi_v),
\end{align*}
\]

where
\[
\begin{align*}
L(s, \pi_v, r_{w_1}) := & \begin{cases} 
L_{k'}(s, \mu_{1,w}^{-1}) & \text{if } v \text{ is inert in } k', \\
L_{k'}(s, \mu_{1,w}^{-1})L_{k'}(s, \mu_{2,w}) & \text{if } v \text{ splits in } k',
\end{cases}
\end{align*}
\]

and
\[
\begin{align*}
\varepsilon(s, \pi_v, r_{w_1}, \psi_v) := & \begin{cases} 
\lambda(k_{w'/k_v}^{-1}, \psi_v) \varepsilon_{k'}(s, \mu_{1,w}^{-1}, \psi_v \circ \text{Tr}_{k'/k_v}) & \text{if } v \text{ is inert in } k', \\
\varepsilon_{k'}(s, \mu_{1,w}^{-1}, \psi_v) & \text{if } v \text{ splits in } k'.
\end{cases}
\end{align*}
\]

2. **The case of the simple reflection $w_2$.** We have $(U_0)_{w_2} = \hat{U}_0 \cap \hat{M}_2$ and $\hat{G}_{\alpha_2, D} = SL(2, k)$. Hence by [KS], §2,
\[
\begin{align*}
\epsilon(s, \pi_v, r_{w_2}) := & \frac{L(0, \pi_v, r_{w_2})}{L(1, \pi_v, r_{w_2})} \varepsilon(0, \pi_v, r_{w_2}, \psi_v),
\end{align*}
\]

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where
\[
L(s, \pi_v, r_{w_2}) := \begin{cases} 
L_k(s, \mu_2, w|_{k'}) & \text{if } v \text{ is inert in } k', \\
L_k(s, \mu_2, w_{1,2}) & \text{if } v \text{ splits in } k', 
\end{cases}
\]
\[
\varepsilon(s, \pi_v, r_{w_2}, \psi_v) := \begin{cases} 
\varepsilon_k(s, \mu_2, w|_{k'}, \psi_v) & \text{if } v \text{ is inert in } k', \\
\varepsilon_k(s, \mu_2, w_{1,2}, \psi_v) & \text{if } v \text{ splits in } k'. 
\end{cases}
\]

(3) The case of general \( w \in W \). Each \( w \in W \) can be written as a product of simple reflections \( w_i \) \( (i = 1, 2) \). We take such an expression \( w = w_{i_1}w_{i_2} \cdots w_{i_k} \) \( (0 \leq k \leq 4) \), which is reduced. Then the normalization factor is given by
\[
r'(w, \pi_v) = r(w_{i_1}, w_{i_2} \cdots w_{i_k}) \pi_v r(w_{i_2}, w_{i_3} \cdots w_{i_k}) \cdots r(w_{i_k}, \pi_v).
\]
This does not depend on the choice of the reduced expression of \( w \).

For \( w \in W \), write \( \text{inv}(w) \) for the set of positive roots \( \alpha \in R^+(P_0, A_0) \) such that \( w_\alpha \notin R^+(P_0, A_0) \). Then the analytic behavior of the local operators \( M(w, \pi_v) \) is stated as follows.

**Lemma 3.1.** (i) \( M(w, \pi_v)\phi_v(\pi_v), r(w, \pi_v) \) are holomorphic and non-zero at \( \pi_v \in \mathcal{P}_v \) such that \( (\text{Re} \pi_v, \alpha^\vee) > 0 \) for any \( \alpha \in \text{inv}(w) \).

(ii) The normalized intertwining operator \( N(w, \pi_v) := r(w, \pi_v)^{-1}M(w, \pi_v) \) is unitary at \( \pi_v \in \mathcal{P}_v \) with \( (\text{Re} \pi_v, \alpha^\vee) = 0 \) for any \( \alpha \in \text{inv}(w) \).

**Proof.** (i) The holomorphy of \( M(w, \pi_v)\phi_v(\pi_v) \) was proved in [BW], IV, 4.3, 4.5, if \( v \) is archimedean and in [loc. cit], XI, 2.6, if \( v \) is non-archimedean. The statement for \( r(w, \pi_v) \) is clear from its definition.

(ii) This is a special case of [Ar2], Theorem 2.1, when \( v \) is archimedean, and of [KS], Theorem 3.1, when \( v \) is non-archimedean. \( \Box \)

**3.1.2. Analytic behavior of global intertwining operators.** We go back to the global situation and define
\[
r(w, \pi) := \prod_v r(w, \pi_v), \quad N(w, \pi) := r(w, \pi)^{-1}M(w, \pi).
\]

For \( \pi = \mu_1 \otimes \mu_2 \in \mathcal{P} \) and \( \alpha \in R^+(P_0, A_0) \), define a quasi-character \( \alpha^\vee(\pi) \) by
\[
\alpha^\vee(\pi) := \mu_1^{-1}, \quad \alpha^\vee(\pi) := \mu_2|_{A_\times}, \quad \beta_1(\pi) := \mu_1(\mu_2 \circ \sigma), \quad \beta_2(\pi) := \mu_1|_{A_\times}.
\]

**Lemma 3.2.** (i) \( N(w, \pi)\phi(\pi) \) is holomorphic at \( \pi \in \mathcal{P} \) with \( (\text{Re} \pi, \alpha^\vee) > 0 \) for any \( \alpha \in \text{inv}(w) \).

(ii) \( r(w, \pi) \) has simple poles at \( \pi \in \mathcal{P} \) with
\[
\alpha^\vee(\pi) = \begin{cases} 
|_{A_\times}, & \text{if } \alpha = \alpha_1 \text{ or } \beta_1, \\
|_{A_\times}, & \text{if } \alpha = \alpha_2 \text{ or } \beta_2, 
\end{cases}
\]
for some \( \alpha \in \text{inv}(w) \), and is holomorphic at other \( \pi \in \mathcal{P} \) with \( (\text{Re} \pi, \alpha^\vee) > 0 \).

**Proof.** (i) For each \( \mathcal{P} \) and \( \phi \in P(\mathcal{M}_0, \mathcal{P}) \), we take a finite set \( S \) of places of \( k \), so that if \( v \notin S \) then \( \pi_v \in \mathcal{P}_v \) is unramified and \( \phi_v(\pi_v) \) equals \( \phi^0_v \), the \( K_v \)-fixed vector in \( \text{Ind}_{P_0(k_v)}^{G(k_v)}[\mathbf{1}_{N_0(k_v)}] \) with \( \phi^0_v(1) = 1 \). Then the Gindikin-Karpelevich formula ([La2], p. 45) gives
\[
M(w, \pi_v)\phi_v(\pi_v) = M(w, \pi_v)\phi^0_v = r(w, \pi_v)\phi^0_v \quad \text{at } v \notin S,
\]
where
\[
L(s, \pi_v, r_{w_2}) := \begin{cases} 
L_k(s, \mu_2, w|_{k'}) & \text{if } v \text{ is inert in } k', \\
L_k(s, \mu_2, w_{1,2}) & \text{if } v \text{ splits in } k', 
\end{cases}
\]
\[
\varepsilon(s, \pi_v, r_{w_2}, \psi_v) := \begin{cases} 
\varepsilon_k(s, \mu_2, w|_{k'}, \psi_v) & \text{if } v \text{ is inert in } k', \\
\varepsilon_k(s, \mu_2, w_{1,2}, \psi_v) & \text{if } v \text{ splits in } k'. 
\end{cases}
\]
and hence
\[ M(w, \pi)\phi(\pi) = \bigotimes_{v \in S} r(w, \pi_v)N(w, \pi_v)\phi_v(\pi_v) \otimes \bigotimes_{v \notin S} r(w, \pi_v)\phi^0_{v}. \]

\[ = r(w, \pi) \bigotimes_{v \in S} N(w, \pi_v)\phi_v(\pi_v) \otimes \bigotimes_{v \notin S} \phi^0_{v}. \]

\[ N(w, \pi)\phi(\pi) = \bigotimes_{v \in S} N(w, \pi_v)\phi_v(\pi_v) \otimes \bigotimes_{v \notin S} \phi^0_{v}. \]

Now the assertion follows from Lemma 3.1.

(ii) This is clear from the definition of \( r(w, \pi) \) and fundamental properties of Hecke \( L \)-functions.

3.1.3. **Singular hyperplanes.** We write \( S_{(M, \mathfrak{P})}^{h, +} \) for the set of elements in \( S_{(M, \mathfrak{P})}^{h} \), which intersect the closure of the “positive chamber”

\[ \{ \pi \in \mathfrak{P} ; \ (\text{Re} \pi, \alpha^\vee) > 0, \ for \ any \ \alpha \in R^+(P, A_M) \}. \]

Then from Lemma 3.2, we have

\[ S_{(M_0, \mathfrak{P})}^{h, +} = \{ \mathfrak{S} ; \ (\text{codim}_{\mathfrak{a}_M^*} \text{Re}(\mathfrak{S})) = 0 \} \]

3.2. Decomposition of the scalar product.

3.2.1. **Notation.** Recall some general notation from [MW], V.2.1. Let \( P = MU \) be a standard parabolic subgroup. Take \( \mathfrak{X} \in \mathfrak{C}_P \) and \( (M, \mathfrak{P}) \in \mathfrak{X} \). \( S_{(M, \mathfrak{P})}^{h, +} \) denotes the set of intersections of elements in \( S_{(M, \mathfrak{P})}^{h, +} \). For \( \mathfrak{S} \in S_{(M, \mathfrak{P})}^{h, +} \), we define its origin \( o(\mathfrak{S}) \in \mathfrak{a}_M^* \) to be \( \text{Re}(\mathfrak{S}) \cap (\text{Re}(\mathfrak{S}^0))^\perp \). Here \((\cdot)^\perp\) denotes the orthogonal complement in \( \mathfrak{a}_M^* \) with respect to the fixed \( W_M \)-invariant inner product. We write \( d_{\mathfrak{S}} \pi \) for the Lebesgue measure on \( \mathfrak{S} \) defined by the \( W_M \)-invariant metric on \( \mathfrak{a}_M^* \).

3.2.2. **The first step.** Going back to the situation \( \mathfrak{X} \in \mathfrak{C}_P \), \( (M_0, \mathfrak{P}) \in \mathfrak{X} \), we have (cf. Figure 1):

\[ S_{(M_0, \mathfrak{P})}^{h} = \{ \mathfrak{P} ; \ (\text{codim}_{\mathfrak{a}_M^*} \text{Re}(\mathfrak{S})) = 0 \} \]

\[ \mathfrak{S} = \mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4 \]

\[ \mathfrak{S}_{1,2} := \mathfrak{S}_1 \cap \mathfrak{S}_2, \mathfrak{S}_{2,4} := \mathfrak{S}_2 \cap \mathfrak{S}_4, \]

\[ \mathfrak{S}_{2,3} := \mathfrak{S}_2 \cap \mathfrak{S}_3, \mathfrak{S}_{1,3} := \mathfrak{S}_1 \cap \mathfrak{S}_3, \]

\[ \mathfrak{S}_{1,4} := \mathfrak{S}_1 \cap \mathfrak{S}_4 \]

\[ o(\mathfrak{P}) = 0, \]

\[ o(\mathfrak{S}_1) = \alpha_1, \ o(\mathfrak{S}_2) = \frac{\alpha_2}{2}, \ o(\mathfrak{S}_3) = \beta_1, \ o(\mathfrak{S}_4) = \frac{\beta_2}{2}, \]

\[ o(\mathfrak{S}_{i,j}) = \text{Re}(\mathfrak{S}_{i,j}), \ (1 \leq i < j \leq 4). \]

Recall that \( \lambda_0 \in \mathfrak{a}_0^* \) in the formula (3.1) is contained in \( \text{Re}(C_{P_0}^{h}(\mathfrak{P})) \). We write \( E_0 \) for the set \( \{ \mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4 \} \). Fix a path \( \Gamma \) from \( \lambda_0 \) to \( o(\mathfrak{P}) \) as in Figure 2. This is **general** in the sense of [MW], V.1.5. We put \( y_{\mathfrak{S}} := \text{Re}(\mathfrak{S}) \cap \Gamma \) for \( \mathfrak{S} \in E_0 \).
Figure 1. $\text{Re}(\mathcal{G})$ ($\mathcal{G} \in S_{(M_0, \psi)}^r$).

Figure 2. The path $\Gamma$. 
The first step in decomposing the scalar product is to move the integration axis from $\Re \pi = a_0$ to $\Re \pi = 0$. But before doing this, we must estimate the integrands to apply the residue theorem. For each $w \in \mathcal{W}$ we write

$$A_w(\phi, \phi')(\pi) := \langle M(w, \pi)\phi(\pi), \phi'(-w\pi) \rangle$$

$$= r(w, \pi)\langle N(w, \pi)\phi(\pi), \phi'(-w\pi) \rangle.$$ 

$r(w, \pi)$ is decomposed into a product of the form

$$r(w, \pi) = (\varepsilon(0, \chi_1) \cdots \varepsilon(0, \chi_p))^{-1} L(0, \chi_1) \cdots L(0, \chi_p) L(1, \chi_1) \cdots L(1, \chi_p)^{-1},$$

where $L(s, \chi)$ are Hecke $L$-functions and $\varepsilon(s, \chi)$ are their root numbers. We write $L_{\infty}(s, \chi)$ and $L_{fin}(s, \chi)$ for the infinite and finite factor of $L(s, \chi)$ respectively. Since $\varepsilon(s, \chi)$ are exponential functions in $s$, they are bounded in every vertical strip. It is well-known that $L_{fin}(s, \chi)$ are of the order of a polynomial in $\Im(s)$ in any vertical strip. Also a classical calculation shows that $L_{fin}(s + 1, \chi)^{-1}$ is of the order of a polynomial in $\log(\Im(s))$ in the region $\Re(s) > -\delta$ for some sufficiently small positive $\delta$. For these facts see [Ay]. Finally we use Stirling’s formula to see that $L_{\infty}(s, \chi)/L_{\infty}(s + 1, \chi)$ is slowly increasing as $\Im(s)$ tends to infinity in every vertical strip in $\Re(s) \geq 0$. Now we recall that we have taken $\phi$ and $\phi'$ to be of Paley-Wiener type, so that they decrease rapidly as $\Im(s)$ goes to infinity. Summing up, the term $A_w(\phi, \phi')(\pi)$ tends to 0 as $\Im(s)$ goes to infinity, uniformly in each vertical strip. Thus we can apply the usual residue theorem to them.

Applying this to the integrand $A(\phi, \phi')(\pi)$ when it gets across each element of $\mathcal{E}_0$, we obtain

$$\langle \theta_\phi, \theta_{\phi'} \rangle_{L^2(G(k)|G(A))} = \int_{\pi \in \mathcal{E}}, \Re \pi = 0 A(\phi, \phi')(\pi) \, d\pi$$

$$+ \sum_{i=1}^{4} \int_{\pi \in \mathcal{E}, \Re \pi = y_i} \operatorname{Res}_{\mathcal{E}_i} A(\phi, \phi')(\pi) d\Theta, \pi.$$  

(3.5)

Here $\operatorname{Res}_{\mathcal{E}_i} A(\phi, \phi')$ denotes the residue of $A(\phi, \phi')$ along $\mathcal{E}$, which is a meromorphic function on $\mathcal{E}$.

3.2.3. Calculation of the residues. To carry out the second step, we must first calculate the residues $\operatorname{Res}_{\mathcal{E}_i} A(\phi, \phi')(\pi)$ in (3.5). We write $L_k(\pi)$ and $\varepsilon_k(\pi)$ for the Dedekind zeta function and its root number of $k$, and $L_{k'}(\pi)$ and $\varepsilon_{k'}(\pi)$ for those of $k'$. $c_k$ and $c_{k'}$ denote the residues

$$c_k := \operatorname{Res}_{s=1} \frac{L_k(s)}{L_k(s + 1)\varepsilon_k(s)}, \quad c_{k'} := \operatorname{Res}_{s=1} \frac{L_{k'}(s)}{L_{k'}(s + 1)\varepsilon_{k'}(s)}.$$ 

Then the residues are calculated as:

**Proposition 3.3.** The residues $\operatorname{Res}_{\mathcal{E}_i} A(\phi, \phi')(\pi)$ are calculated as follows ($\pi \in \mathcal{E}$).

(i) $\operatorname{Res}_{\mathcal{E}_i} A(\phi, \phi')(\pi) = \sqrt{2}c_{k'} \left[ \langle N(w_1, \pi)\phi(\pi), \phi'(-w_1\pi) \rangle + \langle M(w_2, w_1, \pi)N(w_1, \pi)\phi(\pi), \phi'(-w_2w_1\pi) \rangle + \langle M(w_1w_2, w_1\pi)N(w_1, \pi)\phi(\pi), \phi'(-w_1w_2w_1\pi) \rangle + \langle M(w_2w_1w_2, w_1\pi)N(w_1, \pi)\phi(\pi), \phi'(-w_{\pi}) \rangle \right].$
Proof. The proof is straightforward and will be omitted. Note that

\[
\frac{d_{\mathcal{S}, \pi}}{d\alpha^{\nu}(\pi)} = \frac{\|\alpha\|}{\langle \alpha, \alpha^{\nu} \rangle} = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } \alpha \text{ is a short root,} \\ 1 & \text{if } \alpha \text{ is a long root.} \end{cases}
\]

3.2.4. The second step. In the second step, we move the integration axis \( \text{Re} \, \pi = y_{\mathcal{S}} \) in the right hand side of (3.5) to \( \text{Re} \, \pi = z(\mathcal{S}) \), where \( z(\mathcal{S}) \) is general but near \( o(\mathcal{S}) \) in the sense of [MW], V.2.1. Then we take the limit as \( z(\mathcal{S}) \) tends to \( o(\mathcal{S}) \).

1) Contributions from \( \text{Res}_{\mathcal{S}_1}^\Psi A(\phi, \phi')(\pi) \). We begin with the second term in the right hand side of (3.5). We take \( z(\mathcal{S}_1) \) to be \( o(\mathcal{S}_1) \) and apply the residue theorem to obtain

\[
\int_{\pi \in \mathcal{S}_1, \text{Re} \, \pi = y_{\mathcal{S}_1}} \text{Res}_{\mathcal{S}_1}^\Psi A(\phi, \phi')(\pi) \, d_{\mathcal{S}_1, \pi} = \int_{\pi \in \mathcal{S}_1, \text{Re} \, \pi = o(\mathcal{S}_1)} \text{Res}_{\mathcal{S}_1}^\Psi A(\phi, \phi')(\pi) \, d_{\mathcal{S}_1, \pi} + \text{Res}_{\mathcal{S}_{1,2}}^\Psi A(\phi, \phi')(\mathcal{S}_{1,2}) + \text{Res}_{\mathcal{S}_{1,3}}^\Psi A(\phi, \phi')(\mathcal{S}_{1,3}).
\]

We shall study each term in the right hand side.

1-i) The first term: We have to check the well-definedness of

\[
\frac{1}{2\pi \sqrt{-1}} \int_{\pi \in \mathcal{S}_1, \text{Re} \, \pi = o(\mathcal{S}_1)} \text{Res}_{\mathcal{S}_1}^\Psi A(\phi, \phi')(\pi) \, d_{\mathcal{S}_1, \pi}
\]

when \( \mathcal{S}_{1,4} = \mathcal{S}_1 \cap \mathcal{S}_4 \) is non-empty. In this case \( \mathcal{S}_{1,4} = \mu_1 \otimes \mu_2 \) satisfies \( \mu_1 \mu_2^{-1} = |\lambda|_{\mathcal{S}_1} \) and \( \mu_1|_{\mathcal{S}_1} = |\lambda| \), and, in particular, \( \sigma(\mu_1) \mu_2 = \alpha_1^\nu(w_2 w_1 \pi) = 1 \). We know
We calculate the residue $\text{Res}_{k'} \{ A(\phi, \phi')(\pi) \}$

\begin{equation}
\frac{1}{\sqrt{2\pi}} \text{Res}_{k'} \{ A(\phi, \phi')(\pi) \} \\
= \langle N(w_1, \pi)\phi(\pi), \phi'(-w_1 \pi) \rangle + r(w_2 w_1 w_2, w_1 \pi) \langle N(w_-, \pi)\phi(\pi), \phi'(-w_- \pi) \rangle \\
+ r(w_2, w_1 \pi) \left[ \langle N(w_2 w_1, \pi)\phi(\pi), \phi'(-w_2 w_1 \pi) \rangle \right. \\
+ \left. \langle M(w_1, w_2 w_1 \pi)N(w_2 w_1, \pi)\phi(\pi), \phi'(-w_1 w_2 w_1 \pi) \rangle \right].
\end{equation}

First consider

\[ r(w_2 w_1 w_2, w_1 \pi) = (r(w_2, w_1 w_2 w_1 \pi) r(w_2, w_1 \pi)) r(w_1, w_2 w_1 \pi). \]

The simple zero of the first term kills the simple pole of the second term. Hence the first two terms on the right hand side of (3.7) are well-defined at $\mathcal{S}_{1,2}$. As for the other terms, we know from [KS], Proposition 6.3, that $M(w_1, w_2 w_1 \mathcal{S}_{1,4}) = (-1) \text{Id}$. Thus the term inside the brackets in (3.7) has a simple zero at $\pi \in \mathcal{S}_{1,4}$, which kills the simple pole of $r(w_2, w_1 \pi)$. The well-definedness is proved.

(1-ii) The second term. We calculate the residue $\text{Res}_{\mathcal{S}_{1,2}} \langle \mathcal{S}_{1,1,2} \rangle A(\phi, \phi')(\pi)$. The terms $\langle N(w_1, \pi)\phi(\pi), \phi'(-w_1 \pi) \rangle$, $\langle M(w_2, w_1 \pi)N(w_1, \pi)\phi(\pi), \phi'(-w_2 w_1 \pi) \rangle$, $\langle M(w_1, w_2 \pi)N(w_1, \pi)\phi(\pi), \phi(-w_1 w_2 w_1 \pi) \rangle$

are holomorphic at $\mathcal{S}_{1,2}$, while

\[
\text{Res}_{\mathcal{S}_{1,2}} \langle M(w_2 w_1 w_2, w_1 \pi)N(w_1, \pi)\phi(\pi), \phi'(-w_- \pi) \rangle \\
= \frac{d_{\mathcal{S}_{1,2}}}{d_{\mathcal{S}_{1,2}}} \langle N(w_2, w_1 w_2 w_1 \mathcal{S}_{1,2})M(w_1 w_2, w_1 \mathcal{S}_{1,2})N(w_1, \mathcal{S}_{1,2}) \times \phi(\mathcal{S}_{1,2}), \phi'(-w_- \mathcal{S}_{1,2}) \rangle \\
= \sqrt{2} c_k \langle N(w_2, w_1 w_2 w_1 \mathcal{S}_{1,2})M(w_1 w_2, w_1 \mathcal{S}_{1,2})N(w_1, \mathcal{S}_{1,2}) \times \phi(\mathcal{S}_{1,2}), \phi'(-w_- \mathcal{S}_{1,2}) \rangle.
\]

Hence we have

\begin{equation}
\text{Res}_{\mathcal{S}_{1,2}} \text{Res}_{\mathcal{S}_{1,1}} A(\phi, \phi')(\pi) = \sqrt{2} c_k c_{k'} \langle N(w_2, w_1 w_2 w_1 \mathcal{S}_{1,2})M(w_1 w_2, w_1 \mathcal{S}_{1,2})N(w_1, \mathcal{S}_{1,2}) \phi(\mathcal{S}_{1,2}), \phi'(\pi - w \mathcal{S}_{1,2}) \rangle.
\end{equation}

(1-iii) The third term. We calculate $\text{Res}_{\mathcal{S}_{1,2}} \text{Res}_{\mathcal{S}_{1,1}} A(\phi, \phi')(\pi)$. Among the terms on the right hand side of Proposition 3.3 (i),

\[ \langle N(w_1, \pi)\phi(\pi), \phi'(-w_1 \pi) \rangle, \langle M(w_2, w_1 \pi)N(w_1, \pi)\phi(\pi), \phi'(-w_2 w_1 \pi) \rangle \]

are holomorphic. As for the other two terms, we have the following two possibilities. Note that, writing $\mathcal{S}_{1,3} = \mu_1 \otimes \mu_2$, we have $\mu_2 \sigma(\mu_2) = \mu_2 \circ \eta_{k'/k} = 1$.

Case 1. $\mu_2 \land = \eta_{k'/k}$. In this case, noting that $M(w_2, w_1 w_2 w_1 \mathcal{S}_{1,3}) = \text{Id}$ by [KS], Proposition 6.3, we have

\begin{equation}
\text{Res}_{\mathcal{S}_{1,3}} \langle M(w_1 w_2, w_1 \pi)N(w_1, \pi)\phi(\pi), \phi'(-w_1 w_2 w_1 \pi) \rangle \\
= \sqrt{2} c_k' \langle N(w_1, w_2 w_1 \mathcal{S}_{1,3})M(w_2, w_1 \mathcal{S}_{1,3})N(w_1, \mathcal{S}_{1,3})\phi(\mathcal{S}_{1,3}), \phi'(\pi - w \mathcal{S}_{1,3}) \rangle,
\end{equation}
and

\[(3.10)\]
\[
\text{Res}^{E_1}_1 (M(w_2 w_1 w_2, w_1 \pi) N(w_1, \pi) \phi(\pi), \phi'(-w_\pi))
\]
\[
= \sqrt{2} c_k \langle N(w_1, w_2 w_1 \mathcal{S}_{1,3}) M(w_2, w_1 \mathcal{S}_{1,3}) N(w_1, \mathcal{S}_{1,3}) \phi(\mathcal{S}_{1,3}), \phi'(-w_\mathcal{S}_{1,3}) \rangle
\]

**Case 2.** $\mu_2|_{A^x}$ is trivial. The formula (3.9) still holds. But in this case $M(w_2, w_1 w_2 w_1 \mathcal{S}_{1,3}) = (-1) \text{Id}$ by [KS], Proposition 6.3. Thus the right hand side of the formula (3.10) must be multiplied by $-1$.

In both cases, we take summations of the two terms, and get

\[(3.11)\]
\[
\text{Res}^{E_1}_{1,3} \text{Res}^{P}_{E_1} A(\phi, \phi')(\pi)
\]
\[
= \begin{cases} 
4 c_k \langle N(w_1, w_2 w_1 \mathcal{S}_{1,3}) M(w_2, w_1 \mathcal{S}_{1,3}) N(w_1, \mathcal{S}_{1,3}) \phi(\mathcal{S}_{1,3}), \phi'(-w_\mathcal{S}_{1,3}) \rangle & \text{if } \mu_2|_{A^x} = \eta_k/k, \\
0 & \text{if } \mu_2|_{A^x} \text{ is trivial.}
\end{cases}
\]

(2) **Contributions from $\text{Res}^{P}_{E_2} A(\phi, \phi')(\pi)$.** Next comes the third term in the right hand side of (3.5). We take $z(\mathcal{S}_2)$ to be $o(\mathcal{S}_2)$ and apply the residue theorem to see that

\[(3.12)\]
\[
\int_{\pi \in \mathcal{S}_2, \text{Re } \pi = y e_2} \text{Res}^{P}_{E_2} A(\phi, \phi')(\pi) d_\mathcal{S}_2 \pi
\]
\[
= \int_{\pi \in \mathcal{S}_2, \text{Re } \pi = o(\mathcal{S}_2)} \text{Res}^{P}_{E_2} A(\phi, \phi')(\pi) d_\mathcal{S}_2 \pi + \text{Res}^{E_2}_{1,3} \text{Res}^{P}_{E_2} A(\phi, \phi')(\pi)
\]
\[
+ \text{Res}^{E_2}_{2,4} \text{Res}^{P}_{E_2} A(\phi, \phi')(\pi)
\]

We study each term on the right hand side in turn. The first term has no problem.

(2-i) **The second term.** We calculate the residue $\text{Res}^{E_2}_{2,3} \text{Res}^{P}_{E_2} A(\phi, \phi')(\pi)$. First note that $E_{2,3} = \mu_1 \otimes \mu_2$ satisfies $\mu_2|_{A^x} = |A$ and $\mu_1 \sigma(\mu_2) = |A^x$, and hence $\sigma(\mu_1 \mu_2) = \alpha^2_1(w_2 w_1 w_2\pi)$ is trivial. Among the terms in the right hand side of Proposition 3.3 (ii), $\langle N(w_2, \pi) \phi(\pi), \phi'(-w_\pi) \rangle$ is holomorphic at $E_{2,3}$, and

\[
\text{Res}^{E_2}_{2,3} (M(w_1, w_2\pi) N(w_2, \pi) \phi(\pi), \phi'(-w_1 w_2\pi))
\]
\[
= 2 c_k \langle d_\mathcal{S}_2 \pi \rangle \langle N(w_1, w_2, \mathcal{S}_{2,3}) \phi(\mathcal{S}_{2,3}), \phi'(-w_1 w_2\mathcal{S}_{2,3}) \rangle
\]
\[
= 2 c_k \langle N(w_1, \mathcal{S}_{2,3}), \phi'(-w_1 w_2\mathcal{S}_{2,3}) \rangle.
\]

As for the other two terms, we have

\[
\text{Res}^{E_2}_{2,3} \left[ \langle M(w_2 w_1, w_2\pi) N(w_2, \pi) \phi(\pi), \phi'(-w_2 w_1 w_2\pi) \rangle \right.
\]
\[
+ \langle M(w_1 w_2 w_1, w_2\pi) N(w_2, \pi) \phi(\pi), \phi'(-w_\pi) \rangle \right]
\]
\[
= 2 c_k \lim_{\pi \to \mathcal{S}_{2,3}} \int_{\pi \in \mathcal{S}_2} \left[ r(w_2, w_1 w_2\pi) (M(w_1, w_2 w_1 w_2\pi) + \text{Id}) \right]
\]
\[
\times \langle N(w_2 w_1, \mathcal{S}_{2,3}) \phi(\mathcal{S}_{2,3}), \phi'(-w_\mathcal{S}_{2,3}) \rangle.
\]
From [KS], Proposition 6.3, \(M(w_1, w_2 w_1 w_2 \pi) + \text{Id}\) has a simple zero at \(S_{2,3}\), which kills the simple pole of \(r(w_2, w_1 w_2 \pi)\). Thus the limit exists. Summing up, we have

\[
(3.13) \quad \text{Res}_{S_{2,3}} \text{Res}_{S_2} A(\phi, \phi')(\pi) \\
= 2c_k c_k' \left[ \langle N(w_1 w_2, S_{2,3}) \phi(S_{2,3}), \phi'(-w_1 w_2 S_{2,3}) \rangle \\
+ \lim_{\pi \to \phi_{S_{2,3}}} \left[ r(w_2, w_1 w_2 \pi) \left( M(w_1, w_2 w_1 w_2 \pi) + \text{Id} \right) \right] \\
\times \langle N(w_2 w_1 w_2, S_{2,3}) \phi(S_{2,3}), \phi'(-w_2 - S_{2,3}) \rangle \right]
\]

(2-ii) The third term. Calculate the residue \(\text{Res}_{S_{2,4}} \text{Res}_{S_2} A(\phi, \phi')(\pi)\). \(S_{2,4} = \mu_1 \otimes \mu_2\) satisfies \(\mu_1|_{\lambda} = \mu_2|_{\lambda} = |_{\lambda}\). Since the case \(\mu_1 = \mu_2\) has already been treated in (2-i), we assume \(\mu_1 \neq \mu_2\). Then

\[
\text{Res}_{S_{2,4}} \langle N(w_2, \pi) \phi(\pi), \phi'(-w_2 - S_{2,4}) \rangle = 0, \\
\text{Res}_{S_{2,4}} \langle M(w_1, w_2 \pi) N(w_2, \pi) \phi(\pi), \phi'(-w_1 w_2 \pi) \rangle = 0, \\
\text{Res}_{S_{2,4}} \langle M(w_2 w_1, w_2 \pi) N(w_2, \pi) \phi(\pi), \phi'(-w_2 w_1 w_2 \pi) \rangle \\
= c_k \langle N(w_2, w_1 w_2 S_{2,4}) M(w_1, w_2 S_{2,4}) N(w_2, S_{2,4}) \phi(S_{2,4}), \phi'(-w_2 w_1 S_{2,4}), \phi'(-w_2 - S_{2,4}) \rangle, \\
\text{Res}_{S_{2,4}} \langle M(w_1 w_2 w_1, w_2 \pi) N(w_2, \pi) \phi(\pi), \phi'(-w_2 w_1 w_2 \pi) \rangle \\
= c_k \langle M(w_1, w_2 w_1 w_2 S_{2,4}) N(w_2, w_1 w_2 S_{2,4}) M(w_1, w_2 S_{2,4}) N(w_2, S_{2,4}) \phi(S_{2,4}), \phi'(-w_2 - S_{2,4}) \rangle.
\]

Hence

\[
(3.14) \quad \text{Res}_{S_{2,4}} \text{Res}_{S_2} A(\phi, \phi')(\pi) \\
= c_k^2 \langle N(w_2, w_1 w_2 S_{2,4}) M(w_1, w_2 S_{2,4}) N(w_2, S_{2,4}) \phi(S_{2,4}), \phi'(-w_2 w_1 S_{2,4}) \rangle \\
+ c_k \langle M(w_1, w_2 w_1 w_2 S_{2,4}) N(w_2, w_1 w_2 S_{2,4}) M(w_1, w_2 S_{2,4}) N(w_2, S_{2,4}) \phi(S_{2,4}), \phi'(-w_2 - S_{2,4}) \rangle.
\]

(3) Contributions from \(\text{Res}_{S_3} A(\phi, \phi')(\pi)\). We choose \(z(S_3)\) to be any point on the segment tying \(y_{S_3}\) and \(o(S_3)\), but not \(o(S_3)\) itself. We use the principal value integration theorem, at first formally, to get

\[
(3.15) \quad \int_{\pi \in S_3, \text{Re} \pi = y_{S_3}} \text{Res}_{S_3} A(\phi, \phi')(\pi) d_{S_3} \pi \\
= \frac{1}{2} \int_{\pi \in S_3, \text{Re} \pi = z(S_3)} \left[ \text{Res}_{S_3} A(\phi, \phi')(\pi) + \text{Res}_{S_3} A(\phi, \phi')(w_1 \pi) \right] d_{S_3} \pi \\
+ \frac{1}{2} \text{Res}_{S_{2,3}} \text{Res}_{S_3} A(\phi, \phi')(\pi).
\]

(3-i) The first term. First we need to check that the limit

\[
(3.16) \quad \lim_{z(S_3) \to o(S_3)} \int_{\pi \in S_3, \text{Re} \pi = z(S_3)} \frac{1}{2} \left[ \text{Res}_{S_3} A(\phi, \phi')(\pi) + \text{Res}_{S_3} A(\phi, \phi')(w_1 \pi) \right] d_{S_3} \pi
\]
We calculate the residue \( \text{Res}^\Psi_{\mathcal{S}_3}(\phi, \phi')(\pi) + \text{Res}^\Psi_{\mathcal{S}_3}(\phi, \phi')(w_1\pi) \)

(I) \[ = \sqrt{2}c_k \left[ (N(w_1, w_2\pi)(M(w_2, \pi)\phi(\pi) + M(w_2w_1, w_1\pi)\phi(w_1\pi)), \phi'(-w_1w_2\pi)) \right. \]

(II) \[ + \langle N(w_1, w_2w_1\pi)(M(w_2, w_1\pi)\phi(w_1\pi) + M(w_2w_1, \pi)\phi(\pi)), \phi'(-w_1w_2w_1\pi) \rangle \]

(III) \[ + \langle (M(w_2, w_1w_2\pi)N(w_1, w_2\pi)M(w_2, \pi)\phi(\pi), \phi'(-w_2w_1w_2\pi) \rangle \]

+ \langle (M(w_1w_2, w_1w_2w_1\pi)N(w_1, w_2w_1\pi)M(w_2, w_1\pi)\phi(w_1\pi), \phi'(-w_2w_1w_2\pi) \rangle \]

(IV) \[ + \langle (M(w_1w_2, w_1w_2w_1\pi)N(w_1, w_2\pi)M(w_2, \pi)\phi(\pi), \phi'(-w_2w_1w_2\pi) \rangle \]

+ \langle (M(w_2, w_1w_2w_1\pi)N(w_1, w_2w_1\pi)M(w_2, w_1\pi)\phi(w_1\pi), \phi'(-w_2w_1w_2\pi) \rangle \]

As for (I), we have

\[ M(w_2, \pi)\phi(\pi) + M(w_2w_1, w_1\pi)\phi(w_1\pi) \]

\[ = r(w_2, \pi)N(w_2, \pi)[\phi(\pi) + M(w_1, w_1\pi)\phi(w_1\pi)]. \]

Since \( \alpha_1(\mathcal{S}_{2,3}) = 1 \), we see from [KS] Proposition 6.3 that

\[ \phi(\mathcal{S}_{2,3}) + M(w_1, w_1\mathcal{S}_{2,3})\phi(\mathcal{S}_{2,3}) = 0. \]

This kills the simple pole of \( r(w_2, \pi) \), and (I) is well-defined at \( \mathcal{S}_{2,3} \). (II) can be treated in the same manner. The terms (III) and (IV) must be handled together. (III) becomes

\[ \langle (r(w_2, w_1w_2\pi)r(w_2, \pi)N(w_2w_1w_2, w_2\pi)\phi(\pi) + r(w_1w_2, w_1w_2w_1\pi)r(w_2, w_1\pi) \]

\[ N(w_2, w_2\pi)\phi(w_2\pi), \phi'(-w_2w_1w_2\pi) \rangle \]

\[ = \langle r(w_2, w_1w_2\pi)r(w_2, \pi)N(w_2w_1w_2, w_2\pi)(\phi(\pi) + M(w_1, w_1\pi)\phi(w_1\pi)), \phi'(-w_2w_1w_2\pi) \rangle, \]

while (IV) becomes

\[ \langle r(w_1w_2, w_1w_2\pi)r(w_2, \pi)N(w_2w_1w_2, w_2\pi)\phi(w_2\pi), \phi'(-w_2w_1w_2\pi) \rangle \]

\[ = \langle M(w_1, w_2w_1w_2\pi)r(w_2, w_1w_2\pi)r(w_2, \pi)N(w_2w_1w_2, \pi) \]

\[ \times (\phi(\pi) + M(w_1, w_1\pi)\phi(w_1\pi)), \phi'(-w_2w_1w_2\pi) \rangle \]

We now apply Proposition 6.3 of [KS] to \( M(w_1, w_1\pi) \) and \( M(w_1, w_2w_1w_2\pi) \) to see that (III)+(IV) is well-defined at \( \mathcal{S}_{2,3} \). Thus the limit (3.16) exists.

(3-ii) The second term: We calculate the residue \( \text{Res}^\Psi_{\mathcal{S}_{2,3}} \text{Res}^\Psi_{\mathcal{S}_3}(\phi, \phi')(\pi) \). Among those terms in the right hand side of Proposition 3.3 (iii), the first two
have the residue
\[
\text{Res}^{S_3}_{\bar{G}_2,3} \left[ (N(w_1, w_2 \pi)M(w_2, \pi)\phi(\pi), \phi'(-w_1 w_2 \pi)) + \langle N(w_1, w_2 w_1 \pi)M(w_2 w_1, \pi)\phi(\pi), \phi'(-w_1 w_2 w_1 \pi) \rangle \right] = \frac{d\phi}{d\alpha^2}(\pi) c_k \left[ (N(w_1 w_2, G_{2,3})\phi(G_{2,3}), \phi'(-w_1 w_2 \bar{G}_{2,3})) \right.
\]
\[
\left. - \langle N(w_1 w_2, w_1 G_{2,3})M(w_1, G_{2,3})\phi(G_{2,3}), \phi'(-w_1 w_2 \bar{G}_{2,3}) \rangle \right]
\]
\[
= -2\sqrt{2}c_k \langle N(w_1 w_2, G_{2,3})\phi(G_{2,3}), \phi'(-w_1 w_2 \bar{G}_{2,3}) \rangle.
\]
Here we have used $w_1 G_{2,3} = G_{2,3}$ and $M(w_1, G_{2,3}) = (-1) \text{Id}$ (Proposition 6.3 in [KS]). As for the other two terms, we have
\[
\text{Res}^{S_3}_{\bar{G}_2,3} \left[ (M(w_2, w_1 w_2 \pi)N(w_1, w_2 \pi)M(w_2, \pi)\phi(\pi), \phi'(-w_2 w_1 \pi)) + \langle M(w_1, w_2 w_1 \pi)N(w_1, w_2 \pi)M(w_2, \pi)\phi(\pi), \phi'(-w_2 \pi) \rangle \right] = \frac{d\phi}{d\alpha^2}(\pi) c_k \left[ \lim_{\pi \to G_{2,3}} \left[ r(w_2, w_1 w_2 \pi)(M(w_1, w_2 w_1 \pi) + \text{Id}) \right] \right.
\]
\[
\times \langle N(w_2 w_1 w_2, G_{2,3})\phi(G_{2,3}), \phi'(-w_2 \bar{G}_{2,3}) \rangle
\]
\[
= 2\sqrt{2}c_k \lim_{\pi \to G_{2,3}} \left[ r(w_2, w_1 w_2 \pi)(M(w_1, w_2 w_1 \pi) + \text{Id}) \right]
\]
\[
\times \langle N(w_2 w_1 w_2, G_{2,3})\phi(G_{2,3}), \phi'(-w_2 \bar{G}_{2,3}) \rangle.
\]
Note that $M(w_1, w_2 w_1 w_2 G_{2,3}) = (-1) \text{Id}$ by [KS], Proposition 6.3; hence the limit in the above formula exists. Also we have used
\[
\lim_{\pi \to G_{2,3}} \left( \right) = 2 \lim_{\pi \to G_{2,3}} \left( \right).
\]
Now we conclude
\[
(3.17)
\]
\[
\frac{1}{2} \text{Res}^S_{\bar{G}_2,3} \text{Res}^\mathbb{P}_{G_2,3} A(\phi, \phi')(\pi)
\]
\[
= -2c_k c_k \left[ \lim_{\pi \to G_{2,3}} \left[ r(w_2, w_1 w_2 \pi)(M(w_1, w_2 w_1 w_2 \pi) + \text{Id}) \right] \right.
\]
\[
\times \langle N(w_2 w_1 w_2, G_{2,3})\phi(G_{2,3}), \phi'(-w_2 \bar{G}_{2,3}) \rangle
\]
\[
\left. + \langle N(w_1 w_2, G_{2,3})\phi(G_{2,3}), \phi'(-w_1 w_2 \bar{G}_{2,3}) \rangle \right].
\]
(4) Contributions from \( \text{Res}_{\mathcal{E}_4}^\mathfrak{P} A(\phi, \phi')(\pi) \). \( z(\mathcal{E}_4) \) is chosen to be \( \mathfrak{o}(\mathcal{E}_4) \). From Cauchy’s integration theorem, we have

\[
\int_{\pi \in \mathcal{E}_4, \Re \pi = \mathfrak{o}(\mathcal{E}_4)} \text{Res}_{\mathcal{E}_4}^\mathfrak{P} A(\phi, \phi')(\pi) \, d\mathcal{E}_4 \pi = \int_{\pi \in \mathcal{E}_4, \Re \pi = \mathfrak{o}(\mathcal{E}_4)} \text{Res}_{\mathcal{E}_4}^\mathfrak{P} A(\phi, \phi')(\pi) \, d\mathcal{E}_4 \pi.
\]

(3.18)

3.2.5. The final form of the inner product formula. Now we combine (3.6), (3.19), (3.20), (3.12), (3.13), (3.21), (3.15), (3.17) and (3.18) and apply them to (3.5).

**Theorem 3.4.** Take \( \mathfrak{X} \in \mathcal{E}_{P_0} \) and \( (M_0, \mathfrak{P}) \in \mathfrak{X} \). Then the \( L^2 \)-inner product of \( \theta_\phi \) \( (\phi \in P(M_0, \mathfrak{P})) \) and \( \theta_{\phi'} \) \( (\phi' \in P_\mathfrak{X}) \) is given by

\[
\langle \theta_\phi, \theta_{\phi'} \rangle_{L^2(G(\mathfrak{k}) \backslash G(\mathfrak{a}))) = \int_{\pi \in \mathfrak{P}, \Re \pi = 0} A(\phi, \phi')(\pi) \, d\pi
\]

\[
+ \int_{\pi \in \mathcal{E}_1, \Re \pi = \mathfrak{o}(\mathcal{E}_1)} \text{Res}_{\mathcal{E}_1}^\mathfrak{P} A(\phi, \phi')(\pi) \, d\mathcal{E}_1 \pi
\]

\[
+ \int_{\pi \in \mathcal{E}_2, \Re \pi = \mathfrak{o}(\mathcal{E}_2)} \text{Res}_{\mathcal{E}_2}^\mathfrak{P} A(\phi, \phi')(\pi) \, d\mathcal{E}_2 \pi
\]

\[
+ \lim_{\mathfrak{z}(\mathcal{E}_3) \to \mathfrak{o}(\mathcal{E}_3)} \frac{1}{2} \sum_{w=1,w_1} \int_{\pi \in \mathcal{E}_3, \Re \pi = \mathfrak{z}(\mathcal{E}_3)} \text{Res}_{\mathcal{E}_3}^\mathfrak{P} A(\phi, \phi')(w, \pi) \, d\mathcal{E}_3 \pi
\]

\[
+ \int_{\pi \in \mathcal{E}_4, \Re \pi = \mathfrak{o}(\mathcal{E}_4)} \text{Res}_{\mathcal{E}_4}^\mathfrak{P} A(\phi, \phi')(\pi) \, d\mathcal{E}_4 \pi
\]

(3.19)

\[
+ 2c_k c_{k'} \langle N(w_2, w_1 w_2 \mathcal{E}_{1,2}) M(w_1 w_2, w_1 \mathcal{E}_{1,2}) N(w_1, \mathcal{E}_{1,2}) \phi(\mathcal{E}_{1,2}), \phi'(-w_2 - \mathcal{E}_{1,2}) \rangle
\]

(3.20)

\[
+ 4c_{k_1}^2 \langle N(w_1, w_2 w_1 \mathcal{E}_{1,3}) M(w_2, w_1 \mathcal{E}_{1,3}) N(w_1, \mathcal{E}_{1,3}) \phi(\mathcal{E}_{1,3}), \phi'(-w_2 - \mathcal{E}_{1,3}) \rangle
\]

(3.21)

\[
+ 4c_{k_2}^2 \langle [N(w_2, w_1 w_2 \mathcal{E}_{2,4}) M(w_1, w_2 \mathcal{E}_{2,4}) N(w_2, \mathcal{E}_{2,4}) \phi(\mathcal{E}_{2,4}), \phi'(-w_2 w_1 - \mathcal{E}_{2,4})] + \langle M(w_1, w_2 w_1 \mathcal{E}_{2,4}) N(w_1, w_2 \mathcal{E}_{2,4}) M(w_1, w_2 \mathcal{E}_{2,4}) N(w_2, \mathcal{E}_{2,4}) \phi(\mathcal{E}_{2,4}), \phi'(-w_2 - \mathcal{E}_{2,4}) \rangle \rangle.
\]

Here in the term (3.21), \( \mathcal{E}_{2,4} \neq \mathcal{E}_{2,3} \).

3.3. The residual spectrum from \( P_0 \). In this subsection, we describe the contributions of cuspidal data attached to \( P_0 \) to the residual spectrum. Only the terms (3.19), (3.20) and (3.21) contribute to the residual discrete spectrum. To determine these contributions, we shall study the images of intertwining operators appearing in these terms. We need some results from [KS] and [LeZ].

3.3.1. Review of the results of Kudla, Sweet, Lee and Zhu. Their results describe the complete irreducible decomposition of certain degenerate principal series representations of \( U(n, n) \) (inert case) and \( GL(2n) \) (split case) over a local field. Though our review concentrates on the case \( n = 2 \), their results cover the general rank case.

(1) The non-archimedean inert case. Take a non-archimedean place \( v \) of \( k \) which is inert in \( k' \), and let \( w \) be the place of \( k' \) lying over \( v \). For each character
\( \chi_v : k_w^\times \to \mathbb{C} \) and \( s \in \mathbb{C} \), we write \( I(s, \chi_v) \) for the degenerate principal series representation:

\[
I(s, \chi_v) := \text{Ind}_{P_1(k_v)}^{G(k_v)}((\chi_v \circ \det)| \det |_{k_w} \otimes 1_{U_1(k_v)}).
\]

**Proposition 3.5** (Theorem 1.1 in [KSw]). (1) \( I(s, \chi_v) \) is reducible if and only if \( \chi_v \circ N_{k_v/k_w} \) is trivial.

(2) If \( \chi_v |_{k_v^\times} = 1 \), then the points of reducibility are \( s_0 = -1, 0, 1 \).

(3) If \( \chi_v |_{k_v^\times} = \eta_{k_v'/k_v} \), then the points of reducibility are \( s_0 = -1/2, 1/2 \). Here \( \eta_{k_v'/k_v} \) is the quadratic character of \( k_v^\times \) corresponding to \( k_v'/k_v \) by class field theory.

To describe the irreducible constituents of \( I(s, \chi_v) \) at its reducible points, we need oscillator representations for unitary dual pairs. We write a reducible point as \( s_0 = (m - 2)/2 \), where \( m \) equals one of \( 0, 1, 2, 3, 4 \). We first consider the case \( m \neq 0 \). The \( m \)-dimensional non-degenerate Hermitian spaces \( (V, \langle \cdot, \cdot \rangle_V) \) over \( k_w \) are classified as follows.

1. When \( m = 1 \), there are, up to equivalence, two possibilities:

\[
V_{1,i} := k_w' \text{ if } Q_1 = 1 \text{ and } Q_2 = \gamma \in k_v^\times - N_{k_v'/k_v}(k_v^\times) \text{.}
\]

where \( Q_1 = 1 \) and \( Q_2 = \gamma \in k_v^\times - N_{k_v'/k_v}(k_v^\times) \).

2. When \( m = 2 \), there are two possibilities up to equivalence

\[
V_{2,i} := k_w'^{\oplus 2} \text{ if } Q_1 = 1 \text{ and } Q_2 = \gamma \in k_v^\times - N_{k_v'/k_v}(k_v^\times) \text{.}
\]

where

\[
Q_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_2 := \begin{pmatrix} \gamma & 0 \\ 0 & -1 \end{pmatrix} \text{.}
\]

and \( \gamma \) is the same as in (1).

3. If \( 3 \leq m \leq 4 \), then there are two possibilities \( V_{m,1} \) and \( V_{m,2} \), where

\[
V_{m,i} := \begin{cases} V_{1,i} \oplus V_{2,1} & \text{if } m = 3, \\ V_{2,i} \oplus V_{2,1} & \text{if } m = 4, \end{cases} \quad (i = 1, 2) \text{.}
\]

We write \( U(V_{m,i}) \) for the unitary group over \( k_v \) attached to \( (V_{m,i}, \langle \cdot, \cdot \rangle_{V_{m,i}}) \).

\( (W, \langle \cdot, \cdot \rangle_W) \) denotes the anti-Hermitian space attached to \( G_v \):

\[
W := k_w'^{\oplus 4}, \quad (x,y)_W := xJ_2(\gamma y) \text{.}
\]

Then we have the symplectic space

\[
W_{m,i} := W \otimes k_v' \text{ if } m = 3 \text{, and } W_{m,i} := \text{Tr}_{k_v'/k_v}(W \otimes \tilde{\sigma}(\cdot)), \text{if } m = 4.
\]

and write \( Sp(W_{m,i}) \) for its symplectic group. \( G_v \times U(V_{m,i}) \) is a dual reductive pair in \( Sp(W_{m,i}) \).

Fix a non-trivial character \( \psi_v : k_v \to \mathbb{C}^1 \). Then we have the metaplectic group

\[
1 \to \mathbb{C}^1 \to Mp(W_{m,i})_v \to Sp(W_{m,i}, k_v) \to 1, \text{ and its oscillator representation } \omega_{\psi_v} \text{. We identify } Mp(W_{m,i})_v \text{ with } Sp(W_{m,i}, k_v) \times \mathbb{C}^1 \text{ as a set. The multiplication law is given by}
\]

\[
(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2c_v(g_1, g_2)) \quad (g_i \in Sp(W_{m,i}, k_v), \epsilon_i \in \mathbb{C}^1),
\]

where \( c_v(g_1, g_2) \) is the metaplectic two cocycle calculated in [P].
Note that our $\chi_v$ satisfies $\chi_v|_{k_v^\times} = \eta_{k_v}^{|m|}$. Then from [Ku], Theorem 3.1, we have the splitting
\[
\ell_{\chi_v} : G(k_v) \times U(V_{m,i}, k_v) \longrightarrow Mp(W_{m,i})_v
\]
attached to $\chi_v$. Hence we can define the auxiliary oscillator representation $\omega'_{\psi_v, \chi_v}$ of $G(k_v) \times U(V_{m,i}, k_v)$ by $\omega'_{\psi_v, \chi_v} := \omega_{\psi_v} \circ \ell_{\chi_v}$. For simplicity, we twist the action of $U(V_{m,i}, k_v)$ by the character $(\chi_v \circ \det)^{-1}$ and obtain the oscillator representation $\omega_{\psi_v, \chi_v}$. Some explicit formulae for its Schrödinger model $S(V_{m,i})$ are given by
\[
\omega_{\psi_v, \chi_v} \left( \begin{pmatrix} A & 0 \\ 0 & \tilde{\sigma}(A^{-1}) \end{pmatrix} \right) \Phi(x) = \chi_v(\det A)|\det A|^{m/2} \Phi(x, A) \quad (A \in GL(2, k_v')).
\]
(3.22)
\[
\omega_{\psi_v, \chi_v} \left( \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \right) \Phi(x) = \psi_v(\text{tr}(x, x)_{V_{m,i}} B))\Phi(x) \quad (B \in \text{Her}_2(k_v'/k_v)).
\]
(3.23)
\[
\omega_{\psi_v, \chi_v} \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \Phi(x) = \frac{1}{I_{m,i}} \int_{V_{m,i}^\oplus} \Phi(y) \psi_v(\text{Tr}_{k_v'/k_v}(x, y)_{V_{m,i}})) \, dy,
\]
(3.24)
\[
\omega_{\psi_v, \chi_v}(h)\Phi(x) = \Phi(h^{-1} x) \quad (h \in H(k_v)).
\]
(3.25)
Here, $(x, y)_{V_{m,i}}$ denotes the matrix
\[
\begin{pmatrix} \langle x_1, y_1 \rangle_{V_{m,i}} & \langle x_1, y_2 \rangle_{V_{m,i}} \\ \langle x_2, y_1 \rangle_{V_{m,i}} & \langle x_2, y_2 \rangle_{V_{m,i}} \end{pmatrix}, \quad \text{for } x = (x_1, x_2), \ y = (y_1, y_2) \in V_{m,i}^\oplus,
\]
and the measure in (3.24) is chosen to be self-dual with respect to the pairing $\text{tr}(\cdot, \cdot)_{V_{m,i}}$. $\text{Her}_2(k_v'/k_v)$ denotes the space of $2 \times 2$ Hermitian matrices over $k_v'$, and $\gamma_{m,i}$ is the usual Weil constant for $\text{Tr}_{k_v'/k_v}(\cdot, \cdot)_{V_{m,i}}$ with respect to $\psi_v$ (cf. [P]).

Now we write $R(V_{m,i}, \chi_v)$ for the image of the map
\[
S(V_{m,i}^\oplus) \ni \Phi \longrightarrow (g \rightarrow \omega_{\psi_v, \chi_v}(g)\Phi(0)) \in I(s_0, \chi_v)
\]
(recall that $s_0 = (m - 2)/2$). Then Rallis’s coinvariant theorem ([Ra], Theorem II.1.1) extended to the general case in [MWB], Chapter 3-IV, Théorème 7, asserts that
\[
R(V_{m,i}, \chi_v) \simeq (\omega_{\psi_v, \chi_v})_{U(V_{m,i}, k_v)} \quad \text{(the $U(V_{m,i}, k_v)$-coinvariant space)}.
\]
Next we consider the case $m = 0$. $U(1)_{k_v'/k_v}$ denotes the unitary group
\[
U(1)_{k_v'/k_v} := \{ g \in \text{Res}_{k_v'/k_v} G_m : \tilde{\sigma}(g)^{-1} = g \}.
\]
Then we have the $k$-homomorphism $\det : G \rightarrow U(1)_{k_v'/k_v}$. When $m = 0$, $\chi_v$ satisfies $\chi_v|_{k_v^\times} = 1$. This allows us to define a character $\chi_v'$ of $U(1, k_v)_{k_v'/k_v}$ by $\chi_v'(x\tilde{\sigma}(x)^{-1}) := \chi_v(x)$. (Note that every $u \in U(1, k_v)_{k_v'/k_v}$ can be written in the form $x\tilde{\sigma}(x)^{-1}$ for some $x \in k_v^\times$ by Hilbert’s Theorem 90.) We write $\chi_v'$ for the 1-dimensional representation $\chi_v' \circ \det$ of $G(k_v)$. We define $R(0, \chi_v') := \chi_v'^G$.

Proposition 3.6 (Theorem 1.2 and Proposition 5.8 in [KSW]). (1) The representation $R(V_{4,1}, \chi_v)$ equals $I(1, \chi_v)$, and $R(V_{4,2}, \chi_v)$ is the unique irreducible submodule of $I(1, \chi_v')$. The normalized intertwining operator $N(w_2w_1w_2, (\chi_v' \circ \det) |\det| w)$ has image $R(0, \chi_v')$ and kernel $R(V_{4,2}, \chi_v')$. 
(2) $R(V_{1,1}, \chi_v)$ and $R(V_{1,2}, \chi_v)$ are two inequivalent irreducible submodules of $I(-1/2, \chi_v)$. $R(V_{2,1}, \chi_v)$ and $R(V_{3,2}, \chi_v)$ are two distinct maximal submodules of $I(1/2, \chi_v)$. The normalized operator $N(w_2 w_1 w_2, (\chi_v \circ \det)|\det|^{i/2})$ induces isomorphisms

$$R(V_{3,i}, \chi_v)/(R(V_{3,1}, \chi_v) \cap R(V_{3,2}, \chi_v)) \overset{\sim}{\longrightarrow} R(V_{1,i}, \chi_v) \quad (i = 1, 2).$$

(2) The non-archimedean split case. Let $v$ be a finite place of $k$ which splits in $k'$, and let $w_1, w_2$ be the places of $k'$ lying over $v$. Then $G_v = GL(4)_{k_v}$ and $G_v = GL(2)_{k_v}$ and let

$$M_{1,v} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A, D \in GL(2)_{k_v} \right\}, \quad U_{1,v} = \left\{ \begin{pmatrix} 1_2 & B \\ 0_2 & 1_2 \end{pmatrix} \middle| B \in M_2 \right\}.$$

For each character $\chi_v = \chi_{w_1} \otimes \chi_{w_2}$ of $k_v'$ and $s \in \mathbb{C}$, define

$$I(s, \chi_v) := \text{Ind}^G_{P^1(k_v)}(\chi_{w_1} \circ \det) \otimes (\chi_{w_2}^{-1} \circ \det) \otimes \mathbf{1}_{U_{1,v}(k_v)}.$$  

Then we have

**Proposition 3.7** (Theorem 1.3 in [KSw]). (1) $I(s, \chi_v)$ is reducible if and only if $\chi_v \circ N_{k'/k_v} = \chi_{w_1} \chi_{w_2} = 1$ and $s = s_0$ with $s_0 \in \{-1, -1/2, 0, 1/2, 1\}$.

(2) $I(s, \chi_v)$ in (1) at its reducible point $s = s_0$ has a unique irreducible submodule $A$, and the quotient $I(s_0, \chi_v)/A$ is also irreducible. Moreover the irreducible quotient of $I(1/2, \chi_v)$ ($I(1, \chi_v)$ resp.) and the irreducible submodule of $I(-1/2, \chi_v)$ ($I(-1, \chi_v)$ resp.) are both isomorphic to

$$\text{Ind}^{G(k_v)}_{M_{3,1}(k_v)}(\chi_{w_1} \circ \det) \otimes \mathbf{1}_{U_{3,1}(k_v)}, \quad (\chi_{w_1} \circ \det \text{ resp.}).$$

Here $P^{(3,1)} = M_{3,1} U_{3,1}$ is the standard parabolic subgroup of $G_v$ such that $M_{3,1} \simeq GL(3) \times \mathbb{G}_m$.

(3) The above irreducible representations are related to the Weil representation as follows. Write $s_0 = (m-2)/2$ with $m \in \{0, 1, \ldots, 4\}$, and consider the dual reductive pair $G_v \times H_v$ with $H_v := GL(m)_{k_v}$. We can construct an oscillator representation $\omega_{\psi_v, \chi_v}$ of $G(k_v) \times H(k_v)$, for which explicit formulæ for its Schrödinger model $\mathcal{S}(k^{\oplus 4m}_v)$ are given by

$$\omega_{\psi_v, \chi_v} \left( \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right) \Phi(x, y) = \chi_{w_1}(\det(AD))|\det(AD)|^{m/2} \Phi(xA, y'D^{-1}) \quad (A, D \in GL(2, k_v)), \quad (3.26)$$

$$\omega_{\psi_v, \chi_v} \left( \begin{pmatrix} 1_2 & B \\ 0_2 & 1_2 \end{pmatrix} \right) \Phi(x, y) = \psi_v(xB' y) \Phi(x, y) \quad (B \in M_2), \quad (3.27)$$

$$\omega_{\psi_v, \chi_v} \left( \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right) \Phi(x, y) = \int_{p^{V_2}_{\omega}} \Phi(u, v) \psi_v(u^t x + v'y) \, du dv, \quad (3.28)$$

$$\omega_{\psi_v, \chi_v}(h) \Phi(x, y) = \Phi(h^{-1}(x, y)) \quad (h \in H(k_v)). \quad (3.29)$$

We write $R(V_m, \chi_v)$ for the image of the map

$$\mathcal{S}(k^{\oplus 4m}_v) \ni \Phi \longrightarrow \left( g \mapsto \omega_{\psi_v, \chi_v}(g) \Phi(0) \right) \in I(\frac{m-2}{2}, \chi_v).$$

Then

(i) $R(V_3, \chi_v) = I(1/2, \chi_v)$ and $R(V_4, \chi_v) = I(1, \chi_v)$. 

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(ii) \( R(V_1, \chi_v) \) is the unique irreducible submodule of \( I(-1/2, \chi_v) \).

(iii) \( R(0, \chi_v) := \chi_v \circ \det \) is the unique irreducible submodule of \( I(-1, \chi_v) \).

(iv) These irreducible constituents are related by the normalized intertwining operators as follows.

\[
R(V_1, \chi_v) = \text{Im} \, N(w_2 w_1 w_2, (\chi_v \circ \det) \mid \det |^{1/2}) \otimes (\chi_v^{-1} \circ \det) \mid \det |^{-1/2}),
\]

\[
R(0, \chi_v) = \text{Im} \, N(w_2 w_1 w_2, (\chi_v \circ \det) \mid \det | v \otimes (\chi_v^{-1} \circ \det) \mid \det | v^{-1}).
\]

(3) The archimedean inert case. The situation is \( k'/k_v = \mathbb{C}/\mathbb{R} \). For each character \( \chi_v \) of \( \mathbb{C}^\times \) and \( s \in \mathbb{C} \), we construct \( I(s, \chi_v) \) as in the non-archimedean case. We may assume that \( \chi_v \) is of the form \( \chi_v(z) = (z/z^\nu)^{-1/2} (\nu \in \mathbb{Z}) \).

**Proposition 3.8** (Theorem 6.2 in [Le]). The reducible points of \( I(s, \chi_v) \) are \( s_0 \in \nu/2 + \mathbb{Z} \).

Next we describe the irreducible constituents of \( I(s, \chi_v) \) at its reducible points \( s_0 \). For our purpose, it is sufficient to do this at

\[
s_0 = \begin{cases} 
-1/2, 1/2 & \text{if } \nu \not\in 2\mathbb{Z}, \\
-1, 0, 1 & \text{if } \nu \in 2\mathbb{Z}.
\end{cases}
\]

We again write these \( s_0 \) as \( (m-2)/2 \) \( (m = 0, 1, 2, 3, 4) \). For \( m \neq 0 \) we classify the \( m \)-dimensional Hermitian spaces \( V_{m,i} \) \( (i = 1, 2) \) just as in the non-archimedean inert case (by replacing \( \gamma \) with \(-1\)). For each \( V_{m,i} \) we have the oscillator representation \((\omega_{v,\chi_v}, S(V_{m,i}^{\mathbb{R}2}))\) of the dual pair \((G(k_v), U(V_{m,i}, k_v))\) as in the non-archimedean case.

Let \( V_{m,i} = V_{m,i}^+ \oplus V_{m,i}^- \) be the direct sum decomposition such that \( \text{Tr}_{\mathbb{C}/\mathbb{R}}(\langle \cdot, \cdot \rangle_{m,i}) \) is positive (negative resp.) definite (as a \( 2 \times 2 \) symmetric matrix!) on \( V_{m,i}^+ \) \( (V_{m,i}^- \) resp.). We define an inner product \( \langle \cdot, \cdot \rangle_{m,i} \) on \( V_{m,i} \) by

\[
\langle x, y \rangle_{m,i}^+ := \begin{cases} 
\langle x, y \rangle_{m,i} & \text{if } x, y \in V_{m,i}^+,
\langle -x, y \rangle_{m,i} & \text{if } x, y \in V_{m,i}^-,
0 & \text{if } x \in V_{m,i}^+ \text{ and } y \in V_{m,i}^-.
\end{cases}
\]

Using this we define the Gaussian function on \( V_{m,i}^{\mathbb{R}2} \) by

\[
\Phi_0^{\mathbb{R}2}(x) := \exp(-\pi \text{tr}(x, x)_{m,i}^+) \in S(V_{m,i}^{\mathbb{R}2}).
\]

Then the resulting space of “standard functions” with respect to \( \omega_v \),

\[
S^0(V_{m,i}^{\mathbb{R}2}) := \{ \Phi_0^{\mathbb{R}2}(x) P(x) \mid P(x) \text{ is a polynomial} \},
\]

is the underlying \((g_v, K_v) \times (u(V_{m,i})_{\mathbb{C}}, K_{m,i})\)-bimodule of \( \omega_{v,\chi_v} \). Here \( u(V_{m,i})_{\mathbb{C}} \) is the complexified Lie algebra of \( U(V_{m,i}, \mathbb{R}) \) and \( K_{m,i} \) is a certain maximal compact subgroup of \( U(V_{m,i}, \mathbb{R}) \).

**Proposition 3.9** (Theorems 4.2 and 5.9 in [LeZ]). Assume that the character \( \chi_v \) is such that \( \chi_v|_{\mathbb{R}^n} = \text{sgn}^m \) \( (m = 0, \ldots, 4) \).

(1) If we write \( R(V_{m,i}, \chi_v) \) for the image of the map

\[
S^0(V_{m,i}^{\mathbb{R}2}) \ni \Phi \rightarrow (g \rightarrow \omega_{v,\chi_v}(g)\Phi(0)) \in I(m-2, \chi_v),
\]

then this map induces an isomorphism \( S^0(V_{m,i}^{\mathbb{R}2})|_{u(V_{m,i})_{\mathbb{C}}, K_{m,i}} \rightarrow R(V_{m,i}, \chi_v) \).
The normalized intertwining operator

\[ N(w_2w_1w_2, (\chi_v \circ \det)|\det|_C) : I(1, \chi_v) \longrightarrow I(-1, \chi_v) \]

is \( R(V_{4,1}, \chi_v) \) and its image is \( \chi_v^{G} \). Here \( \chi_v^{G} \) is defined as in the non-archimedean case.

(3) \( R(V_{3,i}, \chi_v) \) (\( i = 1, 2 \)) are two distinct maximal submodules of \( I(1/2, \chi_v) \). \( R(V_{1,i}, \chi_v) \) (\( i = 1, 2 \)) are two inequivalent irreducible submodules of \( I(-1/2, \chi_v) \). The normalized intertwining operator \( N(w_2w_1w_2, (\chi_v \circ \det)|\det|_C^{1/2}) \) induces isomorphisms

\[ R(V_{3,i}, \chi_v)/(R(V_{3,1}, \chi_v) \cap R(V_{3,1}, \chi_v)) \cong R(V_{1,i}, \chi_v) \quad (i = 1, 2). \]

(4) Archimedean split case. In this case we have \( G(k_v) = GL(4, \mathbb{R}) \) or \( GL(4, \mathbb{C}) \). For each character \( \chi_v = \chi_{w_2} \otimes \chi_{w_2} \) of \( (k_v')^* \) and \( s \in \mathbb{C} \), we define \( I(s, \chi_v) \) as in the non-archimedean case. Then the following is essentially a classical result of E. M. Stein.

**Proposition 3.10** (Lemma 2.4 in [V]). (1) \( I(s, \chi_v) \) is irreducible unless \( \chi_{w_1} \chi_{w_2} = 1 \).

(2) If \( \chi_{w_1} \chi_{w_2} = 1 \), then it is reducible at \( s = s_0 \) with \( s_0 \in \{-1, -1/2, 1/2, 1\} \). The unique irreducible submodule of \( I(-1/2, \chi_v) \) and the unique irreducible quotient of \( I(1/2, \chi_v) \) are both isomorphic to \( \text{Ind}^{G(k_v)}_{P_{(3,1)}(k_v)} \left[ ((\chi_{w_1} \circ \det) \otimes \chi_{w_2}^{-1}) \otimes 1_{U_{(3,1)}(k_v)} \right] \). Here \( P_{(3,1)} \) is as in Proposition 3.7. An isomorphism between them is given by \( N(w_2w_1w_2, (\chi_{w_1} \circ \det)|\det|_C^{1/2} \otimes (\chi_{w_2}^{-1} \circ \det)|\det|_C^{-1/2}) \).

(3) We adopt the notation of Proposition 3.7 for Weil representations. Then

\[ R(V_3, \chi_v) = I(1/2, \chi_v) \text{ and } R(1, \chi_v) = \text{Ind}^{G(k_v)}_{P_{(3,1)}(k_v)} \left[ ((\chi_{w_1} \circ \det) \otimes \chi_{w_2}^{-1}) \otimes 1_{U_{(3,1)}(k_v)} \right]. \]

### 3.3.2. The contributions of (3.19) and (3.20).

Let \( V_A = (V_v)_v \) be a collection of 1-dimensional Hermitian spaces \( V_v \) over \( k_v' \) at each \( v \). We fix a non-trivial character \( \psi = \otimes_v \psi_v \) of \( A/k \). For each character \( \chi = \otimes_v \chi_v \) of \( \mathbb{A}_k^*/k^* \) such that \( \chi|_{\mathbb{A}_k^*} = \eta_{k'/k} \), we can construct a collection of irreducible representations \( R(V_v, \chi_v) \) of \( G(k_v) \) using \( \psi \). Moreover \( R(V_v, \chi_v) \) is unramified at almost all \( v \), and hence we have a smooth irreducible representation \( R(V_A, \chi) := \otimes_v R(V_v, \chi_v) \) of \( G(A) \).

**Theorem 3.11.** (1) The contribution of (3.19) in Theorem 3.4 to the residual spectrum is multiplicity free and consists of one dimensional representations \( \chi^G = \chi' \circ \det \). Here \( \chi' : U(1, A)_k'/k \rightarrow \mathbb{C}^1 \) is defined by \( \chi'(x\sigma(x)^{-1}) := \chi(x) \), where \( \chi \) runs over characters of \( \mathbb{A}_k^*/k^* \) such that \( \chi|_{\mathbb{A}_k^*} = 1 \).

(2) The contribution of (3.20) in Theorem 3.4 is multiplicity free and consists of \( R(V_A, \chi) \), where \( V_A = \otimes_v V_v \) is obtained from a one dimensional Hermitian space \( V \) over \( k' \) by \( V_v = V \otimes_k k_v' \), and \( \chi \) is a character of \( \mathbb{A}_k^*/k^* \) such that \( \chi|_{\mathbb{A}_k^*} = \eta_{k'/k} \).

The proof of this will occupy the following three subsubsections. We begin by reviewing some material from [KRS].
3.3.3. **Lemmas on coinvariant.** We still fix the nontrivial character \( \psi \) of 3.3.2. For brevity we write

\[
m(A) := \begin{pmatrix} A & 0_2 \\ 0_2 & \sigma(t A^{-1}) \end{pmatrix} \quad (a \in \text{Res}_{k'/k} \text{GL}(2)),
\]

\[
u(B) := \begin{pmatrix} 1_2 & B \\ 0_2 & 1_2 \end{pmatrix} \quad (B \in \text{Her}_2(k'/k)),
\]

and

\[
w := \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}.
\]

**(1) The non-archimedean case.** First take a non-archimedean \( v \) which is inert in \( k' \). For each \( \beta \in \text{Her}_2(k'/k_v) \) we have a character of \( U_1(k_v) \) defined by

\[
\psi_\beta : U_1(k_v) \ni u(B) \mapsto \psi_v(\text{tr}(B\beta)) \in \mathbb{C}^1.
\]

For each smooth representation \( V \) of \( G(k_v) \), we define its twisted coinvariant space by

\[
V_{U_1,\beta} := V / \text{Span}\{u.\xi - \psi_\beta(u)\xi \mid \xi \in V, u \in U_1(k_v)\}.
\]

Now recall the Hermitian space \( V_v = V_{1,i} \) \((i = 1, 2)\) and the oscillator representation \((\omega_{\psi_v,\chi_v}, \mathcal{S}(V_v^{\otimes 2}))\) of \( G(k_v) \times U(V_v, k_v) \). For \( \beta \) as above, we set

\[
\Omega_\beta(V_v) := \{ x \in V_v^{\otimes 2} : \langle x, x \rangle_{V_v} = \beta \}.
\]

**Lemma 3.12** (Lemma 1.3 in [KRS]). *(1) The canonical projection \( \mathcal{S}(V_v^{\otimes 2}) \rightarrow \mathcal{S}(V_v^{\otimes 2})_{U_1,\beta} \) factors through the “restriction to \( \Omega_\beta(V_v) \)” map \( \mathcal{S}(V_v^{\otimes 2}) \rightarrow \mathcal{S}(\Omega_\beta(V_v)) \).

*(2) In particular, if \( \det \beta \neq 0 \), then \( \Omega_\beta(V_v) \) consists of a single \( U(V_v, k_v) \)-orbit, and the projection \( \mathcal{S}(V_v^{\otimes 2}) \rightarrow (\mathcal{S}(V_v^{\otimes 2})_{U_1,\beta})_{U(V_v, k_v)} \) is given by

\[
\Phi \mapsto \int_{\Omega_\beta(V_v)} \Phi(x) \, d\beta x.
\]

Here \( d\beta x \) is the unique (up to a factor) \( U(V_v, k_v) \)-invariant measure.

**Proof.** *(1) is merely a combination of the explicit formula (3.23) and [BZ], Lemma 2.33. As for (2), the first assertion is clear and the second follows from the first and (1). \[ \square \]

The corresponding result at split \( v \) is almost trivial and will be omitted.

**(2) The archimedean case.** Again we treat the inert case only. Thus \( k_v' / k_v = \mathbb{C} / \mathbb{R} \). As in the non-archimedean case, we have a character \( \psi_\beta \) of \( U_1(\mathbb{R}) \) for each \( \beta \in \text{Her}_2(\mathbb{C} / \mathbb{R}) \). We write \( d\psi_\beta : \text{Lie}(U_1(\mathbb{R}))_{\mathbb{C}} \rightarrow \mathbb{C} \) for its differential. \( \Omega_\beta(V_v) \) is defined similarly as in the non-archimedean case. Again recall the Hermitian space \( V_v = V_{1,i} \) \((i = 1, 2)\) and the \((\mathfrak{g}_v, K_v) \times (u(V_v)_{\mathbb{C}}, K_{V_v})\)-module \( (\omega_{\psi_v,\chi_v}, \mathcal{S}'(V_v^{\otimes 2})) \)

and \( R(V_v, \chi_v) \). We write \( \mathcal{S}'(V_v^{\otimes 2})_{\beta} \) for the space of \( U(V_v, \mathbb{R}) \)-invariant tempered distributions \( T \) satisfying

\[
T(d\omega_{\psi_v,\chi_v}(X) \Phi) = d\psi_\beta(X) T(\Phi), \quad \forall X \in \text{Lie}(U_1(\mathbb{R})).
\]

Also, the space of linear functionals \( \mathcal{L} \) on \( R(V_v, \chi_v) \) which satisfy

\[
\mathcal{L}(X.\Phi) = d\psi_\beta(X) \cdot \mathcal{L}(\Phi), \quad \forall X \in \text{Lie}(U_1(\mathbb{R}))
\]

is denoted by \( R(V_v, \chi_v)^* \). Then
Lemma 3.13 (Proposition 1.7 and Corollary 1.8 in [KRS]). (1) If $\Omega_\beta(V_v)$ is empty, then both $(S_0(V_v^{\otimes 2})_\beta)^{U(V_v,\mathbb{R})}$ and $R(V_v,\chi_v)_\beta^0$ are $\{0\}$.

(2) If $\Omega_\beta(V_v)$ is not empty, then $(S_0(V_v^{\otimes 2})_\beta)^{U(V_v,\mathbb{R})}$ is spanned by the orbital integral

$$\Phi \mapsto \int_{\Omega_\beta(V_v)} \Phi(x) \, dx.$$ 

Consequently $\dim R(V_v,\chi_v)_\beta^0 = 1$.

Proof. The proof in the symplectic case was given in [Ra2], Lemma 4.2. The unitary case can be treated similarly using [LeZ].

3.3.4. Generalized Whittaker models for $R(V_\Lambda,\chi)$. Here we review some facts about the generalized Whittaker models from [KRS], §2.

Every character of $U_1(\hat{A}_v)/U_1(k)$ is of the form

$$\psi_\beta : U_1(\hat{A}_v) \ni u(B) \mapsto \psi(\text{tr}(B\beta)) \in \mathbb{C}^1,$$

for some $\beta \in \text{Her}_2(k'/k)$. We restrict ourselves to those $\psi_\beta$ with $\det \beta \neq 0$. Then the space of generalized Whittaker functionals for an irreducible smooth representation $(\pi, V_{\pi})$ of $G(\hat{A}_v)$ is defined by

$$W_\beta(\pi) := \left\{ \mathcal{L} : V_{\pi} \to \mathbb{C} \mid \begin{array}{l}
(\text{i}) \mathcal{L}(\pi(u)f) = \psi_\beta(u)\mathcal{L}(f), \forall u \in U_1(\hat{A}_v), \\
(\text{ii}) \mathcal{L}(d\pi(X)f) = d\psi_\beta(X)\mathcal{L}(f), \forall X \in \text{Lie} U_1(\hat{A}_\infty) \end{array} \right\}.$$ 

Now we suppose that $(\pi, V_{\pi})$ is an automorphic subrepresentation. For each $f \in V_{\pi}$, we define its $\beta$-th Fourier coefficient as

$$W_\beta(f)(g) := \int_{U_1(k)/U_1(\hat{A}_v)} f(ug)\overline{\psi_\beta(u)} \, du,$$

and a Whittaker functional $W_\beta \in W_\beta(\pi)$ by

$$V_{\pi} \ni f \mapsto W_\beta(f)(1) \in \mathbb{C}.$$ 

Let $V_\Lambda = \bigotimes_v V_v$ and $R(V_\Lambda,\chi) = \bigotimes_v R(V_v,\chi_v)$ be as in 3.3.2.

Lemma 3.14 (Lemma 2.5 in [KRS]). If an intertwining map $D$ from $R(V_\Lambda,\chi)$ to the space of $L^2$-automorphic forms $L^2(G(\hat{A}_v)\backslash G(\hat{A}))$ satisfies

$$W_\beta \circ D = 0, \quad \forall \beta \in \text{Her}_2(k'/k) \text{ with } \det \beta \neq 0,$$

then $D$ is zero.

Proof. We write $\pi = \bigotimes_v \pi_v$ for the representation of $G(\hat{A}_v)$ on $R(V_\Lambda,\chi_v)$. Take a finite place $v$ which is unramified or split in $k'$. We write $O_{k'_v}$ for the integral closure of $O_{k_v'}$ in $k'_v$.

Let $\mathcal{D}(V_v) := \{ \beta \in \text{Her}_2(k'_v/k_v) \mid \det \beta \neq 0, \beta = \langle x, x \rangle_{V_v} \text{ for some } x \in V_v^{\otimes 2} \}$. Take a Schwartz function $h$ on $\text{Her}_2(k'_v/k_v)$ such that its Fourier transform

$$\hat{h}(b) := \int_{\text{Her}_2(k'_v/k_v)} h(B)\psi_v(\text{tr}(Bb)) \, dB$$

is the characteristic function of $\mathcal{D}(V_v) \cap \text{M}_2(O_{k'_v})$. We write $f_h$ for the function on $U_1(k_v)$ given by $f_h(u(b)) := h(b)$. Then from the definition of $R(V_v,\chi_v)$, we have
\[ \pi_v(f_h) \varphi_v(u) = \int_{\text{Herm}(k'/k)} \varphi_v(wu(b))h(b) db \]

choosing \( \Phi_v \in \mathcal{S}(V^\oplus 2) \) which projects to \( \varphi_v \),

\[ = \int_{\text{Herm}(k'/k)} \omega_{\psi_v, \chi_v}(wu(b))\Phi_v(0)h(b) db \]

from (3.24) or (3.28),

\[ = \int_{\text{Herm}(k'/k)} \frac{1}{\gamma(\text{Tr}_{k'/k}(\ell \cdot V_v))} \int_{\ell \cdot V_v} h(b)\psi_v(\text{tr}(b\langle x, x \rangle_{V_v})) db \Phi_v(x) dx \]

Thus we can choose \( \Phi_v \) so that this does not vanish.

Now using this specific \( \Phi_v \), we construct \( \Phi = \Phi_v \otimes \otimes_{v' \neq v} \Phi_{v'} \in \mathcal{S}(V^\oplus 2) \) and write \( \varphi = \varphi_v \otimes \otimes_{v' \neq v} \varphi_{v'} \) for its image in \( R(V_h, \chi) \). Then using the Fourier expansion on \( U_1(k) \backslash U_1(k) \), one has

\[ R(f_h)D(\varphi)(g) = \int_{U_1(k)} D(\varphi)(gu)f_h(u) du \]

\[ = \int_{U_1(k)} \sum_{\beta \in \text{Herm}(k'/k)} (W_\beta \circ D)(\varphi)(gu)f_h(u) du \]

If we take \( g_v = 1 \), then this equals

\[ = \int_{\text{Herm}(k'/k)} \sum_{\beta \in \text{Herm}(k'/k)} \psi_v(\text{tr}(b\beta))h(b) (W_\beta \circ D)(\varphi)(g) \]

\[ = \sum_{\beta \in \text{Herm}(k'/k)} \hat{h}(\beta)(W_\beta \circ D)(\varphi)(g). \]

This is zero from our hypothesis. Since this means \( D(\pi(f_h)\varphi) \) restricted to the dense subset \( G(k) \prod_{v' \neq v} G(k_{v'}) \) of \( G(A) \) vanishes identically, \( D(\pi(f_h)\varphi) \) is forced to be identically zero. But our choice of \( \varphi \) asserts that \( \pi(f_h)\varphi \neq 0 \). That is, \( D \) has non-trivial kernel. Hence \( D = 0 \), because \( R(V_h, \chi) \) is irreducible.

\[ \text{dim Hom}_{G(A)}(R(V_h, \chi), L^2(G(k)\backslash G(A))) \]

denotes the space of \( G(A_f) \times (\mathfrak{g}_\infty, K_\infty) \)-equivariant maps from \( R(V_h, \chi) \) to the space of smooth \( K_\infty \)-finite vectors in \( L^2(G(k)\backslash G(A)) \).

**Proposition 3.15** (Theorem 2.2 in [KRS]). Assume \( V_h \) comes from a Hermitian space \( V \) over \( k' \), i.e. \( V_h = V \otimes_k A \). Then \( \text{dim Hom}_{G(A)}(R(V_h, \chi), L^2(G(k)\backslash G(A))) \leq 1 \).

**Proof.** Take two elements \( A, B \in \text{Hom}_{G(A)}(R(V_h, \chi), L^2(G(k)\backslash G(A))) \) and set \( A_\beta := W_\beta \circ A \) and \( B_\beta := W_\beta \circ B \) for \( \beta \in \text{Herm}(k'/k) \). Also set \( \mathcal{O}(V) := \{ \beta \in \text{Herm}(k'/k) \colon \beta(\xi, \xi) \neq 0, \xi \in V \} \). If \( A_\beta \) is not zero, then by applying Lemma 3.12 and Lemma 3.13 to \( A_\beta|_{\mathcal{O}(V_h^\vee)} \) and to \( A(\mathcal{O}(V_h^\vee)) \), we see that \( \beta \in \mathcal{O}(V) \) and there exists a constant \( c_\beta \) such that \( B_\beta = c_\beta A_\beta \). Moreover, since

\[ A_\beta(\pi(m(a))\Phi) = W_\beta(R(m(a)))A(\Phi) \]

\[ = W_{\pi(a)\beta}(A(\Phi)) = A_{\pi(a)\beta}(\Phi) \]

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for $a \in GL(2,k')$ and $D(V)$ is a single $GL(2,k')$ orbit, we know that $c = c_{\beta}$ is independent of $\beta$. Thus if we set $D := B - cA$, we have $W_{\beta} \circ D = 0$ for any $\beta \in \text{Her}_2(k'/k)$ with det $\beta \neq 0$. Now we apply Lemma 3.14 to finish the proof. □

**Lemma 3.16 (Proposition 2.6 in [KRS]).** Suppose there is no Hermitian space $V$ over $k'$ such that $V_{\Delta} = V \otimes_k A$. Then $\text{Hom}_{G(A)}(R(V_{\Delta}, \chi), L^2(G(k)\backslash G(A))) = 0$.

**Proof.** Take an intertwining map $D : R(V_{\Delta}, \chi) \to L^2(G(k)\backslash G(A))$ and consider $W_{\beta} \circ D$ for $\beta \in \text{Her}_2(k'/k)$. Then by Lemma 3.12 and Lemma 3.13, at every $v$ it is necessary that $\Omega_{\beta}(V_v)$ be non-empty for $W_{\beta} \circ D$ restricted to $R(V_v, \chi_v)$ to be non-zero. But this means that the Hermitian space $(V = k', \langle x, y \rangle_V := x\beta\bar{\sigma}(y))$ over $k'$ satisfies $V_{\Delta} = V \otimes_k A$, which contradicts our assumption. Thus $W_{\beta} \circ D = 0$ must hold for any $\beta \in \text{Her}_2(k'/k)$. Now the lemma follows from Lemma 3.14. □

3.3.5. **The proof of Theorem 3.11.** First note that we can choose a character $\chi$ of $\text{Aut}_{k'}/\text{k}^\times$ such that

$$\mathcal{S}_{1,2} = \chi|_{k_v'}^{|\frac{1}{2}} \otimes \chi|_{k_v'}^{|\frac{1}{2}}, \chi|_{k_v'} = 1$$

$$\text{(}\mathcal{S}_{1,3} = \chi|_{k_v'} \otimes \chi, \chi|_{k_v'} = \eta_{k'/k} \text{ resp.)}$$

The contribution to be calculated is the image $\text{Im} N(w_{-1}, \mathcal{S}_{1,2})$ (or $\text{Im} N(w_{-1}, \mathcal{S}_{1,3})$ resp.). But since $w_{-} = (w_{2}w_{1}w_{2})$ and $w_{1}$ and

$$\text{Im} N(w_{1}, \mathcal{S}_{1,2}) = \text{Ind}_{P_{1}(A)}^{G(A)}[\chi \circ \det]|_{k_v'} \otimes 1_{U_{1}(A)}$$

$$\text{(Im} N(w_{1}, \mathcal{S}_{1,3}) = \text{Ind}_{P_{2}(A)}^{G(A)}[\chi \circ \det]|_{k_v'} \otimes 1_{U_{1}(A)} \text{ resp.)}$$

the statement (1) follows immediately from Propositions 3.5 (2), 3.6 (2), 3.7, 3.9, and 3.10.

To prove (2), we have to determine the irreducible constituents of

$$N(w_{2}w_{1}w_{2}, \chi \otimes \chi|_{k_v'}) = \text{Ind}_{P_{1}(A)}^{G(A)}[\chi \circ \det]|_{k_v'} \otimes 1_{U_{1}(A)}).$$

A local component of this consists of at most two irreducible representations $R(V_{1,i}, \chi_v)$ by Propositions 3.5 (2) (3), 3.6 (2), 3.7, 3.9 and 3.10. Thus the irreducible constituents are contained in

$$\left\{ R(V_{\Delta}, \chi) \mid \begin{array}{l}
\text{(i) } V_{\Delta} \text{ is a collection of one-dimensional Hermitian spaces } V_v \text{ over } k_v \text{ at each } v, \\
\text{(ii) } \chi \text{ is such that } \chi|_{k_v'} = \eta_{k'/k}.
\end{array} \right\}$$

But we know from Lemma 3.16 that $R(V_{\Delta}, \chi)$ with $V_{\Delta}$ not coming from a global $V$ must be excluded. Also Proposition 3.15 assures that the multiplicity of $R(V_{\Delta}, \chi)$ is at most 1.

Finally we need to check that $R(V_{\Delta}, \chi)$ with $V_{\Delta} = V \otimes_k A$ for some $V$ over $k'$ really contributes to the residual spectrum (cf. [KRS], Proposition 3.3). For such a Hermitian space $V$, we write its unitary group $U(V)$. We can construct the global oscillator representation $(\omega_{\phi}, \mathcal{S}_{0}(V_{\Delta}))$ of $G(A) \times U(V, A)$ (see 3.3.7 below). Using this, we have the usual theta kernel

$$\theta_{\chi}(g, h; \Phi) := \sum_{\xi \in V} \omega_{\phi, \chi}(g, h)\Phi(\xi) \quad (g \in G(A), h \in U(V, A)).$$

Since $V$ is always anisotropic, the theta integral

$$I_{\chi}(g, \Phi) := \int_{U(V, k) \backslash U(V, A)} \theta_{\chi}(g, h; \Phi) \, dh$$
is well-defined. Moreover the map $I_\chi : S^0(V_{\mathcal{A}}) \to L^2(G(k) \backslash G(\mathcal{A}))$ is non-zero (calculate the $\psi_\beta$-Fourier coefficients) and factors $R(V_{\mathcal{A}}, \chi)$. This completes the proof of Theorem 3.11. (q.e.d.)

3.3.6. The contributions of (3.21)—the result. The contribution of the term (3.21) is relatively easy to describe. First we prove the following.

**Lemma 3.17.** Decompose $\mathfrak{S}_{2,4}$ into a restricted tensor product $\otimes_v (\mathfrak{S}_{2,4})_v$. Then the images

$$\text{Im}[N(w_2, w_1 w_2(\mathfrak{S}_{2,4})_v) M(w_1, w_2(\mathfrak{S}_{2,4})_v) N(w_2, (\mathfrak{S}_{2,4})_v)],$$

$$\text{Im}[M(w_1, w_2 w_1 w_2(\mathfrak{S}_{2,4})_v) N(w_2, (\mathfrak{S}_{2,4})_v) M(w_1, w_2(\mathfrak{S}_{2,4})_v) N(w_2, (\mathfrak{S}_{2,4})_v)]$$

coincide with each other and are irreducible.

**Proof.** We write $\mathfrak{S}_{2,4}$ as $\chi_1 \mid |_{\mathbb{A}_{k^s}}^{|1/2 \otimes \chi_2|} \mid_{\mathbb{A}_{k^s}}^{|1/2}$, where $\chi_i (i = 1, 2)$ are distinct characters of $\mathbb{A}_{k^s}/k^s$ such that $\chi_i|_{k^s} = 1$.

**THE CASE OF INERT $v$.** We take an inert place $v$ of $k$. Then $(\mathfrak{S}_{2,4})_v := \chi_1|_v \mid |_{\mathbb{A}_v}^{|1/2 \otimes \chi_2|} \mid_{\mathbb{A}_v}^{|1/2}$, and hence

$$r(w_2, (\mathfrak{S}_{2,4})_v) = \frac{L_k(1, \chi_1|_v \mid |_{\mathbb{A}_v}^{|1/2 \otimes \chi_2|} \mid_{\mathbb{A}_v}^{|1/2})}{L_k(2, \chi_1|_v \mid |_{\mathbb{A}_v}^{|1/2 \otimes \chi_2|} \mid_{\mathbb{A}_v}^{|1/2})},$$

$$r(w_2, w_1 w_2(\mathfrak{S}_{2,4})_v) = \frac{L_k(1, \chi_2|_v \mid |_{\mathbb{A}_v}^{|1/2 \otimes \chi_1|} \mid_{\mathbb{A}_v}^{|1/2})}{L_k(2, \chi_2|_v \mid |_{\mathbb{A}_v}^{|1/2 \otimes \chi_1|} \mid_{\mathbb{A}_v}^{|1/2})}$$

are both defined and non-zero. Thus we may replace $N(w, \bullet)$ by $M(w, \bullet)$ and study the images

$$\text{Im} M(w_2 w_1 w_2, (\mathfrak{S}_{2,4})_v), \quad \text{Im} M(w_-, (\mathfrak{S}_{2,4})_v).$$

But since $(\pi_v, V_{\mathcal{A}}) := \text{Ind}_{\{P_0 \cap M_1\}(k_v)}^{\{\mathfrak{S}_{2,4}\}(k_v)}(\mathfrak{S}_{2,4})_v \otimes 1_{U_0 \cap M_1}(k_v)$ is essentially tempered, irreducible and $M(w_1, (\mathfrak{S}_{2,4})_v) : V_{\mathcal{A}} \to V_{\mathcal{A}}$ is an isomorphism, we know that these images are both isomorphic to the Langlands’ quotient of $\text{Ind}_{P_0(k_v)}^{G(k_v)}[\pi_v \otimes 1_{U_0(k_v)}]$.

**THE CASE OF SPLIT $v$.** At a split $v$, $(\mathfrak{S}_{2,4})_v$ is of the form

$$\chi_{1, w_1} \mid |_v^{|1/2 \otimes \chi_{2, w_1}|} \mid v \mid -1/2 \otimes \chi_{2, w_2} \mid v \mid 1/2,$$

where $\chi_{1, v} = \chi_{1, w_1} \otimes \chi_{1, w_2}$, $\chi_{2, v} = \chi_{2, w_1} \otimes \chi_{2, w_2}$, and hence

$$r(w_2, (\mathfrak{S}_{2,4})_v) = \frac{L_k(1, \chi_{2, w_1} \chi_{2, w_2})}{L_k(2, \chi_{2, w_1} \chi_{2, w_2})},$$

$$r(w_2, w_1 w_2(\mathfrak{S}_{2,4})_v) = \frac{L_k((1, \chi_{1, w_1} \chi_{1, w_2}))}{L_k((2, \chi_{1, w_1} \chi_{1, w_2}))},$$

These again allow us to replace the normalized intertwining operators by the non-normalized ones. Thus we study

$$\text{Im} M(w_2 w_1 w'_2, (\mathfrak{S}_{2,4})_v), \quad \text{Im} M(w_-, (\mathfrak{S}_{2,4})_v),$$

where $w'_2$ denotes the simple reflection of $G_v \simeq GL(4)_v$ attached to the simple root $\alpha'_1$ (see 2.1.4). Again noting that $M_{1,v} \simeq GL(2)_v \times GL(2)_v$, we see that both images are the Langlands quotient of $\text{Ind}_{P_0(k_v)}^{G(k_v)}[\pi_v \otimes 1_{U_0(k_v)}]$, where $\pi_v := \text{Ind}_{\{P_0 \cap M_1\}(k_v)}^{\{\mathfrak{S}_{2,4}\}(k_v)}(\mathfrak{S}_{2,4})_v \otimes 1_{U_0 \cap M_1(k_v)}$.

$\square$
We may call the resulting irreducible representation
\[
\text{Im } N(w_2w_1w_2, \mathcal{S}_{2,4}) = \text{Im } N(w_-, \mathcal{S}_{2,4})
\]
the global Langlands quotient of \( \text{Ind}_{P_0(k)}^{G(k)}[\mathcal{S}_{2,4} \otimes 1_{U_n(k)}] \).

**Theorem 3.18.** The contribution of the term \((3.21)\) in Theorem 3.4 to the residual spectrum is multiplicity free and consists of irreducible representations corresponds to non-trivial one-dimensional representations of \(U(1,1)_{k'/k}\) by the theta correspondence.

Write \( H \) for the group \( U(1,1)_{k'/k} \) defined by
\[
H := \left\{ h \in \text{Res}_{k'/k} GL(2) \mid h \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bar{\sigma}(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.
\]
All we have to show is that the irreducible representation
\[
\text{Im } N(w_2w_1w_2, \mathcal{S}_{2,4}) = \text{Im } N(w_-, \mathcal{S}_{2,4})
\]
of \( G(k) \) corresponds to the one dimensional representation \((\chi_1 \chi_2^{-1})^H := (\chi_1 \chi_2^{-1})' \circ \det\), where \((\chi_1 \chi_2)'\) is defined similarly as in Theorem 3.11. To do this we calculate the constant terms of theta series on \(G(k) \times H(k)\) (cf. [Ra]).

**3.3.7. Constant terms of theta series.** (1) Global oscillator representations for \(G(k) \times H(k)\). We begin with a short review of theta series on \(G(k) \times H(k)\). As in the local case (3.3.1), we consider \(G \times H\) as a dual reductive pair in \(Sp(W)\).

Here the Hermitian space \(V_{m,i}\) is replaced by
\[
(V := k^{r+2}, (,)_V := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).
\]
As in [I], §1, we have the global metaplectic group \(Mp(W_{\mathfrak{k}})\) of \(Sp(W_{\mathfrak{k}})\). This is a restricted direct product of \(Mp(W_v)\) with respect to their appropriate compact subgroups \(K_v\) divided by the central subgroup \(\{(\epsilon_v)_v \in \bigoplus_v \mathbb{C}^1 : \prod_v \epsilon_v = 1\}\). The group \(Sp(W,k)\) is naturally considered as a closed subgroup of \(Mp(W)\). Also for a character \(\psi\) of \(k'/k\), we have the oscillator representation \(\omega_{\psi}\) of \(Mp(W_{\mathfrak{k}})\) associated to it.

Recall that our \(\mathcal{S}_{2,4}\) was written in the form \(\chi_1 | \chi_1^2 \otimes \chi_2 | \chi_2^2\) of \(\mathbb{A}_{k'}\). The two distinct characters of \(\mathbb{A}_{k'}^\times /k^\times\) whose restriction to \(\mathbb{A}_{k'}^\times\) are trivial. Using this \(\chi_1\) we can construct a system of local splittings (cf. [Ku], Theorem 3.1):
\[
\iota_{\chi_1,v} : G(k_v) \times H(k_v) \longrightarrow Mp(W_v).
\]
Then from explicit formulae (3.22), (3.23), (3.24), (3.25) and (3.26), (3.27), (3.28), (3.29) we see that \(\iota_{\chi_1,v}(K_v) \subset K_v\) at almost all \(v\). This allows us to construct a global splitting
\[
\iota_{\chi_1} : G(k) \times H(k) \ni (g,h) \rightarrow (\iota_{\chi_1,v}(g_v,h_v))_v \in Mp(W_{\mathfrak{k}}).
\]
It follows from the product formula for the Weil constant that \(\iota_{\chi_1}(G(k) \times H(k)) \subset Sp(W,k)\). As in the local case we have the auxiliary oscillator representation \(\omega_{\psi',\chi_1} := \omega_{\psi} \circ \iota_{\chi_1}\), and twist the \(H(k)\)-action by the character \(\chi_1 \circ \det\) to obtain the final oscillator representation \(\omega_{\psi',\chi_1}(S(V_{\mathfrak{k}}^{\otimes 2}))\).

Explicit formulae for \(\omega_{\psi',\chi_1}(S(V_{\mathfrak{k}}^{\otimes 2}))\) are given by
\[
(3.30) \quad \omega_{\psi',\chi_1}(m(A))\Phi(x) = \chi_1(\det A)|\det A|_{\mathbb{A}_{k'}}\Phi(xA) (A \in GL(2, \mathbb{A}_{k'})\).
\[(3.31)\]
\[
\omega_{\psi,\chi_1}(u(B))\Phi(x) = \psi(\text{tr}((x, x)_V B))\Phi(x) \quad (B \in \text{Her}_2(\A_{k'}/\A)),
\]

\[(3.32)\]
\[
\omega_{\psi,\chi_1}(w) = \int_{V^\oplus_2} \Phi(y)\psi(\text{Tr}_{k'/k}(\text{tr}(x, y)_V))\ dy,
\]

\[(3.33)\]
\[
\omega_{\psi,\chi_1}(h)\Phi(x) = \Phi(h^{-1}x) \quad (h \in H(\A)).
\]

If we replace \(S(V^\oplus_2)\) with \(S^0(V^\oplus_2)\) defined in \(3.3.1\) at each archimedean \(v\), then we have the underlying \([G(\A_f) \times (\g(\A_{\infty})_{\C}, K_{\infty})] \times [H(\A_f) \times (\h(\A_{\infty})_{\C}, K^H_{\infty})]-\)
bimodule \(S^0(V^\oplus_2)\). Now for \(\Phi \in S^0(V^\oplus_2)\), define the theta series attached to it by

\[
\theta_\Phi(g, h) := \sum_{\xi \in V^\oplus_2} \omega_{\psi,\chi_1}(g, h)\Phi(\xi).
\]

This is a slowly increasing smooth \(K_\infty \times K^H_\infty\)-finite function on \(G(k) \backslash G(\A) \times H(k) \backslash H(\A)\).

(2) Calculation of the constant term. At this point we identify \(V^\oplus_2\) with \(M_2(k')\), on which \(G (H \text{ resp.})\) acts by right (left resp.) translation. We fix a point \(x_1 \in V^\oplus_2\) as

\[
x_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Lemma 3.19. If we set

\[
F_\Phi(h)(g) := \sum_{\zeta \in \mathcal{Z}(M_1, k)} \omega_{\psi,\chi_1}(\zeta g, h)\Phi(x_1)
\]

for \(\Phi \in S(V^\oplus_2)\), then the constant term of \(\theta_\Phi\) along \(P_0 \subset G\) is given by

\[(3.34)\]
\[
\theta_\Phi(g, h)_{P_0} = \omega_{\psi,\chi_1}(g, h)\Phi(0) + \sum_{\gamma \in B_{H}(k) \backslash H(k)} F_\Phi(\gamma h)(g)
\]

\[
+ \sum_{\gamma \in B_{H}(k) \backslash H(k)} \int_{U_{M_1}(k)} F_\Phi(\gamma h)(w_1^{-1}ug)\ du.
\]

Here \(B_{H}\) denotes the Borel subgroup of \(H\) consisting of lower triangular elements.

Proof. We follow Rallis’s argument in the proof of Theorem I.1.1 in [Ra]. First we calculate the constant term along \(P_1\):

\[
\theta_\Phi(g, h)_{P_1} = \int_{U_1(k)U_1(k)} \sum_{\xi \in V^\oplus_2} \omega_{\psi,\chi_1}(ug, h)\Phi(\xi)\ du
\]

\[
= \int_{\text{Her}_2(k'/k) \backslash \text{Her}_2(\A_{k'}/\A)} \sum_{\xi \in V^\oplus_2} \omega_{\psi,\chi_1}(u(B))\Phi(\xi)\ dB
\]

\[
= \sum_{\xi \in V^\oplus_2} \omega_{\psi,\chi_1}(g, h)\Phi(\xi) \int_{\text{Her}_2(k'/k) \backslash \text{Her}_2(\A_{k'}/\A)} \psi(\text{tr}([\xi, \xi]_V B))\ dB
\]

\[
= \sum_{\xi \in V^\oplus_2} \langle \xi, \xi \rangle_V = 0 \omega_{\psi,\chi_1}(g, h)\Phi(\xi).
\]
Next set \( X_0 := \{ \xi \in V_k^{\otimes 2} \mid \langle \xi, \xi \rangle_V = 0 \} \). Then there are two \( M_1(k) \times H(k) \)-orbits in \( X_0 \):

\[
X_{0,0} := \emptyset, \quad X_{0,1} := \{ (\xi, \eta) \in X_0 \mid \dim \text{Span}\{\xi, \eta\} = 1 \}.
\]

The element \( x_1 \) above is a representative of \( X_{0,1} \). If we write \( S_1 \) for the stabilizer of \( x_1 \) in \( M_1(k) \times H(k) \), then

\[
\theta_{\Phi}(g, h)_{P_1} = \omega_{\psi, x_1}(g, h)\Phi(0) + \sum_{(\delta, \gamma) \in S_1 \setminus H(k) \times M_1(k)} \omega_{\psi, x_1}(\delta g, \gamma h)\Phi(x_1).
\]

Next we proceed to

\[
\theta_{\Phi}(g, h)_{P_0} := \int_{U_0^{M_1(k)} \setminus U_0^{M_1(k)}} \theta_{\Phi}(ug, h)_{P_1} \, du,
\]

where \( P_0^{M_1} = M_0 U_0^{M_1} \) is the intersection of \( P_0 \) and \( M_1 \). Note that

\[
S_1 = \left\{ m\left( \begin{pmatrix} x & y \\ 0 & (a^{-1}) \end{pmatrix} \right), \left( \begin{pmatrix} a & b \\ 0 & (a^{-1}) \end{pmatrix} \right) \in M_1(k) \times H(k) \right\}.
\]

This combined with the Bruhat decomposition yields

\[
\theta_{\Phi}(g, h)_{P_1} = \omega_{\psi, x_1}(g, h)\Phi(0) + \sum_{\gamma \in B_0^\chi(k) \setminus H(k)} \sum_{\delta \in P(M_1(k)) \setminus M_1(k)} \omega_{\psi, x_1}(\delta g, \gamma h)\Phi(x_1)
\]

where \( P(M_1) := \left\{ m\left( \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \right) \in M_1 \right\} \),

\[
= \sum_{\gamma \in B_0^\chi(k) \setminus H(k)} \left[ \omega_{\psi, x_1}(g, h)\Phi(0) + \sum_{\zeta \in Z(M_1, k)} \omega_{\psi, x_1}(\zeta g, \gamma h)\Phi(x_1) \right]
\]

\[
+ \sum_{\zeta \in Z(M_1, k)} \sum_{\epsilon \in U_0^{M_1(k)}} \omega_{\psi, x_1}(w_1^{-1} \zeta \epsilon x g, \gamma h)\Phi(x_1) \right].
\]

On the other hand,

\[
\int_{U_0^{M_1(k)} \setminus U_0^{M_1(k)}} \left( \omega_{\psi, x_1}(ug, h)\Phi(0) + \sum_{\zeta \in Z(M_1, k)} \omega_{\psi, x_1}(\zeta ug, h)\Phi(x_1) \right) \, du
\]

\[
= \omega_{\psi, x_1}(g, h)\Phi(0) + \sum_{\zeta \in Z(M_1, k)} \omega_{\psi, x_1}(\zeta g, \gamma h)\Phi(x_1),
\]

\[
\int_{U_0^{M_1(k)} \setminus U_0^{M_1(k)}} \sum_{\zeta \in Z(M_1, k)} \sum_{\epsilon \in U_0^{M_1(k)}} \omega_{\psi, x_1}(w_1^{-1} \zeta \epsilon u g, \gamma h)\Phi(x_1) \, du
\]

\[
= \int_{U_0^{M_1(k)}} \sum_{\zeta \in Z(M_1, k)} \omega_{\psi, x_1}(\zeta w_1^{-1} u g, \gamma h)\Phi(x_1) \, du.
\]

Hence the lemma follows. \( \square \)
3.3.8. Proof of Theorem 3.18. For \( s \in \mathbb{C} \), define

\[
F_{\Phi}(h, s)(g) := \int_{\mathcal{A}} \omega_{\psi, \chi_1}(g, h) \Phi(ax_1) \chi_1 \chi_2^{-1}(a)|a|^s_{h, k}, \quad \Phi \in \mathcal{S}^0(V^{\mathbb{R}, 2}).
\]

Since the main part of the integral is \( L_k'(s, \chi_1 \chi_2^{-1}) \) and \( \chi_1 \neq \chi_2 \), this converges absolutely for \( \text{Re}(s) > 0 \) and extends to an entire function on \( s \). Moreover, it satisfies

\[
F_{\Phi}(\left(\begin{array}{c} t \\ \sigma(t)^{-1} \end{array}\right) h, s)(g) = \tilde{\sigma}(\chi_1^{-1} \chi_2)(t)|t|^{-s}_{h, k} F_{\Phi}(h, s)(g),
\]

\[
F_{\Phi}(h, s)(m(\left(\begin{array}{c} x \\ 0 \end{array}\right) y)) = \chi_1(x) \chi_2(y)|x|_{h, k} |y|^{-s}_{h, k} F_{\Phi}(h, s)(g).
\]

That is, \( F_{\Phi}(h, s)(g) \) is a holomorphic section of \( A(B^{-}_{\mathcal{H}}(\mathbb{A}) \backslash H(\mathbb{A}))(\mathbb{H}^{-1} \mathbb{H}^{-1}) \mathbb{H}^{-1} |h, k| \mathbb{H}^{-1} |h, k| \mathbb{H}^{-1} \mathbb{H}^{-1} \) and of \( A(P_0(\mathbb{A}) \backslash M_1(\mathbb{A}))(\chi_1 |1/2 \otimes \chi_2 |3/2 - s)|h, k| \mathbb{H}^{-1} |h, k| \mathbb{H}^{-1} \mathbb{H}^{-1} \mathbb{H}^{-1} \mathbb{H}^{-1} \).

Recall that:

1. The Eisenstein series

\[
E(F_{\Phi}, h; s)(g) := \sum_{\gamma \in B_{\mathcal{H}}(k) \backslash H(k)} F_{\Phi}(\gamma h, s)(g)
\]

on \( H(k) \backslash H(\mathbb{A}) \) converges absolutely for \( \text{Re}(s) > 1 \) and can be meromorphically continued to the whole plane.

2. It has a simple pole at \( s = 1 \), and its residue there spans the one dimensional representation \( (\chi_1 \chi_2^{-1})^H \).

Write \( \Theta_{\chi_1, P_0} \) for the space of functions on \( M_0(k) \backslash M_0(\mathbb{A}) \times H(k) \backslash H(\mathbb{A}) \) spanned by \( \theta_{\Phi}(g, h)P_0 (\Phi \in \mathcal{S}^0(V^{\mathbb{R}, 2})) \). We consider the integral

\[
\int_{Z(M_1, k) \backslash Z(M_1, k)} \sum_{\gamma \in B_{\mathcal{H}}(k) \backslash H(k)} F_{\Phi}(\gamma h)(zg)
\]

\[
+ \sum_{\gamma \in B_{\mathcal{H}}(k) \backslash H(k)} \int_{U_0 \leq M_1(\mathbb{A})} F_{\Phi}(\gamma h)(w_1^{-1} uzg) du \chi_1 \chi_2^{-1}(z)|z|^s_{h, k} dz.
\]

If \( \text{Re}(s) > 1 \), then

\[
\sum_{\gamma \in B_{\mathcal{H}}(k) \backslash H(k)} \int_{Z(M_1, k) \backslash Z(M_1, k)} F_{\Phi}(\gamma h)(zg) \chi_1 \chi_2^{-1}(z)|z|^s_{h, k} dz
\]

\[
= \sum_{\gamma \in B_{\mathcal{H}}(k) \backslash H(k)} \int_{\mathbb{H}^\times / k^\times} \sum_{z \in Z(M_1, k)} \omega_{\psi, \chi_1}(\zeta g, h) \Phi(ax_1) \chi_1 \chi_2^{-1}(a)|a|^s_{h, k}, \quad da
\]

\[
= \sum_{\gamma \in B_{\mathcal{H}}(k) \backslash H(k)} F_{\Phi}(\gamma h, s)(g) = E(F_{\Phi}, h; s)(g)
\]
and
\[\sum_{\gamma \in B_H(k)\backslash H(k)} \int_{Z(M_1,k)\backslash Z(M_1,k)} F_{\Phi}(\gamma h)(w_1^{-1}u g) \, dz \, du = \sum_{\gamma \in B_H(k)\backslash H(k)} \int_{U^{M_1}_0(k)\backslash U^{M_1}_0(k)} F_{\Phi}(\gamma h, s)(w_1^{-1}u g) \, du = \sum_{\gamma \in B_H(k)\backslash H(k)} M^{M_1}(w_1, \chi_1 | 1/2 \otimes \chi_2 | 3/2 - s) F_{\Phi}(\gamma h, s)(g) \]

are absolutely convergent and are equal to
\[\int_{Z(M_1,k)\backslash Z(M_1,k)} \sum_{\gamma \in B_H(k)\backslash H(k)} F_{\Phi}(\gamma h)(z g) \chi_1 \chi_2^{-1}(z) |z|^{-s}_{k_{H', k}}, \, dz, \]

respectively. Thus (3.35) is well-defined for Re(s) > 1 and can be meromorphically continued to define an intertwining operator
\[A(s) : \Theta_{X_1, P_0} \otimes \Theta_{\Phi, h} P_0 \rightarrow E(F_{\Phi}, h; s)(g) + E(M^{M_1}_1 \otimes \chi_2 | 1/2 \otimes \chi_2 | 3/2 - s) F_{\Phi}, h; s)(g) \]

in \(A(H(k)\backslash H(A))\), for \(s \in \mathbb{C}\).

From (2) above, we know that \(A(s)\) has a simple pole at \(s = 1\), and the image of the residue \(\text{Res}_{s=1} A(s)\) is the constant term of the automorphic representation corresponding to \((\chi_1 \chi_2^{-1})^H\) by the Howe duality. But as a function on \(G(k)\backslash G(A)\), Im \(\text{Res}_{s=1} A(s)\) is still of the form
\[\phi(g) + M(w_1, \chi_1 | 1/2 \otimes \chi_2 | 1/2 - s) \phi(g)\]

for some \(\phi \in A(P_0(k)\backslash M_1(A)) \chi_1 | 1/2 \otimes \chi_2 | 1/2 - s). \) This is precisely the constant term of the global Langlands quotient of \(\text{Ind}_{P_0(k)\backslash G(A)}^{G(k)} [S_{2, \chi} \otimes 1_{V_0(k)}]\), and we are done. (q.e.d.)

4. The contributions of cuspidal data attached to \(P_1\)

In this section we describe the contribution of cuspidal data attached to the Siegel parabolic subgroup \(P_1 = M_1 U_1\) to the residual spectrum.

4.1. Singular hyperplanes. We retain the notation defined at the beginning of the previous section. In particular we take \(X \in E_{P_1}\) and \((M_1, \Psi) \in X\). For \(\phi \in P(M_1, \Psi)\) and \(\phi' \in P_X\) we consider
\[\langle \theta_{\phi}, \theta_{\phi'} \rangle_{L^2(G(k)\backslash G(A))} = \int_{\pi \in \mathcal{P}, \text{Re} \pi = \lambda_0} A(\phi, \phi')(\pi) \, d\pi, \]

where
\[A(\phi, \phi') = \sum_{(M_1, \Psi') \in X} \left( \sum_{w \in W(\Psi, \Psi')} M(w, \pi) \phi(\pi), \phi'(-w\pi) \right). \]
In the Siegel parabolic case $W_{M_1}$ consists of 1 and $w(P_1) := w_2 w_1 w_2$ hence the singular hyperplanes for $A(\phi, \phi')(\pi)$ are those of $M(w(P_1), \pi) \phi(\pi)$.

4.1.1. **Analytic behavior of the local intertwining operator.** Take $\pi = \bigotimes_v \pi_v \in \mathfrak{P}$. We begin by reviewing the normalization factor for $M(w(P_1), \pi_v)$ ([Sh]). Recall that our $M_1, v$ is isomorphic to $\text{Res}_{k'/k_v} GL(2)$. We have the following two cases.

1. **At inert $v$.** Let $w | v$ be as before and write $\Gamma_v := \text{Gal}(k'_w / k_v)$. We identify the generator of $\Gamma_v$ with $\sigma$. Just as in the global case, one has $L(G_v = \hat{G} \rtimes_{\theta_2} W_{k_v})$, where the Weil group $W_{k_v}$ acts on $\hat{G}$ through $\Gamma_v$ by $\sigma = \theta_2$. Hence we have the $L$-isomorphism

$$L \longrightarrow \frac{L_{M_1} \cong \begin{pmatrix} A & 0_2 \\ 0_2 & D \end{pmatrix}}{w} \longrightarrow (A, t D^{-1}) \times w \in L(\text{Res}_{k'_w / k_v} GL(2)),$$

and we identify $L_{M_1}$ with $L(\text{Res}_{k'_w / k_v} GL(2))$ by this.

In the notation of [Sh], we have $(U_1)_{w(P_1)} = U_1$. The adjoint representation $r_{w(P_1)}$ of $L_{M_1}$ on $\text{Lie}((U_1)_w^{\vee}(P_1)) \cong \mathfrak{m}_2(C)$ is given by

$$r_{w(P_1)} = \text{St}_2 \otimes \text{St}_2, \quad r_{w(P_1)}(1 \times \sigma)(v_1 \otimes v_2) = v_2 \otimes v_1,$$

where $\text{St}_2$ denotes the standard representation of $GL(2, C)$. The $L$-function attached to $\pi_v$ and $r_{w(P_1)}$ is the twisted tensor $L$-function $L_{\text{Asai}(s, \pi_v)} ([As], [HLR])$. The normalization factor $r(w(P_1), \pi_v)$ for $M(w(P_1), \pi_v)$ is defined by

$$r(w(P_1), \pi_v) := \frac{L_{\text{Asai}(0, \pi_v)}}{L_{\text{Asai}(1, \pi_v) \in \text{Asai}(0, \pi_v, \psi_v)}}.$$

2. **At split $v$.** Let $w_1, w_2 | v$ be as before and write

$$\pi_v = \pi_{w_1} \otimes \pi_{w_2} : M_1(k_v) \cong \begin{pmatrix} A & 0_2 \\ 0_2 & D \end{pmatrix} \longrightarrow \pi_{w_1}(A) \otimes \pi_{w_2}(t D^{-1}) \in GL(V_{\pi_v}).$$

The normalization factor in this case is well-known and given by

$$r(w(P_1), \pi_v) := \frac{L(0, \pi_{w_1} \times \pi_{w_2})}{L(1, \pi_{w_1} \times \pi_{w_2}) \in (0, \pi_{w_1} \times \pi_{w_2}, \psi_v)},$$

where $L(s, \pi_{w_1} \times \pi_{w_2})$ and $\varepsilon(s, \pi_{w_1} \times \pi_{w_2}, \psi_v)$ are the Rankin product $L$-factor and its root number for $\pi_{w_1}$ and $\pi_{w_2}$ defined in [Ja].

**Lemma 4.1.** The local normalized intertwining operator

$$N(w(P_1), \pi_v) := r(w(P_1), \pi_v)^{-1} M(w(P_1), \pi_v)$$

is holomorphic on the closed positive chamber $\{\pi_v \in \mathfrak{P}_v : (\text{Re} \pi_v, \beta_v) \geq 0\}$. (Note that $\beta_1$ is the only positive root of $A_{M_1}$ in $P_1$.)

**Proof.** At splitting $v$, this was proved in [MW2]. Hence we assume $v$ is inert in $k'$. When $\pi_v$ is essentially tempered, this follows from [Ar2], Theorem 2.1, if $v$ is archimedean, and from Proposition 7.2 in [Sh] if $v$ is finite. Thus we may assume that $\pi_v$ is in the complementary series. Then $\pi_v$ is of the form

$$\pi_v = \text{Ind}_{(P_0 \cap M_1)(k_v)}^{M_1(k_v)} [(\mu_w | \mu_w) \otimes |\mu_w|^2 / |\mu_w|, 1_{(U_0 \cap M_1)(k_v)}],$$
where $\mu_w$ is a quasi-character of $k^*_{w}$ and $0 < \lambda < 1$. From 3.1.1, $M(w(P), \pi_v)$ and its normalization factor can be written as

$$M(w(P), \pi_v) = M(w_2, \overline{\sigma}(\mu_w^{-1}) |_{w_2}^{1/2} \otimes \mu_w |_{w_2}^{1/2})M(w_1, \mu_w |_{w_1}^{1/2} \otimes \overline{\sigma}(\mu_w^{-1}) |_{w_1}^{1/2}) \times M(w_2, \mu_w |_{w_2}^{1/2} \otimes \mu_w |_{w_2}^{1/2}),$$

$$r(w(P), \pi_v) = r(w_2, \overline{\sigma}(\mu_w^{-1}) |_{w_2}^{1/2} \otimes \mu_w |_{w_2}^{1/2})r(w_1, \mu_w |_{w_1}^{1/2} \otimes \overline{\sigma}(\mu_w^{-1}) |_{w_1}^{1/2}) \times r(w_2, \mu_w |_{w_2}^{1/2} \otimes \mu_w |_{w_2}^{1/2}).$$

Since $(\Re \pi_v, \beta'_1) \geq 0$ is equivalent to $\Re \mu_w \geq 0$, both of

$$r(w_2, \overline{\sigma}(\mu_w^{-1}) |_{w_2}^{1/2} \otimes \mu_w |_{w_2}^{1/2})^{-1}M(w_2, \overline{\sigma}(\mu_w^{-1}) |_{w_2}^{1/2} \otimes \mu_w |_{w_2}^{1/2}),$$

$$r(w_1, \mu_w |_{w_1}^{1/2} \otimes \overline{\sigma}(\mu_w^{-1}) |_{w_1}^{1/2})^{-1}M(w_1, \mu_w |_{w_1}^{1/2} \otimes \overline{\sigma}(\mu_w^{-1}) |_{w_1}^{1/2})$$

are holomorphic on the region $(\Re \pi_v, \beta'_1) \geq 0$. On the other hand, it follows from [Sh] Corollary 7.6, that both

$$r(w_2, \mu_w |_{w_2}^{1/2} \otimes \mu_w |_{w_2}^{1/2})^{-1}$$

and $M(w_2, \mu_w |_{w_2}^{1/2} \otimes \mu_w |_{w_2}^{1/2})$

have a simple pole at $\mu_w = |_{w_2}^{1/2}$, and are holomorphic and non-zero at other $\pi_v$ in our positive cone.

4.1.2. Analytic behavior of the global intertwining operator. As in §3, we set

$$r(w(P), \pi) := \prod_v r(w(P), \pi_v), \quad N(w(P), \pi) := r(w(P), \pi)^{-1}M(w(P), \pi).$$

The following proposition will be proved in Appendix A.

**Proposition 4.2.** For an irreducible representation $\pi$ of $GL(2, k_v)$, let $\omega_\pi$ be its central character, and let $H$ be the diagonal subgroup $GL(2)_k$ of $Res_{k/k} GL(2)$. Then the only pole of $r(w(P), \pi)$ in the region $(\Re \pi_v, \beta'_1) \geq 0$ occurs at

$$\mathcal{G}(P) := \left\{ \pi \in \mathfrak{g} \mid (i) \omega_\pi |_{\Delta^*} = |_{\mathfrak{h}^*}^2, \quad (ii) \int_{H(k)}|Z(H, k) \backslash H(k)} f(h) |\det(h)|^{-1}_{\mathfrak{h}} dh \neq 0 \right\},$$

and it is simple.

4.1.3. Singular hyperplanes. From Proposition 4.2, we have $S^h_{(M, \mathfrak{g})} = \{ \mathcal{G}(P) \}$.

4.2. Decomposition of the scalar product. In this case $S^\pi_{(M, \mathfrak{g})} = \{ \mathfrak{g}, \mathcal{G}(P) \}$, and we set $o(\mathfrak{g}) := 0$.

**Theorem 4.3.** For $\phi \in P(M, \mathfrak{g})$ and $\phi' \in P_X$, we have

$$\langle \theta_0, \theta_{\phi'} \rangle_{L^2(G(k) \backslash G(A))} = \int_{\pi \in \mathfrak{g}, \Re \pi = 0} A(\phi, \phi')(\pi) d\pi$$

$$+ c_1 \langle N(w(P), \mathcal{G}(P)), \phi(\mathcal{G}(P)), \phi'(-w(P), \overline{\mathcal{G}(P)}) \rangle$$

for some non-zero constant $c_1$.

**Proof.** Since our $P_1$ is a maximal parabolic subgroup, we can apply Lemma 101 in [HC]. This allows us to apply the usual residue theorem to $A(\phi, \phi')$, and the assertion is obvious. \qed
4.3. The residual spectrum from $P_1$. By Theorem 4.3, it is enough to determine the image

$$\text{Im } N(w(P_1), \mathfrak{S}(P_1)) = \bigotimes_v \text{Im } N(w(P_1), \mathfrak{S}(P_1)_v).$$

If $\mathfrak{S}(P_1)_v$ is essentially tempered, then $\text{Im } N(w(P_1), \mathfrak{S}(P_1)_v)$ is irreducible by the Langlands classification. Next we handle $\mathfrak{S}(P_1)_v$ in the complementary series. Then its exponent is of the form

$$\left\{ \begin{array}{ll}
\| (1+\lambda)/2 \| \otimes \| (1-\lambda)/2 \| (0 < \lambda < 1) & \text{if } v \text{ is inert}, \\
\| (1+\lambda)/2 \| \otimes \| (1-\lambda)/2 \| \otimes \| (1+\mu)/2 \| \otimes \| (1-\mu)/2 \| (0 < \lambda, \mu < 1) & \text{if } v \text{ splits}.
\end{array} \right.$$ 

In both cases this is regular, and

$$\text{Im } N(w(P_1), \mathfrak{S}(P_1)_v) = \text{Ind}_{P_1(k_v)}^{G(k_v)} [\mathfrak{S}(P_1)_v \otimes \mathbf{1}_{U_1(k_v)}]$$

is irreducible. We call the irreducible representation $\text{Im } N(w(P_1), \mathfrak{S}(P_1))$ the global Langlands quotient of $\text{Ind}_{P_1(k)}^{G(k)} [\mathfrak{S}(P_1) \otimes \mathbf{1}_{U_1(k)}]$. Then our conclusion is

**Theorem 4.4.** The contribution of the cuspidal data attached to $P_1$ to the residual discrete spectrum is multiplicity free and consists of the global Langlands quotients of $\text{Ind}_{P_1(k)}^{G(k)} [\mathfrak{S}(P_1) \otimes \mathbf{1}_{U_1(k)}]$, where $\mathfrak{S}(P_1)$ was defined in Proposition 4.2.

5. The contributions of cuspidal data attached to $P_2$

In this section we study the contributions of cuspidal data attached to the non-Siegel maximal parabolic subgroup $P_2$ to the residual spectrum. For this purpose, it is more convenient to redefine our group $G = U(2, 2)_{k'/k}$ by replacing $J_2$ in 2.1.3 by

$$J_2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$ 

Then the Borel subgroup $P_0$ consists of upper triangular elements in $G$ and the Cartan subgroup $M_0$ consists of elements of the form

$$d(x_1, x_2) := \text{diag}(x_1, x_2, \sigma(x_1^{-1}), \sigma(x_2^{-1})).$$

Also $P_2 = M_2 U_2$ and $\hat{P}_2 = \hat{M}_2 \hat{U}_2$ become

$$M_2 = \left\{ \begin{pmatrix} x & A \\ \sigma(x^{-1}) & \end{pmatrix} \Bigg| x \in \text{Res}_{k'/k} \mathbb{G}_m, A \in U(1, 1)_{k'/k} \right\},$$

$$U_2 = \left\{ \begin{pmatrix} 1 & * & * \\ * & 1 \\ * & 1 \end{pmatrix} \in G \right\},$$

$$\hat{M}_2 = \left\{ \begin{pmatrix} x & A \\ y \end{pmatrix} \Bigg| x, y \in \mathbb{G}_m(\mathbb{C}), A \in GL(2, \mathbb{C}) \right\},$$

$$\hat{U}_2 = \left\{ \begin{pmatrix} 1 & * & * \\ * & 1 \end{pmatrix} \in \hat{G} \right\}.$$
Here the group $U(1,1)_{k'/k}$ is defined by

$$U(1,1)_{k'/k} := \{ g \in \text{Res}_{k'/k} GL(2) : g J_1 \overline{\sigma}(g) = J_1 \},$$

where $J_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We conventionally write $G_2$ for this group.

5.1. Singular hyperplanes. We take $\mathfrak{X} \in \mathfrak{E}_{P_2}$, $(M_2, \mathfrak{P}) \in \mathfrak{X}$ and consider

$$(\theta_{\phi}, \theta_{\phi'})_{L^2(G(k) \backslash G(\mathfrak{A}))} = \int_{\pi \in \mathfrak{P}, \text{Re } \pi = \lambda_0} A(\phi, \phi')(\pi) \, d\pi,$$

where

$$A(\phi, \phi')(\pi) := \sum_{(M_2, \mathfrak{P'}) \in \mathfrak{X}} \langle \sum_{w \in W(\mathfrak{P}, \mathfrak{P'})} M(w, \pi) \phi(w), \phi'(w) \rangle$$

for $\phi \in P_{(M_2, \mathfrak{P})}$, $\phi' \in P_{\mathfrak{X}}$. Since $W_{M_2}$ consists of 1 and $w(P_2) := w_1 w_2 w_1$, the analytic behavior of $A(\phi, \phi')(\pi)$ is determined by that of $M(w(P_2), \pi)$.

5.1.1. Analytic behavior of the local intertwining operator. Take $\pi = \otimes_v \pi_v \in \mathfrak{P}$ and consider $M(w(P_2), \pi)\phi(\pi) = \otimes_v M(w(P_2), \pi_v)\phi_v(\pi_v)$. We first recall the normalization factor $r(w(P_2), \pi_v)$ defined in [Sh]. We write $\pi \in \mathfrak{P}$ as $\chi \otimes \tau$, where $\chi$ is a quasi-character of $\mathbb{A}_k^\times / k^\times$ and $\tau$ is an irreducible cuspidal automorphic representation of $G_2(\mathbb{A})$.

(1) At inert $v$. Let $w|u$ be as before and $\Gamma_v$ be as in 4.1.1. We use the isomorphism

$$L M_2 \cong \begin{pmatrix} x & \ A \\ y & \end{pmatrix} \times w \xrightarrow{\sim} [(x, y^{-1}) \times A] \times w \in L(\text{Res}_{k'/k} \mathbb{G}_m \times G_2)$$

to identify these $L$-groups. Then the adjoint representation $r_{w(P_2)}$ of $L M_2$ on $\text{Lie}(U_2)^0_{w(P_2)} = \text{Lie} \hat{U}_2$ decomposes into the direct sum $r_{w(P_2),1} \oplus r_{w(P_2),2}$, where

$$r_{w(P_2),1}|_{\hat{M}_2} = [(\text{St}_1 \otimes 1) \otimes \hat{S}_2] \oplus [(1 \otimes \text{St}_1) \otimes S_2],$$

$$r_{w(P_2),1}(\sigma)(v_1 \otimes v_2) = v_2 \otimes v_1, r_{w(P_2),2}|_{\hat{M}_2} = (\text{St}_1 \otimes \text{St}_2) \otimes 1,$$

$$r_{w(P_2),2}(\sigma)(v_1 \otimes v_2 \otimes u) = (v_2 \otimes v_1) \otimes u.$$ 

Here $\text{St}_1$ denotes the standard representation of $\mathbb{G}_m(\mathbb{C})$ and $\hat{S}_2$ is the contragredient of $S_2$. The $L$-factor attached to $\pi_v = \chi_v \otimes \tau_v$ and $r_{w(P_2),1}$ is the product $L$-factor of $\tau_v \chi_v$;

$$L(s, \pi_v, r_{w(P_2),1}) = L(s, \tau_v \times \chi_v).$$

The $L$-factor attached to $\pi_v$ and $r_{w(P_2),2}$ is simply $L_k(s, \chi_v | k^\times)$. Thus from [Sh], §7, we have

$$r(w(P_2), \pi_v) = \frac{L(0, \tau_v \times \chi_v)}{L(1, \tau_v \times \chi_v)\varepsilon(0, \tau_v \times \chi_v, \psi_v) \overline{L_k(1, \chi_v k^\times)} \varepsilon_k(0, \chi_v k^\times, \psi_v)}.$$

(2) At splitting $v$. Again let $w_1$, $w_2|v$ be as before. We identify $M_{2,v}$ with $(\mathbb{G}_m \times \mathbb{G}_m) \times GL(2)$ by

$$M_2 \cong \begin{pmatrix} x & \ A \\ y & \end{pmatrix} \xrightarrow{\sim} (x, y^{-1}) \times A \in (\mathbb{G}_m \times \mathbb{G}_m) \times GL(2).$$
Accordingly $\pi_v = \chi_v \otimes \tau_v$ is written as $(\chi_{w_1} \otimes \chi_{w_2}) \otimes \tau_v$. The normalization factor in this case is well-known and is given by

$$r(w(P_2), \pi_v) = \frac{L(0, \pi_v, r_w(P_2), \psi_v)}{L(1, \pi_v, r_w(P_2), 1, \psi_v) L(1, \pi_v, r_w(P_2), 2, \psi_v)},$$

where, writing $\bar{\tau}_v$ for the contragredient of $\tau_v$,

$$L(s, \pi_v, r_w(P_2), 1) = L(s, \bar{\tau}_v \otimes \chi_{w_1}) L(s, \pi_v \otimes \chi_{w_2}),$$

$$\varepsilon(s, \pi_v, r_w(P_2), 1, \psi_v) = \varepsilon(s, \bar{\tau}_v \otimes \chi_{w_1}, \psi_v) \varepsilon(s, \tau_v \otimes \chi_{w_2}, \psi_v),$$

$$L(s, \pi_v, r_w(P_2), 2) = L_k(s, \chi_{w_1} \chi_{w_2}), \quad \varepsilon(s, \pi_v, r_w(P_2), 2, \psi_v) = \varepsilon_k(s, \chi_{w_1} \chi_{w_2}, \psi_v).$$

**Lemma 5.1.** The normalized operator

$$N(w(P_2), \pi_v) := r(w(P_2), \pi_v)^{-1} M(w(P_2), \pi_v,v)$$

is holomorphic on the region $\langle \Re \pi, \beta^\vee_2 \rangle \geq 0$. (Note that $\beta_2$ is the only positive root of $A_{M_2}$ in $P_2$.)

**Proof.** If $v$ splits in $k'$, this was proved in [MW2]. Hence we assume $v$ is inert in $k'$, and let $w|v$. First assume $\pi_v$ is essentially tempered. The archimedean case is a special case of Theorem 2.1 in [Ar2]. The non-archimedean case follows immediately from [Sh], Proposition 7.2 (b). Next comes $\pi_v$, with $\tau_v$ in the complementary series. Then $\tau_v$ is of the form

$$\tau_v = \Ind_{B_2(k_v)}^{G_2(k_v)}[\nu_v \otimes |\lambda/2 \otimes 1_{\mathcal{N}_2(k_v)}] \quad (0 < \lambda < 1),$$

where $\nu_v$ is a character of $k_v$ such that $|\nu_v|_{k_v} = 1$. $B_2 = T_2 N_2$ is the Borel subgroup consisting of upper triangular elements in $G_2$. It follows from

$$L(s, \pi_v, r_w(P_2), 1) = L(s - \lambda/2, \chi_v \nu_v^{-1}) L(s + \lambda/2, \bar{\tau}(\chi_v) \nu_v)$$

that $r(w(P_2), \pi_v)$ has its only pole in the region $\langle \Re \pi_v, \beta^\vee_2 \rangle \geq 0$ at $\chi_v = \nu_v |\lambda/2$, and it is simple. On the other hand, we apply Corollary 7.6 of [Sh] to $M(w_1, w_2 w_1 \pi_v)$ and note that

$$M(w(P_2), \pi_v) = M(w_1, w_2 w_1 \pi_v) M(w_2, w_1 \pi_v) M(w_1, \pi_v),$$

to see that $M(w(P_2), \pi_v)$ has its only pole in the positive region at $\chi_v = \nu_v |\lambda/2$, and it is simple. The lemma is proved. \hfill $\Box$

**5.1.2. Analytic behavior of the global intertwining operator.** For $\pi \in \mathfrak{P}$, we define

$$r(w(P_2), \pi) := \prod_v r(w(P_2), \pi_v), \quad N(w(P_2), \pi) := r(w(P_2), \pi)^{-1} M(w(P_2), \pi).$$

The following proposition will be proved in Appendix B.

**Proposition 5.2.** Here again, we write $\pi \in \mathfrak{P}$ as $\chi \otimes \tau$. The central character of $\tau$ is denoted by $\omega_{\tau}$. For each character $\mu$ of $U(1, A)_{k'/k}$, we write $\Theta(\xi, \psi)_{\mu}$ for the theta-lift of $\mu$ to $G_2(A)$ under the Weil representation $\omega_{\tau}, \xi$ of $G_2(A) \times U(1, A)_{k'/k}$. (See Appendix B for more details.) Then the only poles of $M(w(P_2), \pi)$ in the region $\langle \Re \pi, \beta^\vee_2 \rangle \geq 0$ are located at

$$\mathcal{S}(P_2, \eta_{k'/k}) := \{ \pi = \chi \otimes \tau \in \mathfrak{P} \colon \chi|_{A^\times} = |\frac{1}{2} \eta_{k'/k}, \tau = \Theta(\chi^{-1})|_{A_{k'}}, \psi_{\omega_{\tau}} \},$$

$$\mathcal{S}(P_2, 1) := \{ \pi = \chi \otimes \tau \in \mathfrak{P} \colon \chi|_{A^\times} = |\frac{1}{2}, L(0, \tau \times \chi) \neq 0 \},$$
and they are simple. Here in the definition of $\mathcal{G}(P_2, \eta_{k'/k})$, $\psi$ has been chosen so that $\tau_v$ is generic with respect to it.

Remark 5.3. If we change $\psi$ in the definition of $\mathcal{G}(P_2, \eta_{k'/k})$, then $\tau$ changes to another irreducible cuspidal representation in the same global $L$-packet (cf. [LL]).

5.1.3. Singular hyperplanes. From Proposition 5.2, we have

$$S^+_{(M_2, \Psi)} := \{ \mathcal{G}(P_2, \eta_{k'/k}), \mathcal{G}(P_2, 1) \}.$$  

5.2. Decomposition of the scalar product. We have

$$S^+ = \{ \Psi, \mathcal{G}(P_2, \eta_{k'/k}), \mathcal{G}(P_2, 1) \},$$

and we set $o(\Psi) := 0$. Then the following is proved in the same manner as Theorem 4.3.

Theorem 5.4. For $\phi \in P_{(M_2, \Psi)}$ and $\phi' \in P_\chi$, we have

\[ \langle \phi, \phi' \rangle_{L^2(G(k) \backslash G(A))} = \int_{\pi \in \Psi, \Re \pi = 0} A(\phi, \phi')(\pi) \, d\pi \]

\[ + c_2 N(w(P_2), \mathcal{G}(P_2, \eta_{k'/k}))(\phi(\mathcal{G}(P_2, \eta_{k'/k})), \phi'(w(P_2)\mathcal{G}(P_2, \eta_{k'/k}))) \]

\[ + c'_2 N(w(P_2), \mathcal{G}(P_2, 1))(\phi(\mathcal{G}(P_2, 1)), \phi'(w(P_2)\mathcal{G}(P_2, 1))) \]

for some non-zero constants $c_2$ and $c'_2$.

5.3. The residual spectrum from $P_2$. By Theorem 5.4, it is enough to determine the images

$$\text{Im } N(w(P_2), \mathcal{G}(P_2, \eta_{k'/k})) = \bigotimes_v \text{Im } N(w(P_2), \mathcal{G}(P_2, \eta_{k'/k})_v),$$

$$\text{Im } N(w(P_2), \mathcal{G}(P_2, 1)) = \bigotimes_v \text{Im } N(w(P_2), \mathcal{G}(P_2, 1)_v).$$

(1) $\text{Im } N(w(P_2), \mathcal{G}(P_2, \eta_{k'/k})_v)$. If $\mathcal{G}(P_2, \eta_{k'/k})_v$ is essentially tempered, then $\text{Im } N(w(P_2), \mathcal{G}(P_2, \eta_{k'/k})_v)$ is irreducible by the Langlands classification. Next we handle $\mathcal{G}(P_2, \eta_{k'/k})_v = \chi_v \otimes \tau_v$ with $\tau_v$ in the complementary series. Then its exponent is of the form

$$\left\{ \begin{array}{ll}
|w|^{\lambda/2} & (0 < \lambda < 1) \\
|v|^{\mu/2} & (0 < \mu < 1)
\end{array} \right. \quad \text{if } v \text{ inert,}
$$

$$\left\{ \begin{array}{ll}
|w|^{\lambda/2} & (0 < \lambda < 1) \\
|v|^{\nu/2} & (0 < \mu < 1)
\end{array} \right. \quad \text{if } v \text{ splits.}
$$

In both cases this is regular, and hence $\text{Im } N(w(P_2), \mathcal{G}(P_2, \eta_{k'/k})_v)$ is irreducible.

(2) $\text{Im } N(w(P_2), \mathcal{G}(P_2, 1)_v)$. Again the tempered case is covered by the Langlands classification. In the non-tempered case, the exponent of $\mathcal{G}(P_2, 1)_v$ is of the form

$$\left\{ \begin{array}{ll}
|w|^{\lambda/2} & (0 < \lambda < 1) \\
|v|^{\mu/2} & (0 < \mu < 1)
\end{array} \right. \quad \text{if } v \text{ inert,}
$$

$$\left\{ \begin{array}{ll}
|w|^{\lambda/2} & (0 < \lambda < 1) \\
|v|^{\nu/2} & (0 < \mu < 1)
\end{array} \right. \quad \text{if } v \text{ splits.}
$$

Again this is regular, and hence

$$\text{Im } N(w(P_2), \mathcal{G}(P_2, 1)_v) = \text{Ind}^{G(k_v)}_{P_2(k_v)}[\mathcal{G}(P_2, 1)_v \otimes 1_{U_2(k_v)}]$$

is irreducible.
The resulting irreducible representations
\[ \text{Im } N(w(P_2), \mathcal{S}(P_2, \eta_{k'/k})) \quad \text{and} \quad \text{Im } N(w(P_2), \mathcal{S}(P_2, 1)) \]
will be called the global Langlands quotients of \( \text{Ind}^{G(A)}_{P_2(A)}[\mathcal{S}(P_2, \eta_{k'/k}) \otimes 1_{U_2(A)}] \) and \( \text{Ind}^{G(A)}_{P_2(A)}[\mathcal{S}(P_2, 1) \otimes 1_{U_2(A)}] \) respectively.

**Theorem 5.5.** The contribution of cuspidal data attached to \( P_2 \) to the residual discrete spectrum is multiplicity free and consists of the global Langlands quotients defined above of \( \text{Ind}^{G(A)}_{P_2(A)}[\mathcal{S}(P_2, \eta_{k'/k}) \otimes 1_{U_2(A)}] \) and \( \text{Ind}^{G(A)}_{P_2(A)}[\mathcal{S}(\mathbb{Q}_2, 1) \otimes 1_{U_2(A)}] \), where \( \mathcal{S}(P_2, \eta_{k'/k}) \) and \( \mathcal{S}(P_2, 1) \) are as in Proposition 5.2.

**Appendix A. Poles of the twisted tensor \( L \)-function.**

In this appendix we determine the poles of the twisted tensor \( L \)-function in the half plane \( \text{Re}(s) > 0 \), and prove Proposition 4.2 as a corollary. In fact this is little more than Satz 3.13 in [HLR]. We begin with the integral representation of \( L_{A,s}(s, \pi) \).

**A.1. The zeta integral.**

**A.1.1. The zeta integral \( Z(f, \Phi, s) \) and its unfolding.** Let \( k'/k \) and \( \Gamma \) be as before. We write \( G_1 \) for \( \text{Res}_{k'/k} GL(2) \) and \( H \) for the \( k \)-subgroup \( GL(2)_k \) of \( G_1 \). We fix a Borel subgroup \( B = TN \) consisting of upper triangular elements, together with the Levi factor \( T \) consisting of diagonal elements in \( G_1 \). Set \( B_H := B \cap H, T_H := T \cap H \) and \( N_H := N \cap H \).

For \( \Phi \in \mathcal{S}(A^{\oplus 2}) \) and a quasi-character \( \omega \) of \( \mathbb{A}_k^x/k^x \) we define a meromorphic section \( f_\Phi(\omega, s; h) \) for \( \text{Ind}^{H(\mathbb{A})}_{B(\mathbb{A})}(|(s-1)/2 \otimes |(1-s)/2 \omega^{-1}) \otimes 1_{N_H(\mathbb{A})} \) as
\[
  f_\Phi(\omega, s; h) := |\det(h)|^{s/2} \int_{\mathbb{A}^x} \Phi([0, t]h)(\omega(t)t)^s dt^x \quad (h \in H(\mathbb{A})).
\]

Using this, we define the Eisenstein series
\[
  E_\Phi(\omega, s; h) := \sum_{\gamma \in B_H(k) \backslash H(k)} f_\Phi(\omega, s; h) \quad (h \in H(\mathbb{A})).
\]

This converges absolutely for \( \text{Re } \omega + \text{Re}(s) \gg 0 \), and can be meromorphically continued to the whole \( s \)-plane.

Now let \( \pi \) be an irreducible cuspidal automorphic representation of \( G_1(\mathbb{A}) \), and write \( \omega_\pi \) for its central character. We define the zeta integral \( Z(f, \Phi, s) \), where \( f \) is a cusp form in the space of \( \pi \) and \( \Phi \in \mathcal{S}(A^{\oplus 2}) \), by
\[
  (A.1) \quad Z(f, \Phi, s) := \int_{H(\mathbb{A}) \backslash Z(H, \mathbb{A})} f(h) E_\Phi(\omega_\pi, 2s; h) \, dh.
\]

Here \( Z(H) \) is the center of \( H \). Since \( f \) is rapidly decreasing and \( E_\Phi(\omega_\pi, 2s; h) \) is slowly increasing, this converges absolutely for \( \text{Re } \omega + \text{Re}(s) \gg 0 \) and can be meromorphically continued to the whole plane. Also note that the functional equation for \( Z(f, \Phi, s) \) can be deduced from that of \( E_\Phi(\omega_\pi, 2s; h) \) as was done in [As].

The zeta integral (A.1) is decomposed into the direct product of local zeta integrals as follows (cf. [HLR], pp. 76–77). We fix a non-trivial character \( \psi_{k'} \) of \( \mathbb{A}_{k'}/k' \).
such that $\psi_{k'}|_{A^\times} = 1$, and take the Whittaker model $W(\pi, \psi_{k'})$ for $\pi$. Then we have the Fourier expansion on $N_H(k) \setminus N_H(1)$:

$$f(h) = \sum_{\alpha \in k' \times} W_f(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} h) \quad \text{(for some } W_f \in W(\pi, \psi_{k'})).$$

Using this and the definition of $E_\Phi(\omega_\pi, 2s; h)$, we have

$$Z(f, \Phi, s) = \int_{N_H(k)Z(H, k) \setminus H(k)} \sum_{\alpha \in k \times} W_f(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} h) f_\Phi(\omega_\pi, 2s; h) \, dh.$$

We note that $W(h) := \sum_{\alpha \in k \times} W_f(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} h)$ is contained in $W(\pi, \psi_{k'})$. Further, we may assume that $W(h)$ and $\Phi$ are of the form

$$W(h) = \bigotimes_v W_v(h_v), \ W_v \in W(\pi_v, \psi_{k'}, v), \ \Phi(x) = \bigotimes_v \Phi_v(x_v), \ \Phi_v \in \mathcal{S}(k_v^{\times 2})$$

(restricted tensor products). Now we have

(A.2) $$Z(f, \Phi, s) = \prod_v Z_v(W_v, \Phi_v, s),$$

where

$$Z_v(W_v, \Phi_v, s) := \int_{N_H(k_v)Z(H, k_v) \setminus H(k_v)} W_v(h) f_\Phi(\omega_{\pi_v}, 2s; h) \, dh,$$

$$f_\Phi(\omega_v, s; h) := |\det(h)|^{s/2} \int_{k_v^\times} \Phi_v([0, t]h \omega_v(t)|t|^s dt^\times \in \text{Ind}_{B_H(k_v)}^H(k_v) [|] |(s-1)/2 \otimes \omega_{\pi_v}^{-1}|(1-s)/2 \otimes 1_{N_H(k_v)}]).$$

These local zeta integrals converge absolutely for $\text{Re}(\omega_{\pi_v}) + \text{Re}(s) > 0$, and can be meromorphically continued to the whole plane. (A.2) should be considered as a equality of these meromorphic functions.

A.2. Comparisons of local factors. Here we shall check that $Z_v(W_v, \Phi_v, s)$ has the same poles as the $L$-factor $L_{Asai}(s, \pi)$, at least in the region $\langle \Re \pi, \beta_1^\dagger \rangle > 0$. We treat various types of $\pi_v$ separately.

A.2.1. The case of splitting $v$. Let $w_1$ and $w_2$ be the places of $k'$ lying over $v$. By our choice of $\psi_{k'}$ we can take a non-trivial character $\psi_v$ of $k_v$ such that $\psi_{k', w_1} = \psi_v$ and $\psi_{k', w_2} = \psi_{\bar{v}}$. We identify $G(k_v)$ with $GL(2, k_v) \times GL(2, k_v)$ and write $\pi_v$ as $\pi_v = \pi_{w_1} \otimes \pi_{w_2}$ accordingly. Then we may assume $W_v = W_{w_1} \otimes W_{w_2}$, where $W_{w_1} \in W(\pi_{w_1}, \psi_v)$ and $W_{w_2} \in W(\pi_{w_2}, \bar{\psi}_v)$. Noting that $W(h) \simeq W(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h)$ gives an isomorphism from $W(\pi_{w_2}, \psi_v)$ to $W(\pi_{w_2}, \bar{\psi}_v)$, we have

$$Z_v(W_v, \Phi_v, s)$$

$$= \int_{N_H(k_v)Z(H, k_v) \setminus H(k_v)} W_{w_1}(h) W'_{w_2}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h) z(|2s \omega_{\pi}, h, \Phi)|\det(h)|^s dh.$$
(2) (Unramified situation.) We assume that \( \pi_v = \pi_{w_1} \otimes \pi_{w_2} \) is unramified, \( \Phi_v \) is \( \Phi_v^0 \), the characteristic function of \( \mathcal{O}_v \oplus \mathcal{O}_v \), and \( \psi_v \) is of order 0. Then for the class-1 Whittaker function \( W_v^0 \in \mathcal{W}(\pi_v, \psi_{k', v}) \) with \( W_v^0(1) = 1 \), we have
\[
Z_v(W_v, \Phi_v, s) = L(s, \pi_{w_1} \times \pi_{w_2}).
\]

A.2.2. The case of inert unramified \( v \). Let \( w | v \) be the place of \( k' \) lying over \( v \). We assume that
1. \( k'_{w_0} / k_v \) is an unramified quadratic extension,
2. \( \pi_v \) is of the form \( \text{Ind}_{B(k_v)}^{G_1(k_v)}((\mu \otimes \nu) \otimes 1_{S(k_v)}) \), where \( \mu \) and \( \nu \) are unramified quasi-characters of \( k_v \),
3. \( \Phi_v \) equals \( \Phi_v^0 \) (see A.2.1), and \( W_v^0 \in \mathcal{W}(\pi_v, \psi_{k', v}) \) is the class-1 Whittaker function for \( \pi_v \);
\[
W_v^0(1) = 1, \quad W_v^0(gk) = W_v^0(g) \quad \text{for all } k \in \text{GL}(2, \mathcal{O}_w).
\]

Lemma A.1 (Lemma 3.14 in [HLR]). Under these assumptions
\[
Z_v(W_v^0, \Phi_v^0, s) = L_{\text{Asai}}(s, \pi_v).
\]

A.2.3. The case of non-archimedean inert \( v \). Let \( w | v \) be as before. It follows from the well-known integration formula that
\[
Z_v(W_v, \Phi_v, s) = \int_{k_v} \int_{GL(2, \mathcal{O}_v)} W_v\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) f_{\Phi_v}(\omega_{\pi_v}, 2s; \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k)|a|^{-1} dk \, da^\times.
\]
But since
\[
f_{\Phi_v}(\omega_{\pi_v}, 2s; \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k) = |a|^s L(2s, \omega_{\pi_v}) P(q_w^s, q_v^{-s})
\]
with \( P(X, Y) \) a polynomial, and \( W_v(h) \) is \( \text{GL}(2, \mathcal{O}_v) \)-finite, it is enough to consider
\[
Z_v(\varphi_v, \Phi_v, s) := \int_{k_v} \varphi_v(a)|a|^{-1} da^\times L(2s, \omega_{\pi_v}) P(q_w^s, q_v^{-s}).
\]
Here \( \varphi_v \) is in the Kirillov model \( K(\pi_v, \psi_{k, v}) \) of \( \pi_v \) with respect to \( \psi_{k, v} \).
Recall that the space \( K(\pi_v, \psi_{k, v}) \) was calculated by Godement as follows.
1. If \( \pi_v = \text{Ind}_{B(k_v)}^{G_1(k_v)}((\mu \otimes \nu) \otimes 1_{S(k_v)}) \), then
\[
K(\pi_v, \psi_{k, v}) = \begin{cases} \{ f(a)\mu(a)|a|^{1/2} + g(a)\nu(a)|a|^{1/2} \mid f, g \in S(k'_w) \} & \text{if } \mu \neq \nu, \\
\{ f(a)\mu(a)|a|^{1/2} + g(a)\mu(a) \text{ord}_{k'_w}(a)|a|^{1/2} \mid f, g \in S(k'_w) \} & \text{if } \mu = \nu.
\end{cases}
\]
2. If \( \pi_v \) is a special representation embedded in \( \text{Ind}_{B(k_v)}^{G_1(k_v)}((\mu \otimes \nu) \otimes 1_{S(k_v)}) \) as a submodule, then
\[
K(\pi_v, \psi_{k, v}) = \{ f(a)\mu(a)|a|^{1/2} \mid f \in S(k'_w) \}.
\]
3. If \( \pi_v \) is supercuspidal, then \( K(\pi_v, \psi_{k, v}) = S(k'_w) \).
In the case (1), we see that
\[
Z_v(\varphi_v, \Phi_v, s) = L_k(s, \mu|k'_w) L_k(s, \nu_v|k'_w) L_k(2s, \omega_{\pi_v}) P(q_w^s, q_v^{-s})
\]
for some polynomial \( P'(X, Y) \), while we have from 3.1.1
\[
L_{\text{Asai}}(s, \pi_v) = L_k(s, \mu|k'_w) L_k(s, \nu_v|k'_w) L_k'(s, \omega_{\pi_v}).
\]
But both $L_k(2s, \omega_{\pi_v} | k_v^\times)$ and $L_k(s, \omega_{\pi_v})$ are holomorphic and non-zero on the region where $\langle \text{Re} \pi_v, \beta \rangle > 0$ and $\text{Re}(s) > 0$. Thus $Z_v(\varphi_v, \Phi_v, s)$ has the same poles as $L_{\text{Asai}}(s, \pi_v)$ in that positive region.

In the case (2), we have

$$Z_v(\varphi_v, \Phi_v, s) = L_k(s, \mu | k_v^\times)L_k(2s, \omega_{\pi_v} | k_v^\times) P'(q_v^s, q_v^{-s}).$$

On the other hand, the twisted tensor $L$-factor in this case was calculated in [G], Theorem 5.6:

$$L_{\text{Asai}}(s, \pi_v) = L_k(s, \mu | k_v^\times)L_k(s-1, (\mu | k_v^\times) \otimes \eta_{k_v'/k_v}).$$

Here again, both $L_k(2s, \omega_{\pi_v} | k_v^\times)$ and $L_k(s-1, (\mu | k_v^\times) \otimes \eta_{k_v'/k_v})$ are holomorphic and non-zero on the region where $\langle \text{Re} \pi_v, \beta \rangle = \text{Re}(\mu^2 | [-1]_w) > 0$ and $\text{Re}(s) > 0$. Hence $Z_v(\varphi_v, \Phi_v, s)$ and $L_{\text{Asai}}(s, \pi_v)$ have the same poles in this region.

In case (3), the G.C.D. of $Z_v(\varphi_v, \Phi_v, s)$ is $L_k(2s, \omega_{\pi_v} | k_v^\times)$. This is holomorphic on our positive region. On the other hand, [Sh], Proposition 7.2 (a), asserts that $L_{\text{Asai}}(s, \pi_v)$ is also holomorphic in that region.

A.2.4. The case of archimedean ramified $v$. Since $k_v'/k_v = \mathbb{C}/\mathbb{R}$, we may assume $\pi_v = \text{Ind}_{G(k_v)} \left[ (\mu \otimes \nu) \otimes 1_{\text{N}(k_v)} \right]$. As usual, we write $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2)$ and $L(s, \chi) := \Gamma_{\mathbb{R}}(s+r+m)$ for $\chi = | \det \nu |_{\mathbb{R}}^{s/2}$ $(r \in \mathbb{R}, m \in \{0,1\})$. We may assume

$$f_{\varphi_v}(\omega_{\pi_v}, 2s; \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) = e^{-\sqrt{-1}\pi \nu \theta} a_\mathbb{R} L(2s, \omega_{\pi_v} | k_v^\times) P(s),$$

$$W_v \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k(\theta) \right) = e^{-\sqrt{-1}\pi \nu \theta} W_v \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right), \quad k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},$$

where $P(s)$ is a holomorphic function. Then it is enough to consider

$$Z(\Phi_w, \Phi_v, s) := \int_{\mathbb{R}^x} \varphi_{\Phi_w}(a) a_\mathbb{R}^{-1} da^x L(2s, \omega_{\pi_v} | k_v^\times),$$

with

$$\varphi_{\Phi_w}(a) = W_{\Phi_w} \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right), \quad \Phi_w \in \mathcal{S}(\mathbb{C}^{\oplus 2}),$$

$$W_{\Phi_w}(g) := \mu(\det(g)) |\det(g)|^{1/2} \int_{\mathbb{C}^x} |r(g)\Phi_w|(z, z^{-1}) \mu \nu^{-1}(z) dz^x.$$
decomposing $\mathbb{C}^\times$ into $\mathbb{R}^\times \times (\mathbb{C}^1/\pm 1)$,

$$
\begin{align*}
&= \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \int_{\mathbb{C}^1/\pm 1} \mu(ax^{-1}u^{-1})|a|_R^s \Phi_w(ax^{-1}u^{-1}, xu)\nu(xu) \, du \, dx^\times \, da^\times \\
&= \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \left( \int_{\mathbb{C}^1/\pm 1} \mu^{-1}(u)\Phi_w(au^{-1}, xu) \, du \right) \mu(a)|a|_R^s \nu(x)|x|_R^a \, da^\times \, dx^\times 
\end{align*}
$$

writing $\Phi'_w(a, x)$ for the quantity inside the large parentheses,

$$
\begin{align*}
&= \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \Phi'_w(a, x)\mu(a)|a|_R^s \nu(x)|x|_R^a \, da^\times \, dx^\times
\end{align*}
$$

Since the $\Phi'_w(a, x)$'s span a dense subspace of $\mathcal{S}(\mathbb{R}^{\mathbb{S}^2})$, the G.C.D. of these integrals is $L(s, \mu|_{k^+})L(s, \nu|_{k^+})$. Again noting that $L_k(2s, \omega_{\pi}, |R^\times|)$ and $L_k(s, \omega_{\pi^v})$ are holomorphic on the region where $(\text{Re } \pi, \beta''_1) > 0$ and $\text{Re}(s) > 0$, we conclude that $Z(\Phi_w, \Phi_v, s)$ has the same pole as $L_{\text{Asai}}(s, \pi_v)$ has.

**A.3. Poles of the twisted tensor $L$-function.** We now state the main result of this appendix.

**Proposition A.2.** Let $\pi$ be an irreducible unitary cuspidal automorphic representation of $G_1(\mathbb{A})$, and write $\omega_\pi$ for its central character. Then the only possible pole of $L_{\text{Asai}}(s, \pi)$ in the region

$$(\text{Re } \pi, \beta''_1) > 0, \quad \text{Re}(s) > 0$$

is located at $s = 1$ with $\omega_\pi|_{A^\times} = 1$. It is a pole if and only if

$$
\int_{H(\mathbb{A})^\times} f(h) \, dh \neq 0, \quad \text{for some } f \in V_\pi.
$$

**Proof.** By A.2, it is sufficient to determine the poles of $Z(f, \Phi, s)$ in the positive domain of the proposition. This was done in [HLR], Satz 3.13. \qed

**Corollary A.3** (Proposition 4.2). Let the notation be as above. Then the only pole of $r(w(P_1), \pi)$ in the region $(\text{Re } \pi, \beta''_1) \geq 0$ occurs at

$$
\mathcal{S}(P_1) := \left\{ \pi \in A_0(G_1(\mathbb{A}) \backslash G_1(\mathbb{A})) \mid \begin{array}{l}
(i) \omega_\pi|_{A^\times} = |\beta''_1|_A, \\
(ii) \int_{H(\mathbb{A})^\times} f(h) |\det(h)|^{-1} \, dh \neq 0
\end{array} \right\},
$$

and it is simple.

**Proof.** Recall that

$$
r(w(P_1), \pi) = \frac{L_{\text{Asai}}(0, \pi)}{L_{\text{Asai}}(1, \pi)} \varepsilon_{\text{Asai}}(0, \pi).
$$

The infinite product $\varepsilon_{\text{Asai}}(s, \pi) = \prod_v \varepsilon_{\text{Asai}}(s, \pi_v, \psi_v)$ is in fact a finite product ([Sh], Theorem 3.5 (1)), and each local factor is an exponential function. Thus $\varepsilon_{\text{Asai}}(0, \pi)$ contributes nothing to the analytic property of $r(w(P_1), \pi)$. On the other hand, $L_{\text{Asai}}(1, \pi)$ does not vanish in our positive region by Shahidi's non-vanishing theorem ([Sh3], Theorem 5.1). Thus the poles of $r(w(P_1), \pi)$ are the same as those of $L_{\text{Asai}}(0, \pi)$, and the corollary follows from Proposition A.2. \qed
APPENDIX B. POLES OF THE PRODUCT L-FUNCTION OF $U(1,1) \times \text{Res}_{E/F} \mathbb{G}_m$

Here we shall determine the poles of the product $L$-function of $G_2 \times \text{Res}_{E/F} \mathbb{G}_m$ and deduce Proposition 5.2 as a corollary. For this we use the Shimura type integral representation (cf. [GJ]), and hence we begin with a review of theta series on $G_2(A)$.

B.1. Oscillator representations for $U(1,1) \times U(1)$.

B.1.1. Preliminaries for the local theory. Let $F$ be a local field of characteristic 0, $E$ a 2-dimensional abelian semisimple algebra over $F$, and $\{1, \epsilon\}$ a basis of $E$ over $F$. We may assume that $\epsilon^2 = -\delta \in F^\times$. We write $\sigma$ for the non-trivial involution of $E$ trivial on $F$. We write $|.|_F$ for the modulus of $F$. If $F$ is non-archimedean we write $\mathcal{O}_F$, $\mathfrak{p}_F$, $\mathcal{W}_F$ and $q_F$ for the maximal compact subring of $F$, the maximal ideal in $\mathcal{O}_F$, a generator of $\mathfrak{p}_F$ and the cardinality of the residue field of $F$, respectively. $\eta_{E/F}$ denotes the quadratic character of $F^\times$ which corresponds to $E/F$ by class field theory.

The group $G_2$ to be considered is defined by

$$G_2 = U(1,1)_E/F := \{g \in \text{Res}_{E/F} \text{GL}(2); g \delta \bar{\sigma}(g) = J_1\},$$

where $J_1$ is as in the beginning of §5. This is attached to the skew-Hermitian space $(V,\Phi) = (E^\otimes 2, J_1)$. We define the symplectic space $(W,\phi)$ over $F$ by $W := \text{Res}_{E/F} V$ and $\phi(\cdot,\cdot) := \text{Tr}_{E/F} \Phi(\cdot,\cdot)$, and write $Sp(W)$ for its symplectic group. Of course we have the natural embedding $\iota : G_2 \hookrightarrow Sp(W)$. We fix a Borel pair $(B_2,T_2)$ of $G_2$ such that $B_2$ consists of upper triangular elements and $T_2$ consists of $d(a) := \text{diag}(a,\bar{\sigma}(a)^{-1})$ ($a \in \text{Res}_{E/F} \mathbb{G}_m$). Then $G_2$ is generated by $H := SL(2)_F$ and $T_2$. We take the Borel pair $(B_H,T_H)$ to be the intersection of $(B_2,T_2)$ with $H$. We choose a suitable symplectic basis of $W$ and identify $Sp(W)$ with $Sp(2)$, so that the above embedding $\iota$ is given by

$$\iota\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 & 2b \\ 0 & a & 2d \\ c/2 & 0 & d \end{pmatrix}, \quad \iota(x) = \begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 \delta & \xi_1 \\ 0 & \eta_1 / \eta_2 \delta \end{pmatrix} \begin{pmatrix} 0 \\ \eta_1 \eta_2 \delta \end{pmatrix} \begin{pmatrix} x = \xi_1 + \xi_2 \epsilon \in \text{Res}_{E/F} \mathbb{G}_m \end{pmatrix}.$$

B.1.2. Splitting of the metaplectic 2-cocycle and local oscillator representations. It is known that the metaplectic 2-cocycle on $Sp(W) = Sp(2)$ splits over $\iota(G_2(F))$. The explicit splitting is calculated in [MVW], Chapter 3.14 (see also [Ku], Theorem 3.1). To state the result, we fix a character $\xi$ of $E^\times$ such that $\xi|_{F^\times} = \eta_{E/F}$. We define three 1-cocycles as follows.

(1) Define a 1-cocycle $\lambda_1$ on $H(F)$ by

$$\lambda_1\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{cases} \langle c/2,-\delta \rangle_F & \text{if } c \neq 0, \\ \langle d,-\delta \rangle_F & \text{otherwise}, \end{cases}$$

where $\langle \cdot,\cdot \rangle_F$ denotes the Hilbert symbol for $F$. 

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(2) Define a 1-cocycle \( \lambda_2 \) on \( T_2(F) \) by
\[
\lambda_2(d(x)) := \frac{\gamma(N_{E/F}(x))}{\gamma(1)}.
\]

Here \( \gamma \) is the Weil constant (cf. [P]).

(3) Define a 1-cocycle \( \lambda_3 \) on \( G_2(F) \) for \( h \in H(F) \) and \( d(x) \in T_2(F) \),
\[
\lambda_3(h \cdot d(x)) = \lambda_3(d(x) \cdot h) := \begin{cases} 
\langle N_{E/F}(x), \delta \rangle_F & \text{if } c \neq 0, \\
1 & \text{otherwise}.
\end{cases}
\]

Then we define a 1-cocycle \( \lambda_\xi \) on \( G_2(F) \) by
\[
\lambda_\xi(g = h \cdot d(x)) = \lambda_\xi(d(x) \cdot h) := \lambda_1(h) \lambda_2(d(x)) \xi(x) \lambda_3(g), \quad (h \in H(F), \ d(x) \in T_2(F)).
\]

This is independent of the decomposition \( g = h \cdot d(x) \), and the metaplectic 2-cocycle \( c(g_1, g_2) \) ([P]) restricted to \( \iota(G_2(F)) \) becomes
\[
c(\iota(g_1), \iota(g_2)) = \frac{\lambda_\xi(g_1 g_2)}{\lambda_\xi(g_1) \lambda_\xi(g_2)}.
\]

Thus we have the splitting
\[
\iota_\xi : G_2(F) \ni g \longrightarrow (\iota(g), \lambda_\xi(g)) \in Mp(W).
\]

We fix a non-trivial character \( \psi \) of \( F \) and write \( \omega_\psi \) for the oscillator representation of \( Mp(W) \) associated with \( \psi \). The above splitting gives the oscillator representation \( \omega_{\psi, \xi} := \omega_\psi \circ \iota_\xi \) on \( S(E) \). We have explicit formulae:

\[
\omega_{\psi, \xi}(d(a)) \Phi(x) = \xi(a) |a|_{E}^{-1/2} \Phi(ax) \quad (a \in E^\times),
\]

\[
\omega_{\psi, \xi}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Phi(x) = \psi(bN_{E/F}(x)) \Phi(x),
\]

\[
\omega_{\psi, \xi}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Phi(x) = \frac{1}{\gamma(2N_{E/F})} \int_{E} \Phi(y) \psi(\text{Tr}_{E/F}(y\sigma(x))) dy.
\]

B.1.3. **Local Howe duality for \( U(1, 1) \times U(1) \).** Here we assume that \( E/F \) is a quadratic extension of fields. Note that the group \( U(1)(E) \) is contained in \( G_2 \) as its center \( Z_2 \). Since \( Z_2(F) \) is compact, we can define the local theta integral
\[
\Pr_{\chi} \Phi(x) := \int_{Z_2(F)} \overline{\chi(z)} \Phi(xz) \, dz.
\]

This defines a projection on \( S(E) \) whose image \( (\omega_{\psi, \xi}(\chi), S(E)_{\chi}) \) is the \( \chi \)-isotypic subspace of \( (\omega_{\psi, \xi}, S(E)) \).

The following lemma was essentially proved in [ST], 1.1.

**Lemma B.1.** (i) \( \Pr_{\chi} \) commutes with \( \omega_{\psi, \xi}(g) \), and hence
\[
(\omega_{\psi, \xi}, S(E)) = \bigoplus_{\chi \in \text{Hom}(Z_2(F), \mathbb{C}^\times)} (\omega_{\psi, \xi}(\chi), S(E)_{\chi}).
\]

(ii) Each \( (\omega_{\psi, \xi}(\chi), S(E)_{\chi}) \) is non-zero and irreducible.

(iii) Suppose \( F \) is non-archimedean. If \( \chi \neq \xi \), then \( \omega_{\psi, \xi}(\chi) \) is supercuspidal. If \( \chi = \xi \) then \( \omega_{\psi, \xi}(\chi) \) is a limit of discrete series representation. Another limit of discrete series representation in the same \( L \)-packet will be obtained by replacing \( \psi \).
(iv) When $F = \mathbb{R}$, (iii) still holds if the term supercuspidal is replaced by discrete series.

**Proof.** (i) is clear since $Z_2(F)$ is compact. For (ii) we use a classical argument in [ST]. We first consider the non-archimedean case. Take

$$A \in \text{Hom}_{G_2(F)}(\omega_{\psi,\xi}(\chi), \omega_{\psi,\xi}(\chi)).$$

Then, from (B.2),

$$\psi(bN_{E/F}(x))A\Phi(x) = \omega_{\psi,\xi}(\chi; \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})A\Phi(x) = A\omega_{\psi,\xi}(\chi; \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})\Phi(x) = A\psi(bN_{E/F}(x))\Phi(x).$$

But since $(\omega_{\psi,\xi}(\chi), S(E))$ extends to a continuous representation on $L^2(E)$, $A$ has a kernel function $K(x, y)$;

$$A\Phi(x) = \int_E K(x, y)\Phi(y) dy.$$ 

Then we must have

$$\int_E |\psi(bN_{E/F}(x)) - \psi(bN_{E/F}(y))| K(x, y)\Phi(y) dy = 0, \quad \forall \Phi \in S(E).$$

It follows that $K(x, \bullet) = k(x)\delta_x$, where $\delta_x$ is the Dirac distribution supported at $x$ and $k(x)$ is a bounded measurable function; $A\Phi(x) = k(x)\Phi(x)$. Moreover it follows from (B.1) that $k(xa) = k(x)$ for any $a \in E^\times$. Now we have two $E^\times$-orbits in $E$, the open orbit $E^\times$ and the closed one $\{0\}$. If $\chi \neq \xi$, then

$$\text{Pr}_\chi \Phi(0) = \int_{Z_2(F)} \nabla\xi(z) dz \Phi(0) = 0.$$ 

Hence $\dim \text{Hom}_{G_2(F)}(\omega_{\psi,\xi}(\chi), \omega_{\psi,\xi}(\chi)) = 1$ and $\omega_{\psi,\xi}(\chi)$ is irreducible. Next we assume $\chi = \xi$. Then we apply Rallis’s invariant distribution theorem, and get

$$\text{Pr}_\xi \mathcal{S}(E) \simeq \text{Span}\{\omega_{\psi,\xi}(g)\delta_0\}.$$ 

Again this asserts that $\dim \text{Hom}_{G_2(F)}(\omega_{\psi,\xi}(\xi), \omega_{\psi,\xi}(\xi)) = 1$ and $\omega_{\psi,\xi}(\xi)$ is irreducible. This proof is also valid in the archimedean case, but the notation will become more complicated.

(iii) and (iv) were proved in [JL].

B.1.4. Local Howe duality for $GL(2) \times \mathbb{G}_m$. Next we assume $E = F \oplus F$. Then $G_2 = GL(2)_F$, and $\xi$ is identified with a character of $F^\times$ by $\xi(xy^{-1}) := \xi(x, y)$. Then the explicit formulae for $(\omega_{\psi,\xi}, S(F^\otimes 2))$ become

(B.5) \begin{equation} \omega_{\psi,\xi}
\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \Phi(x, y) = \xi(a_1a_2)|a_1a_2|^{1/2}F(a_1x, a_2^{-1}y), \end{equation}

(B.6) \begin{equation} \omega_{\psi,\xi}
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Phi(x, y) = \psi(bxy)\Phi(x, y), \end{equation}

(B.7) \begin{equation} \omega_{\psi,\xi}
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi(x, y) = \int_E \Phi(u, v)\psi(uy + vx) du dv. \end{equation}

The following results were proved in [JL].
Lemma B.2. (i) The partial Fourier transform
\[ F_2 \Phi(x,y) := \int_F \Phi(x,u)\psi(yu) \, du \quad (\Phi \in \mathcal{S}(E)) \]
gives a $G_2(F)$-equivariant isomorphism
\[ (\omega_\psi,\xi,\mathcal{S}(E)) \rightarrow ((\xi \circ \det)|\det|^{1/2} \otimes R, \mathcal{S}(F^{\otimes 2})). \]
Here $R$ denotes the right translation action.

(ii) The map
\[ \mathcal{S}(F^{\otimes 2}) \ni \Phi \mapsto \chi(\det(g))|\det(g)|^{1/2} \Phi([0,1],g) \in \text{ind}_{B_2(F)}^{G_2(F)}[\xi]|_{F^*} \otimes 1_{N_2(F)} \]
is a $G_2(F)$-equivariant isomorphism. Here $B_2^0$ denotes the subgroup of $B_2$ consisting of elements of the form $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$. $\text{ind}_{B_2(F)}^{G_2(F)}$ assigns the non-normalized induction.

(iii) We define a projector $\text{Pr}^\chi_\eta$ on $\text{ind}_{B_2(F)}^{G_2(F)}[\xi]|_{F^*} \otimes 1_{N_2(F)}$ by
\[ \text{Pr}^\chi_\eta \phi(g) := \int_{E_2(F)} \chi(z)\phi(zg) \, dz. \]
This is a surjection to $\text{Ind}_{B_2(F)}^{G_2(F)}[(\xi \otimes \chi^{-1}) \otimes 1_{N_2(F)}]$. Hence the $\chi$-isotypic quotient $\omega_\psi,\xi(\chi)$ of $\omega_\psi,\xi$ is isomorphic to $\text{Ind}_{B_2(F)}^{G_2(F)}[(\xi \otimes \chi^{-1}) \otimes 1_{N_2(F)}]$.

B.1.5. The global theory. Now let $k'/k$ be a quadratic extension of number fields.

We use notation defined in 2.1 and in §5. As in the local case, we have the skew-Hermitian space $(V,\Phi)$ over $k'$ and obtain the symplectic space $(W,\phi)$ over $k$ by the restriction of scalars. We again write $Sp(W)$ for the symplectic group of $(W,\phi)$.

Fix a non-trivial character $\psi = \bigotimes_v \psi_v$. As in [I], §1, we can construct the global metaplectic group $Mp(W)_k$ and its oscillator representation $(\omega_\psi,\xi,\mathcal{S}(k'))$.

We also fix a character $\xi = \bigotimes_v \xi_v$ of $\mathbb{A}^\times_{k'}/k'^\times$ whose restriction to $\mathbb{A}^\times_k$ equals $\eta_{k'/k}$. Then, using the arguments of 3.3.7 (1), we have the global splitting $\iota_\xi : G_2(A) \rightarrow Mp(W)_k$. This time we define the oscillator representation $\omega_\psi,\xi$ of $G_2(\mathbb{A})$ by $\omega_\psi,\xi := \omega_\psi \circ \iota_\xi$.

The explicit formulae are given as follows:

(B.8) \[ \omega_\psi,\xi(d(a))\Phi(x) = \xi(a)|a|^{1/2}\Phi(ax) \quad (a \in \mathbb{A}_{k'}^\times), \]

(B.9) \[ \omega_\psi,\xi(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})\Phi(x) = \psi(bN_{k'/k}(x))\Phi(x), \]

(B.10) \[ \omega_\psi,\xi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\Phi(x) = \int_{\mathbb{A}_{k'}} \Phi(y)\psi(Tr_{k'/k}(y\sigma(x))) \, dy. \]

Now we define a tempered distribution, called the $\theta$-distribution, on $S(\mathbb{A}_{k'})$ by
\[ S(\mathbb{A}_{k'}) \ni \Phi \mapsto \theta(\Phi) := \sum_{\delta \in \mathfrak{a}} \Phi(\delta) \in \mathbb{C}. \]

The theta series $\theta_\Phi(g)$ attached to $\Phi \in S(\mathbb{A}_{k'})$ is defined by $\theta_\Phi(g) := \theta(\omega_\psi,\xi(g)\Phi)$ ($g \in G_2(\mathbb{A})$). We write the space of automorphic forms spanned by $\theta_\Phi$'s with $\Phi \in S^0(\mathbb{A}_{k'})$ as $\Theta(\xi,\psi)$. Here $S^0(\mathbb{A}_{k'})$ is defined similarly as in 3.3.7 (1).

Next we take a character $\chi = \bigotimes_v \chi_v$ of $Z_2(k') \setminus Z_2(\mathbb{A})$. At inert $v$ we have the projection $\text{Pr}_\chi$. At split $v$, $\text{Pr}_\chi$ can still be defined by (B.4) if $x \in k'_v$ satisfies $N_{k'_v/k_v}(x) \neq 0$. This defines $\text{Pr}_\chi \Phi_v(x)$ ($\Phi_v \in S(k'_v)$) at such $x$, and we define
PrχvΦv ∈ S(k′v) to be its extension to k′v by zero. We now define the global
projector Prχ on S(ÅkK) by

\[ \text{Pr}_\chi : S(ÅkK) \ni \bigotimes_v \Phi_v \longrightarrow \bigotimes_v \text{Pr}_{\chi,v} \Phi_v \in S(ÅkK). \]

This is clearly well-defined and G2(Å)-equivariant. We write S(ÅkK)χ for its
image and ωψ,ξ(χ) for the restriction of ωψ,ξ to S(ÅkK)χ. On the other hand, since
Z2(k)\Z2(Å) is compact, the distribution

\[ \Phi \longrightarrow \theta_\chi(\Phi) := \int_{Z2(k)\Z2(Å)} \chi(z) \theta(\Phi(\bullet z)) dz \]

is also tempered. We define θφ,ξ(g) := θχ(ωψ,ξ(g)Φ) (g ∈ G2(Å)). The space of
automorphic forms spanned by θφ,ξ (Φ ∈ S0(ÅkK)) is denoted by Θ(ξ, ψ)χ.

**Proposition B.3.** (i) We have the direct sum decomposition

\[ \Theta(ξ, ψ) = \bigoplus_{\chi \in A(Z2(k)\Z2(Å))} \Theta(ξ, ψ)_\chi. \]

(ii) Each Θ(ξ, ψ)_χ is an irreducible automorphic representation. Moreover if
χ ≠ ξ, then Θ(ξ, ψ)_χ is cuspidal, while Θ(ξ, ψ)_ξ is not cuspidal.

**Proof.** (i) is clear. The first statement and the cuspidality in (ii) were proved in
[ST], Proposition 4.9. The last statement follows from the fact that θφ,ξ has the
non-trivial constant term ωψ,ξ(g)Φ(0) along B2.

B.2. The zeta integral—definition and unfolding. Here we fix an irreducible
cuspidal automorphic representation τ = \bigotimes_v τ_v of G2(Å) and give an integral
representation for L(s, τ × χ).

B.2.1. Classification of τ_v at inert v. Let w|v be as before. The results of [JL]
combined with [Si], Theorem 2.5.9, and [Sh], Corollary 7.6, give

**Lemma B.4** (Non-archimedean case), (i) If τ_v is not supercuspidal, then we can
choose a character ν0 of k_w× and s ∈ C such that τ_v is a submodule of I(ν0, s) :=

\[ \text{Ind}_{B^*(k_v)}^{G^*(k_v)}(\nu_0)^{\otimes s} \otimes 1_{N_2(k_v)} \].

(ii) I(ν0, s) is irreducible unless ν0|k_w× = ηk_w/k_v and s = 0, or ν0|k_w× = 1 and
s ∈ \{-1, 1\}.

(iii) If ν0|k_w× = ηk_w/k_v, then I(ν0, 0) is a direct sum of two distinct limits of
discrete series representations.

(iv) If ν0|k_w× = 1, then I(ν0, 1) has a unique irreducible submodule, the Steinberg
representation St(ν0) with the central character ν0. The quotient of I(ν0, 1) by
St(ν0) is the one-dimensional representation ν0G2 defined as in 3.3.1. I(ν0, −1) has
ν0G2 as its unique submodule, and the quotient of I(ν0, −1) by it is St(ν0).

The corresponding archimedean results are very well-known and will be omitted
(cf. [JL], Lemma 5.7).
B.2.4. Whittaker models for $\tau_v$ at inert $v$. Note that in the classification above, finite dimensional irreducible representations can never be a $\tau_v$. We begin with the non-archimedean case. Take a non-trivial character $\psi_v$ and consider it as a character of $N_2(k_v)$. The space of Whittaker functionals on $\tau_v$ is defined by

$$W_{\psi_v}(\tau_v) := \text{Hom}_{N_2(k_v)}(\tau_v|_{N_2(k_v)}, \psi_v) = (\tau_v)^*_{N_2(k_v), \psi_v}.$$  

From [LL], we know that $W_{\psi_v}(\tau_v) \neq 0$ with some suitable $\psi_v$. For this suitable $\psi_v$, we choose a non-zero $\xi_v \in W_{\psi_v}(\tau_v)$ and define the Whittaker model of $\tau_v$ with respect to $\psi_v$ by

$$W(\tau_v, \psi_v) := \{ W_f(\psi_v)(g) := \xi_v(\tau_v(g)f) \mid f \in V_{\tau_v} \}.$$  

Finally we remark that if $\tau_v$ is in the principal series, $W_{\psi_v}(\tau_v) \neq 0$ for any $\psi_v$.

Next assume $k'_v/k_v = \mathbb{C}/\mathbb{R}$. The non-trivial characters $\psi_v = \psi_\mathbb{R}$ or $\psi_\mathbb{R}^{-1}$ are considered as a character of $N_2(\mathbb{R})$. Its differential is denoted by $d\psi_v$. Define

$$W_{\psi_v}(\tau_v) := \begin{cases} \xi_v : V_{\tau_v} \to \mathbb{C} \quad \text{linear functional} \\
\text{(i) } \xi_v \text{ is continuous with respect to} \\
\text{the Schwartz topology,} \\
\text{(ii) } \xi_v(d\tau_v(X)f) = d\psi_v(X)\xi_v(f), \\
\text{ X } \in \text{Lie}(N_2(\mathbb{R})) \end{cases}.$$  

Then again $W_{\psi_v}(\tau_v) \neq 0$ for $\psi_v = \psi_\mathbb{R}$ or $\psi_\mathbb{R}^{-1}$. With this suitable $\psi_v$, we choose a non-zero $\xi_v \in W_{\psi_v}(\tau_v)$. Extend $\xi_v$ to a map on a unitary completion $(\tau_v, \nabla_{\tau_v})$. Then, using this extended $\xi_v$, we define the Whittaker model $W(\tau_v, \psi_v)$ as in the non-archimedean case. By the uniqueness of the Whittaker model, this does not depend on the choice of the unitary completion.

B.2.3. Whittaker models for $\tau$. By the above, we can choose a non-trivial character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A} / k$ so that $W_{\psi_v}(\tau_v) \neq 0$ at any $v$. Moreover at almost all $v$, $\tau_v$ is unramified and $\psi_v$ is of order 0. At such $v$ we write $W^0_v$ for the element of $W(\tau_v, \psi_v)$ such that $W^0_v$ is right $K_{2,v}$-invariant and $W^0_v(1) = 1$. Here $K_{2,v}$ is a suitable maximal compact subgroup of $G_2(k_v)$. The global Whittaker model $W(\tau, \psi)$ of $\tau$ is given by the restricted tensor product of $W(\tau_v, \psi_v)$'s with respect to $W^0_v$. It was shown by Shalika that $W(\tau, \psi)$ is unique for a fixed $\psi$.

Our final remark is about the Fourier expansion. For a cuspidal form $\varphi \in V_\tau$, we can define an element of $W(\tau, \psi)$ by

$$W^\varphi_\psi(g) := \int_{N_2(k) \setminus N_2(\mathbb{A})} \varphi(ng)\overline{\psi(n)} \, dn \quad (g \in G_2(\mathbb{A})).$$  

If we write $\psi^\delta(x) := \psi(\delta x)$ for $\delta \in k$, then the Fourier expansion on $N_2(k) \setminus N_2(\mathbb{A})$ reads

$$\varphi(g) = \sum_{\delta \in k^\times} W^\varphi_\psi(\psi^\delta).$$  

B.2.4. The zeta integral and its unfolding. We retain the notation defined above. In particular, fix a character $\xi$ of $\mathbb{A}_k^\times / k^\times$ such that $\xi^\mathbb{A}_k^\times = \eta_k^\times / k$. Additionally, for each character $\chi$ of $\mathbb{A}_k^\times / k^\times$, we choose a holomorphic section $F_s(\chi)\xi$ of $A(N_2(\mathbb{A})T_2(k) \setminus G_2(\mathbb{A}))_\chi(\xi^\mathbb{A}_k^\times / k^\times)$ $(s \in \mathbb{C})$. Then we have the Eisenstein series

$$E(F(\xi^\chi, s), g) := \sum_{\gamma \in B_2(k) \setminus G_2(k)} F_s(\chi^\chi; \gamma g).$$
on $G_2(\mathbb{A})$. The properties of $E(F(\chi\xi), s)$ were given in 3.3.8. We choose a non-trivial character $\psi$ of $\mathbb{A}/k$ so that $\tau$ has a global Whittaker model with respect to it. Then we construct the oscillator representation $\omega_{\psi, \xi}$ and $\theta$-series using this $\psi$. Now for fixed data $\varphi \in V_\tau$, $\Phi \in \mathcal{S}(\mathbb{A}_k)$ and $F_s(\chi\xi)$, we define the zeta integral by

$$Z(\varphi, \theta_\Phi, \chi; s) := \int_{G_2(k) \backslash G_2(\mathbb{A})} \varphi(g)\overline{\theta_\Phi(g)}E(F(\chi\xi), s - 1/2)(g) \, dg.$$  

**Proposition B.5.** Suppose that the data in the definition of the zeta integral are chosen to be of the form

$$\Phi = \bigotimes_v \Phi_v \in \mathcal{S}(\mathbb{A}_k), \quad F_s(\chi\xi)(g) = \bigotimes_v F_{s,v}(\chi_v\xi_v)(g_v), \quad W^\psi_\varphi(g) = \bigotimes_v W_v(g_v).$$

If we define the local zeta integral $Z_v(W_v, \Phi_v, \chi_v; s)$ by

$$Z_v(W_v, \Phi_v, \chi_v; s) := \prod_v Z_v(W_v, \Phi_v, \chi_v; s),$$  

then

$$Z(\varphi, \theta_\Phi, \chi; s) = \prod_v Z_v(W_v, \Phi_v, \chi_v; s).$$

**Proof.** Since the analogous argument in the case of $SL(2, \mathbb{A})$ was given in [GJ], we only give the formal outline of the proof, and the convergence argument will be omitted. From the definition of $E(F(\chi\xi), s)$, we have

$$Z(\varphi, \theta_\Phi, \chi; s) = \int_{B_2(k) \backslash G_2(\mathbb{A})} \varphi(g)\overline{\theta_\Phi(g)}F_{s-1/2}(\chi\xi; g) \, dg$$

$$= \int_{K_2} \int_{k^* \backslash \mathbb{A}_k} \varphi(k)F_{s-1/2}(\chi\xi)(\left(\begin{array}{c} 1 \\
0 \end{array}\right) \left(\begin{array}{c} x \\
1 \end{array}\right) d(a)k) \, \frac{da^\times}{|a|_{\mathbb{A}_k}} \, dk$$

$$= \sum_{\alpha \in \mathbb{A}_k^*} \omega_\psi, \xi(g) \varphi(\left(\begin{array}{c} 1 \\
0 \end{array}\right) g) \psi(xN_{k'/k}(\alpha)) \Phi(\alpha) \, dx.$$  

Here $K_2$ is a certain maximal compact subgroup of $G_2(\mathbb{A})$. From (B.9), our inner integral becomes

$$\int_{k^* \backslash \mathbb{A}_k} \varphi(\left(\begin{array}{c} 1 \\
0 \end{array}\right) g) \, dx$$

$$= \sum_{\alpha \in \mathbb{A}_k^*} \omega_\psi, \xi(g) \varphi(\left(\begin{array}{c} 1 \\
0 \end{array}\right) g) \psi(xN_{k'/k}(\alpha)) \Phi(\alpha) \, dx.$$  

But notice that

$$\int_{k^* \backslash \mathbb{A}_k} \varphi(\left(\begin{array}{c} 1 \\
0 \end{array}\right) g) \psi(xN_{k'/k}(\alpha)) \, dx = \int_{k^* \backslash \mathbb{A}_k} \varphi(\left(\begin{array}{c} 1 \\
0 \end{array}\right) N_{k'/k}(\alpha)^{-1} x) \psi(x) \, dx$$

$$= \int_{k^* \backslash \mathbb{A}_k} \varphi(\left(\begin{array}{c} \alpha^{-1} \\
0 \end{array}\right) \left(\begin{array}{c} 1 \\
0 \end{array}\right) \alpha \left(\begin{array}{c} 0 \\
\sigma(\alpha)^{-1} \end{array}\right) g) \psi(x) \, dx$$

$$= W^\psi_\varphi(d(\alpha)g).$$
Thus we conclude that
\[
Z(\varphi, \theta_v, \chi_v; s) = \int_{K_2} \int_{k'^{\times} \backslash L^{\times}_v} F_{s-1/2}(\chi\xi; k) \chi(\alpha)|a|^{s-1/2} \sum_{\alpha \in k'^{\times}} \frac{\omega_{\psi, \xi}(d(a)k)\Phi(\alpha)W_{s,\xi}^{\psi}(d(a)k)\chi(\alpha)|a|^{s-1/2}}{|a|^{s-1/2} \Delta_v} da \text{ } dk
\]
putting \(a := \alpha a\) and using (B.8),
\[
= \int_{K_2} \int_{\Delta_v^{\times}} F_{s-1/2}(\chi\xi; k)\omega_{\psi, \xi}(k)\Phi(\alpha)W_{s,\xi}^{\psi}(d(a)k)\chi(\alpha)|a|^{s-1/2} \frac{da}{|a|^{s-1/2}} \text{ } dk
\]
\[
= \prod_v Z_v(W_v, \Phi_v, \chi_v; s).
\]

\[\square\]

B.3. Comparisons of local factors. Again we shall see that \(Z_v(W_v, \Phi_v, \chi_v; s)\) is equal to a certain quotient of \(L(s, \tau_v \times \chi_v)\) at almost all \(v\). Then we shall prove that, at every \(v\) in the region \(\text{Re}(s) > 0\), the G.C.D. has the same set of poles as \(L(s, \tau_v \times \chi_v)\) has.

B.3.1. Unramified calculations. We begin with the inert case, so let \(k_v'/k_v\) be an unramified quadratic extension. Further assume that \(\psi_v\) and \(\xi_v\) are unramified, \(\psi_v\) is of order 0, \(\Phi_v = \Phi_v^0\) (the characteristic function of \(O_w\)) and \(F_{s,v}(\chi\xi)\) is the spherical vector such that \(F_{s,v}(\chi\xi; k) = 1\) for \(k \in K_{2,v}\). The spherical Whittaker function \(W_v^0\) was defined in B.2.3.

**Lemma B.6.** Under these assumptions we have
\[
Z_v(W_v^0, \Phi_v^0, \chi_v; s) = \frac{L_k(k, \chi_v \nu_v)\epsilon(k, \chi_v \nu_v^{-1})}{L_k(2s, \chi_v \xi_v k^\times)} = \frac{L(s, \tau_v \times \chi_v)}{L_k(2s, \chi_v \xi_v k^\times)}.
\]

**Proof.** We have
\[
Z_v(W_v^0, \Phi_v^0, \chi_v; s) = \int_{k_v^\times} \Phi_v^0(a)W_v^0(d(a))\chi_v(a)|a|^{s-1/2} da
\]
\[
= \int_{\Omega_v \cap k_v^\times} W_v^0(d(a))\chi_v(a)|a|^{s-1/2} da
\]
\[
= \sum_{n=0}^{\infty} W_v^0(d(\omega_v^n))\chi_v(\omega_v)\omega_v^{-n(s-1/2)}.
\]
We need to calculate \(W_v^0(d(\omega_v^n))\). We first restrict \(\tau_v = \text{Ind}_{B(k_v)}^{G_2(k_v)}[\nu_v \otimes 1_{N_2(k_v)}]\) to \(\tau_v|_{H(k_v)} = \text{Ind}_{B(k_v)}^{H(k_v)}[\nu_v|_{k_v^\times} \otimes 1_{N_2(k_v)}]\), then extend it to a representation
\[
\tilde{\tau}_v = \text{Ind}_{B(k_v)}^{GL(2, k_v)}[(\nu_v|_{k_v^\times} \otimes 1) \otimes 1_{N(k_v)}]
\]
of \(GL(2, k_v)\). Here \(B = TN\) denotes the upper triangular Borel subgroup of \(GL(2, k_v)\). Accordingly \(W_v^0\) is restricted to \(H(k_v)\) and then extended to the spherical Whittaker function for \(\tilde{\tau}_v\). For this an explicit formula is available (cf. [JL], p.
Here $\text{Ch}_{\mathcal{O}_v} \oplus \mathcal{O}_v$ denotes the characteristic function of $\mathcal{O}_v \oplus \mathcal{O}_v$. Hence we have

\[
Z_v(W_v^0, \Phi_v^0, \chi_v; s) = \frac{1}{\nu_v(\mathcal{w}_v) - 1} \sum_{n=0}^{\infty} \left( \chi_v \nu_v(\mathcal{w}_v^n) \nu_v(\mathcal{w}_v) - \chi_v \nu_v^{-1}(\mathcal{w}_v^n) \right) q_w^{-ns} \\
= \frac{1}{\nu_v(\mathcal{w}_v) - 1} \left( \frac{\nu_v(\mathcal{w}_v)}{1 - \chi_v \nu_v(\mathcal{w}_v)q_w^{-s}} - 1 \right) \\
= \frac{1 + \chi_v(\mathcal{w}_v)q_w^{-s}}{(1 - \chi_v \nu_v^{-1}(\mathcal{w}_v)q_w^{-s})(1 - \chi_v \nu_v^{-1}(\mathcal{w}_v)q_w^{-s})}
\]

noting that $\xi_v(\mathcal{w}_v) = \eta_{k_v'}/k_v(\mathcal{w}_v) = -1$,

\[
= \frac{L_{k'}(s, \chi_v \nu_v) L_k'(s, \chi_v \nu_v^{-1})}{L_k(2s, (\chi_v \xi_v)|_{k_v})}.
\]

Next comes a split $v$. Let $w_1, w_2|v$ be as before. Assume that $\chi_v = \chi_{w_1} \otimes \chi_{w_2}$ and $\xi_v(x, y) = \xi_v(xy^{-1})$ are unramified, $\psi_v$ is of order 0, $\Phi_v = \Phi_v^0$ (the characteristic function of $\mathcal{O}_v \oplus \mathcal{O}_{w_2}$) and $F_{x,v}(\chi_x \xi_v)$ is the spherical vector such that $F_{x,v}(\chi_x \xi_v; k) = 1$ for $k \in K_{2,v}$. Further assume that $\tau_v$ is of the form

\[
\text{Ind}^{G_2(k_v)}_{B_2(k_v)}(\nu_{w_1} \otimes \nu_{w_2} \otimes 1_{\text{Ind}(k_1(k_v))}),
\]

where $\nu_{w_i}$ are unramified characters of $k_v^*$. The spherical Whittaker function $W_v^0$ is as in B.2.3.

**Lemma B.7.** Under these assumptions we have

\[
Z_v(W_v^0, \Phi_v^0, \chi_v; s) = \frac{L(s, \tau_v \otimes \chi_{w_1}) L(s, \tau_v \otimes \chi_{w_2})}{L(2s, (\chi_v \xi_v)|_{k_v})} = \frac{L(s, \tau_v \times \chi_v)}{L_k(2s, (\chi_v \xi_v)|_{k_v})}.
\]

**Proof.** In this case $G_2(k_v) = GL(2, k_v)$, and from [JL, p.123] we have

\[
W_v^0(d(\mathcal{w}_v^n, \mathcal{w}_v^m)) = \nu_{w_1} \nu_{w_2}^{-1}(\mathcal{w}_v^{-m}) \nu_{w_1}(\mathcal{w}_v^{n+m}) |_{\mathcal{w}_v^{n+m} l_0}^{1/2} \\
\times \int_{k_v^*} \text{Ch}_{\mathcal{O}_v \oplus \mathcal{O}_v}(\mathcal{w}_v^{n+m} t, t^{-1}) \nu_{w_1} \nu_{w_2}(t) dt^x \\
= \nu_{w_1}(\mathcal{w}_v^n) \nu_{w_2}(\mathcal{w}_v^m) q_w^{-(n+m)/2} \sum_{j=0}^{n+m} \nu_{w_1} \nu_{w_2}(\mathcal{w}_v)^{-j} \\
= \begin{cases} 
q_w^{-(n+m)/2} \sum_{j=-m}^{n} \nu_{w_1}(\mathcal{w}_v)^j \nu_{w_2}(\mathcal{w}_v)^{j+m-n} & \text{if } m + n \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\]
Thus one has
\[
Z_v(W^0_v, \Phi^0_v; \chi_v; s) = \int_{k_v^0} \int_{k_v^0} \Phi^0_v(a, b) W^0_v(d(a, b)) \chi_{w_1}(a) \chi_{w_2}(b) |ab|^{s-1/2} \, da \times
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} W^0_v(d(\varpi^n_v, \varpi^m_v)) \chi_{w_1}(\varpi^n_v) \chi_{w_2}(\varpi^m_v) q_v^{-(n+m)(s-1/2)}
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=-m}^{n} \nu_{w_1}(\varpi)^j \nu_{w_2}(\varpi)^{j-m} \chi_{w_1}(\varpi^n_v) \chi_{w_2}(\varpi^m_v) q_v^{-(n+m)s}.
\]

We now use the abbreviations
\[
\alpha := \nu_{w_1}(\varpi), \quad \beta := \nu_{w_2}(\varpi), \quad A := \chi_{w_1}(\varpi)q_v^{-s}, \quad B := \chi_{w_2}(\varpi)q_v^{-s}.
\]

Then
\[
Z_v(W^0_v, \Phi^0_v, \chi_v; s) = \sum_{n,m \geq 0} A^n B^m (\alpha^n \beta^m + \alpha^{n-1} \beta^{m-1} + \cdots + \alpha^{-m} \beta^{-n})
\]
\[
= \sum_{n,m \geq 0} A^n B^m \frac{\alpha^{n+1} \beta^{m+1} - \alpha^{-m} \beta^{-n}}{\alpha \beta - 1}
\]
\[
= \frac{\alpha \beta}{\alpha \beta - 1} \sum_{n,m \geq 0} (A\alpha)^n (B\beta)^m - \frac{1}{\alpha \beta - 1} \sum_{n,m \geq 0} (A\beta^{-1})^n (B\alpha^{-1})^m
\]
\[
= \frac{\alpha \beta}{(\alpha \beta - 1)(1 - A\alpha)(1 - B\beta)} - \frac{1}{(\alpha \beta - 1)(1 - A\beta^{-1})(1 - B\alpha^{-1})}
\]
\[
= \frac{\alpha \beta (1 - A\beta^{-1} - B\alpha^{-1} + A\beta^{-1}B\alpha^{-1}) - (1 - A\alpha - B\beta + A\beta B\alpha)}{(1 - A\alpha)(1 - A\beta^{-1})(1 - B\alpha^{-1})(1 - B\beta)(\alpha \beta - 1)}
\]
\[
= \frac{1 - AB}{(1 - A\alpha)(1 - A\beta^{-1})(1 - B\alpha^{-1})(1 - B\beta)}
\]
\[
= \frac{L(\sigma, \tau_v \otimes \chi_{w_1}) L(\sigma, \tau_v \otimes \chi_{w_2})}{L(2s, \chi_v \xi_v \chi_{w_1} \chi_{w_2})}.
\]

\[
\square
\]

B.3.2. Ramified split case. We want to show that the G.C.D. of \( Z_v(W_v, \Phi_v, \chi_v; s) \) has the same analytic behavior as that of \( L(s, \tau_v \otimes \chi_v) \) in the region \( \Re(s) > 0 \). At a split \( v \) this reduces to the corresponding results in [Ja]. The key point is the following lemma.

Lemma B.8. If we set
\[
\tau_2 := \text{Ind}_{B(k_2)}^{G(k_2)} [(\chi_{w_1} \otimes \omega_{\tau_v} \chi_{w_2}^{-1}) \otimes 1_{N(k_v)}]
\]
\[
= \text{Ind}_{B(k_v)}^{GL(2, k_v)} [(\chi_{w_1} \otimes \omega_{\tau_v}^{-1} \chi_{w_2}) \otimes 1_{N(k_v)}],
\]

then we can choose \( W_2 \in \mathcal{W}(\tau_2, \psi_v) \) and \( \Phi_2 \in \mathcal{S}(k_v^{\oplus 2}) \) \( \langle \mathcal{S}(k_v^{\oplus 2}) \rangle \) if \( v \) is archimedean so that
\[
Z_v(W_v, \Phi_v, \chi_v; s) = \Psi(s, W_v, W_2, \Phi_2).
\]

Here the right hand side is the Rankin product local zeta integral defined in [Ja].
Proof. We may assume \( \text{Re}(s) \gg 0 \) throughout the proof. By definition, we have

\[
Z_v(W_v, \Phi_v, x_v; s) = \int_{k_v} F_{s-1/2}(x_v x_v; k) \omega_{x_v x_v} \Phi_v(a, b)
\]

putting \( x := ab, y = b^{-1} \),

\[
= \int_{k_v} F_{s-1/2}(x_v x_v; k) \omega_{x_v x_v} \Phi_v(x y, y^{-1})
\]

\[
\times W_v \left( \begin{pmatrix} x y & 0 \\ 0 & y \end{pmatrix} \right) \chi_{x_v}^{-1}(x) |x|^{-s/2} dx dy \, dk.
\]

Here we note that

\[
\int_{k_v} [\omega_{x_v x_v} \Phi_v(x y, y^{-1}) \omega_{x_v}^{-1} \chi_{x_v}^{-1} \chi_{v}^{-1} (y) dy
\]

\[
= \int_{k_v} \omega_{x_v}^{-1} \chi_{v}^{-1} \chi_{x_v}^{-1} \chi_{v}^{-1} (y) |\omega_{x_v x_v} \Phi_v(x y, y^{-1})| dy
\]

\[
= \text{Pr}_{\omega_{x_v} \chi_{x_v} \chi_{x_v} \chi_{x_v}} (\omega_{x_v x_v} \Phi_v)(x, 1)
\]

\[
= \omega_{x_v x_v} \Phi_v(k) \chi_{x_v}^{-1} \chi_{x_v}^{-1} \chi_{v}^{-1} (1)
\]

and hence

\[
Z_v(W_v, \Phi_v, x_v; s)
\]

\[
= \int_{k_v} \omega_{x_v x_v} (k) \chi_{x_v}^{-1} \chi_{x_v}^{-1} \chi_{v}^{-1} (1)
\]

\[
\times W_v \left( \begin{pmatrix} x y & 0 \\ 0 & y \end{pmatrix} \right) \chi_{x_v}^{-1}(x) |x|^{-s/2} dx dy \, dk.
\]

Next we specify the integrand. We may assume \( F_{s-1/2}(x_v x_v; g) \) is of the form

\[
f_{\phi_2}(x_v x_v; g) = \chi_{x_v} \xi_v \det(g) \int_{k_v} \Phi_2([0, t] g) \chi_{v} \chi_{x_v} (t) |t|^{2s} dt
\]

where \( \Phi_2 \in \mathcal{S}(k_v^{-2}) \) and \( z(\omega, \Phi) \) was defined in the introduction of [Ja]. Moreover, if we set

\[
\tau'_2 := \text{Ind}_{B(k_v)}^{GL(2, k_v)} \left( \xi_v \otimes \omega_{x_v} \chi_{x_v}^{-1} \chi_{v}^{-1} \right) \otimes 1_{N(k_v)}.
\]
then from Lemma B.2 we know that \( \omega_{\psi_v, \xi_v}(\omega_{\tau, \chi_{w_1} \chi_{w_2}^{-1}}) \simeq \tau_2' \) and
\[
W^*(g) := \omega_{\psi_v, \xi_v}(g) \Pr_{(\omega_{\tau, \chi_{w_1} \chi_{w_2}^{-1}})}(1, 1)
\]
is in the Whittaker model \( \mathcal{W}(\tau_2', \psi_v) \) (cf. [JL]). Note that \( W^*(g) \) spans a dense subspace of \( \mathcal{W}(\tau_2', \psi_v) \) as \( \Phi_v \) changes. Now the zeta integral becomes
\[
Z_v(W_v, \Phi_v, \chi_v; s) = \int_{L_2(k_v) \setminus G_2(k_v)} W_v(g) \overline{W^*(g)} \chi_w \xi_v(dg) z(\sigma(2) \chi_{w_1} \chi_{w_2} g, \Phi_v, \psi_v) \det(g)^{\sigma_v} dg.
\]
On the other hand, since \( \overline{W^*(g)} \) runs over the Whittaker functions \( W_2'(1 \begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix}) g \) \((W_2' \in \mathcal{W}(\tau_2', \psi_v))\), \( W_2(g) := \chi_{w_1} \xi_v(\det(g)) W_2'(g) \) runs over Whittaker functions for
\[
\tau_2 = (\chi_{w_1} \xi_v \circ \det) \otimes \tau_2' = \text{Ind}_{B(k_v)}^{GL(2, k_v)}[(\chi_{w_1} \chi_{w_2}^{-1}) \otimes 1_{N(k_v)}]
\]
with respect to \( \psi_v \). Hence we conclude that
\[
Z_v(W_v, \Phi_v, \chi_v; s) = \int_{L_2(k_v) \setminus G_2(k_v)} W_v(g) W_2(1 \begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix} g) z(\sigma(2) \chi_{w_1} \chi_{w_2} g, \Phi_v, \psi_v) \det(g)^{\sigma_v} dg = \Psi(s, W_v, W_2, \Phi_v).
\]

\[ \square \]

**Corollary B.9.** The G.C.D. of \( Z_v(W_v, \Phi_v, \chi_v; s) \) equals
\[
L(s, \tau_v \otimes \chi_{w_1}) L(s, \tau_v \otimes \chi_{w_2}) = L(s, \tau_v \times \chi_v).
\]

**B.3.3. Ramified inert non-archimedean case.** Let \( w|v \) be as before. We first state the result.

**Proposition B.10.** \( Z_v(W_v, \Phi_v, \chi_v; s) \) has the same poles in the region \( \Re(s) > 0 \) as
\[
\frac{L(s, \tau_v \times \chi_v)}{L(2s, (\chi_v \xi_v)_{k_v^{\times}})}.
\]

1) **A key lemma.** The proof of the proposition is a case-by-case argument and quite lengthy. We start with a key lemma which will be commonly used in all cases.

**Lemma B.11.** \( Z_v(W_v, \Phi_v, \chi_v; s) \) has the same set of poles as
\[
H(k, s) := \int_{K_v^{\times}} \omega_{\psi_v, \xi_v}(k) \Phi(a) W_v(d(a)k) \chi_v(a) |a|_{w_v}^{-1/2} da \quad (k \in K_{2,v}).
\]

**Proof.** Since
\[
Z_v(W_v, \Phi_v, \chi_v; s) = \int_{B_2(k_v) \cap K_{2,v} \setminus K_{2,v}} F_{s-1/2,v}(\chi_v \xi_v; k) H(k, s) dk,
\]
all poles of \( Z_v(W_v, \Phi_v, \chi_v; s) \) come from those of \( H(k, s) \).

Conversely if we set
\[
K_{2,v}(p_v^{\times}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{2,v} \middle| \begin{array}{c} a, d \in 1 + p_v^{\times} \\ b, c \in p_v^{\times} \end{array} \right\},
\]
\[
\overline{K}_{2,v}(p_v^{\times}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{2,v} \middle| c \in p_v^{\times} \right\},
\]

then we have, for sufficiently large $n$,

\[(i) \quad \widetilde{K}_{2,v}(p_v^n) = (B_2(k_v) \cap K_{2,v})K_{2,v}(p_v^n),
(ii) \quad H(kl) = H(k, s) \quad (\forall l \in K_{2,v}(p_v^n), \ k \in K_{2,v}).\]

Thus if we choose

\[F_{s-1/2,v}(\chi_v \xi_v; k) = \begin{cases} \chi_v \xi_v(a) & \text{if } k = (a \ b) \in \widetilde{K}_{2,v}(p_v^n), \\ 0 & \text{otherwise,} \end{cases}\]

then $Z_v(W_v, \Phi_v, \chi_v; s)$ becomes

\[
\int_{B_2(k_v) \cap K_{2,v}(p_v^n) \setminus K_{2,v}(p_v^n)} F_{s-1/2,v}(\chi_v \xi_v; k)H(k, s) \, dk = c \cdot H(1, s).
\]

Hence every pole of $H(1, s)$ is a pole of $Z_v(W_v, \Phi_v, \chi_v; s)$.

\[\square\]

(2) Whittaker functions for generalized principal series. To prove the proposition for non-supercuspidal $\tau_v$, we need explicit formulae for their Whittaker functions. For this we calculate the Whittaker functions for the induced module $I(\nu_v) = \text{Ind}_{B_2(k_v)}^{G_2(k_v)}[\nu_v \otimes 1_{N_2(k_v)}], \ \text{where } \nu_v \text{ is a quasi-character of } k_v^{\times}.$

We define the Whittaker functional on $I(\nu_v)$ by the principal value integral

\[W_{\psi_v}(f)(g) := \sum_{n \in \mathbb{Z}} \int_{p_v^n - p_v^{n+1}} f(w^{-1} \left( \begin{array}{cc} x & 1 \\ 0 & 1 \end{array} \right) g) \psi_v(x) \, dx,\]

where $w = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$. If this is well-defined, then it gives a Whittaker function at least for an irreducible quotient of $I(\nu_v)$. We formally have

\[W_{\psi_v}(f)(d(a)) = \sum_{n \in \mathbb{Z}} \int_{p_v^n - p_v^{n+1}} f(\left( \begin{array}{cc} \sigma(a)^{-1} & 0 \\ 0 & a \end{array} \right) w^{-1} \left( \begin{array}{cc} 1 & N_{k_v/v}(a)^{-1}x \\ 0 & 1 \end{array} \right)) \psi_v(x) \, dx\]

\[= \nu_v(\sigma(a))^{-1}[a]_{w^{-1}}^{1/2} \sum_{n \in \mathbb{Z}} \int_{p_v^n - p_v^{n+1}} f(w^{-1} \left( \begin{array}{cc} x & 1 \\ 0 & 1 \end{array} \right)) \psi_v(N_{k_v/v}(a)x) \, dx.\]

But since

\[f(w^{-1} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)) = f(\left( \begin{array}{cc} x^{-1} & -1 \\ 0 & x \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ x^{-1} & 1 \end{array} \right)) = \nu_v(1 - |x|^{-1} f(1)\]

for $|x|_v \gg 1$, the principal value integral converges if $\text{Re } \nu_v > -1$ (cf. [JL]). Moreover, using the explicit formulae for the Kirillov models on $GL(2, k_v)$ (see A.2.3), we see that the Kirillov functions

\[\sum_{n \in \mathbb{Z}} \int_{p_v^n - p_v^{n+1}} f(w^{-1} \left( \begin{array}{cc} x & 1 \\ 0 & 1 \end{array} \right)) \psi_v(N_{k_v/v}(a)x) \, dx\]

are contained in

\[
\left\{ \begin{array}{ll}
\{ f_1(N_{k_v/v}(a)) \nu_v(N_{k_v/v}(a)) + f_2(N_{k_v/v}(a)) \mid f_1, f_2 \in S(k_v) \} & \text{if } \nu_v |_{k_v^{\times}} \neq |_{k_v^{\times}}^{-1}, 1, \\
\{ f_1(N_{k_v/v}(a)) \text{ord}_v(N_{k_v/v}(a)) + f_2(N_{k_v/v}(a)) \mid f_1, f_2 \in S(k_v) \} & \text{if } \nu_v |_{k_v^{\times}} = 1,
\end{array} \right.
\]

\[
\left\{ \begin{array}{ll}
\{ f_1(N_{k_v/v}(a)) \mid f_1 \in S(k_v) \} & \text{if } \nu_v |_{k_v^{\times}} = |_{k_v^{\times}}^{-1}.
\end{array} \right.
\]
Hence \( W_{\psi_v}(f)(d(a)) \) is of the form

\[
(B.14) \quad \begin{cases}
(f_1(N_{k_v^\infty/k_v}(a))\nu_v(a) + f_2(N_{k_v^\infty/k_v}(a))\nu_v^{-1}(\sigma(a)))|a|_{k_v^\infty}^{1/2} \\
(f_1(N_{k_v^\infty/k_v}(a))\nu_v(a) \text{ord}_v(N_{k_v^\infty/k_v}(a)) + f_2(N_{k_v^\infty/k_v}(a))\nu_v^{-1}(\sigma(a)))|a|_{k_v^\infty}^{1/2}
\end{cases}
\]

if \( \nu_v|_{k_v^\infty} \neq |_{k_v^\infty}^{-1},1 \),

\[
(f_1(N_{k_v^\infty/k_v}(a))\nu_v(a) \text{ord}_v(N_{k_v^\infty/k_v}(a)) + f_2(N_{k_v^\infty/k_v}(a))\nu_v^{-1}(\sigma(a)))|a|_{k_v^\infty}^{1/2}
\]

if \( \nu_v|_{k_v^\infty} = |_{k_v^\infty}^{-1} \),

where \( f_1 \) and \( f_2 \) are in \( S(k_v) \).

**3) Proof of Proposition B.10 for non-supercuspidal \( \tau_v \).** First consider \( \tau_v \) in the principal or complementary series. It equals some \( I(\nu_v) \) for which either (i) \( \nu_v \) is unitary and \( \nu_v|_{k_v^\infty} \neq \eta_{k_v^\infty/k_v} \), or (ii) \( \nu_v|_{k_v^\infty} = |_{k_v^\infty}^{-\lambda} \) with \( 0 < \lambda < 1 \). Hence, by (B.14), \( H(1,s) \) is of the form

\[
\int_{k_v^\infty \times k_v^\infty} \Phi_1(a)\chi_v(\nu_v)(a)|a|_{k_v^\infty}^s da \times + \int_{k_v^\infty \times k_v^\infty} \Phi_2(a)\chi_v(\nu_v \circ \sigma)|a|_{k_v^\infty}^s da
\]

if \( \nu_v|_{k_v^\infty} \neq 1 \), or

\[
\int_{k_v^\infty \times k_v^\infty} \Phi_1(a)\chi_v(\nu_v)(a) \text{ord}_v(N_{k_v^\infty/k_v}(a))|a|_{k_v^\infty}^s da \times + \int_{k_v^\infty \times k_v^\infty} \Phi_2(a)\chi_v(\nu_v \circ \sigma)|a|_{k_v^\infty}^s da
\]

if \( \nu_v|_{k_v^\infty} = 1 \), with \( \Phi_i \in S(k_v) \) \( (i = 1,2) \). In both cases the G.C.D. of \( H(1,s) \) is \( L_k(s,\chi_v \nu_v)L_k(s,\chi_v(\nu_v \circ \sigma)^{-1}) \), which equals \( L(s,\tau_v \times \chi_v) \).

Next we treat \( \tau_v \) in the limit of discrete series. This \( \tau_v \) is a direct summand of \( I(\nu_v) \) with \( \nu_v|_{k_v^\infty} = \eta_{k_v^\infty/k_v} \). As in the principal case, we have from (B.14) that the G.C.D. of \( H(1,s) \) equals

\[
L_k(s,\chi_v \nu_v)L_k(s,\chi_v(\nu_v \circ \sigma)^{-1}).
\]

This is holomorphic for \( \text{Re}(s) > 0 \), while \( L(s,\tau_v \times \chi_v) \) is also holomorphic in that region by [Sh], Proposition 7.2 (b).

Finally we consider \( \tau_v \) in the discrete series. It is a unique irreducible quotient of \( I(\nu_v) \) with \( \nu_v|_{k_v^\infty} = |_{k_v^\infty}^{-1} \). Hence from (B.14),

\[
H(1,s) = \int_{k_v^\infty \times k_v^\infty} \Phi(a)\chi_v(\nu_v \circ \sigma)|a|_{k_v^\infty}^s da
\]

and their G.C.D. is \( L_k(s,\chi_v(\nu_v \circ \sigma)^{-1}) \). This is holomorphic for \( \text{Re}(s) > 0 \), and again we use [Sh], Proposition 7.2 (b), to finish the proof.

**4) Whittaker functions for supercuspidal \( \tau_v \).** To obtain an explicit formula for the Kirillov space of such \( \tau_v \), we use the following variant of the Gel’fand-Kazhdan theory (cf. [BZ], Chapter III). We freely use the notation and terminology of [BZ]. In particular, \( Ind_H^G \) denotes the induction functor from \( Alg(H) \) to \( Alg(G) \), and \( ind_H^G \) denotes the finite induction functor between these categories.

We write the space of characters of \( N_2(k_v) \) as \( \widetilde{N_2(k_v)} \). Then our \( \psi_v \) allows us to identify \( \widetilde{N_2(k_v)} \) with the \( \ell \)-space \( k_v \) and \( C_\infty^\infty(N_2(k_v)) \) with \( S(\widetilde{N_2(k_v)}) \) (by the Fourier transform). This gives the category equivalence

\[
(\dagger) \quad Alg(N_2(k_v)) \sim \ell-\text{sheaves}/\widetilde{N_2(k_v)}.
\]
If $E$ is an $\ell$-sheaf on $N_2(k_v)$ corresponding to an object $E$ of $\text{Alg}(N_2(k_v))$ by the equivalence $(\dagger)$, then $E_{N_2(k_v)} \isin \mathcal{E}$ as $S(N_2(k_v))$-modules for any $\psi \in \widetilde{N}_2(k_v)$ (cf. [BZ], 5.9 and 5.10).

Fix $\alpha \in k_v \setminus N_{k_v/(k_v^2,w)}$ and set $\psi^+ := \psi_v$, $\psi^- := \psi_v^0$. Then we have six functors:

$\Phi^\pm : \text{Alg}(B_2(k_v)) \ni E \longrightarrow E_{N_2(k_v),\psi^\pm} \in \text{Alg}(Z_2(k_v))$,

$\Phi^+ = \text{ind}_{Z_2(k_v)}^{B_2(k_v)}(\rho \otimes \psi^+) \in \text{Alg}(B_2(k_v))$,

$\Psi^- : \text{Alg}(B_2(k_v)) \ni E \longrightarrow E_{N_2(k_v)} \in \text{Alg}(T_2(k_v))$,

$\Psi^+ : \text{Alg}(T_2(k_v)) \ni \mu \longrightarrow \mu \otimes 1_{N_2(k_v)} \in \text{Alg}(B_2(k_v))$.

The main properties of these functors are as follows.

Lemma B.12. (a) All functors are exact.

(b) $\Psi^-$ is left adjoint to $\Psi^+$, and $\Phi^\pm$ is left adjoint to $\Phi^\pm$.

(c) $\Phi^\pm \circ \Psi^+ = 0$ and $\Psi^- \circ \Phi^\pm = 0$.

(d) The adjunction maps $i^\pm : i \rightarrow \Phi^\pm \Phi^+ \Phi^\pm$ and $j : \Psi^- \Psi^+ \rightarrow \text{id}$ are isomorphisms.

(e) We have an exact sequence

$0 \longrightarrow \Phi^\pm \Phi^+ \Phi^\pm \longrightarrow \text{id} \longrightarrow \Psi^- \Psi^+ \longrightarrow 0$.

Proof. (a) is clear. (b)–(c) will be proved from the following considerations.

1. For $(\pi, E) \in \text{Alg}(B_2(k_v))$, we can take an $\ell$-sheaf $(\widetilde{N}_2(k_v), \mathcal{E})$ such that $E \gin \mathcal{E}_c$ as $S(N_2(k_v))$-modules. Using this isomorphism we transport the $B_2(k_v)$-module structure to $\mathcal{E}_c$. Then

$\pi(n)\varphi(\theta) = \theta(n)\varphi(\theta)$ \hspace{1em} $(\theta \in N_2(k_v), n \in N_2(k_v), \varphi \in \mathcal{E}_c)$.

2. Once we have realized $(\pi, E)$ on $\mathcal{E}_c$, we analyze the $B_2(k_v)$-orbits in $\widetilde{N}_2(k_v)$. We have three orbits:

$Z := \{1_{N_2(k_v)}\}$, \hspace{1em} $Y^\pm := \{(\psi^\pm)^{B_2(k_v)}\}$.

Let $\pi^\pm := \pi|_{\mathcal{E}_c(Y^\pm)}$. Then since the representation of $Z_2(k_v)N_2(k_v)$, the stabilizer of $\psi^\pm$ in $B_2(k_v)$, on the stalk $\mathcal{E}_c(Y^\pm)\psi^\pm = \mathcal{E}_c\psi^\pm$ is equivalent to $\Phi^\pm(\pi) \otimes \psi^\pm$, we have from [BZ], 2.23 (b), that

$E(N_2(k_v), 1) = \pi^+ \oplus \pi^-$, \hspace{1em} $\Phi^\pm \Phi^\pm (\pi) = \pi^\pm$.

Thus from $(\tau \rightarrow \Phi^\pm(\pi)) \in \text{Hom}(\mathcal{E}_c(Y^\pm), \Phi^\pm(\pi))$ we can construct an element of $\text{Hom}(\mathcal{E}_c(Y^\pm), \Phi^\pm(\pi))$ by the composition

$\Phi^\pm(\pi) \longrightarrow [\Phi^\pm \Phi^\pm (\pi) = \pi^\pm] \rightarrow \pi$.

3. Next we construct the inverse $\text{Hom}(\mathcal{E}_c(Y^\pm), \Phi^\pm(\pi)) \rightarrow \text{Hom}(\mathcal{E}_c(Y^\pm), \Phi^\pm(\pi))$. By [BZ], 2.23 (a), we have an $\ell$-sheaf $\mathcal{F}^\pm$ over $Y^\pm$ such that $\Phi^\pm(\pi) \simeq \mathcal{F}^\pm \otimes \mathcal{S}(Y^\pm)$-modules. We extend this to an $\ell$-sheaf $\mathcal{F}^\pm$ on $\widetilde{N}_2(k_v)$ by $\mathcal{F}^\pm_{|Z \times Y^\pm} := \{0\}$. Then $\mathcal{F}^\pm$ is the $\ell$-sheaf associated to $\Phi^\pm(\pi)$ by the construction of (1). Hence

$\Phi^\pm(\pi) \rightarrow (\Phi^\pm(\pi))_{N_2(k_v),\psi^\pm} \simeq \mathcal{F}^\pm_{|\psi^\pm} = \tau$. 


Also we have $\Phi_+^\pm \circ \Phi_+^\pm = 0$. This gives the natural isomorphism $i_\pm : \text{id} \sim \Phi_+^\pm \Phi_+^\pm$, and for $\Phi_+^\pm(\tau) \to \pi$ we have an element
\[
\tau \sim \Phi_+^\pm \Phi_+^\pm(\tau) \to \Phi_+^\pm(\pi)
\]
of $\text{Hom}_{Z_2(k_v)}(\tau, \Phi_+^-(\pi))$.

(4) The left adjointness of $\Psi^-$ to $\Psi^+$ is obvious. Since $F_\pm|_Z = \{0\}$, we have $\Psi^- \Phi_+^\pm = 0$, and $\Psi^+|_{N_2(k_v)} = 1_{N_2(k_v)}$ gives $\Phi_+^\pm \Psi^+ = 0$. Finally, the exact sequence in (e) is given by
\[
0 \longrightarrow E(N_2(k_v), 1) \longrightarrow E \longrightarrow E_{N_2(k_v), 1} \longrightarrow 0.
\]

\[\square\]

**Corollary B.13** (Kirillov models for supercuspidal $\tau_v$). Assume that $\tau_v$ is supercuspidal. Then for $W_v \in W(\tau_v, \psi_v)$, $W_v(d(a))$ is in $S(k_v^\infty)$ as a function of $a \in k_v^\times$.

**Proof.** Define $\pi_\pm := \text{Ind}_{B_2(k_v)} Z_2(k_v) N_2(k_v)(\omega_{\tau_v} \otimes \psi^\pm)$ and $\pi_0^\pm := \text{Ind}_{Z_2(k_v) N_2(k_v)}(\omega_{\tau_v} \otimes \psi^\pm)$. By definition we have the natural embedding $\pi_0^\pm \hookrightarrow \pi_\pm$. On the other hand,
\[
\text{Hom}_{B_2(k_v)}(\pi_0^\pm, \pi_\pm) = \text{Hom}_{Z_2(k_v) N_2(k_v)}(\pi_0^\pm|_{Z_2(k_v) N_2(k_v)}, \omega_{\tau_v} \otimes \psi^\pm)
\]
is 1-dimensional.

Since $\tau_v$ is supercuspidal, it follows by Lemma B.12 that $\tau_v|_{B_2(k_v)} = \Phi_+^\pm \Phi_+^-(\tau_v) \oplus \Phi_+^\pm \Phi_+^-(\tau_v)$. But $W(\tau_v, \psi_v) \ni W_v \to W_v|_{B_2(k_v)} \in \pi_\pm$ is non-zero by our assumption (recall $\psi_v = \psi^+$), and hence $\Phi_+^\pm \Phi_+^-(\tau_v) = \pi_0^\pm$.

We still have to show that $\Phi_+^\pm \Phi_+^-(\tau_v) = \{0\}$. If this is not $\{0\}$, then it is a direct sum of copies of $\pi_0^\pm$. But since $\text{Hom}_{B_2(k_v)}(\pi_0^-, \pi_\pm) = 0$, $\Phi_+^\pm \Phi_+^-(\tau_v)$ goes to $\{0\}$ under the "restriction to $B_2(k_v)$"-functor. \[\square\]

**Proof of Proposition B.10 in the supercuspidal case.** From Corollary B.13, the G.C.D. of $H(1, s)$ is 1. On the other hand, from [Sh], Proposition 7.2 (b), we know that $L(s, \tau_v \times \chi_v)$ is also holomorphic on the region $\text{Re}(s) > 0$. (q.e.d)

### B.3.4. $K$-types of the oscillator representation.

We now proceed to the archimedean inert case, $k_v^\times / k_v = \mathbb{C}/\mathbb{R}$. To calculate $Z_v(W_v, \Phi_v, \chi_v; s)$ explicitly, we need to know the $K$-types of the oscillator representation $\omega_{\psi_v, \xi_v}$.

We begin with the explicit formulae for the differential $d\omega_{\psi_v, \xi_v}$. Noting that $\eta_{\mathbb{C}/\mathbb{R}} = \text{sgn}$, we can write $\xi_v$ as
\[
\xi_v(z) = (z/\overline{z})^{k/2}, \quad (k \in 2\mathbb{Z} + 1).
\]
Write $g_{2, \mathbb{R}}$ for the $\mathbb{R}$-Lie algebra of $G_2(\mathbb{R})$. It is generated by the elements
\[
X_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}.
\]

**Lemma B.14.** The explicit formulae for the differential of $(\omega_{\psi_v, \xi_v}, S(\mathbb{C}))$ are given as follows:

\[
d\omega_{\psi_v, \xi_v}(X_+) \Phi(z = x + \sqrt{-1}y) = \pm 2\pi \sqrt{-1}(x^2 + y^2)\Phi(z) \quad \text{(as } \psi_v = \psi_v^{\pm 1},
\]
\[
d\omega_{\psi_v, \xi_v}(H) \Phi(z) = \Phi(z) + x \frac{\partial}{\partial x} \Phi(z) + y \frac{\partial}{\partial y} \Phi(z),
\]
\[
d\omega_{\psi_v, \xi_v}(U) \Phi(z) = \Phi(z) + y \frac{\partial}{\partial y} \Phi(z).
\]

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\[
d\omega_{\psi, \xi}(Z) \Phi(z) = \sqrt{-1} k \Phi(z) - y \frac{\partial}{\partial x} \Phi(z) + x \frac{\partial}{\partial y} \Phi(z),
\]
\[
d\omega_{\psi, \xi}(U) \Phi(z) = \pm \left[ \frac{1}{8\pi \sqrt{-1}} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi(z) + 2\pi \sqrt{-1}(x^2 + y^2) \Phi(z) \right]
\]
for the choice of \( \psi_\nu = \psi_{\nu, \xi}^{\pm 1} \).

Proof. These can be easily deduced from (B.1), (B.2) and (B.3).

Next we change the polarization.

**Lemma B.15** (Change of the polarization). Define the partial Fourier transform \( F_2 \) by
\[
F_2 \Phi(x + \sqrt{-1}y) = \int_R \Phi(x + \sqrt{-1}v) \psi_v(yv) dv.
\]
Then we have
\[
F_2(d\omega_{\psi, \xi}(U) \Phi)(z) = \pm \left[ \frac{1}{8\pi \sqrt{-1}} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) F_2 \Phi(z) + 2\pi \sqrt{-1}(x^2 + y^2) F_2 \Phi(z) \right],
\]
(B.15)
\[
F_2(d\omega_{\psi, \xi}(Z) \Phi)(z) = \left[ \sqrt{-1} k \pm \left( \frac{1}{4\pi \sqrt{-1}} \frac{\partial^2}{\partial x \partial y} + 4\pi \sqrt{-1} xy \right) \right] F_2 \Phi(z).
\]
(B.16)
Here the signs \( \pm \) are according to \( \psi_\nu = \psi_{\nu, \xi}^{\pm 1} \).

Proof. The proof is again a simple computation.

The final preliminary is the change of variables \( u := x + y \) and \( v = x - y \). Then we have
\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v},
\]
\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right), \quad \frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}.
\]
Hence if we write \( \phi(u, v) := F_2 \Phi(z) \in S(\mathbb{R}^2) \) and set
\[
D_t := \frac{1}{4\pi \sqrt{-1}} \frac{\partial^2}{\partial t^2} + \pi \sqrt{-1} t^2 \quad (t = u \text{ or } v),
\]
then (B.15) and (B.16) become
\[
d\omega_{\psi, \xi}(U) \phi(u, v) = \pm (D_u + D_v) \phi(u, v), \tag{B.17}
\]
\[
d\omega_{\psi, \xi}(Z) \phi(u, v) = (\sqrt{-1} k \pm (D_u - D_v)) \phi(u, v). \tag{B.18}
\]

We now describe the \( \mathbb{K}_2 \)-types in \( \omega_{\psi, \xi} \). First we solve the differential equation
\[
D_t \phi(t) = \sqrt{-1} \lambda \phi(t) \quad (\lambda \in \mathbb{R}, \ \phi \in S(\mathbb{R})).
\]

**Lemma B.16** (cf. [W], Chapt. 5, §6). \( D_t \phi(t) = \sqrt{-1} \lambda \phi(t) \) has a non-trivial solution in \( S(\mathbb{R}) \) (or in \( L^2(\mathbb{R}) \)) if and only if \( \lambda \in \mathbb{Z}_{\geq 1/2} \). If this is the case, the solution space is spanned by
\[
\phi_{\lambda}(t) = e^{-\pi t^2} H_{\lambda-1/2}(\sqrt{2\pi} t).
\]
Here $H_n \ (n \in \mathbb{Z}_{\geq 0})$ denotes the classical Hermite function:

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}.$$  

As usual, we identify each irreducible representation of $K_{2,v}$ with a pair of integers:

$$(m, n) : K_{2,v} \ni e^{\sqrt{-1} \varphi} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mapsto e^{\sqrt{-1} m \theta} e^{\sqrt{-1} n \varphi} \in \mathbb{C}^1.$$  

**Proposition B.17** ($K_{2,v}$-types of $\omega_{\psi_v, \xi_v}$). Using the notation defined above, the $K_{2,v}$-types of $\omega_{\psi_v, \xi_v}$ are

$$\left\{ \begin{array}{l} (m, n) \in \mathbb{Z} \oplus \mathbb{Z} \mid m = n \ (\text{mod } 2), k - m < n < k + m \quad \text{if } \psi_v = \psi_R, \\ (m, n) \in \mathbb{Z} \oplus \mathbb{Z} \mid m = n \ (\text{mod } 2), k + m < n < k - m \quad \text{if } \psi_v = \psi_L. \end{array} \right.$$  

**B.3.5.** $Z_v(W_v, \Phi_v, \chi_v; s)$ in the principal case. We treat $\tau_v$ in the principal and complementary series at the same time. We may assume that

$$\tau_v = \text{Ind}_{\mathcal{B}_2(\mathbb{R})}^{G_2(\mathbb{R})}[\nu_v \otimes 1_{N_2(\mathbb{R})}], \quad \chi_v \xi_v(z) = |z|^{\sigma_v(z/\overline{z})\nu_v/2}, \quad \nu_v(z) = |z|^{\sigma_v(z/\overline{z})\nu_v/2},$$

where

$$\text{Re}(\sigma_v) = 0, \quad 0 \leq \text{Re}(\sigma_v) \leq 1, \quad l_{\mu}, l_{\nu} \in \mathbb{Z}.$$  

As before, we write $S^0(\mathbb{C})$ for the space of standard functions with respect to $\psi_v$:

$$S^0(\mathbb{C}) := \{ e^{-2\pi(x^2+y^2)} P(x, y) \mid P \text{ is a polynomial function} \}.$$  

We may assume $\Phi_v \in S^0(\mathbb{C}).$

**Proposition B.18.** Under these assumptions, $Z_v(W_v, \Phi_v, \chi_v; s)$ is a linear combination of functions of the form

$$\Gamma_{\mathbb{R}}(s + \sigma_{\mu} + \sigma_{\nu} + j_1 + \frac{n}{2} + 1) \Gamma_{\mathbb{R}}(s + \sigma_{\mu} - \sigma_{\nu} + j_2 + \frac{n}{2} + 1)$$

$$\times \Gamma_{\mathbb{R}}(s + \sigma_{\mu} + \sigma_{\nu} + j_1 + \frac{n}{2}) \Gamma_{\mathbb{R}}(s + \sigma_{\mu} - \sigma_{\nu} + j_2 + \frac{n}{2}) / \Gamma_{\mathbb{R}}(2s + 2\sigma_{\mu} + 1)$$

with $j_1, j_2, n \in \mathbb{Z}_{\geq 0}$. Moreover, the term with $j_1 = j_2 = n = 0$ appears if and only if $k = l_{\mu} \pm l_{\nu}$ and $l_{\mu}$ is odd.

Assuming the proposition for a moment, we shall prove the following.

**Corollary B.19.** The G.C.D. of $Z_v(W_v, \Phi_v, \chi_v; s)$ has the same set of poles in the region $\text{Re}(s) > 0$ as

$$\frac{L_\mathbb{C}(s, \chi_v \nu_v) L_\mathbb{C}(s, \chi_v \nu_v^{-1})}{L_{\mathbb{R}}(2s, (\chi_v \xi_v)|_{\mathbb{R}^+})}.$$  

**Proof.** Recall from [Sh2], §3, that

$$L_\mathbb{C}(s, \chi_v \nu_v) = L_\mathbb{C}(s, |z|^{\sigma_v(z/\overline{z}) (l_{\mu} + l_{\nu} - k)/2}) = \Gamma_{\mathbb{C}}(s + \sigma_{\mu} + \sigma_{\nu} + \frac{|l_{\mu} + l_{\nu} - k|}{2}),$$

$$L_\mathbb{C}(s, \chi_v \nu_v^{-1}) = \Gamma_{\mathbb{C}}(s + \sigma_{\mu} - \sigma_{\nu} + \frac{|l_{\mu} - l_{\nu} - k|}{2}),$$

$$L_{\mathbb{R}}(2s, (\chi_v \xi_v)|_{\mathbb{R}^+}) = \Gamma_{\mathbb{R}} \left( s + \sigma_{\mu} + \frac{1}{2} \right).$$
where $\epsilon = 0$ or 1 is such that $\epsilon \equiv l_\mu \pmod{2}$. Thus (B.20) has its only pole in the region $\Re(s) > 0$ at $s = \sigma_\mu - \sigma_\nu$ with
\[
k = l_\mu - l_\nu, \quad 0 < \sigma_\nu < 1/2, \quad l_\nu \text{ even}.
\]
On the other hand, (B.19) in Proposition B.18 has its only pole in the same region at $s = \sigma_\mu - \sigma_\nu$ with
\[
f_1 = f_2 = n = 0.
\]
We apply the last statement of Proposition B.18 to verify that these two conditions are equivalent.

**Proof of Proposition B.18.** We may assume that $F_{-1/2,v}(\chi_v \xi_v)$ and $W_v$ are of the form
\[
F_{s-1/2,v}^{(m_1)}(\chi_v \xi_v; e^{\sqrt{-1}\varphi}(\cos \theta - \sin \theta \cos \theta)) = e^{\sqrt{-1}m_1 \varphi} e^{\sqrt{-1}m_1 \theta}, \quad m_1 \in l_\mu + 2\mathbb{Z},
\]
\[
W_v^{(m_2)}(e^{\sqrt{-1}\varphi}(\cos \theta - \sin \theta \cos \theta)) = e^{\sqrt{-1}m_2 \varphi} e^{\sqrt{-1}m_2 \theta}, \quad m_2 \in l_\nu + 2\mathbb{Z}.
\]
Let $(n_1, n_2)$ be a $K_{2,v}$-type in $\omega_{\psi_{v,\xi_v}}$ (cf. Proposition B.17). Using the notation of Lemma B.16, we set
\[
\phi_{n_1, n_2}(x + \sqrt{-1}y) := \phi_{\pm, n_1 + n_2, \pm, \pm} (x + y) \phi_{\pm, n_1 + n_2, \pm, \pm} (x - y) \in \mathcal{S}^0(\mathbb{C})
\]
\[
\Phi_{n_1, n_2}(x + \sqrt{-1}y) := \mathcal{F}_2^{-1} \phi_{n_1, n_2}(x + \sqrt{-1}y),
\]
where the inverse Fourier transform $\mathcal{F}_2^{-1}$ is defined by
\[
\mathcal{F}_2^{-1} \phi(x + \sqrt{-1}y) := \int_{\mathbb{R}} \phi(x + \sqrt{-1}v) \psi_v(vy) dv.
\]
From Lemma B.16, it is clear that $\Phi_{n_1, n_2}$ is the $(n_1, n_2)$-weight vector in $\omega_{\psi_{v,\xi_v}}$, and it is enough to calculate
\[
Z_v(W_v^{(m_2)}, \Phi_{n_1, n_2}, \chi_v; s) = \int_{\mathbb{R}^+} \int_{\mathbb{C}^\times} F_{s-1/2,v}^{(m_1)}(\chi_v \xi_v; k) \omega_{\psi_{v,\xi_v}}(k) \Phi_{n_1, n_2}(a) W_v^{(m_2)}(d(a)k) \chi_v(a) |a|_{\mathbb{C}}^{-1/2} da dk
\]
writing $k(\theta) := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$,
\[
= \int_{\mathbb{R}^+} \int_{0}^{2\pi} F_{s-1/2,v}^{(m_1)}(\chi_v \xi_v; d(e^{\sqrt{-1}\varphi})k(\theta)) \omega_{\psi_{v,\xi_v}}(d(e^{\sqrt{-1}\varphi})k(\theta)) \Phi_{n_1, n_2}(a) W_v^{(m_2)}(d(ae^{\sqrt{-1}\varphi})k(\theta)) \chi_v(a) |a|_{\mathbb{R}}^{2s-1} d\varphi da d\theta
\]
\[
= \int_{\mathbb{R}^+} \Phi_{n_1, n_2}(a) W_v^{(m_2)}(d(a)) \chi_v(a) |a|_{\mathbb{R}}^{2s-1} da \times \int_{0}^{2\pi} e^{\sqrt{-1}(l_\nu + l_\mu - n_2)} \varphi e^{\sqrt{-1}(m_1 + m_2 - n_1) \theta} d\varphi d\theta.
\]
This equals
\[
\int_{\mathbb{R}^+} \Phi_{n_1, n_2}(a) W_v^{(m_2)}(d(a)) \chi_s(a) |a|^{2s-1} da^x
\]
if \( n_2 = l_\mu + l_\nu \), \( n_1 = m_1 + m_2 \), and it is zero otherwise.

Now we set \( n_2 = l_\mu + l_\nu \), and take \( n_1, m_1 \) and \( m_2 \) so that \( n_1 = m_1 + m_2 \) and the conditions of Proposition B.17 are satisfied. Then
\[
Z_v(W_v^{(m_2)}, \Phi_{n_1, n_2}, \chi_s; s) = \int_{\mathbb{R}^+} \Phi_{n_1, n_2}(a) W_v^{(m_2)}(d(a)) |a|^{2(s+\sigma_\mu-\sigma_\nu)} da^x.
\]
Next we extend the restriction \( \tau_v|_{H(\mathbb{R})} \) to a representation
\[
\tilde{\tau}_v := \text{Ind}_{B(\mathbb{R})}^{GL(2, \mathbb{R})} ([\nu_v]_{\mathbb{R}^x} \otimes 1) \otimes 1_{N(\mathbb{R})}
\]
of \( GL(2, \mathbb{R}) \) just as in the proof of Lemma B.6. Accordingly \( W_v^{(m_2)} \) restricted to \( H(\mathbb{R}) \) is extended to a Whittaker function of \( \tilde{\tau}_v \), and we have
\[
W_v^{(m_2)}(d(a)) = \nu_v(a)^{-1} W_v^{(m_2)} \left( \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix} \right).
\]
Thus
\[
Z_v(W_v^{(m_2)}, \Phi_{n_1, n_2}, \chi_s; s) = 2 \int_{\mathbb{R}^+} \Phi_{n_1, n_2}(\sqrt{a}) W_v^{(m_2)} \left( \begin{pmatrix} a \nu_s & 0 \\ 0 & 1 \end{pmatrix} \right) |a|^{s+\sigma_\mu-\sigma_\nu-1/2} da^x.
\]
To compute this we need following lemmas.

**Lemma B.20.** (i) The Mellin transform of \( \Phi_{n_1, n_2}(\sqrt{a}) \),
\[
\int_{\mathbb{R}^+} \Phi_{n_1, n_2}(\sqrt{a}) |a|^{s-1/2} da^x,
\]
is a linear combination of terms \( \Gamma(\mathbb{R})(s + \frac{n+1}{2}) \Gamma(\mathbb{R})(s + \frac{n-1}{2}) \) \((n \in \mathbb{Z}_{\geq 0})\).

(ii) The term \( \Gamma(\mathbb{R})(s + \frac{1}{2}) \Gamma(\mathbb{R})(s - \frac{1}{2}) \) appears if and only if \( k = l_\mu + l_\nu \).

**Proof.** Recall that
\[
\Phi_{n_1, n_2}(r) = F_{2r}^{-1} \phi_{n_1, n_2}(r) = \int_{\mathbb{R}} \phi_{n_1, n_2}(r + \sqrt{-1}y) dy
\]
\[
= \int_{\mathbb{R}} \phi_{\frac{n_1-n_2+k}{2}, \frac{n_1+n_2-k}{2}}(r + y) \phi_{\frac{n_1+n_2+k}{2}, \frac{n_1-n_2-k}{2}}(r - y) dy
\]
\[
= \int_{\mathbb{R}} e^{-2\pi r(y^2+1)} H_{\frac{n_1-n_2+k}{2}}(\sqrt{2\pi}(r+y)) H_{\frac{n_1+n_2-k}{2}}(\sqrt{2\pi}(r-y)) dy
\]
\[
= e^{-2\pi r^2} \int_{\mathbb{R}} e^{-2\pi x} H_{\frac{n_1-n_2+k}{2}}(x+\sqrt{2\pi}r) H_{\frac{n_1+n_2-k}{2}}(\sqrt{2\pi}r-x) \frac{dx}{\sqrt{2\pi}}
\]
\[
= (-1)^{\frac{n_1+n_2-k}{2}} e^{-2\pi r^2}
\]
\[
\times \int_{\mathbb{R}} e^{-2\pi x} H_{\frac{n_1-n_2+k}{2}}(x+\sqrt{2\pi}r) H_{\frac{n_1+n_2-k}{2}}(x-\sqrt{2\pi}r) \frac{dx}{\sqrt{2\pi}}.
\]
Clearly this is a linear combination of terms of the form \( e^{-2\pi r^2} r^n \) \((n \in \mathbb{Z}_{\geq 0})\). The orthogonality relation of Hermite polynomials asserts that the term \( e^{-2\pi r^2} \) appears
if and only if \( \pm \frac{m_n - m_2 + k}{2} = \pm \frac{m_n + m_2 - k}{2} \), or equivalently \( k = n_2 = l_\mu + l_\nu \). Now our lemma follows from the well-known formula
\[
\int_0^\infty e^{-2\pi a |a|^s} a^{s+n/2-1/2} da = \pi^{-s-n/2} \Gamma(s+(n+1)/2) \Gamma(s+(n-1)/2).
\]

\( \square \)

Lemma B.21. (i) The Mellin transform of \( \psi_{v}^{(m_2)} \),
\[
\int_{\mathbb{R}^x} W_{v}^{(m_2)}\left(\begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix}\right)|a|^{s-1/2} \, da \times,
\]
is a linear combination of terms \( \Gamma_{\mathbb{R}}(s+j_1+2\sigma_\nu)\Gamma_{\mathbb{R}}(s+j_2) \) \((j_1, j_2 \in \mathbb{Z}_{\geq 0})\).

(ii) The term \( \Gamma_{\mathbb{R}}(s+2\sigma_\nu)\Gamma_{\mathbb{R}}(s) \) appears if and only if \( m_2 \) is even.

Proof. We may assume \( \Re(\sigma_\nu) \gg 0 \) and \( \Re(s) \gg 0 \) throughout the proof. Let
\[
\Phi^{(m_2)}(x, y) := e^{-\pi(x^2+y^2)}(x+i y)^{m_2} \in \mathcal{S}^0(\mathbb{R}^2),
\]
\[
f^{(m_2)}_{\nu_v}(g) := \nu_v(\det(g))|\det(g)|^{1/2} \int_{\mathbb{R}^x} \Phi^{(m_2)}([0, t]g) \nu_v(t) \, dt \, dx.
\]
This integral converges absolutely for \( \Re(\sigma_\nu) > -m_2 + 1 \), and \( f^{(m_2)}_{\nu_v} \) is an \( m_2 \)-weight vector in \( \mathcal{T}_v \). Thus from [JL], §5, we may take \( W_{v}^{(m_2)} \) to be
\[
W_{v}^{(m_2)}(g) = \int_{\mathbb{R}} f^{(m_2)}_{\nu_v} (w^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) \psi_v(x) \, dx
\]
by definition,
\[
= \nu_v(\det(g))|\det(g)|^{1/2} \int_{\mathbb{R}^x} \Phi^{(m_2)}([t, tx]g) \psi_v(t) \, dt \, \overline{\psi_v(x)} \, dx
\]
putting \( x := tx \),
\[
= \nu_v(\det(g))|\det(g)|^{1/2} \int_{\mathbb{R}^x} \Phi^{(m_2)}([t, x]g) \psi_v(t^{-1}x) \, dx \, \nu_v(t) \, dt \times
\]
writing \( R(g) \) for the right translation by \( g \),
\[
= \nu_v(\det(g))|\det(g)|^{1/2} \int_{\mathbb{R}^x} \mathcal{F}_2^{-1}[R(g)\Phi^{(m_2)}](t, t^{-1}) \psi_v(t) \, dt \times
\]
using Proposition 1.6 (ii) in [JL],
\[
= \nu_v(\det(g))|\det(g)|^{1/2} \int_{\mathbb{R}^x} [\omega_{\psi_v}(g)\mathcal{F}_2^{-1}\Phi^{(m_2)}](t, t^{-1}) \psi_v(t) \, dt \times.
\]
Here \( \omega_{\psi_v} \) denotes the Weil representation of \( SL(2, \mathbb{R}) \) defined in [JL], Proposition 1.3, extended to \( GL(2, \mathbb{R}) \) by Proposition 1.6, loc.cit.

Using this, we have
\[
W_{v}^{(m_2)}\left(\begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix}\right) = \int_{\mathbb{R}^x} \mathcal{F}_2^{-1}\Phi^{(m_2)}(at, t^{-1}) \psi_v(at) \, |a|^{1/2} \, dt \times.
\]
Hence the Mellin transform becomes
\[
\int_0^\infty W^{(m_2)}_x\left(\begin{array}{c} a \\ 0 \\ 1 \end{array}\right)|a|^s da_x
\]
\[
= \frac{1}{2} \left( \int_{\mathbb{R}^+} W^{(m_2)}_x\left(\begin{array}{c} a \\ 0 \\ 1 \end{array}\right)|a|^{s-1/2} da_x \\
+ \int_{\mathbb{R}^+} W^{(m_2)}_x\left(\begin{array}{c} a \\ 0 \\ 1 \end{array}\right)|a|^{s-1/2} \text{sgn}(a) da_x \right)
\]
\[
= \frac{1}{2} \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} F^{-1}_2 \Phi^{(m_2)}(at, t^{-1})|\nu_v(0)|a|^s \text{dt} \times da_x \\
+ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} F^{-1}_2 \Phi^{(m_2)}(at, t^{-1})|\nu_v(0)|a|^s \text{sgn}(a) \text{dt} \times da_x \right)
\]
\[
= \frac{1}{2} \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} F^{-1}_2 \Phi^{(m_2)}(x, y)|\nu_v(x)|y|^s \text{dx} \times dy_x \\
+ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} F^{-1}_2 \Phi^{(m_2)}(x, y)|\nu_v(x)|y|^s \text{sgn}(y) \text{dx} \times dy_x \right)
\]

On the other hand,
\[
F^{-1}_2 \Phi^{(m_2)}(x, y) = \int_{\mathbb{R}} e^{-\pi(x^2+u^2)}(x + \sqrt{-1}u)m_2\psi_v(-uy) du
\]
\[
= \sum_{j=0}^{m_2} \sqrt{-1}^j x^{m_2-j} \binom{m_2}{j} e^{-\pi x^2} \int_{\mathbb{R}} e^{-\pi u^2} u^j \psi_v(-uy) du.
\]
This is a linear combination of terms \(e^{-\pi(x^2+y^2)}x^{j_1}y^{j_2}\) \((j_1, j_2 \in \mathbb{Z}_{\geq 0})\), and the term \(e^{-\pi(x^2+y^2)}\) appears if and only if \(m_2\) is even. Hence the Mellin transform is a linear combination of
\[
\frac{1}{2} \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} e^{-\pi(x^2+y^2)}x^{j_1}y^{j_2}|\nu_v(0)|y|^s \text{dx} \times dy_x \\
+ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} e^{-\pi(x^2+y^2)}x^{j_1}y^{j_2}|\nu_v(0)|y|^s \text{sgn}(y) \text{dx} \times dy_x \right)
\]
\[
= \frac{1}{2} \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} e^{-\pi x^2}|x|^{2\nu + j_1}|\nu_v(0)|y|^s \text{dx} \times dy_x \\
+ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} e^{-\pi x^2}|x|^{2\nu + j_1+1}|\nu_v(0)|y|^s \text{sgn}(y) \text{dx} \times dy_x \right)
\]
\[
+ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} e^{-\pi y^2}|y|^{2\nu + j_2}|\nu_v(0)|y|^s \text{dx} \times dy_x \\
+ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} e^{-\pi y^2}|y|^{2\nu + j_2+1}|\nu_v(0)|y|^s \text{sgn}(y) \text{dx} \times dy_x \right)
\]
This equals
\[
2 \int_0^\infty e^{-\pi x^2}x^{2\nu + j_1+s} \text{dx} \times \int_0^\infty e^{-\pi y^2}y^{j_2+s} \text{dy} = \frac{1}{2} \Gamma(s + j_1 + 2\nu) \Gamma(s + j_2)
\]
if \(j_1 + l_\nu \equiv j_2 \mod 2\), and is zero otherwise. The term with \(j_1 = j_2 = 0\) appears if and only if \(l_\nu\) (or equivalently \(m_2\)) is even.

We now return to the proof of Proposition B.18. We apply the Barnes-Mellin lemma ([Ja], Lemma 17.3.2) to the result of Lemmas B.20 and B.21 to see that
\[
\int_{\mathbb{R}_+^2} \Phi_{n_1, n_2}(a) W^{(m_2)}_x\left(\begin{array}{c} a \\ 0 \\ 1 \end{array}\right)|a|^{s-1} da_x
\]
is a linear combination of terms
\[
\frac{\Gamma_{\mathbb{R}}(s + j_1 + 2\sigma_{\nu} + \frac{n+1}{2})\Gamma_{\mathbb{R}}(s + j_2 + \frac{n+1}{2})\Gamma_{\mathbb{R}}(s + j_1 + 2\sigma_{\nu} + \frac{n-1}{2})\Gamma_{\mathbb{R}}(s + j_2 + \frac{n-1}{2})}{\Gamma_{\mathbb{R}}(2s + j_1 + j_2 + 2\sigma_{\nu} + n)},
\]
and the term corresponding to \( j_1 = j_2 = n = 0 \) appears if and only if \( k = l_{\mu} + l_{\nu} \) and \( l_{\nu} \) is even. Since \( k \) is odd, this is equivalent to \( k = l_{\mu} + l_{\nu} \) and \( l_{\mu} \) is odd. Replacing \( s \) with \( s + \sigma_{\mu} - \sigma_{\nu} + 1/2 \) will yield the proposition. The statement for \( k = l_{\mu} - l_{\nu} \) will be obtained by replacing \( \psi_{\nu} \) with \((\psi_{\nu} \circ \sigma)^{-1}\). (q.e.d.)

B.3.6. \( Z_v(W_v, \Phi_v, \chi_v; s) \) in the discrete case. Here we take care of \( \tau_v \) in the discrete series. We embed such \( \tau_v \) into a generalized principal series:
\[
\tau_v \leftrightarrow \text{Ind}_{B_{2}(\mathbb{R})}^{G_{2}(\mathbb{R})}[\nu_v \otimes 1_{N_{2}(\mathbb{R})}].
\]
Here \( \nu_v(z) = |z|_{\mathbb{C}}^{k_{\nu}/2}(z/\pi)^{s_{\nu}/2} \) satisfies the following condition: if \( \epsilon_{\nu} = 0 \) or 1 is such that \( l_{\nu} \equiv \epsilon_{\nu} (\text{mod} \ 2) \), then \( k_{\nu} - \epsilon_{\nu} \in 2\mathbb{N} - 1 \).

**Proposition B.22.** (i) Under these assumptions, the G.C.D. of \( Z_v(W_v, \Phi_v, \chi_v; s) \) is given by
\[
\frac{\Gamma_{\mathbb{C}}(s + \sigma_{\mu} + k_{\nu}/2)\Gamma_{\mathbb{C}}(s + \sigma_{\mu} + k_{\nu}/2 + 1)}{\Gamma_{\mathbb{R}}(2s + 2\sigma_{\mu} + k_{\nu} + 2)}.
\]
(ii) In particular, \( Z_v(W_v, \Phi_v, \chi_v; s) \) is holomorphic in the region \( \text{Re}(s) > 0 \).

**Proof.** Let the notation be as in the proof of Proposition B.18. Then the local zeta integral \( Z_v(W_v^{(m_2)}, \Phi_{n_1,n_2}, \chi_v; s) \) equals
\[
\int_{\mathbb{R}_+} \mathcal{F}_{n_1,n_2}(a)W_v^{(m_2)}(d(a))\chi_v(a)|a|_{\mathbb{R}}^{2s-1} \, da
\]
if \( n_2 = l_{\mu} + l_{\nu} \) and \( n_1 = m_1 + m_2 \), and is zero otherwise. We concentrate on the former case. We again extend \( W_v^{(m_2)} \) to a Whittaker function of \( \tau_v \),
\[
\tau_v = \text{Ind}_{B(\mathbb{R})}^{GL(2,\mathbb{R})}[\nu_v|_{\mathbb{R}} \times 1] \otimes 1_{N(\mathbb{R})}],
\]
and obtain
\[
Z_v(W_v^{(m_2)}, \Phi_{n_1,n_2}, \chi_v; s) = 2 \int_{\mathbb{R}_+} \mathcal{F}_{n_1,n_2}(\sqrt{a})W_v^{(m_2)}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})|a|_{\mathbb{R}}^{s+\sigma_{\mu}-(k_{\nu}+1)/2} \, da.
\]

The analogue of Lemma B.21 in this case is much simpler. \( \square \)

**Lemma B.23.** The Mellin transform of \( W_v^{(m_2)} \),
\[
\int_{\mathbb{R}_+} W_v^{(m_2)}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})|a|_{\mathbb{R}}^{-1/2} \, da,
\]
is a linear combination of terms \( \Gamma_{\mathbb{R}}(s + k_{\nu} + j + 1)\Gamma_{\mathbb{R}}(s + k_{\nu} + j) \) \((j \in \mathbb{Z}_{\geq 0})\).

**Proof.** From the explicit calculation in [JL], pp. 187–189, we know that
\[
W_v^{(2n+k_{\nu}+1)}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) = \begin{cases} P_n(a)a^{k_{\nu}+1/2}e^{-2\pi a} & \text{if } a > 0, \\ 0 & \text{if } a \leq 0, \\ 0 & \text{if } a \geq 0, \\ P_n(-a)(-a)^{k_{\nu}+1/2}e^{2\pi a} & \text{if } a < 0, \\ \end{cases} \text{ if } \psi_{\nu} = \psi_{\mathbb{R}}.
\]
Here $P_n(a)$ is a certain polynomial. The formulae for $W^{(-2n-\kappa_{\nu}^{-1})}_\nu$ are obtained by replacing $\psi_\nu = \psi_R$ and $\psi_{\nu} = \psi_R^{-1}$ in the above. Thus our Mellin transform is a linear combination of
\[
\int_{R^\times} e^{-2\pi a d^x k_{\nu} + j} \, da^x = \Gamma_R(s + k_{\nu} + j) \Gamma(s + k_{\nu} + j) \quad (j \in \mathbb{Z}_{\geq 0}).
\]
\[\square\]

We again apply the Barnes-Mellin lemma to the result of Lemmas B.20 and B.23 to see that $\varepsilon_\nu(W^{(mz)}_\nu, \Phi_{n_1, n_2}, \chi; s)$ is a linear combination of
\[
\frac{\Gamma_R(s + \sigma_{\mu} + \frac{k_{\mu}}{2} + j + \frac{n}{2} + 2) \Gamma_R(s + \sigma_{\mu} + \frac{k_{\mu}}{2} + j + \frac{n}{2} + 1)^2 \Gamma_R(s + \sigma_{\mu} + \frac{k_{\mu}}{2} + j + \frac{n}{2}) \Gamma_R(2s + 2\sigma_{\mu} + k_{\nu} + 2j + n + 2)}{\Gamma_R(2s + 2\sigma_{\mu} + k_{\nu} + 2j + n + 2)}
\]
$(j, n \in \mathbb{Z}_{\geq 0})$. Finally, we note that
\[
\Gamma_R(s + 1) \Gamma_R(s) = \int_0^\infty e^{-2\pi x^x} \, dx = \frac{1}{2} \Gamma_C(s),
\]
which gives us the proposition. \[\square\]

### B.4. Poles of $L(s, \tau \times \chi)$

We finally come to the main result of this appendix.

**Theorem B.24** (Poles of $L(s, \tau \times \chi)$). $L(s, \tau \times \chi)$ has its only possible pole in $\text{Re}(s) > 0$ at $s = 1$, and it is a pole if and only if

1. $\chi|_{\mathbb{H}^x} = \eta e^{k_{\nu}/k}$,
2. $\tau = \Theta(\chi^{-1}, \psi)_{\omega}$.

**Proof.** Lemma B.6, Lemma B.7, Corollary B.9, Proposition B.10, Corollary B.19 and Proposition B.22 combined with Proposition B.5 assert that we have only to determine the poles of
\[
Z(\varphi, \theta_\psi, \chi; s) := \int_{G_2(k) \setminus G_2(A)} \varphi(g) \theta_\psi(g) E(F(\chi \xi), s - 1/2)(g) \, dg.
\]

First note that $\varphi$ is rapidly decreasing while $\theta_\psi$ and $E(F(\chi \xi), s - 1/2)(g)$ are slowly increasing. Hence the integral converges absolutely (if $E(F(\chi \xi), s - 1/2)(g)$ is defined), and its poles come from those of $E(F(\chi \xi), s - 1/2)(g)$. But it is well-known that

1. The only pole of $E(F(\chi \xi), s - 1/2)(g)$ in $\text{Re}(s) > 0$ occurs if and only if $\chi|_{\mathbb{H}^x} = 1$, and it is located at $s = 1$.
2. The residue of $E(F(\chi \xi), s - 1/2)(g)$ at the pole spans the 1-dimensional representation $(\chi \xi)^{G_{2}}$.

Thus $Z(\varphi, \theta_\psi, \chi; s)$ has its only possible pole in $\text{Re}(s) > 0$ at $s = 1$, and it is a pole if and only if $\chi|_{\mathbb{H}^x} = \eta e^{k_{\nu}/k}$ and
\[
\text{B.21) } \int_{G_2(k) \setminus G_2(A)} \varphi(g) \theta_\psi(g) (\chi \xi)^{G_2}(g) \, dg \neq 0.
\]

But the left hand side reads
\[
\int_{G_2(k) \setminus G_2(A)} \varphi(zg) \theta_\psi(zg) (\chi \xi)^{G_2}(zg) \, dz \, dg
\]
\[
= \int_{G_2(k) \setminus G_2(A)} \varphi(g) (\chi \xi)^{G_2}(g) \int_{Z_2(k) \setminus Z_2(A)} \omega_{-\chi \xi}(z) \theta_\psi(zg) \, dz \, dg
\]
\[
= \int_{G_2(k) \setminus G_2(A)} (\chi \xi)^{G_2}(g) \varphi(g) \theta_\psi(\omega_{-\chi \xi}(g)) \, dg.
\]
Of course this means that (B.21) is equivalent to  

\[(\chi \xi)_{G^2}^{-1} \otimes \Theta(\xi, \psi)_{\omega, \chi \xi},\]

since the multiplicity one theorem is true for $G_2$.

We now analyze the representation $(\chi \xi)_{G^2}^{-1} \otimes \Theta(\xi, \psi)_{\omega, \chi \xi}$. By definition this space is spanned by functions of the form  

\[\int_{Z_2(k) \backslash Z_2(\mathbb{A})} \overline{\omega_{\tau}(z)} \theta(\omega_{\psi, \xi}(g) \Phi(z)) \, dz\]

\[= \int_{Z_2(k) \backslash Z_2(\mathbb{A})} (\chi \xi)_{G^2}^{-1}(zg) \chi(z) \overline{\omega_{\tau}(z)} \theta(\xi(z)^{-1} \omega_{\psi, \xi}(zg) \Phi(z)) \, dz\]

\[= \int_{Z_2(k) \backslash Z_2(\mathbb{A})} \omega_{\tau}(z) \theta((\chi \xi)_{G^2}^{-1}(zg) \omega_{\psi, \xi}(zg) \Phi(z)) \, dz.\]

On the other hand, it follows from the explicit formulae (B.8), (B.9), (B.10) that  

\[((\chi \xi)_{G^2}^{-1} \otimes \omega_{\psi, \xi}, S(k)) \simeq (\omega_{\psi, \chi^{-1}}, S(k')).\]

Hence $(\chi \xi)_{G^2}^{-1} \otimes \Theta(\xi, \psi)_{\omega, \chi \xi}$ is spanned by  

\[\int_{Z_2(k) \backslash Z_2(\mathbb{A})} \omega_{\tau}(z) \theta(\omega_{\psi, \chi^{-1}}(zg) \Phi(z)) \, dz\]

\[= \int_{Z_2(k) \backslash Z_2(\mathbb{A})} \omega_{\tau}(z) \theta(\omega_{\psi, \chi^{-1}}(z) \omega_{\psi, \chi^{-1}}(g) \Phi(z)) \, dz\]

\[= \int_{Z_2(k) \backslash Z_2(\mathbb{A})} \omega_{\tau}(z) \theta(\omega_{\psi, \chi^{-1}}(g) \Phi(z)) \, dz.\]

These span $\Theta(\chi^{-1}, \psi)_{\omega, \chi}$.

We now apply Shahidi’s non-vanishing theorem to $L(s, \pi, w(P_2))$, and get the following corollary.

**Corollary B.25** (Proposition 5.2). The poles of $r(w(P_2), \pi)$ with $\langle \Re \pi, \beta_{\mathcal{P}} \rangle \geq 0$ are located at  

\[\mathcal{S}(P_2, \eta_{\mathcal{K}/k}) := \{ \pi = \chi \otimes \tau \in \mathcal{P} : \chi|_{\mathcal{A}^*} = |_{\mathcal{A}^*}, \tau = \Theta(\chi^{-1}) |_{\mathcal{A}^*}, s, \psi \},\]

\[\mathcal{S}(P_2, 1) := \{ \pi = \chi \otimes \tau \in \mathcal{P} : \chi|_{\mathcal{A}^*} = |_{\mathcal{A}^*}, L(0, \tau \times \chi) \neq 0 \} \]

Here in the definition of $\mathcal{S}(P_2, \eta_{\mathcal{K}/k})$, $\psi$ has been chosen so that $\tau_0$ is generic with respect to it.

Note that $\mathcal{S}(P_2, 1)$ is the only pole of $L(s, \pi, w(P_2), 2) = L(k(s, \chi|_{\mathcal{A}^*})$.  

**References**


THE RESIDUAL SPECTRUM OF $U(2,2)$


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