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LINEARIZATION, DOLD-PUPPE STABILIZATION, AND MAC LANE'S Q-CONSTRUCTION

BRENDA JOHNSON AND RANDY MCCARTHY

Abstract. In this paper we study linear functors, i.e., functors of chain complexes of modules which preserve direct sums up to quasi-isomorphism, in order to lay the foundation for a further study of the Goodwillie calculus in this setting. We compare the methods of Dold and Puppe, Mac Lane, and Goodwillie for producing linear approximations to functors, and establish conditions under which these methods are equivalent. In addition, we classify linear functors in terms of modules over an explicit differential graded algebra. Several classical results involving Dold-Puppe stabilization and Mac Lane's Q-construction are extended or given new proofs.

0. Introduction

Derived functors are a fundamental tool in homological algebra. However, to have a good notion of derived functors, one imposes certain conditions (additivity, right exactness) on the functor with which one starts. These conditions raise some natural questions—for example, can one give a well-defined notion of a derived functor in the absence of these conditions? One solution to this problem is described by Dold and Puppe in [D-P]. For a non-additive functor F from an additive category (with enough projectives) to an abelian category, Dold and Puppe replace F by a functor that preserves direct sums up to quasi-isomorphism and agrees with F up to a certain degree in homology. With these properties, the Dold-Puppe *stabilization* of F can be treated as a linear approximation to F. Some logical next questions to raise are "Can we construct degree n approximations to F, and what kind of functors would such approximations be?"

The answer to the existence question is provided in a series of papers by Tom Goodwillie ([G1], [G2], [G3]) in which he develops the theory of calculus of homotopy functors. In particular, for a functor F of *topological spaces* satisfying certain connectivity conditions, Goodwillie constructs a tower of functors and natural transformations as shown in Figure 1, in which each P_nF is a degree n functor and approximates F in a range that tends to infinity with n. Although this is a result about functors of topological spaces, rather than functors of additive categories, we will see that any functor of R-modules for a ring R can be extended to a functor of spaces for which such a tower exists. In this way, we can establish that the degree n approximations to such functors exist by virtue of existence in the topological case.

Unfortunately, this means of proving the existence of the degree n approximations is not very satisfactory to the algebraist. Our goal is to develop Goodwillie's

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$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ & & P_nF \\ & \nearrow & \downarrow \\ F & \rightarrow & P_{n-1}F \\ & \searrow & \downarrow \\ & & \vdots \\ & & \downarrow \\ & & P_0F = F(\star) \end{array}$$

FIGURE 1

calculus of functors in a purely algebraic fashion. In this paper, we will concentrate on the case where $n = 1$ and will provide two descriptions of P_1F as well as a classification of them. In the first of two sequels to this paper ([J-M1]), we will prove the existence and provide a description of P_nF for all $n \geq 0$. In the second ([J-M2]), we will use the model in [J-M1] to prove a classification theorem for all degree n functors.

In the case when $n = 1$, Goodwillie's P_1F is a functor with the following properties:

1) P_1F is *linear*.
2) There is a natural transformation $F \rightarrow P_1F$ such that for any space X, $F(X) \rightarrow P_1F(X)$ is an isomorphism (roughly) on the first $2k$ homotopy groups, where k is the connectivity of X.
3) P_1F is universal in the homotopy category with respect to properties 1) and 2).

There are two classical constructions that are equivalent to P_1F when F is a functor from an additive category to an abelian category, the Dold-Puppe stabilization described in the first paragraph, and Mac Lane's Q-construction. In this paper we describe these constructions, develop the algebraic analogs of properties 1) and 2), and prove that the constructions satisfy these properties. (The proof of 3) is a formality and is covered in [J-M1].) We also show that linear functors (i.e., functors that preserve direct sums up to quasi-isomorphism) are determined up to homotopy by modules over a certain differential graded algebra, $QS(R^{op})$, described in section 8. This results in the following classification theorem for linear functors.

Corollary 8.8. *For any rings with identity R and S, there is an equivalence of homotopy categories*

$$Ho(F_{lin}(Ch_{\geq 0}\mathcal{P}_R, Ch_{\geq 0}\mathcal{M}_S)) \simeq Ho(Mod - QS(R^{op}))$$

where $F_{lin}(Ch_{\geq 0}\mathcal{P}_R, Ch_{\geq 0}\mathcal{M}_S)$ is the category of linear functors from chain complexes of finitely generated projective R-modules to chain complexes of S-modules and $Mod - QS(R^{op})$ is the category of right modules over the differential graded algebra $QS(R^{op})$.

The equivalence of Dold-Puppe stabilization, Mac Lane's Q-construction and Goodwillie's linearization is generally known, but has not been written up. We felt

that it was important to do so, especially in preparation for the development of the higher degree approximations in [J-M1] and [J-M2]. In the process of proving these results, we also show that the Dold-Puppe stabilization is linear for a larger class of functors than was done by Dold and Puppe or Simson and Tyc ([S-T]). This is an immediate consequence of

Theorem 3.17. *If F is a reduced functor ($F(0) = 0$) from $\mathcal{M}_R$ to $\mathcal{M}_S$ and $P'' \to P \to P'$ is a cofibration sequence in $Ch_{\geq 0}\mathcal{M}_R$, then $D_1F(P'') \to D_1F(P) \to D_1F(P')$ is a quasi-exact sequence in $Ch_{\geq 0}\mathcal{M}_S$, i.e., it yields a natural long exact sequence in homology (or equivalently, forms a distinguished triangle in the triangulated category of $Ch_{\geq 0}\mathcal{M}_R$).*

We have also provided a new proof that Dold-Puppe stabilization and Mac Lane's Q-construction are equivalent for certain classes of functors. This is the goal of section 7 and is established in theorem 7.5:

Theorem 7.5. *For a reduced functor F from $\mathcal{M}_R$ to $\mathcal{M}_S$, the functors D_1F (the Dold-Puppe stabilization of F) and* **QF** *(Mac Lane's Q-construction applied to the prolongation of F) are naturally quasi-isomorphic as functors from $Ch_{\geq 0}\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$.*

The material in this paper should be accessible to anyone with a basic knowledge of homological algebra, and we have endeavored to write with that goal in mind. This has meant, in some cases, including more detail than is necessary for the expert, and in other cases, reviewing some standard definitions and results. Attaining this level of accessibility has occasionally come at the expense of not employing more general terminology (e.g., Quillen model category), but those familiar with such terminology should be able to interpret the results in their favorite categorical language without too much difficulty.

The paper is organized as follows. In section 1 we review the definition and properties of classical derived functors and point out the problems inherent in extending these results to the non-additive case. In sections 2 and 3 we review Dold and Puppe's solution to these problems and prove that their functors have the "nice" properties of the classical derived functors. We also review basic definitions and results about simplicial objects that will be used throughout the paper. In section 4 we summarize some calculations of stable derived functors of the symmetric and exterior power functors done by Dold and Puppe ([D-P]) and Simson and Tyc ([S-T], [S]). In section 5 we define linearity for our functors, prove that the Dold-Puppe stable derived functors are linear and establish some basic properties of linear functors. Sections 6 and 7 are devoted to Mac Lane's Q-construction, in particular to defining the Q-construction, showing that it agrees with the Dold-Puppe stabilization in certain cases and determining the extent to which the Q-construction of a functor agrees with the original functor. In section 8 we prove our classification theorem for linear functors, and in section 9 we explain how Dold-Puppe stabilization is related to the linearization of a functor of spaces.

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1. Classical Derived Functors

Given an additive functor F from $\mathcal{A}$ to $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $\mathcal{A}$ has enough projectives, one can define the left derived functors of F. These functors are highly important to homological algebra, but their construction and use depend on the fact that the original functor is additive. Dold and Puppe ([D-P]) provide a way of extending the definition of derived functors to non-additive functors that preserves the useful properties present in the additive case. Our first goal in this paper is to outline their construction. We will begin by reviewing the definition and some properties of derived functors in the additive case in order to indicate the problems that arise in the non-additive case, as well as to indicate the properties that the Dold-Puppe functors should have. The material that follows can be found in any standard text on homological algebra (e.g., [We] or [M1]).

We begin with some notation. For a ring R, let $\mathcal{M}_R$ be the category consisting of all right R-modules and $\mathcal{P}_R$ the full subcategory of $\mathcal{M}_R$ consisting of projective R-modules. Let $Ch_{\geq n}\mathcal{M}_R$ be the category of chain complexes over $\mathcal{M}_R$ that are concentrated in degrees $\geq n$. The objects of this category are chain complexes $\{M_*;\partial\}$ where $\{M_*\}_{*\in Z}$ is a Z-graded module, $M_* = 0$ for $* < n$, and $\partial : M_* \to M_{*-1}$ is a differential. Similarly, $Ch_{\geq n}\mathcal{P}_R$ is the subcategory of $Ch_{\geq n}\mathcal{M}_R$ whose objects are chain complexes of projective modules. One can identify the objects in $\mathcal{M}_R$ with those objects in $Ch_{\geq 0}\mathcal{M}_R$ that are concentrated in degree 0 only. Similarly, objects in $\mathcal{P}_R$ can be identified with objects in $Ch_{\geq 0}\mathcal{P}_R$. We will make use of these identifications throughout this paper. For rings R and S, we will be studying functors F from $\mathcal{M}_R$ to $\mathcal{M}_S$. In this section, we focus our attention on additive functors, i.e., functors for which $F(\alpha + \alpha') = F(\alpha) + F(\alpha')$ for all morphisms $\alpha, \alpha' \in Hom_{\mathcal{M}_R}(M, M')$, or, equivalently, functors that preserve finite direct sums of modules. Throughout this section, unless otherwise indicated, F will be an additive functor.

To define derived functors, one replaces objects in $\mathcal{M}_R$ with approximations by equivalent objects in $Ch_{\geq 0}\mathcal{P}_R$. By an approximation of an R-module M, we will mean a chain complex $\{P_*;\partial\}$ in $Ch_{\geq 0}\mathcal{P}_R$ such that there is a natural chain map $P_* \to M$ which is a quasi-isomorphism, i.e., a chain map that induces an isomorphism on homology. (Note that an approximation of M is equivalent to a projective resolution of M.) One can easily determine the existence of such a resolution by noting that every object in $\mathcal{M}_R$ admits a surjection from a free one.

In order to apply our additive functor of modules, F, to these approximations, we must enlarge the domain and codomain of F to the category $Ch_{\geq 0}\mathcal{M}_R$. We do so by defining the prolongation of F to be the functor $\mathbf{F} : Ch_{\geq 0}\mathcal{M}_R \to Ch_{\geq 0}\mathcal{M}_S$, where $\mathbf{F}(\{M_*;\partial\}) = \{F(M_*); F(\partial)\}$.

With this, we are ready to recall the definition of (left) derived functors.

Definition 1.1. Let F be an additive functor from $\mathcal{M}_R$ to $\mathcal{M}_S$. Let M be an object in $\mathcal{M}_R$ and let P be an approximation of M. The *nth left derived functor* of F is the functor L_nF defined on M by $L_nF(M) = H_n\mathbf{F}(P)$.

Recall that for L_nF to be well-defined, it must be independent of the choice of projective resolution of M. This follows from the fact that if P and Q are two different approximations to the module M, then $\mathbf{F}(P)$ and $\mathbf{F}(Q)$ are chain homotopy equivalent. More specifically, the comparison theorem ([We], p. 35) guarantees the existence of chain maps $f : P \to Q$ and $g : Q \to P$ such that $g \circ f$

and $f \circ g$ are liftings of id_M to P and Q respectively. Moreover, these liftings are unique up to a chain homotopy, and so $g \circ f$ and $f \circ g$ are chain homotopic to id_P and id_Q, respectively. Since F is additive, it preserves these chain homotopies, and thus $F(P)$ is chain homotopic to $F(Q)$. Furthermore, if F preserves cokernels (i.e., is right exact) then $H_0\mathbf{F}(P) \cong H_0\mathbf{F}(Q) \cong F(M)$.

An important feature of derived functors is that they associate to any short exact sequence of modules a long exact sequence of the L_nF's.

Theorem 1.2. *Let F be an additive functor from $\mathcal{M}_R$ to $\mathcal{M}_S$. For any short exact sequence*

$$0 \to M'' \to M \to M' \to 0,$$

there is a long exact sequence of left derived functors

$$\ldots \to L_nF(M'') \to L_nF(M) \to L_nF(M') \to L_{n-1}F(M'') \to \ldots.$$

Proof. This is a standard result, and the proof can be found in any text on homological algebra (see, for example, Lemma 2.2.8 of [We]). Here, we will outline the proof and provide only enough detail to indicate why F must be additive. The first step in the proof is to approximate $0 \to M'' \to M \to M' \to 0$ by a short exact sequence, $0 \to P'' \to P \to P' \to 0$, in $Ch_{\geq 0}\mathcal{P}_R$. Roughly speaking, this can be done by choosing approximations P'' and P' for M'' and M'. Then, P is constructed as a chain complex that in degree n is $P''_n \oplus P'_n$ with differentials carefully defined so as to produce a resolution that fits into an exact sequence $P'' \to P \to P'$.

Next, one needs to know how $\mathbf{F}$ behaves with respect to short exact sequences in $Ch_{\geq 0}\mathcal{P}_R$. To start, F preserves short exact sequences of projective modules. To see this, recall that every short exact sequence of projective modules is split exact. Since F is additive, it preserves such sequences. Since a short exact sequence in $Ch_{\geq 0}\mathcal{P}_R$ is a sequence of chain complexes which in each degree is a short exact sequence in $\mathcal{P}_R$, and F was prolonged to $Ch_{\geq 0}\mathcal{P}_R$ degreewise, it follows that $\mathbf{F}$ also preserves short exact sequences in $Ch_{\geq 0}\mathcal{P}_R$. Note that F need not preserve arbitrary short exact sequences in $\mathcal{M}_R$ or $Ch_{\geq 0}\mathcal{M}_R$. For example, given an $R-S$ bimodule A, the functor $G(-) = - \otimes_R A$ is additive but need not preserve short exact sequences.

Finally, since $0 \to \mathbf{F}(P'') \to \mathbf{F}(P) \to \mathbf{F}(P') \to 0$ is a short exact sequence in $Ch_{\geq 0}\mathcal{M}_S$, it yields a long exact sequence in homology, and the result follows.

We conclude this section by noting the problems that arise when F is not additive. First, if F is not additive, two approximations of M need not be taken by $\mathbf{F}$ to quasi-isomorphic complexes in $Ch_{\geq 0}\mathcal{M}_S$. To see this, suppose that $f, g : P \to Q$ are two chain maps and $h : f \simeq g$ is a chain homotopy between them. Then

$$\partial^Q_{n+1} \circ h_n + h_{n-1} \circ \partial^P_n = f_n - g_n$$

for all n. However, if F does not preserve addition of morphisms, then although $F(\partial^Q_{n+1} \circ h_n + h_{n-1} \circ \partial^P_n) = F(f_n - g_n)$, $F(\partial^Q_{n+1} \circ h_n) + F(h_{n-1} \circ \partial^P_n)$ may not equal $F(f_n) - F(g_n)$, and hence $F(f)$ and $F(g)$ may no longer be chain homotopic. If F does not preserve chain homotopies, then two approximations of a module may not be sent to quasi-isomorphic chain complexes. Secondly, if F is not additive then it will not preserve direct sums and, more specifically, it is no longer true that F preserves short exact sequences of projective modules. To overcome these problems, and produce a reasonable definition of derived functor in the non-additive

case, we will provide (in section 2) a means of prolonging F in such a way that the prolongation will preserve chain homotopy equivalences, and define (in section 3) a functor arising from F that takes short exact sequences to long exact sequences in homology.

2. Simplicial Modules and Chain Complexes

To define a prolongation of F that preserves quasi-isomorphisms, we move to the category of simplicial R-modules. In this section we review some basic facts about simplicial objects and simplicial homotopies, including the Dold-Kan theorem. (For more details the reader is referred to May [Ma], chapters 1 and 5, or Weibel [We], chapter 8.) The Dold-Kan theorem establishes an isomorphism between $Ch_{\geq 0}\mathcal{M}_R$ and the category of simplicial R-modules. With this, we will be able to define our prolongation of F via simplicial modules.

We begin by recalling the definition of simplicial object. Let Δ denote a category equivalent to the category of finite ordered sets. That is, Δ is the category with one object $[n] = \{0 < 1 < \cdots < n\}$ for each cardinality, and whose morphisms are the order-preserving set maps. Given a category $\mathcal{C}$, a simplicial object in $\mathcal{C}$ is a functor from Δ^{op} to $\mathcal{C}$, i.e., a contravariant functor from Δ to $\mathcal{C}$. We let $Simp_{\mathcal{C}}$ denote the category of simplicial objects in $\mathcal{C}$ with morphisms the natural transformations. We will use $X.$, $Y.$, etc., to represent objects in $Simp_{\mathcal{C}}$.

It is useful to distinguish particular generating morphisms of Δ. We will represent the morphisms in Δ by lower case Greek letters and let $Inj([m],[n]) \subseteq Hom_{\Delta}([m],[n])$ and $Sur([m],[n]) \subseteq Hom_{\Delta}([m],[n])$ be the subsets of injective and surjective morphisms, respectively. For a fixed n and each $0 \leq i \leq n+1$, we let $\delta_i^n \in Inj([n],[n+1])$ be the morphism defined by

$$\delta_i^n(j) = \begin{cases} j, & \text{if } j < i, \\ j+1, & \text{if } j \geq i. \end{cases}$$

Similarly, for a fixed n and $0 \leq i \leq n$, we let $\sigma_i^n \in Sur([n+1],[n])$ be defined by

$$\sigma_i^n(j) = \begin{cases} j, & \text{if } j \leq i, \\ j-1, & \text{if } j > i. \end{cases}$$

The upper index n for these maps will be omitted when the domains are clear. The δ_i's are referred to as face maps and the σ_i's as degeneracies. For a simplicial object in $\mathcal{C}$, we will let d_i and s_i represent δ_i^* and σ_i^* respectively. It is important to note that every $\alpha \in Hom_{\Delta}([m],[n])$ can be *uniquely* written as $\alpha = \delta \circ \sigma$ for some $\sigma \in Sur([m],[k])$ and $\delta \in Inj([k],[n])$. Furthermore, every injective map is a composite of face maps and every surjective map is a composite of degeneracies.

With these definitions, one can show that a simplicial object $C.$ in $\mathcal{C}$ is equivalent to a family of objects $\{C_q\}_{q\geq 0}$ together with two families of morphisms of $\mathcal{C}$:

$$d_i : C_q \to C_{q-1}, \quad s_i : C_q \to C_{q+1}, \quad i = 0, \ldots, q,$$

satisfying

$$
\begin{aligned}
&d_i d_j = d_{j-1} d_i \quad \text{if } i < j,\\
&s_i s_j = s_{j+1} s_i \quad \text{if } i \le j,\\
&d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } \ i < j,\\ \text{identity} & \text{if } i = j, i = j+1,\\ s_j d_{i-1} & \text{if } \ i > j+1. \end{cases}
\end{aligned}
\tag{2.1}
$$

Two objects in $Simp_{\mathcal{C}}$ are considered equivalent if there is a simplicial homotopy between them. To define simplicial homotopy, let $\Delta(1)$ be the simplicial set $Hom_{\Delta}(\star, [1])$ and (0) be the simplicial set contained in $\Delta(1)$ consisting of $\delta_0\sigma_0$ and all its degeneracies, i.e., all elements of the form $\sigma\delta_0\sigma_0$, where σ is any composition of degeneracies. Similarly, let (1) denote the simplicial set contained in $\Delta(1)$ consisting of $\delta_1\sigma_0$ and all its degeneracies. With this we have

Definition 2.2. A *simplicial homotopy*, h, between two simplicial maps $f, g : X. \to Y.$ is a simplicial map $h : X. \times \Delta(1) \to Y.$ such that $h|_{X.\times(0)} = f$ and $h|_{X.\times(1)} = g$.

One can also describe a simplicial homotopy as consisting of a set of maps $h_i(q) \in Hom(X_q, Y_{q+1})$, for $0 \le i \le q$, which satisfy the relations:

$$
\begin{aligned}
&d_0 h_0 = f, \qquad d_{q+1} h_q = g,\\
&d_i h_j = \begin{cases} h_{j-1} d_i & \text{if } i < j,\\ d_{j+1} h_{j+1} & \text{if } i = j+1,\\ h_j d_{i-1} & \text{if } i > j+1, \end{cases}\\
&s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \le j,\\ h_j s_{i-1} & \text{if } i > j. \end{cases}
\end{aligned}
\tag{2.3}
$$

As indicated at the beginning of the section, simplicial objects will be of use to us because of the following theorem.

Theorem 2.4 (Dold [D], Kan [K]). *The categories $Ch_{\ge 0}\mathcal{M}_R$ and $Simp_{\mathcal{M}_R}$ are naturally isomorphic in such a manner that the chain homotopies in $Ch_{\ge 0}\mathcal{M}_R$ correspond to the simplicial homotopies in $Simp_{\mathcal{M}_R}$. Moreover, objects in $Ch_{\ge 0}\mathcal{P}_R$ correspond to objects in $Simp_{\mathcal{P}_R}$.*

Many good proofs of this theorem exist in the literature (e.g., [Ma], pp. 93-98, or [We], pp. 270-274). Rather than repeat them, we will simply define the functors $N : Simp_{\mathcal{M}_R} \to Ch_{\ge 0}\mathcal{M}_R$ and $\Gamma : Ch_{\ge 0}\mathcal{M}_R \to Simp_{\mathcal{M}_R}$, that yield the isomorphisms, and leave it to the interested reader to check the necessary relations.

The functor N is called the normalization functor and is defined for an object $X.$ in $Simp_{\mathcal{M}_R}$ by

$$
N(X.)_n = \bigcap_{i=1}^{n} ker(d_i) \cong X_n / \sum_{i=1}^{n} im(s_i), \quad \partial_n = d_0.
$$

We note that $\partial_n \partial_{n+1} = 0$ by the relations (2.1) above. Furthermore, to see that objects in $Simp_{\mathcal{P}_R}$ correspond to objects in $Ch_{\ge 0}\mathcal{P}_R$, we can define, for each $1 \le k \le n$, $f_k : X_n \to ker(d_k)$ by $f_k(a) = a - s_k d_k a$ and use this map and the relations (2.1) to show that $ker(d_k)$ is a direct summand of X_n. By restricting f_k to $\bigcap_{i=1}^{k-1} ker(d_i)$, we can also show that for each k, $\bigcap_{i=1}^{k} ker(d_i)$ is a direct summand

of $\bigcap_{i=1}^{k-1} ker(d_i)$. It then follows that $N(X.)_n$ is a direct summand of X_n, and hence, if X_n is projective, $N(X.)_n$ must be as well.

The functor Γ is defined as follows. Let M_* be a chain complex in $\mathcal{M}_R$. Then $\Gamma(M_*)$ is the simplicial object such that at n,

$$\Gamma(M_*)_n = \bigoplus_{k=0}^{n} \bigoplus_{\sigma \in Sur([n],[k])} M_k$$

with the simplicial structure maps defined on $(m;\sigma) \in M_k \times Sur([n],[k])$ by

$$d_i(m;\sigma) = \begin{cases} (m;\sigma\delta_i) & \text{if } \sigma\delta_i \in Sur([n-1],[k]), \\ (\partial_k(m);\hat{\sigma}) & \text{if } \sigma\delta_i = \delta_0\hat{\sigma} \text{ (where } \hat{\sigma} \in Sur([n-1],[k-1])), \\ 0 & \text{if } \sigma\delta_i = \delta_i\hat{\sigma} \text{ with } i \geq 1, \end{cases}$$

$$s_i(m;\sigma) = (m;\sigma\sigma_i), \qquad \sigma\sigma_i \in Sur([n+1],[k-1]).$$

Clearly, Γ takes objects in $Ch_{\geq 0}\mathcal{P}_R$ to objects in $Simp_{\mathcal{P}_R}$, since each $\Gamma(M)_n$ is a finite direct sum of M_k's.

There is another functor $C : Simp_{\mathcal{M}_R} \to Ch_{\geq 0}\mathcal{M}_R$. For an object $X.$ in $Simp_{\mathcal{M}_R}$, $C(X.)$ is the chain complex with

$$C(X.)_n = X_n \qquad \text{and} \qquad \partial_n = \sum_{i=0}^{n} (-1)^i d_i.$$

The natural transformation from C to N induced by projection is a chain homotopy equivalence (see Corollary 22.3 of [Ma]). We will use this fact to replace $N(X.)$ by $C(X.)$ when convenient.

Because of the Dold-Kan theorem, we can now prolong a functor F by extending it degreewise on simplicial objects. The advantage in doing so is that this process will preserve simplicial homotopies, since the relations necessary for a simplicial homotopy involve compositions rather than sums or differences. To preserve these relations we simply need a functor.

Although we have only considered functors from $\mathcal{M}_R$ to $\mathcal{M}_S$ up to this point, we include in our definition below the prolongation of a functor from $\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$. This version of the definition will be used in later sections. To prolong such a functor, we use the total complex of the normalization of a bicomplex. Doing so entails establishing some conventions for bicomplexes and total complexes.

Remark 2.5. By a bicomplex, we mean a complex of complexes, i.e., a collection of R-modules $P_{k,n}$ and R-module homomorphisms, $\partial^h_{k,n} : P_{k,n} \to P_{k-1,n}$ and $\partial^v_{k,n} : P_{k,n} \to P_{k,n-1}$ such that ∂^v and ∂^h are both differentials, and for each k and n the square below is commutative:

$$\begin{array}{ccc} P_{k,n} & \xrightarrow{\partial^h_{k,n}} & P_{k-1,n} \\ \partial^v_{k,n} \downarrow & & \downarrow \partial^v_{k-1,n} \\ P_{k,n-1} & \xrightarrow{\partial^h_{k,n-1}} & P_{k-1,n-1}. \end{array}$$

This differs from the standard use of the term by a sign, i.e., Weibel ([We]) and Mac Lane ([M1]) require that the above square be anti-commutative. We use the term in this way so that it will describe the result of the normalization in both simplicial directions of a bisimplicial module. Of course, this use of the term bicomplex necessitates the introduction of a sign when taking the total complex

of a bicomplex. Our convention will be as follows. Let $P_{\cdot,\star}$ be a bicomplex with differentials as above. Then the total complex of P will be the chain complex, $Tot(P)$, that in degree n is

$$\bigoplus_{p+q=n} P_{p,q}$$

with differential

$$\bigoplus_{p+q=n} \partial^h_{p,q} + (-1)^p \partial^v_{p,q}.$$

The total complex of an n-dimensional chain complex is defined inductively. Let A be an n-dimensional chain complex and assume we have defined Tot for $n-1$ chain complexes. For each $p > 0$, $A_{p,\star,\ldots,\star}$ is an $n-1$ chain complex, $Tot(A_{p,\star,\ldots,\star})$ is a chain complex, and $Tot(A_{\bullet,\star,\ldots,\star})$ is a bicomplex. We set

$$Tot(A) = Tot(Tot(A_{\bullet,\star,\ldots,\star})).$$

Definition 2.6. For an arbitrary functor, F, from $\mathcal{M}_R$ to $\mathcal{M}_S$ we define the *prolongation* of F to be the functor $\mathbf{F}$ from $Ch_{\geq 0}\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$ given by

$$\mathbf{F} = N \circ F \circ \Gamma,$$

where F is applied degreewise to a simplicial module. If F is a functor from $\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$, then the prolongation of F is the functor

$$\mathbf{F} = Tot(N \circ F \circ \Gamma).$$

That is, $\mathbf{F}$ is the total complex of the bicomplex obtained by prolonging F degreewise.

Note that if F is an additive functor from $\mathcal{M}_R$ to $\mathcal{M}_S$, this definition is equivalent to the definition used in section 1. Moreover, since F preserves simplicial homotopies and simplicial homotopies correspond to chain homotopies, we have

Proposition 2.7. *Let F be a functor from $\mathcal{M}_R$ to $\mathcal{M}_S$ or $Ch_{\geq 0}\mathcal{M}_S$. Let M be an R-module and P and Q be projective approximations of M. Then $\mathbf{F}(P)$ and $\mathbf{F}(Q)$ are chain homotopic.*

3. Dold-Puppe Stable Derived Functors

In this section we review Dold and Puppe's definition of derived functors for non-additive functors and give a new proof of the fact that these functors take short exact sequences to long exact sequences in homology. Throughout this section F will be a *reduced* functor, i.e., a functor for which $F(0) = 0$.

We begin by defining some functors and natural transformations needed for the construction of the Dold-Puppe functors. For $k \in Z$, let sh_k be the functor from $Ch_{\geq i}\mathcal{M}_R$ to $Ch_{\geq (i+k)}\mathcal{M}_R$ defined by sending $\{P_*; \partial^P\}$ to $\{Q_*; \partial^Q\}$, where $Q_* = P_{*-k}$ and $\partial^Q_* = \partial^P_{*-k}$. For a functor $F : \mathcal{M}_R \to \mathcal{M}_S$ and $n \geq 0$ we define a natural transformation

$$Sus^n : sh_n \circ \mathbf{F} \to \mathbf{F} \circ sh_n$$

as follows. We first note that for a chain complex P,

$$[sh_n \circ \mathbf{F}(P)]_p = \bigcap_{j=1}^{p-n} ker(d_j)|_{F(\bigoplus_{l=0}^{p-n} \bigoplus_{Sur([p-n],[l])} P_l)}$$

and

$$[\mathbf{F} \circ (sh_n P)]_p = \bigcap_{i=1}^{p} ker(d_i)|_{F(\bigoplus_{k=0}^{p} \bigoplus_{Sur([p],[k])} P_{k-n})}.$$

We define Sus^n in degree p to be the map induced by the natural inclusion

$$\bigoplus_{l=0}^{p-n} \bigoplus_{Sur([p-n],[l])} P_l \to \bigoplus_{k=0}^{p} \bigoplus_{Sur([p],[k])} P_{k-n},$$

where

$$(l, \sigma, p) \mapsto (l+n, \widehat{\sigma}, p)$$

with $\widehat{\sigma} \in Sur([p],[l+n])$ defined by

$$\widehat{\sigma}(i) = \begin{cases} \sigma(i) & \text{if } 0 \le i \le p-n, \\ l+i-p+n & \text{if } i > p-n, \end{cases}$$

and so obtain our natural map from $sh_n \circ \mathbf{F}(P)$ to $\mathbf{F} \circ sh_n(P)$. Moreover, Sus^n is a chain map. This follows on checking that the diagram

$$\begin{array}{ccc} \bigoplus_{l=0}^{p-n} \bigoplus_{Sur([p-n],[l])} P_l & \xrightarrow{Sus^n} & \bigoplus_{k=0}^{p} \bigoplus_{Sur([p],[k])} P_{k-n} \\ d_0 \downarrow & & \downarrow d_0 \\ \bigoplus_{l=0}^{p-n-1} \bigoplus_{Sur([p-n-1],[l])} P_l & \xrightarrow{Sus^n} & \bigoplus_{k=0}^{p-1} \bigoplus_{Sur([p-1],[k])} P_{k-n} \end{array}$$

commutes and observing that the differentials in $sh_n\mathbf{F}(P)$ and $\mathbf{F}sh_n(P)$ are both given by $F(d_0)$. Similarly, it is straightforward to check that Sus^n preserves chain homotopy equivalences.

Once Sus^n is defined for functors from $\mathcal{M}_R$ to $\mathcal{M}_S$, the definition can be extended to an arbitrary functor F from $\mathcal{M}_R$ to $Ch_{\ge 0}\mathcal{M}_S$. To do so, we consider F as a sequence of functors and natural transformations, $F_0 \overset{\partial_1}{\leftarrow} F_1 \overset{\partial_2}{\leftarrow} F_2 \overset{\partial_3}{\leftarrow} \dots$, and apply Sus^n to each $sh_n \circ \mathbf{F}_i$ individually. Since Sus^n is natural, this process gives us a map of bicomplexes, $sh_n NF\Gamma(P) \to NF\Gamma sh_n(P)$, where we must be careful to note that $sh_n NF\Gamma(P)$ means that we have shifted $\mathbf{F}(P)$ in the direction of the chain complex P. Applying Tot, this yields a natural transformation from $Tot(sh_n NF\Gamma(P))$ to $Tot(NF\Gamma sh_n(P))$. Moreover, since sh_n is being applied in the direction of the chain complex P, Tot and sh_n commute, and so we have produced a natural transformation from $sh_n\mathbf{F}(P)$ to $\mathbf{F}(sh_n P)$.

In addition, one can show (by looking at the composition of the inclusions defined above) that $Sus^n \circ Sus^m = Sus^{n+m}$. Clearly, the analogous statement for sh ($sh_n \circ sh_m = sh_{n+m}$) holds. With this, we are ready to consider Dold and Puppe's work.

Definition 3.1. The *Dold-Puppe stabilization* of F is the functor D_1F from $Ch_{\ge 0}\mathcal{M}_R$ to $Ch_{\ge 0}\mathcal{M}_S$ defined by

$$D_1F = \varinjlim_n sh_{-n} \circ \mathbf{F} \circ sh_n,$$

where

$$sh_{-n} \circ Sus^1 : sh_{1-n} \circ \mathbf{F} \circ sh_{n-1} \to sh_{-n} \circ \mathbf{F} \circ sh_n.$$

Clearly, each $sh_{-n} \circ \mathbf{F} \circ sh_n$ preserves chain homotopies, and since the transformation Sus^n also preserves chain homotopies, D_1F does as well. As discussed in section 1, the other property that D_1F should have is to take short exact sequences in $Ch_{\geq 0}\mathcal{P}_R$ to long exact sequences in homology. Our next goal is to prove this statement, but doing so requires several preliminary definitions and results. The first set of results has to do with the second order cross effect of a functor. This is a bifunctor that measures the extent to which a functor fails to be additive, and as a consequence, the extent to which a functor fails to preserve short exact sequences. In relation to D_1F, we will see that as n increases, the cross effect of $sh_{-n}\mathbf{F}sh_n$ becomes increasingly negligible and in some sense will go to zero in the limit. As a result, D_1F will behave as desired on short exact sequences.

The second cross effect of F is defined to be any bifunctor, cr_2F, from $\mathcal{M}_R \times \mathcal{M}_R$ to $\mathcal{M}_S$ determined by

$$cr_2F(A, B) \oplus F(A) \oplus F(B) \cong F(A \oplus B)$$

using the standard inclusions and projections for $A \oplus B$. Technically, cr_2F is only defined up to natural isomorphism. So, we simply let cr_2F represent some choice. Note that cr_2F satisfies the properties:

(3.2) $cr_2F(0, A) \cong 0$ for all modules A in $\mathcal{M}_R$.
(3.3) $cr_2F(A, B) \cong cr_2F(B, A)$ for all modules A, B in $\mathcal{M}_R$.
(3.4) F is additive if and only if $cr_2F \cong 0$.

For any two simplicial objects $X.$ and $Y.$, their direct sum is the simplicial object that in degree n is $X_n \oplus Y_n$ with face and degeneracy operations being the direct sum of those from $X.$ and $Y.$. Hence, we prolong cr_2F to $Simp_{\mathcal{M}_R} \times Simp_{\mathcal{M}_R}$ by $cr_2F(X., Y.)_n = cr_2F(X_n, Y_n)$ with morphisms defined diagonally. The prolongation of cr_2F to $Ch_{\geq 0}\mathcal{M}_R \times Ch_{\geq 0}\mathcal{M}_R$ is the functor composite: $\mathbf{cr_2F} = N \circ cr_2F \circ (\Gamma \times \Gamma)$. With these conventions, it is straightforward to check that for any chain complexes P and Q

$$\mathbf{cr_2F}(P, Q) \oplus \mathbf{F}(P) \oplus \mathbf{F}(Q) \cong \mathbf{F}(P \oplus Q).$$

In addition to cross effects, we will be working with the following types of chain complexes.

Definition 3.5. A chain complex P is *n-reduced* if $P_i = 0$ for $i \leq n$. We will call a complex *n-acyclic* if it is quasi-isomorphic to an n-reduced complex and simply *acyclic* if it is quasi-isomorphic to the trivial chain complex. A chain map $f : P \to Q$ is said to be *n-connected* if $f_k : H_k(P) \to H_k(Q)$ is an isomorphism for $k \leq n$.

It is not difficult to check that for all $P \in Ch_{\geq 0}\mathcal{P}_R$, $H_i(P) = 0$ for $0 \leq i \leq n$ if and only if P is chain homotopic to an n-reduced complex. We can predict the behavior of F on n-reduced complexes to a certain degree.

Lemma 3.6. *If P and Q are objects in $Ch_{\geq 0}\mathcal{M}_R$ such that P is m-reduced and Q is n-reduced, then $\mathbf{F}(P)$ is also m-reduced and $\mathbf{cr_2F}(P, Q)$ is $m + n + 1$-acyclic.*

Proof. The fact that $\mathbf{F}(P)$ is m-reduced follows immediately from the definitions and the fact that $F(0) = 0$. To show that $\mathbf{cr_2F}(P, Q)$ is $m + n + 1$-acyclic, we note that $cr_2F(\Gamma P, \Gamma Q)$ is the diagonal simplicial module of the bisimplicial module (i.e., the functor from $\Delta^{op} \times \Delta^{op}$ to $\mathcal{M}_S$)

$$[s] \times [t] \mapsto cr_2F(\Gamma(P)_s, \Gamma(Q)_t).$$

Let F_{PQ} denote this bisimplicial module, and let NF_{PQ} denote the bicomplex obtained from F_{PQ} by normalizing it in each simplicial direction. The Eilenberg-Zilber theorem (see [E-Z] for the original version of the theorem, or [We], pp. 275-278 for the version used here) implies that the normalization of the diagonalization of F_{PQ} is naturally chain homotopic to the total chain complex of NF_{PQ}. So, $\mathbf{cr_2F}(P,Q)$ is quasi-isomorphic to the total chain complex of NF_{PQ}. By property (3.2), $cr_2F(\Gamma(P)_s, \Gamma(Q)_t) = 0$ for $t + s \leq m + n + 1$. Hence, the total complex of NF_{PQ} is $m + n + 1$-reduced. The result follows.

Next, we need some terminology and results involving certain types of sequences of chain complexes and bicomplexes. We will use the sign conventions for Tot described in remark 2.5.

Definition 3.7. For a map of chain complexes, $g : Y \to Z$, let the *fiber* of g, denoted $fib(g)$, be the chain complex defined by

$$fib(g) = sh_{-1}Tot\begin{pmatrix} Y \\ \downarrow g \\ Z \end{pmatrix}.$$

Here, $Tot\begin{pmatrix} Y \\ \downarrow g \\ Z \end{pmatrix}$ is the total complex of the first quadrant bicomplex that has Z in its first row, Y in its second row and 0's elsewhere. It is easy to see that in degree n, $fib(g)_n = Y_n \oplus Z_{n+1}$. One should further note that the fiber of g is simply the mapping cone of g shifted down one degree. We use the term fiber here to reflect the fact that $fib(g)$ plays a role similar to that of the homotopy fiber of a map in topology. The fiber of g measures the exactness of a sequence of chain maps.

Definition 3.8. A sequence of chain maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ is *q-quasi-exact* (respectively, *quasi-exact*) if

1) $g \circ f = 0$
2) The natural chain map $\widehat{f} : X \to fib(g)$ given by $X_n \xrightarrow{f_n \oplus 0} Y_n \oplus Z_{n+1}$ is q-connected (respectively, a quasi-isomorphism.)

A quasi-exact sequence is a fibration sequence up to homotopy in the sense of Quillen ([Q]). The important features of q-quasi-exact sequences are described in the following two lemmas.

Lemma 3.9. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is n-quasi-exact for some $n \geq 0$, then there is a natural long exact sequence*

$$H_{n+1}(Y) \xrightarrow{g} H_{n+1}(Z) \xrightarrow{\partial} H_n(X) \xrightarrow{f} H_n(Y) \to \ldots \xrightarrow{f} H_0(Y) \xrightarrow{g} H_0(Z) \to 0.$$

Proof. Recall that the mapping cone fits into a short exact sequence which in our notation becomes $Z \to sh_1 fib(g) \to sh_1 Y$. This yields a long exact sequence in homology

$$\ldots \to H_k(Z) \to H_{k-1}(fib(g)) \to H_{k-1}(Y) \to H_{k-1}(Z)$$
$$\to \ldots \to H_0(fib(g)) \to H_0(Y) \to H_0(Z) \to 0.$$

The result follows from a diagram chase using the fact that $X \xrightarrow{\widehat{f}} fib(g)$ is n-connected.

Lemma 3.10. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a map of first quadrant bicomplexes such that for each n,*

$$X_{\star,n} \xrightarrow{f_{\star,n}} Y_{\star,n} \xrightarrow{g_{\star,n}} Z_{\star,n}$$

is a q-quasi-exact sequence of chain complexes, then

$$Tot(X) \xrightarrow{Tot(f)} Tot(Y) \xrightarrow{Tot(g)} Tot(Z)$$

is a q-quasi-exact sequence of chain complexes.

Proof. Clearly, $Tot(g) \circ Tot(f) = 0$. To show that $\widehat{Totf}$ is q-connected, we will compare it with a second chain map determined by f whose connectivity is easier to determine. To understand this second map, first consider, for a fixed n, the map $g_{*,n} : Y_{*,n} \to Z_{*,n}$ of chain complexes determined by g. Since $g_{*,n}$ is a map of chain complexes, we can form a new chain complex $fib(g_{*,n})$ for each n. These fibers form a bicomplex by using the differentials in Y and Z in the direction perpendicular to that of n. Let $fib(g)$ represent this bicomplex. There is a map $\overline{f} : X \to fib(g)$ of bicomplexes defined in bidegree k, n by $\overline{f_{k,n}} : X_{k,n} \to fib(g)_{k,n} = Y_{k,n} \oplus Z_{k+1,n}$ with $\overline{f_{k,n}}(x) = (f_{k,n}(x), 0)$. It follows from the definition of $\overline{f}$ and the fact that $X_{*,n} \to Y_{*,n} \to Z_{*,n}$ is q-quasi-exact that $\overline{f}$ is a map of bicomplexes. Moreover, since $\overline{f_{*,n}}$ is a q-connected map of chain complexes for each n, it follows that $Tot(\overline{f})$ is q-connected as well.

Now, to finish the proof it suffices to show that $Tot(\overline{f})$ and $\widehat{Totf}$ agree up to a chain isomorphism. But, considering $g : Y \to Z$ as a tricomplex (a complex of bicomplexes), one can see that $Tot(fib(g))$ and $fib(Tot(g))$ are both obtained from $g : Y \to Z$ by taking total complexes of two different bicomplexes within $g : Y \to Z$, and then taking the total complexes of the resulting bicomplexes. Since Tot's commute up to chain isomorphism, $Tot(fib(g))$ and $fib(Tot(g))$ are isomorphic. By this isomorphism, $Tot(\overline{f})$ corresponds to $\widehat{Tot(f)}$, and so $\widehat{Tot(f)}$ is q-connected.

With these two lemmas, we are now in a position to prove a key proposition; this proposition gives conditions under which a functor takes a certain type of short exact sequence to k-quasi-exact sequences.

Definition 3.11. A short exact sequence of chain complexes $P'' \to P \to P'$ is a *cofibration sequence* if for all n, $P''_n \to P_n \to P'_n$ is split exact.

Cofibration sequences are the same as Simson's ([S]) *normal sequences.*

Proposition 3.12. *Let F be a reduced functor from $\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$ such that for all R-modules M and N, $cr_2F(M,N)$ is k-acyclic. Then for every cofibration sequence $P'' \xrightarrow{f} P \xrightarrow{g} P'$ in $Ch_{\geq 0}\mathcal{M}_R$, the sequence*

$$\mathbf{F}(P'') \xrightarrow{\mathbf{F}(f)} \mathbf{F}(P) \xrightarrow{\mathbf{F}(g)} \mathbf{F}(P')$$

is k-quasi-exact.

Proof. That $\mathbf{F}(g) \circ \mathbf{F}(f) = 0$ follows immediately from the fact that F is reduced. The remainder of the theorem follows from the previous lemmas, though we will take some care in applying them, as we must use the prolongation of F, and check that N and Γ preserve the desired properties.

Let $P'' \xrightarrow{f} P \xrightarrow{g} P'$ be a cofibration sequence. Applying Γ to this sequence, we obtain a sequence of simplicial R-modules $\Gamma(P'') \to \Gamma(P) \to \Gamma(P')$. Since the original sequence was split exact in each degree and in each simplicial degree n, $\Gamma(P'')$ (respectively $\Gamma(P)$, $\Gamma(P')$) is a direct sum of copies of P''_k's (respectively P_k's, P'_k's), it is easy to see that $\Gamma(P'')_n \to \Gamma(P)_n \to \Gamma(P')_n$ is split exact for each n. Applying F degreewise to the sequence yields a sequence of simplicial chain complexes,

$$F(\Gamma(P'')) \to F(\Gamma(P)) \to F(\Gamma(P')), \tag{3.13}$$

which in each simplicial degree n gives us a sequence of chain complexes

$$F(\Gamma(P'')_n) \to F(\Gamma(P)_n) \to F(\Gamma(P')_n). \tag{3.14}$$

Since $\Gamma(P'')_n \to \Gamma(P)_n \to \Gamma(P')_n$ is split exact for each n, the sequence of chain complexes (3.14) is equivalent to

$$F(\Gamma(P'')_n) \to F(\Gamma(P'')_n \oplus \Gamma(P')_n) \to F(\Gamma(P')_n), \tag{3.15}$$

and this in turn is equivalent to

$$\begin{aligned} F(\Gamma(P'')_n) &\xrightarrow{F(\Gamma(f)_n)} F(\Gamma(P'')_n) \oplus F(\Gamma(P')_n) \oplus cr_2F(\Gamma(P'')_n, \Gamma(P')_n) \\ &\xrightarrow{F(\Gamma(g)_n)} F(\Gamma(P')_n). \end{aligned} \tag{3.16}$$

Since the middle term of (3.16) is a direct sum of chain complexes, it follows from the definition of fiber that the fiber of $F(\Gamma(g)_n)$ is quasi-isomorphic to $F(\Gamma(P'')_n) \oplus cr_2F(\Gamma(P'')_n, \Gamma(P')_n)$. But, $cr_2F(\Gamma(P'')_n, \Gamma(P')_n)$ is k-acyclic, and so the sequence (3.14) is k-quasi-exact. Applying the functor C defined in the proof of theorem 2.4 to (3.13), we get a sequence of bicomplexes that in each degree n in the former simplicial direction is equal to (3.14) and thus is k-quasi-exact. Hence, by Lemma 3.10,

$$Tot(CF\Gamma(P'')) \to Tot(CF\Gamma(P)) \to Tot(CF\Gamma(P'))$$

is k-quasi-exact. Since N and C are naturally chain homotopy equivalent as functors from $Simp_{\mathcal{M}_S}$ to $Ch_{\geq 0}\mathcal{M}_S$, we know that $CF\Gamma(P'')$ and $NF\Gamma(P'')$ have quasi-isomorphic columns and as a result are quasi-isomorphic. The same is true for $CF\Gamma(P)$ and $NF\Gamma(P)$, and for $CF\Gamma(P')$ and $NF\Gamma(P')$. Thus, by considering the commutative diagram

$$\begin{array}{ccccc} Tot(CF\Gamma(P'') & \xrightarrow{Tot(CF\Gamma(f))} & Tot(CF\Gamma(P)) & \xrightarrow{Tot(CF\Gamma(g))} & Tot(CF\Gamma(P')) \\ \simeq\downarrow & & \downarrow\simeq & & \downarrow\simeq \\ \mathbf{F}(P'') & \xrightarrow{\mathbf{F}(f)} & \mathbf{F}(P) & \xrightarrow{\mathbf{F}(g)} & \mathbf{F}(P') \end{array}$$

we see that $\mathbf{F}(P'') \to \mathbf{F}(P) \to \mathbf{F}(P')$ is a k-quasi-exact sequence.

Theorem 3.17. *If F is a reduced functor from $\mathcal{M}_R$ to $\mathcal{M}_S$ and $P'' \to P \to P'$ is a cofibration sequence in $Ch_{\geq 0}\mathcal{M}_R$, then $D_1F(P'') \to D_1F(P) \to D_1F(P')$ is a quasi-exact sequence in $Ch_{\geq 0}\mathcal{M}_S$.*

Although the proof of the theorem that follows is original, several versions of the theorem exist in the literature. Dold and Puppe ([D-P]) and Simson and Tyc ([S-T]) prove the result for functors of finite degree, while Pirashvili ([P1]) proves the general result stated here.

Proof. Let $G : \mathcal{M}_R \to Ch_{\geq 0}\mathcal{M}_S$ be the functor given by $G = \mathbf{F}sh_k$ for some $k > 0$. We claim that cr_2G satisfies the hypotheses of proposition 3.12. To see this, let A' and A'' be two R-modules, considered as chain complexes concentrated in degree 0, and note that

$$\begin{aligned} G(A' \oplus A'') &= \mathbf{F}(sh_k(A' \oplus A'')) \\ &= \mathbf{F}(sh_kA' \oplus sh_kA'') \\ &\cong \mathbf{F}(sh_kA') \oplus \mathbf{F}(sh_kA'') \oplus cr_2\mathbf{F}(sh_kA', sh_kA'') \\ &= G(A') \oplus G(A'') \oplus cr_2\mathbf{F}(sh_kA', sh_kA''). \end{aligned}$$

Thus, $cr_2G(A', A'') \cong cr_2\mathbf{F}(sh_kA', sh_kA'')$. By lemma 3.6, $\mathbf{cr_2F}(sh_kA', sh_kA'')$ is $(2k-1)$-acyclic since sh_kA' and sh_kA'' are each $(k-1)$-reduced. By Proposition 3.12,

$$\mathbf{G}(P'') \to \mathbf{G}(P) \to \mathbf{G}(P')$$

is $(2k-1)$-quasi-exact. One would like to conclude at this point that

$$\mathbf{F}sh_kP'' \to \mathbf{F}sh_kP \to \mathbf{F}sh_kP'$$

is also $(2k-1)$-quasi-exact. This is true, although some work is involved since $\mathbf{F}sh_k$ and $\mathbf{G}$ are defined differently as functors of chain complexes. ($\mathbf{F}sh_k$ is $NF\Gamma sh_k$ and $\mathbf{G}$ is $TotN(NF\Gamma sh_k)\Gamma$.) The proof that these functors are equivalent is not especially enlightening and can be found in the appendix.

Since $\mathbf{F}sh_kP'' \to \mathbf{F}sh_kP \to \mathbf{F}sh_kP'$ is $(2k-1)$-quasi-exact, it follows that

$$sh_{-k}\mathbf{F}sh_k(P'') \to sh_{-k}\mathbf{F}sh_k(P) \to sh_{-k}\mathbf{F}sh_k(P')$$

is $(k-1)$-quasi-exact. Thus, in the limit

$$D_1F(P'') \to D_1F(P) \to D_1F(P')$$

is a quasi-exact sequence, as desired.

4. Stable Derived Functors of the Symmetric and Exterior Power Functors

With the definition of D_1F and results of section 3, one can define the *stable derived functors* of a non-additive functor just as one defined the derived functors of an additive functor. More specifically, we have

Definition 4.1. Let $F : \mathcal{M}_R \to \mathcal{M}_S$ be a reduced functor. For a module M and a projective resolution P of M, the *qth left stable derived functor* of F at M is given by

$$L_q^sF(M) = H_qD_1F(P).$$

Note that since we are resolving M by an object in $Ch_{\geq 0}\mathcal{P}_R$, $L_q^sF = 0$ for $q < 0$. The fact that $\mathbf{F}$ and Sus^n preserve chain homotopy equivalences ensures that L_q^sF is well-defined. Furthermore, it follows directly from theorem 3.17 that for any short exact sequence of modules

$$0 \to M'' \to M \to M' \to 0$$

and reduced functor $F : \mathcal{M}_R \to \mathcal{M}_S$, there is a long exact sequence of stable derived functors

$$\ldots \to L_q^sF(M'') \to L_q^sF(M) \to L_q^sF(M') \to L_{q-1}^sF(M'') \to \ldots$$

For the examples we describe below the following long exact sequence of stable derived functors is also useful.

Lemma 4.2. *Let $F, F', F'' : \mathcal{M}_R \to \mathcal{M}_S$ be reduced functors with natural transformations*

$$F'' \to F \to F'$$

that form a short exact sequence when evaluated on any projective R-module. Then there is a long exact sequence of stable derived functors

$$\ldots \to L_q^s F'' \to L_q^s F \to L_q^s F' \to L_{q-1}^s F'' \to \ldots$$

Proof. Let M be an R-module and P be a projective resolution of M. It is straightforward to check that for all $n \geq 0$, the sequence

$$0 \to sh_{-n}\mathbf{F}''sh_nP \to sh_{-n}\mathbf{F}sh_nP \to sh_{-n}\mathbf{F}'sh_nP \to 0$$

is exact, and so in the limit we get an exact sequence

$$0 \to D_1F''(P) \to D_1F(P) \to D_1F'(P) \to 0.$$

The result follows.

Simson and Tyc ([S], [S-T]) apply this result to two short exact sequences of functors to completely determine the stable derived functors of the second symmetric and exterior power functors for commutative rings with identity. We summarize below the approach taken in [S].

Definition 4.3. Let R be a commutative ring with identity and $\mathcal{M}_R$ the category of unital R-modules. For $n \geq 1$, the functors SP^n, $\Lambda^n : \mathcal{M}_R \to \mathcal{M}_R$ are defined for an R-module M by

$$SP^n(M) = \otimes^n M/T(M),$$

$$\Lambda^n(M) = \otimes^n M/V(M),$$

where $T(M)$ is the submodule of M generated by all elements of the form $m_1 \otimes m_2 \otimes \cdots \otimes m_n - m_{\sigma(1)} \otimes m_{\sigma(2)} \otimes \cdots \otimes m_{\sigma(n)}$ for $\sigma \in \Sigma_n$, the nth symmetric group, and $V(M)$ is the submodule generated by all elements of the form $m_1 \otimes m_2 \otimes \cdots \otimes m_n$ with $m_i = m_j$ for some $i \neq j$. The functor SP^n is the nth symmetric power functor and Λ^n is the nth exterior power functor.

When $n = 2$, Simson ([S], Corollary 1.3) constructs a diagram of exact sequences

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & & & \\
 & & & & 0 & \longrightarrow & \uparrow & & & & \\
 & & & & & & \Lambda^2 & \longrightarrow & \otimes^2 & \longrightarrow & SP^2 \longrightarrow 0 \\
 & & & & & & \uparrow & & & & \\
 & & & & & & \otimes^2 & & & & \\
 & & & & & & \uparrow & & & & \\
0 & \longrightarrow & W & \longrightarrow & SP^2 & \longrightarrow & \Gamma & \longrightarrow & U & \longrightarrow & 0 \\
 & & & & & & \uparrow & & & & \\
 & & & & & & 0 & & & &
\end{array} \tag{4.4}$$

where Γ is J.H.C. Whitehead's universal quadratic functor ([Wh]), and the functors W and U are the kernel and cokernel, respectively, of the natural transformation $SP^2 \to \Gamma$. For descriptions of the natural transformations in (4.4) see section 1 of [S].

The sequences in (4.4) are useful because one can show that for $F = \otimes^2$, $L^s_q F = 0$ for all q. To see this, let P be a chain complex. Then $F\Gamma sh_n P$ is the diagonal of the bisimplicial R-module $\Gamma sh_n P \otimes \Gamma sh_n P$ that in bidegree p, q is $(\Gamma sh_n P)_p \otimes (\Gamma sh_n P)_q$. By the Eilenberg-Zilber theorem it follows that

$$\mathbf{F} sh_n P \simeq Tot(sh_n P \otimes sh_n P),$$

and since $sh_n P$ is $(n-1)$-reduced, $Tot(sh_n P \otimes sh_n P)$ is $(2n-1)$-reduced. Hence, $sh_{-n}\mathbf{F} sh_n P$ is $(n-1)$-reduced and, in the limit, $D_1 F$ is acyclic. In a similar fashion one can show that the stable derived functors of $\otimes^n$ vanish for all $n \geq 2$. Thus, from (4.4) and lemma 4.2 one obtains

Lemma 4.5. *For all q, and R-modules M,*

$$L^s_q SP^2(M) \cong L^s_{q-1}\Lambda^2(M) \cong L^s_{q-2}\Gamma(M).$$

Furthermore, $L^s_1 SP^2(M) = L^s_0 SP^2(M) = L^s_0 \Lambda^2(M) = 0$.

To complete the calculation of the stable derived functors of SP^2 and Λ^2, Simson and Tyc make the following observations about U and W, obtained by showing that $cr_2 SP^2 \simeq cr_2 \Gamma$, and as a consequence, U and W are additive and determined by their values at R.

Lemma 4.6 ([S-T], 8.13-15). *The functors U and W are additive, right exact functors. Furthermore, for any free R-module M with a well-ordered basis,*

$$U(M) = coker(M \xrightarrow{2} M) \cong M \otimes_R R/2R,$$

$$W(M) = ker(M \xrightarrow{2} M) = Tor^R_1(M, R/2R),$$

where $R/2R$ is the R-R bimodule with the usual right action, and with the left action given by $r[r'] = [r^2 r']$ for $r, r' \in R$ and $[r]$ the equivalence class of r in $R/2R$.

As a result, we have

Theorem 4.7 ([S, 2.3], [S-T, 10.8]). *If R is a commutative ring, M is a free R-module with well-ordered basis, and $R/2R$ has the $R-R$ bimodule structure described in lemma 4.6, then for all q*

$$L^s_q SP^2(M) \cong L^s_{q-1}\Lambda^2(M) \cong L^s_{q-2}\Gamma(M) \cong \begin{cases} 0 & \text{if } q \leq 1, \\ M \otimes_R R/2R & \text{if } q = 2l, l \geq 1, \\ Tor^R_1(M, R/2R) & \text{if } q = 2l+1, l \geq 1. \end{cases}$$

Proof. Consider the short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & SP^2 & \longrightarrow & K & \longrightarrow & 0, \\ 0 & \longrightarrow & K & \longrightarrow & \Gamma & \longrightarrow & U & \longrightarrow & 0, \end{array} \tag{4.8}$$

where K is the image of the natural transformation from Γ to SP^2. The fact that U and W are additive, right exact functors implies that their stable derived functors are equivalent to their derived functors in the classical sense. Hence, on projective objects, we have for $q \geq 1$

$$L^s_q U \simeq L_q U \cong 0,$$
$$L^s_q W \simeq L_q W \cong 0.$$

Then using (4.8) one deduces that $L^s_q SP^2 \cong L^s_q K$ for $q \geq 2$ and $L^s_q \Gamma \cong L^s_q K$ for $q \geq 1$. Thus, $L^s_q SP^2 \cong L^s_q \Gamma$ for $q \geq 2$ and, by lemma 4.4, $L^s_{q+2}\Gamma \cong L^s_q \Gamma$ for $q \geq 0$. Thus, it suffices to know $L^s_0\Gamma$ and $L^s_1\Gamma$. However, from the first sequence and the fact that $L^s_1 SP^2 = L_0 SP^2 = 0$ we see that $L^s_0 K = 0$ and $L^s_1\Gamma = L^s_1 K = W$. Then from the second sequence it follows that $L^s_0\Gamma = U$.

For $n > 2$, the stable derived functors of Λ^n and SP^n were determined in part by Dold and Puppe ([D-P], also described in [S-T]). The key to their approach is the following version of G. Whitehead's homology suspension theorem ([GW]).

Theorem 4.9 ([D-P, 6.11]). *Let P be an n-reduced chain complex of R-modules and $F : \mathcal{M}_R \to \mathcal{M}_S$ a reduced functor. For $q \leq 3n+1$ there are morphisms $\alpha_{q-1}, \alpha_q, \beta_{q+1}$, and σ_q such that the following sequence is exact*:

$$H_q \mathbf{cr_2F}(P,P) \xrightarrow{\alpha_q} H_q\mathbf{F}(P) \xrightarrow{\sigma_q} H_{q+1}\mathbf{F}sh_1P \xrightarrow{\beta_{q+1}} H_{q-1}\mathbf{cr_2F}(P,P) \xrightarrow{\alpha_{q-1}} H_{q-1}\mathbf{F}(P).$$

We will sketch the proof here, although it relies on results from section 6 and the appendix.

Proof. Using the functor B defined in the appendix (definition A.1), consider, for any simplicial R-module X, the natural transformation $\sigma : BF \to FB$ induced by the natural inclusions

$$\bigoplus_{i=1}^{n} F(X) \to F(\bigoplus_{i=1}^{n} X).$$

Using lemmas 7.1 and 7.2, one can determine that when X is an n-reduced simplicial R-module the sequence

$$C\Delta BF(X) \xrightarrow{\sigma} C\Delta FB(X) \longrightarrow sh_2 Ccr_2F(X,X)$$

is $(3n+2)$-quasi-exact. When passing to chain complexes via Γ, σ becomes the transformation Sus^1 described in the construction of D_1F. It follows from lemma A.3 that for any n-reduced chain complex P, we have a $(3n+2)$-quasi-exact sequence

$$sh_1\mathbf{F}(P) \xrightarrow{Sus^1} \mathbf{F}(sh_1P) \longrightarrow sh_2\mathbf{cr_2F}(P,P).$$

The result then follows by lemma 3.9.

As a consequence of this theorem, we have

Corollary 4.10. *For any R-module M, projective resolution $P \to M$, and reduced functor $F : \mathcal{M}_R \to \mathcal{M}_S$,*

$$L^s_k F(M) \cong H_{2k+1}\mathbf{F}(sh_{k+1}P) \cong H_{2k}\mathbf{F}(sh_kP)/Im\alpha_{2k}.$$

Proof. Consider sh_kP. By lemma 3.6, $\mathbf{cr_2F}(sh_kP, sh_kP)$ is $(2k-1)$-acyclic. From theorem 4.9 it follows that

$$H_{2k+1}\mathbf{F}(sh_{k+1}P) \cong H_{2k}\mathbf{F}(sh_kP)/Im\alpha_{2k}.$$

For $t \geq 1$, $\mathbf{cr_2F}(sh_{k+t}P, sh_{k+t}P)$ is $(2k+2t-1)$-acyclic, and so by the theorem

$$H_{2k+t}\mathbf{F}(sh_{k+t}P) \cong H_{2k+t+1}\mathbf{F}(sh_{k+t+1}P).$$

Thus

$$H_k sh_{-k-t}\mathbf{F}(sh_{k+t}P) \cong H_k sh_{-k-t-1}\mathbf{F}(sh_{k+t+1}P),$$

and so $H_kD_1F(M) \cong H_k sh_{-k-t}\mathbf{F}(sh_{k+t}P) = H_{2k+t}\mathbf{F}(sh_{k+t}P)$ for any $t \geq 1$.

In the case where $F = \Lambda^n$ or SP^n, Dold and Puppe determine α explicitly ([D-P], 10.9) and obtain the following general results.

Theorem 4.11 ([D-P], [S]). *Let R be a commutative ring with identity and $q \geq 0$. Then*

$$L_q^s SP^n = L_q^s \Lambda^n = 0 \quad \textit{if } n \textit{ is not a power of a prime},$$
$$p \cdot L_q^s SP^n = p \cdot L_q^s \Lambda^n = 0 \quad \textit{if } n = p^r \textit{ for some prime } p \textit{ and } r \geq 1.$$

In the case when $R = Z$, Dold and Puppe also identify the first non-trivial stable derived functor of SP^n.

Theorem 4.12 ([D-P], [S]). *Let $n = p^r$, where p is a prime number and $r \geq 1$. Then for $R = Z$ and any abelian group M,*

$$L_q^s SP^n(M) = \begin{cases} 0 & \textit{if } q < 2(n-1), \\ M \otimes_Z Z/pZ & \textit{if } q = 2(n-1). \end{cases}$$

5. Linear Functors

With the results of section 3 we may now begin establishing properties for D_1F that are similar to the properties described in the introduction for Goodwillie's linearization. In particular, in this section we define what is meant by a linear functor in our setting and indicate that D_1F is linear. We also prove some results about linear functors that will be needed in later sections. All functors in this section are assumed to be *reduced*, i.e., $F(0) = 0$.

Definition 5.1. Let $\mathcal{A}$ be an additive category (e.g., $\mathcal{M}_R$ or $\mathcal{P}_R$). A functor F from $Ch_{\geq 0}\mathcal{A}$ to $Ch_{\geq 0}\mathcal{M}_S$ is *linear* if it satisfies the following conditions:

1) $F(0) = 0$.
2) F preserves chain homotopy equivalences.
3) For any short exact sequence of chain complexes $P'' \to P \to P'$, the sequence $F(P'') \to F(P) \to F(P')$ is quasi-exact.

Clearly, proposition 2.7 and theorem 3.17 imply that for any reduced functor $F : \mathcal{M}_R \to \mathcal{M}_S$, D_1F is linear as a functor of chain complexes of *projective* R-modules. However, this depends in an essential way on the fact that we are using the prolongation of F. We need lemma 3.6 to prove theorem 3.17, and this lemma does not hold for an arbitrary functor of chain complexes. Consider the following example.

Example 5.2. For a functor $F : Ch_{\geq 0}\mathcal{M}_R \to Ch_{\geq 0}\mathcal{M}_S$, let $DF = \varinjlim_n sh_{-n}Fsh_n$. Let $\widetilde{S} : Ch_{\geq 0}\mathcal{M}_R \to Ch_{\geq 0}\mathcal{M}_S$ be the functor defined for a chain complex P by $\widetilde{S}[P]_k = S[P_k]/S[0]$ where $S[-]$ is the free functor from $\mathcal{M}_R$ to $\mathcal{M}_S$. It is straightforward to check that lemma 3.6 does not hold for $\widetilde{S}$. In particular, for R-modules M and N, one can show that

$$cr_2\widetilde{S}(sh_nM, sh_nN) \cong sh_n(\widetilde{S}[M] \otimes_S \widetilde{S}[N]),$$

which is only n-reduced. Therefore $\varinjlim_n sh_{-n}cr_2\widetilde{S}(sh_nM, sh_nN)$ is 0-reduced and $D\widetilde{S}$ does not preserve direct sums.

Hence if F is a functor of chain complexes that is not the prolongation of another functor, then DF may not be linear unless we impose some condition on F. For

those familiar with Goodwillie's work, this condition is analogous to stable excision. We will discuss this further in [J-M1].

It is also fairly easy to see that linearity is closely connected to additivity. In particular, we have

Proposition 5.3. *Let $\mathcal{A}$ be an additive category. If $F : Ch_{\geq 0}\mathcal{A} \to Ch_{\geq 0}\mathcal{M}_S$ is linear, then it preserves direct sums up to quasi-isomorphism.*

Proof. Let P and Q be chain complexes in $\mathcal{A}$. Consider the short exact sequence $P \to P \oplus Q \to Q$. Applying F produces a quasi-exact sequence $F(P) \to F(P \oplus Q) \to F(Q)$. The inclusion $F(Q) \to F(P \oplus Q)$ produces a splitting in each degree in the homology sequence. It follows that the natural inclusions induce isomorphisms $H_*F(P) \oplus H_*F(Q) \to H_*F(P \oplus Q)$, and thus $F(P) \oplus F(Q) \to F(P \oplus Q)$ is a quasi-isomorphism.

As a partial converse, we can show that additivity implies linearity for the prolongation of a functor. One can prove this directly, but we state the result as a corollary to the following lemma which is needed in section 7.

Lemma 5.4. *Let F be a reduced functor from $\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$ and let $\widetilde{F}$ be the prolongation of F as defined in section 1, i.e., for a chain complex P, $(\widetilde{F}(P))_k = F(P_k)$. Then the following are equivalent.*

a) *For all modules A and B, the map $F(A) \oplus F(B) \to F(A \oplus B)$ induced by inclusions is a natural quasi-isomorphism.*
b) *$\mathbf{F}$ is naturally quasi-isomorphic to $Tot(\widetilde{F})$.*
c) *The natural transformation from $\mathbf{F}$ to D_1F is a natural quasi-isomorphism.*

Proof. To show that a) implies b), recall that for a chain complex P, ΓP is a direct sum of modules in each simplicial degree. Since F preserves direct sums up to quasi-isomorphism, $\Gamma\widetilde{F}(P)$ and $F\Gamma(P)$ are simplicial chain complexes that are quasi-isomorphic in each simplicial degree. Applying the functor C in the simplicial direction to $\Gamma\widetilde{F}(P)$ and $F\Gamma(P)$ preserves these quasi-isomorphisms, since it does not change the chain complexes. As a result, the total complexes of the bicomplexes $C\Gamma\widetilde{F}(P)$ and $CF\Gamma(P)$ are quasi-isomorphic. Now consider the diagram

$$\begin{array}{ccc} Tot\ C\Gamma\widetilde{F}(P) & \longrightarrow & Tot\ CF\Gamma(P) \\ \downarrow & & \downarrow \\ Tot\ N\Gamma\widetilde{F}(P) & \longrightarrow & Tot\ NF\Gamma(P). \end{array}$$

The upper horizontal arrow is a quasi-isomorphism by the above. The vertical arrows are quasi-isomorphisms because of the chain homotopy equivalence between C and N. Thus, the lower horizontal arrow is a quasi-isomorphism. Since $N\Gamma \cong id$, we may conclude that $Tot\ \widetilde{F}$ is naturally quasi-isomorphic to $\mathbf{F}$.

Part c) is a direct consequence of b). We have

$$\begin{aligned} D_1F &= \lim_{\vec{n}} sh_{-n}\mathbf{F}sh_n \\ &\simeq \lim_{\vec{n}} sh_{-n}Tot\widetilde{F}sh_n \\ &\simeq \lim_{\vec{n}} sh_{-n}Tot\ sh_n\widetilde{F} \end{aligned}$$

(where sh_n commutes with $\widetilde{F}$ since $\widetilde{F}$ is applied degreewise). Moreover, sh_n and Tot commute, so we also have

$$\begin{aligned}\lim_{\vec{n}} sh_{-n}Tot\ sh_n\widetilde{F} &= \lim_{\vec{n}} sh_{-n}sh_nTot\widetilde{F}\\ &= \lim_{\vec{n}} Tot\widetilde{F}\\ &= Tot\widetilde{F}\\ &\simeq \mathbf{F}.\end{aligned}$$

Since all of the quasi-isomorphisms above are natural, D_1F is naturally quasi-isomorphic to $\mathbf{F}$.

Finally, to see that c) implies a), note that since $\mathbf{F} \cong D_1F$ and D_1F is linear, we know by 5.3 that $\mathbf{F}$ preserves direct sums of modules up to quasi-isomorphism. By the definitions of N and Γ, F and $\mathbf{F}$ agree on $\mathcal{M}_R$, and thus F preserves direct sums up to quasi-isomorphism.

Corollary 5.5. *The prolongation of a functor F from $\mathcal{P}_R$ to $\mathcal{M}_S$ is linear if and only if F is additive.*

Proof. If F is additive, then $\mathbf{F} \simeq D_1F$. Since D_1F is linear on $Ch_{\geq 0}\mathcal{P}_R$, $\mathbf{F}$ is as well. Conversely, we know that F and $\mathbf{F}$ agree on $\mathcal{P}_R$, and thus, if $\mathbf{F}$ is linear, F is additive.

We conclude this section with a property about linear functors that will be used for the classification of linear functors in section 8.

Proposition 5.6. *Let $\mathcal{A}$ be an additive category. Let F and G be linear functors from $Ch_{\geq 0}\mathcal{A}$ to $Ch_{\geq 0}\mathcal{M}_S$ and let $\eta : F \to G$ be a natural transformation. If η is an isomorphism on all objects in $\mathcal{A}$, then η is a quasi-isomorphism for all objects in $Ch_{\geq 0}\mathcal{A}$.*

Proof. We first prove the result for chain complexes concentrated in degree n for some $n \geq 0$. For an object A, consider it as a chain complex concentrated in degree 0. As such, it fits into a short exact sequence of chain complexes

$$0 \to A \to CA \to sh_1A \to 0$$

where CA is the cone of A. Applying F and G yields two long exact sequences in homology that can be compared via η:

$$\begin{array}{ccccccccc} \dots & \to & H_*F(A) & \to & H_*F(CA) & \to & H_*F(sh_1A) & \to & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \to & H_*G(A) & \to & H_*G(CA) & \to & H_*G(sh_1A) & \to & \dots . \end{array}$$

Since $H_*F(CA) \cong 0 \cong H_*G(CA)$ and $\eta_A : F(A) \to G(A)$ is a quasi-isomorphism, it follows that $\eta_{sh_1A} : F(sh_1A) \to G(sh_1A)$ is a quasi-isomorphism. A similar argument, applied inductively, shows that η induces a quasi-isomorphism from $F(sh_nA)$ to $G(sh_nA)$ for all n and all A in $\mathcal{A}$.

To complete the proof we proceed by induction on the length of a chain complex. Assume η is an isomorphism on all chain complexes concentrated in degrees 0 through n, and let P be a chain complex concentrated in degrees 0 through $n+1$.

Let Q be the chain complex obtained by truncating P at degree n, i.e.,

$$Q_k = \begin{cases} P_k & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise,} \end{cases}$$

with differentials inherited from P. Then P and Q fit into a short exact sequence $Q \to P \to sh_{n+1}P_{n+1}$. Since F and G are linear, we obtain two long exact sequences in homology as above:

$$\begin{array}{ccccccccc} \dots & \to & H_*(F(Q)) & \to & H_*(F(P)) & \to & H_*F(sh_{n+1}P_{n+1}) & \to & \dots \\ & & \eta\downarrow & & \eta\downarrow & & \eta\downarrow & & \\ \dots & \to & H_*(G(Q)) & \to & H_*(G(P)) & \to & H_*G(sh_{n+1}P_{n+1}) & \to & \dots. \end{array}$$

Since η is a quasi-isomorphism for $sh_{n+1}P_{n+1}$ and Q, it is for P as well. By induction, η is a quasi-isomorphism for all chain complexes in $Ch_{\ge 0}\mathcal{A}$.

6. Mac Lane's Q-Construction

To prove the remaining results described in the introduction for D_1F, we use a different model for D_1F. The advantage of this new model will be that, when evaluated on an R-module, it can be given the structure of a differential graded module over a certain differential graded algebra related to R and S. This structure will enable us to prove a classification theorem for linear functors in section 8. Moreover, the new model will be an explicit chain complex, rather than the direct limit of a sequence of chain complexes.

The model for D_1F that we introduce in this section is an extension of Saunders Mac Lane's Q-construction, originally introduced in [E-M1] (and also in [M2]). Given an abelian group A and a ring S, Mac Lane constructed a chain complex $Q(S;A)$ whose homology is isomorphic to the stable homology of Eilenberg-Mac Lane spaces corresponding to A. Using Mac Lane's construction as our guide, we will produce, for a functor F and an abelian group A, a chain complex $QF(A)$ and show that it is quasi-isomorphic to $D_1F(A)$. The Q-construction was brought to our attention for this purpose by the work of Teimuraz Pirashvili in [J-P]. Our exposition below is based on Section 2 of [J-P].

Let A be an R-module and F be a functor from R-modules to S-modules. Constructing $QF(A)$ requires several steps. We will begin by defining some sets and homomorphisms used to build a chain complex $Q'F(A)$; $QF(A)$ will then be defined as a quotient of $Q'F(A)$. The careful reader will note that $QF(A)$ could just as easily be defined for a functor between any two additive categories. In the spirit of Section 1 and in preparation for our classification theorem, we will restrict our attention to modules and chain complexes over R and S. We will define QF for $F : \mathcal{M}_R \to \mathcal{M}_S$ and discuss the construction for functors of chain complexes at the end of the section.

For each positive integer n, let C_n denote the set consisting of all n-tuples $(\epsilon_1, \dots, \epsilon_n)$, where $\epsilon_i = 0$ or 1. Let C_0 denote the set consisting of the 0-tuple $()$. Let $0_i, 1_i : C_n \to C_{n+1}, 1 \le i \le n+1$, be the maps defined by the equations

$$0_i(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1, \dots, \epsilon_{i-1}, 0, \epsilon_i, \dots, \epsilon_n),$$
$$1_i(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1, \dots, \epsilon_{i-1}, 1, \epsilon_i, \dots, \epsilon_n).$$

For a set S, let $A[S]$ be the sum $\bigoplus_{s\in S} A$. For $Q'F(A)$, we are interested in the groups $A[C_n]$. Since C_n is finite, $A[C_n]$ can be identified with the group of all set maps $t : C_n \to A$. We can then use 0_i and 1_i to define group homomorphisms

$\overline{R_i}, \overline{S_i}, \overline{P_i} : A[C_n] \to A[C_{n-1}]$ that will be used to construct the differential for $Q'(F;A)$. These homomorphisms are defined for $e \in C_{n-1}$ and $t \in A[C_n]$ by

$$\begin{aligned}(\overline{R_i}t)(e) &= t(0_i e),\\ (\overline{S_i}t)(e) &= t(1_i e),\\ (\overline{P_i}t)(e) &= t(0_i e) + t(1_i e).\end{aligned}$$

Definition 6.1. For a functor F, $Q'F(A)$ is the chain complex that in degree n is the module

$$Q'F(A)_n = F(A[C_n])$$

with the boundary homomorphism, $\partial : Q'F(A)_n \to Q'F(A)_{n-1}$, given by

$$\partial = \sum_{i=1}^{n} (-1)^i (P_i - R_i - S_i)$$

where

$$R_i = F(\overline{R_i}), \qquad S_i = F(\overline{S_i}), \qquad P_i = F(\overline{P_i}).$$

Essentially, $P_i - R_i - S_i$ measures the failure of F to be additive. It is straightforward to check that $\partial \circ \partial = 0$.

For $QF(A)$ we must set up some additional notation. For a set S and a collection $S_1, \ldots, S_n$ of subsets of S, we write F_S and $F_{S/\{S_i\}}$ for the functors from $\mathcal{M}_R$ to $\mathcal{M}_S$ defined by

$$F_S(A) = F(A[S])$$

and

$$F_{S/\{S_i\}} = Coker(\bigoplus_{i=1}^{n} F_{S_i} \to F_S),$$

where $F_{S_i} \to F_S$ is induced by the injection $S_i \hookrightarrow S$, for $i = 1, \ldots, n$. The particular collection of subsets that we will use consists of the subsets of C_n described below:

$$\begin{aligned} S_i &= \{(\epsilon_1, \ldots, \epsilon_n) \in C_n | \epsilon_i = 0\}, && 1 \le i \le n,\\ L_j &= \{(\epsilon_1, \ldots, \epsilon_n) \in C_n | \epsilon_j = 1\}, && 1 \le j \le n,\\ D_k &= \{(\epsilon_1, \ldots, \epsilon_n) \in C_n | \epsilon_k = \epsilon_{k+1}\}, && 1 \le k < n.\end{aligned}$$

Definition 6.2. $QF(A)$ is the chain complex that in degree n is the module

$$QF(A)_n = \begin{cases} F_{C_n/\{\emptyset\}}(A) & \text{if } n = 0,\\ F_{C_n/\{S_i, L_j\}}(A) & \text{if } n = 1,\\ F_{C_n/\{S_i, L_j, D_k\}}(A) & \text{if } n > 1,\end{cases}$$

with boundary operator ∂, as defined for $Q'F(A)$. (It follows directly from the definitions that ∂ is well-defined on $QF(A)$.)

When F is the free functor $S[\ ]$, i.e., the functor that takes an R-module A to the free S-module $S[A]$ generated by A (considered as a set), Eilenberg and Mac Lane show that $QF(A)$ is a chain complex whose homology is the same as the stable homology groups of A with coefficients in S, i.e., the homotopy groups of the smash product of the Eilenberg-Mac Lane spectra **HA** and **HS**.

If $QF(-)$, or more precisely, its prolongation, is indeed quasi-isomorphic to D_1F, then it must behave nicely with respect to direct sums. In fact, we can show at this point that $QF(-)$ preserves direct sums up to quasi-isomorphism. We will use this result in the next section to show that $QF(-)$ and D_1F are naturally quasi-isomorphic.

Proposition 6.3 [E-M1, 12.1]. *For any R-modules A and B and functor $F : \mathcal{M}_R \to \mathcal{M}_S$, the natural transformation $QF(A) \oplus QF(B) \to QF(A \oplus B)$ induced by the inclusions is a quasi-isomorphism.*

Proof. Let i_A and π_A (respectively, i_B and π_B) be the natural inclusion and projection for A (respectively, B) in $A \oplus B$. To show that $QF(A) \oplus QF(B)$ and $QF(A \oplus B)$ are naturally quasi-isomorphic, we show that the composite

$$QF(A \oplus B) \longrightarrow QF(A) \oplus QF(B) \longrightarrow QF(A \oplus B)$$

induced by the natural projections and inclusions is a chain homotopy equivalence.

To do so, let $h : (A \oplus B)[C_n] \to (A \oplus B)[C_{n+1}]$ be the group homomorphism defined for

$$e = (\epsilon_1, \ldots, \epsilon_{n+1}) \in C_{n+1} \quad \text{and} \quad t \in (A \oplus B)[C_n]$$

by

$$h(t)(e) = \begin{cases} i_A \circ \pi_A \circ t(\epsilon_2, \ldots, \epsilon_{n+1}) & \text{if } \epsilon_1 = 0, \\ i_B \circ \pi_B \circ t(\epsilon_2, \ldots, \epsilon_{n+1}) & \text{if } \epsilon_1 = 1, \end{cases}$$

where $(i_A \circ \pi_A)[C_n]$ (respectively $(i_B \circ \pi_B)[C_n]$) denotes the map from $(A \oplus B)[C_n]$ to itself induced by $i_A \circ \pi_A$ (respectively $i_B \circ \pi_B$). Then, the following relations are immediate consequences of the definitions:

$$\begin{aligned} \overline{R}_1 \circ h =& (i_A \circ \pi_A)[C_n]; & \overline{R}_i \circ h &= h \circ \overline{R}_{i-1} \quad \text{for } i > 1, \\ \overline{S}_1 \circ h =& (i_B \circ \pi_B)[C_n]; & \overline{S}_i \circ h &= h \circ \overline{S}_{i-1} \quad \text{for } i > 1, \\ \overline{P}_1 \circ h =& id_{(A \oplus B)[C_n]}; & \overline{P}_i \circ h &= h \circ \overline{P}_{i-1} \quad \text{for } i > 1. \end{aligned}$$

Thus, if we let $H = F(h)$, then on $Q'F(A \oplus B)$, we have

$$\partial \circ H + H \circ \partial = Q'F(i_A \circ \pi_A) + Q'F(i_B \circ \pi_B) - id_{Q'F(A \oplus B)}.$$

Since the chain homotopy defined above passes to the quotient, it also gives a chain homotopy on $QF(A \oplus B)$. Hence, the composite $QF(A \oplus B) \to QF(A \oplus B)$ is a quasi-isomorphism.

For any functor $G : Ch_{\geq 0}\mathcal{M}_R \to Ch_{\geq 0}\mathcal{M}_S$, definitions 6.1 and 6.2 may be applied to obtain a functor $QG : Ch_{\geq 0}\mathcal{M}_R \to Ch_{\geq 0}(Ch_{\geq 0}\mathcal{M}_S)$ (or $Ch_{\geq 0}\mathcal{M}_S$ after taking the total complex) for which proposition 6.3 still holds. In particular, we will be most interested in the case where G is the prolongation of some functor $F : \mathcal{M}_R \to \mathcal{M}_S$. Note that, using the Q-construction and F, one can obtain another functor from $Ch_{\geq 0}\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$, namely $\mathbf{QF}$. However, we will see in the next section that $Q\mathbf{F}$ and $\mathbf{QF}$ are naturally homotopic.

7. Higher Order Cross Effects

The key to showing that QF and D_1F are naturally quasi-isomorphic is to use some basic facts about higher order cross effects. We begin this section by defining higher order cross effects and recalling some results from [E-M2]. En route to proving the result about QF and D_1F we will also establish the second of the properties outlined in the introduction for D_1F.

Recall that for a functor F of R-modules, its second order cross effect was the bifunctor defined via the isomorphism

$$F(A_1 \oplus A_2) \cong F(A_1) \oplus F(A_2) \oplus cr_2F(A_1, A_2),$$

where A_1 and A_2 are arbitrary R-modules. The higher order cross effects of F are defined inductively for R-modules $A_1, A_2, \ldots, A_n, A_{n+1}$ by choosing $cr_{n+1}F$ to be a functor of $n+1$ variables that makes $cr_nF(A_1, \ldots, A_n \oplus A_{n+1})$ isomorphic to

$$\begin{aligned} cr_nF(A_1, \ldots, A_{n-1}, A_n) &\oplus cr_nF(A_1, \ldots, A_{n-1}, A_{n+1}) \\ &\oplus cr_{n+1}F(A_1, \ldots, A_n, A_{n+1}). \end{aligned}$$

Higher order cross effects satisfy the following properties, analogous to the properties of cr_2F.

1) $cr_nF(0, A_2, \ldots, A_n) \cong 0$ for any R-modules $A_2, \ldots, A_n$.
2) $cr_nF(A_1, A_2, \ldots, A_n) \cong cr_nF(A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(n)})$ for any R-modules A_1, $A_2, \ldots, A_n$ and $\sigma \in \Sigma_n$, the nth symmetric group.

In actuality, all that is needed to define cross effects is a functor between two additive categories. In particular, for a functor $G : Ch_{\geq 0}\mathcal{M}_R \to Ch_{\geq 0}\mathcal{M}_S$, its cross effects can be defined exactly as above, and properties 1) and 2) will still hold. When G is the prolongation of a functor F from $\mathcal{M}_R$ to $\mathcal{M}_S$ or $Ch_{\geq 0}\mathcal{M}_S$, its cross effects are equivalent to the prolongation of the cross effects of F. That is, as in section 3, cr_nF can be prolonged to $Simp^{\times n}_{\mathcal{M}_R}$ by $cr_nF(X^1_\cdot, X^2_\cdot, \ldots, X^n_\cdot)_k = cr_nF(X^1_k, X^2_k, \ldots X^n_k)$ with morphisms defined diagonally, and the prolongation of cr_nF to chain complexes is given by $\mathbf{cr_nF} = Ncr_nF(\Gamma \times \cdots \times \Gamma)$. It follows from the Eilenberg-Zilber theorem that the inductive relationship used to define cr_nF holds for the prolongation as well. That is, $\mathbf{cr_{n-1}F}(X^1_\cdot, X^2_\cdot, \ldots, X^{n-1}_\cdot \oplus X^n_\cdot)$ is homotopic to

$$\mathbf{cr_{n-1}F}(X^1_\cdot, X^2_\cdot, \ldots, X^{n-1}_\cdot) \oplus \mathbf{cr_{n-1}F}(X^1_\cdot, X^2_\cdot, \ldots, X^n_\cdot) \oplus \mathbf{cr_nF}(X^1_\cdot, X^2_\cdot, \ldots, X^n_\cdot),$$

and so $cr_n\mathbf{F} \simeq \mathbf{cr_nF}$. Furthermore, Lemma 3.6 can be generalized using an n-dimensional version of the Eilenberg-Zilber theorem.

Lemma 7.1. *If $P_1, \ldots, P_n$ are chain complexes where P_i is k_i-reduced for each $1 \leq i \leq n$, then $\mathbf{cr_nF}(P_1, \ldots P_n)$ is $(n - 1 + \sum_{i=1}^n k_i)$-reduced.*

The advantage gained from cross effects is a means of decomposing QF as a direct sum of pieces whose connectivity can be determined by the previous lemma. This decomposition (which follows from Lemma 7.3 below) is an easy consequence of the next proposition of Eilenberg and Mac Lane.

Lemma 7.2 ([E-M2]). *Given a finite set S, write $|S|$ for its cardinality and $\mathcal{P}(S)$ for its power set. Let $F : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are additive categories. For an object A in $\mathcal{A}$,*

$$F(A[S]) \cong \bigoplus_{L \in \mathcal{P}(S)} cr_{|L|}F(A, \ldots, A).$$

Lemma 7.3. *Given a finite set S and a collection $S_1, \ldots, S_n$ of subsets of S, a functor F and an object A as in lemma 7.2, there is a natural isomorphism*

$$F_{S\setminus\{S_i\}}(A) \cong \bigoplus_{L\in\Omega} cr_{|L|}F(A, \ldots, A),$$

where Ω is the set consisting of those subsets L of S that are not contained in any of the S_i.

It follows from these lemmas that QF is naturally a direct summand of $Q'F$. However, although ∂ preserves the quotient, it does not preserve this splitting of $Q'F$. In addition, we have a new description of QF_n for $n \geq 1$, that is, $QF_n \cong \bigoplus_{L\in\Omega} cr_{|L|}F$, where Ω is the set of subsets of C_n that are not contained in any of the L_i, S_j, or D_k. As a result, we see that $\mathbf{QF}$ and $Q\mathbf{F}$ are naturally homotopic since the prolongations of the cross effects of F are equivalent to the cross effects of the prolongations.

Since $QF_0 = F()/F(0)$, we have a natural transformation from F to QF_0 and, as a result, a natural transformation from F to QF. With the previous two lemmas, we are able to prove that QF serves as an approximation to F via this natural transformation, in the sense that the homology of $\mathbf{F}(P)$ and $\mathbf{QF}(P)$ will agree in a range dependent on the acyclicity of P.

Proposition 7.4. *If F is a reduced functor from $\mathcal{M}_R$ to $\mathcal{M}_S$ and P is a k-reduced chain complex over $\mathcal{M}_R$, then the natural transformation $\mathbf{F}(P) \to \mathbf{QF}(P)$ is at least $(2k+1)$-connected.*

Proof. If $n \geq 1$, then it follows from lemma 7.3 that $\mathbf{QF}_n$ is the prolongation of

$$\bigoplus_{L\in\Omega} cr_{|L|}F,$$

where Ω is the set consisting of all subsets of C_n that are not contained in any of the S_i, L_j, or D_k. Since prolongation preserves direct sums, we have

$$\mathbf{QF}_n \cong \bigoplus_{L\in\Omega} \mathbf{cr}_{|\mathbf{L}|}\mathbf{F}.$$

However, if $L \in \Omega$ then $|L| \geq 2$, since every n-tuple in C_n is contained in an S_i or L_j for some i or j. Hence $\mathbf{QF}_n$ is the direct sum of prolongations of cross effects of order ≥ 2. Since P is k-reduced, lemma 7.1 implies that $\mathbf{QF}_n(P)$ is at least $(2k+1)$-reduced for all $n \geq 1$.

For $n = 0$, recall that $\mathbf{QF}_0() = \mathbf{F}()/\mathbf{F}(0)$, and so $\mathbf{QF}_0 = \mathbf{F}$ since $\mathbf{F}$ is reduced. Hence $\mathbf{QF}(P)$ is a first quadrant bicomplex that has $\mathbf{F}(P)$ as its bottom row and in which all other rows are $(2k+1)$-reduced. Accordingly, $\mathbf{F}(P)$ and $\mathbf{QF}(P)$ agree in homology through degree $2k+1$.

Proposition 7.4 enables us to show that D_1F and D_1QF are naturally quasi-isomorphic, and in turn that D_1F and $\mathbf{QF}$ are also naturally quasi-isomorphic.

Theorem 7.5. *For a reduced functor F from $\mathcal{M}_R$ to $\mathcal{M}_S$, D_1F and $\mathbf{QF}$ are naturally quasi-isomorphic as functors from $Ch_{\geq 0}\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$.*

A version of this theorem also appears in [P2] (Proposition 4.1).

Proof. Consider the diagram of natural transformations

$$\begin{array}{ccc} \mathbf{F} & \longrightarrow & \mathbf{QF} \\ \downarrow & & \downarrow \\ D_1F & \longrightarrow & D_1QF. \end{array}$$

By proposition 6.3 and lemma 5.4, $\mathbf{QF} \to D_1QF$ is a quasi-isomorphism. In addition, proposition 7.4 guarantees that $\mathbf{F}(sh_nP) \to \mathbf{QF}(sh_nP)$ is at least $(2n-1)$-connected for all chain complexes P in $\mathcal{M}_R$. Hence $sh_{-n}\mathbf{F}(sh_nP) \to sh_{-n}\mathbf{QF}sh_n$ is at least $(n-1)$-connected, and in the limit, $D_1F \to D_1QF$ becomes a quasi-isomorphism. Since $\mathbf{QF}$ and D_1F are both quasi-isomorphic to D_1QF, the result follows.

As an immediate consequence of theorem 7.5 and proposition 5.4 we have

Corollary 7.6. *If F is a reduced functor from $\mathcal{M}_R$ to $\mathcal{M}_S$, then $\mathbf{QF}$ is linear as a functor from $Ch_{\geq 0}\mathcal{P}_R$ to $Ch_{\geq 0}\mathcal{M}_S$.*

Theorem 7.5 depends on the fact that we are using the prolongation of a functor of modules. If F is an arbitrary functor from $Ch_{\geq 0}\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$, then the functors $DF = \varinjlim_n sh_{-n}Fsh_n$ and QF do not agree in general. In particular, as discussed in section 5 (example 5.2), DF may not preserve direct sums. However, QF will always preserve direct sums, since the analog of proposition 6.3 holds.

8. Classification of Linear Functors

Eilenberg and Watts ([E], [W]) showed that all right continuous functors from $\mathcal{M}_R$ to $\mathcal{M}_S$ can be classified by $R^{op} \otimes_Z S$-modules. More specifically, for a right continuous functor F, they show that there is an $R^{op} \otimes_Z S$-module X such that $F(M) \cong M \otimes_R X$ for all R-modules M. Goodwillie ([G3]) proves a similar statement for linear functors of spaces—a linear functor L (that satisfies the limit axiom) has the form $L(-) \simeq \Omega^\infty(C \wedge -)$, where C is a spectrum obtained by evaluating L on spheres. In this section we will prove an analogous result for linear functors on $Ch_{\geq 0}\mathcal{M}_R$, i.e., we will classify linear functors in terms of a differential graded module over a differential graded algebra related to S and R. To motivate our approach to this result, we first review the classification of additive functors.

Let F be an additive functor from R-modules to S-modules. Since F is additive, there is a group homomorphism

$$R \xrightarrow{\cong} Hom_R(R,R) \xrightarrow{F} Hom_S(F(R),F(R)).$$

This homomorphism gives the right S-module $F(R)$ the structure of a left R-module and a right $R^{op} \otimes S$-module. Using this structure, we can define a second functor $G : \mathcal{M}_R \to \mathcal{M}_S$ by

$$G(M) = M \otimes_R F(R)$$

for any right R-module M. This functor is naturally isomorphic to F in most cases.

Theorem 8.1 (Eilenberg [E], Watts [W]). *There is a natural transformation $\eta : G \to F$ that is an isomorphism on all finitely generated projective R-modules. If F is right continuous (preserves cokernels and filtered direct limits), then η is an isomorphism on all modules.*

Proof. To define η we first note that the left R-module structure on $F(R)$ induces a right R-module structure on $Hom_S(F(R), F(M))$ for any R-module M. This allows us to define a right R-module homomorphism α_M by means of the composite

$$M \cong Hom_R(R, M) \xrightarrow{F} Hom_S(F(R), F(M)).$$

As an element of $Hom_R(M, Hom_S(F(R), F(M))$, α_M corresponds to an element of $Hom_S(M \otimes_R F(R), F(M))$ under the canonical isomorphism. Let η_M be the image of α_M under this isomorphism. Since α_M is natural with respect to M, this process defines a natural transformation $\eta : - \otimes_R F(R) \to F(-)$.

With η defined, it remains to describe the conditions under which it is an isomorphism. We do so in stages, taking care to indicate the correspondence between conditions on F and types of R-modules for which η is an isomorphism. In general, η will be an isomorphism for all finitely generated projective R-modules. To see this, first consider free modules. For R, $\eta_R : R \otimes_R F(R) \to F(R)$ is the canonical isomorphism. Since F and G are additive, it follows that η is an isomorphism on all finitely generated free R-modules. Now consider a finitely generated projective R-module P. Since P is a finitely generated projective module, there exist R-modules Q and M with M a finitely generated free module such that $P \oplus Q \cong M$. Then $\eta_P \oplus \eta_Q \cong \eta_M$, and by the above η_M is an isomorphism. It follows that η_P is as well.

In order for η to be an isomorphism on any larger class of modules, some further conditions must be placed on F. More to the point, since G is a right continuous functor, we cannot expect η to be an isomorphism for all R-modules without assuming F is right continuous as well. Assuming that F preserves filtered direct limits is enough to guarantee that η is an isomorphism for all free R-modules. To see this, recall that every free R-module is the filtered direct limit of finitely generated free R-modules. Since η is an isomorphism for all finitely generated free R-modules and F and G preserve filtered direct limits, it follows that η is an isomorphism for all free R-modules. Finally, if we assume that F also preserve cokernels, then η is an isomorphism for all R-modules, since every R-module admits a free resolution.

Thus, we see that any additive functor on finitely generated projective R-modules is classified in terms of a single $R^{op} \otimes S$-module $F(R)$. In the case of a linear functor F we will show that there is a natural transformation

$$QF(R) \otimes_{QS(R^{op})} \mathbf{QS}(-) \to QF(-) \tag{8.2}$$

which is a quasi-isomorphism on all finitely generated chain complexes of projective R-modules. At first glance, this looks different from the Eilenberg-Watts classification theorem, until one recalls that for an additive functor F, there is a natural transformation

$$F(R) \otimes_{R^{op} \otimes S} S[-] \to - \otimes_R F(R)$$

which is an isomorphism on all finitely generated free R-modules. One obtains this natural transformation as follows. For an R-module M, give $S[M]$ the structure of an S-S bimodule by setting

$$s(\sum_i s_i[m_i]) = \sum_i (ss_i)[m_i]$$

and

$$(\sum_i s_i[m_i])s = \sum_i (s_i s)[m_i]$$

for $s \in S$ and $\sum_i s_i[m_i] \in S[M]$ with $s_i \in S$ and $m_i \in M$ for all i. Then one can define an S-module homomorphism

$$F(R) \otimes_{R^{op} \otimes S} S[M] \xrightarrow{\alpha} M \otimes_R F(R)$$

by $\alpha(f \otimes \sum_i s_i[m_i]) = \sum_i m_i \otimes f s_i$ for $f \in F(R)$ and $\sum_i s_i[m] \in S[M]$. Clearly, this is natural with respect to M, and so one obtains the desired natural transformation.

As was done with $F(R)$ in the additive case, we must define a ring structure on $QS(R^{op})$ and a $QS(R^{op})$-module structure on $QF(R)$ and $QS(M)$ for an R-module M to define the natural transformation (8.2). In essence, this will be similar to what was done in the additive case, though more structure is involved since $QS(R^{op})$ is a chain complex rather than a module. That is, we must show that the product we define on $QS(R^{op})$ gives it the structure of a differential graded algebra and that $QF(R)$ and $QS(M)$ are differential graded modules over $QS(R^{op})$. We start this process by looking at $QS(-)$.

Let A be a right R-module, $S(-)$ be the free functor from $\mathcal{M}_R$ to $\mathcal{M}_S$, and $QS(-)$ be the functor that results from applying the Q-construction to the functor $S(-)$. To avoid confusion, we will always use the notation $S(-)$ when referring to the free functor and reserve the symbol S for the ring. Dixmier, in a letter to Mac Lane (see [M2]), defines a pairing

$$Q'S(A) \otimes_Z Q'S(R) \to Q'S(R)$$

as follows. For $t \in A[C_m]$ and $u \in R[C_n]$, considered as set maps from C_m to A and C_n to R, respectively, the product $tu \in A[C_{m+n}]$ is the map defined by

$$tu(\epsilon_1, \epsilon_2, \ldots, \epsilon_{m+n}) = t(\epsilon_1, \epsilon_2, \ldots, \epsilon_m)u(\epsilon_{m+1}, \epsilon_{m+2}, \ldots, \epsilon_{m+n}).$$

Extending this by linearity produces a pairing from $S(A[C_m]) \otimes_Z S(R[C_n])$ to $(S \otimes_Z S)(A[C_{m+n}])$, and, in general, from $Q'S(A) \otimes_Z Q'S(R)$ to $Q'(S \otimes_Z S)(A)$. Composing with the product from $S \otimes_Z S$ to S yields a pairing

$$Q'S(A) \otimes_Z Q'S(R) \xrightarrow{\mu'} Q'S(A).$$

It is straightforward to verify that μ' preserves the quotient in $QS(-)$. Thus we have defined a product $\mu : QS(A) \otimes_Z QS(R) \to QS(A)$. As claimed, μ gives $QS(R)$ a differential graded algebra structure.

Proposition 8.3. *With the multiplication defined by the pairing $\mu : QS(A) \otimes_Z QS(R) \to QS(A)$, $QS(R)$ is a differential graded S-algebra and $QS(A)$ is a right $QS(R)$-module. In addition, μ gives $QS(A)$ the structure of a left $QS(R^{op})$-module.*

Proof. Clearly, μ makes $QS(R)$ a graded ring and $QS(A)$ a right graded module over $QS(R)$, since it is defined for a basis and extended linearly. To show that $QS(R)$ is a differential graded ring and $QS(A)$ is a module over the differential graded ring $QS(R)$, we must show that for $t \in QS(A)_m$ and $u \in QS(R)_n$,

$$\partial(\mu(t, u)) = \mu(\partial t, u) + (-1)^m \mu(t, \partial u).$$

Since ∂ and μ preserve the quotient, it suffices to verify the formula for μ'. Moreover, since μ' is extended linearly from $A[C_m] \times R[C_n]$ to $QS(A)_m \otimes_Z QS(R)_n$, it is enough to show that the formula holds on a basis. Let $t \in A[C_m]$ and $u \in R[C_n]$.

For $\overline{X_i} = \overline{R_i}, \overline{S_i}$, or $\overline{P_i}$, the following identities are straightforward consequences of the definitions:

$$\overline{X_i}(tu) = \begin{cases} \overline{X_i}t(u) & \text{if } 1 \leq i \leq n, \\ t\overline{X_{i-m}}(u) & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

From these identities, we see that $\partial(tu) = \sum_{i=1}^{m+n}(-1)^i(P_i - R_i - S_i)(tu)$. This is equal to

$$\sum_{i=1}^{m}(-1)^i(S[\overline{P_i}] - S[\overline{R_i}] - S[\overline{S_i}])(tu) + \sum_{i=m+1}^{m+n}(-1)^i(S[\overline{P_i}] - S[\overline{R_i}] - S[\overline{S_i}])(tu),$$

which in turn equals

$$\sum_{i=1}^{m}(-1)^i(S[\overline{P_i}t] - S[\overline{R_i}t] - S[\overline{S_i}t])(u) + (-1)^m t(\sum_{j=1}^{n} S[\overline{P_j}u] - S[\overline{R_j}u] - S[\overline{S_j}u]),$$

which equals $(\partial t)u + (-1)^m t(\partial u)$, as desired.

Now, let F be an arbitrary functor from $\mathcal{M}_R$ to $\mathcal{M}_S$, M be a right R-module and A be an $R \otimes_Z R^{op}$-module. The discussion and proposition above give us the differential graded algebra structure on $QS(R^{op})$ and the differential graded module structure on $QS(M)$ needed to produce the pairing in (8.2), but they do not tell us how $QS(R^{op})$ acts on $QF(R)$. To define the right $QS(R^{op})$-module structure on $QF(R)$ and the natural transformation $QF(R) \otimes_{QS(R^{op})} QS(-) \to QF(-)$, we will define a pairing

$$QF(A) \otimes_Z QS(M) \xrightarrow{\widehat{\mu}} QF(M \otimes_R A).$$

We begin by noting that there is a right R-module isomorphism $\sigma : M[C_n] \otimes_R A[C_m] \to (M \otimes_R A)[C_{m+n}]$ defined for $m \in M[C_n]$ and $a \in A[C_m]$ by

$$\sigma(m \otimes a)(\epsilon_1, \epsilon_2, \ldots, \epsilon_{m+n}) = m(\epsilon_{m+1}, \epsilon_{m+2}, \ldots, \epsilon_{m+n}) \otimes a(\epsilon_1, \epsilon_2, \ldots, \epsilon_m).$$

Note that the full R-R bimodule structure of A is used here—the left structure for the tensor product and the right structure for the right R-module structure of $M[C_n] \otimes_R A[C_m]$ and $(M \otimes_R A)[C_{m+n}]$. Furthermore, the transpose in the coordinates $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{m+n})$ is needed later to guarantee that we have produced a $QS(R^{op})$-module structure on $QF(A)$. Now, let τ be the right R-module homomorphism from $M[C_n]$ to $Hom_R(A[C_m], M \otimes_R A[C_{m+n}])$ defined by the composite

$$\begin{aligned} M[C_n] &\cong Hom_R(R, M[C_n]) \\ &\xrightarrow{-\otimes_R A[C_m]} Hom_R(A[C_m], M[C_n] \otimes_R A[C_m]) \\ &\overset{\sigma_*}{\cong} Hom_R(A[C_m], M \otimes_R A[C_{m+n}]). \end{aligned}$$

For $u \in M[C_n]$, the homomorphism $\tau(u) \in Hom_R(A[C_m], M \otimes_R A[C_{m+n}])$ is a right R-module homomorphism, and so $F(\tau(u))$ is a right S-module homomorphism from $F(A[C_m])$ to $F(M \otimes_R A[C_{m+n}])$. For $x \in F(A[C_m])$ and $u \in M[C_n]$ define $\widehat{\mu}(x, u) \in F(M \otimes_R A[C_{m+n}])$ by

$$\widehat{\mu}(x, u) = F(\tau(u))(x).$$

Extending by linearity produces a pairing

$$F(A[C_m]) \otimes_Z S(M[C_n]) \xrightarrow{\widehat{\mu}'} F \otimes_Z S(M \otimes_R A[C_{m+n}])$$

or, more generally,

$$Q'F(A) \otimes_Z Q'S(M) \xrightarrow{\widehat{\mu}'} Q'(F \otimes_Z S)(M \otimes_R A),$$

by $\widehat{\mu}(x, \sum s_i u_i) = \sum F(\tau(u_i))(x) \otimes s_i$ for $s_i \in S$. (The functor $F(-) \otimes_Z S$ takes a right R-module K to the right S-module $F(K) \otimes_Z S$.) Composing with the natural transformation from $F \otimes_Z S$ to F induced by multiplication, one obtains a pairing

$$Q'F(A) \otimes_Z Q'S(M) \xrightarrow{\widehat{\mu}'} Q'F(M \otimes_R A).$$

One can show that $\widehat{\mu}'$ is well defined on the quotient QF by noting that for $X_j = R_j, S_j$, or D_j (by abuse of notation considered as subsets of C_n, C_m, or C_{n+m}),

$$\sigma : M[X_j] \otimes_R A[C_m] \to (M \otimes_R A)[X_{m+j}]$$

and

$$\sigma : M[C_n] \otimes_R A[X_j] \to (M \otimes_R A)[X_j].$$

Thus, we have produced a pairing

$$QF(A) \otimes_Z QS(M) \xrightarrow{\widehat{\mu}} QF(M \otimes_R A)$$

which when $A = R$ becomes

$$QF(R) \otimes_Z QS(M) \xrightarrow{\widehat{\mu}} QF(M \otimes_R R) \cong QF(M).$$

Proposition 8.4. *For any $R \otimes_Z R^{op}$-module A, the pairing $\widehat{\mu}$ gives $QF(A)$ the structure of a right $QS(R^{op})$-module.*

Proof. That $\widehat{\mu}$ is linear follows from the facts that for any $u \in R^{op}[C_n]$, $F(\tau(u))$ is an S-module homomorphism, and that $\widehat{\mu}$ was defined on a basis for $S(R^{op}[C_m])$ and extended linearly.

We must also check that $\widehat{\mu}$ is associative and the Leibniz rule holds. To verify the former, it suffices to check it for basis elements. Let $x \in F(A[C_i])$, $a \in R^{op}[C_j]$ and $u \in R^{op}[C_k]$. Then

$$\begin{aligned} \widehat{\mu}(\widehat{\mu}(x, a), u) &= \widehat{\mu}(F(\tau(a))x, u) \\ &= F(\tau(u))F(\tau(a))x \\ &= F(\tau(u) \circ \tau(a))(x), \end{aligned}$$

and

$$\begin{aligned} \widehat{\mu}(x, \widehat{\mu}(a, u)) &= F(\tau(\widehat{\mu}(a, u))(x) \\ &= F(\tau(\tau(u)a))(x). \end{aligned}$$

Thus, it is enough to prove that $\tau(u) \circ \tau(a) = \tau(\tau(u)a)$. But, for $y \in R[C_i]$ and $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{i+j+k}) \in C_{i+j+k}$,

$$\tau(\tau(u)a)(y)(\epsilon_1, \ldots, \epsilon_{i+j+k}) = \tau(u)a(\epsilon_{i+1}, \ldots, e_{i+j+k}) \otimes y(\epsilon_1, \ldots, \epsilon_i).$$

This is equal to

$$(u(\epsilon_{i+k+1}, \ldots, \epsilon_{i+k+j}) \otimes a(\epsilon_{i+1}, \ldots, \epsilon_{i+k})) \otimes y(\epsilon_1, \ldots \epsilon_i),$$

which becomes $a(\epsilon_{i+1},\dots,\epsilon_{i+k})u(\epsilon_{i+k+1},\dots,\epsilon_{i+k+j})y(\epsilon_1,\dots,\epsilon_i)$ under the isomorphism $(R^{op}\otimes_R R^{op})\otimes_R R\to R^{op}$. On the other hand,

$$\tau(u)\tau(a)y(\epsilon_1,\epsilon_2,\dots,\epsilon_{i+j+k}) = u(\epsilon_{i+j+1},\dots,\epsilon_{i+j+k})\otimes\tau(a)(y)(\epsilon_1,\dots,\epsilon_{i+j}).$$

This is equal to

$$u(\epsilon_{i+j+1},\dots,\epsilon_{i+j+k})\otimes(a(\epsilon_{i+1},\dots,\epsilon_{i+j})\otimes y(\epsilon_1,\dots,\epsilon_i)),$$

which also equals $a(\epsilon_{i+1},\dots\epsilon_{i+k})u(\epsilon_{i+k+1},\dots,\epsilon_{i+k+j})y(\epsilon_1,\dots,\epsilon_i)$ under the isomorphism $R^{op}\otimes_R(R^{op}\otimes_R R)\to R^{op}$. Thus $\widehat{\mu}$ is associative.

We must also prove, as in the previous proposition, that for $x\in F(R[C_m])$ and $u\in R^{op}[C_k]$, $\partial(\widehat{\mu}(x,u)) = \widehat{\mu}(\partial x,u)+(-1)^m\widehat{\mu}(\tau(\partial u))(x)$. Note that for $F=S(-)$, $\widehat{\mu}$ is simply μ, so the proof is just a generalization of the proof for μ once one recognizes that for $\widehat{\mu}$ the following identities hold for $\overline{X_i}=\overline{R_i},\overline{S_i}$, or $\overline{D_i}$:

$$\overline{X_i}\tau u = \begin{cases}(\tau u)\overline{X_i} & \text{if } 1\le i\le m,\\ \tau(\overline{X_{i-m}}u) & \text{if } i>m.\end{cases}$$

We leave the rest of the proof to the reader.

Letting $A=R$ in the above proposition, we see that $\widehat{\mu}$ gives $QF(R)$ a right $QS(R^{op})$-module structure. In addition, since $QS(M)$ is a left $QS(R^{op})$-module, we may form the tensor product $QF(R)\otimes_{QS(R^{op})}QS(M)$. Moreover, the pairing $\widehat{\mu}$ enables us to define a natural map from this tensor product to $QF(M)$.

Proposition 8.5. *Let F be a functor from $\mathcal{M}_R$ to $\mathcal{M}_S$. There is a natural transformation*

$$QF(R)\otimes_{QS(R^{op})}QS(-)\to QF(-)$$

that is a quasi-isomorphism on all finitely generated projective R-modules.

Proof. Consider the composite

$$QF(R)\otimes_Z QS(-)\xrightarrow{\widehat{\mu}} QF(-\otimes_R R)\xrightarrow{\gamma} QF(-).$$

One can show that this composition is $QS(R^{op})$-bilinear. (The proof is similar to the one used to establish the linearity and associativity conditions in proposition 8.4.) Hence, $\gamma\circ\widehat{\mu}$ determines a natural transformation

$$\beta: QF(R)\otimes_{QS(R^{op})}QS(-)\to QF(-).$$

To see that β is an isomorphism on all finitely generated free R-modules, we look first at R and R^{op}. As right R-modules they are isomorphic, and hence $QS(R)\cong QS(R^{op})$. We have a commutative diagram

$$\begin{tikzcd}
QF(R)\otimes_{QS(R^{op})}QS(R) \arrow[r, "\beta_R"] \arrow[d, "\cong"'] & QF(R) \arrow[d, "="] \\
QF(R)\otimes_{QS(R^{op})}QS(R^{op}) \arrow[r, "\gamma\circ\hat{\mu}"] & QF(R^{op}) \arrow[r] & QF(R)
\end{tikzcd}$$

since the pairing $\widehat{\mu}$ only uses the right R-module structure of R and R^{op}. Moreover, the lower composition is simply the multiplication map and hence is a quasi-isomorphism. Thus, β_R is a quasi-isomorphism as well. Both $QF(R)\otimes_{QS(R^{op})}QS(-)$ and $QF(-)$ preserve direct sums up to quasi-isomorphism, and so β is a quasi-isomorphism on all finitely generated free R-modules. Moreover, since every

finitely generated projective has a resolution by finitely generated free R-modules, β is a quasi-isomorphism on all finitely generated projective R-modules, as claimed.

This result is readily extended to functors from $\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$.

Corollary 8.6. *Let $F : \mathcal{M}_R \to Ch_{\geq 0}\mathcal{M}_S$. There is a natural transformation from $QF(R) \otimes_{QS(R^{op})} QS(-)$ to $QF(-)$ which is an isomorphism on all finitely generated projective R-modules.*

Proof. Consider F as a sequence of functors and natural transformations, $\ldots \to F_{n+1} \to F_n \to F_{n-1} \to \ldots$, where each F_n is a functor from $\mathcal{M}_R$ to $\mathcal{M}_S$. For each n, we have a natural transformation

$$QF_n(R) \otimes_{QS(R^{op})} QS(-) \to QF_n(-)$$

that is a quasi-isomorphism on all finitely generated projective R-modules. These transformations can be assembled into a natural transformation between the bicomplexes obtained by using all of the F_n's and the natural transformations between them

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ QF_n(R) \otimes_{QS(R^{op})} QS(-) & & QF_n(-) \\ \downarrow & \longrightarrow & \downarrow \\ QF_{n-1}(R) \otimes_{QS(R^{op})} QS(-) & & QF_{n-1}(-) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array} = QF(-).$$

By the previous proposition, this natural transformation induces a quasi-isomorphism on each row and hence a quasi-isomorphism on the total complexes for all finitely generated projective R-modules. The result follows.

We now have the essential ingredients, i.e., the ring and module structures as well as a natural transformation out of the tensor product, to state and prove the classification theorem for linear functors.

Theorem 8.7. *Let $\mathcal{P}_R$ be the category of finitely generated projective R-modules. Every linear functor from $Ch_{\geq 0}\mathcal{P}_R$ to $Ch_{\geq 0}\mathcal{M}_S$ is naturally quasi-isomorphic to a functor of the form $X \otimes_{QS(R^{op})} \mathbf{QS}(-)$ for some right $QS(R^{op})$-module X.*

Proof. Let F be such a functor and let X be the $QS(R^{op})$-module $QF(R)$. We have constructed a natural transformation from $QF(R) \otimes_{QS(R^{op})} \mathbf{QS}(-)$ to $QF(-)$ which is a quasi-isomorphism on all finitely generated projective R-modules. The functors $QF(R) \otimes_{QS(R^{op})} QS(-)$ and QF both preserve direct sums up to quasi-isomorphism, and so lemma 5.4 guarantees that they can be prolonged degreewise to linear functors on $Ch_{\geq 0}\mathcal{P}_R$. Since they are quasi-isomorphic on all finitely generated projective R-modules, they are quasi-isomorphic on all chain complexes over $\mathcal{P}_R$.

Now consider F and QF. Since F is linear, it preserves direct sums up to quasi-isomorphism, and so its cross effects vanish. Thus F is quasi-isomorphic to QF, and the result follows.

As in the case of additive functors, placing further conditions on the functor F enables one to extend the quasi-isomorphism to a larger category. For example, if

F is right continuous, the natural transformation will be a quasi-isomorphism on all of $Ch_{\geq 0}\mathcal{M}_R$.

As an immediate consequence, we have the following equivalence of homotopy categories.

Corollary 8.8. *For any rings with identity R and S, the correspondence $F \longleftrightarrow QF(R)$ induces an equivalence of homotopy categories*

$$Ho(F_{lin}(Ch_{\geq 0}\mathcal{P}_R, Ch_{\geq 0}\mathcal{M}_S)) \simeq Ho(Mod - QS(R^{op}))$$

where $F_{lin}(Ch_{\geq 0}\mathcal{P}_R, Ch_{\geq 0}\mathcal{M}_S)$ is the category of linear functors from $Ch_{\geq 0}\mathcal{P}_R$ to $Ch_{\geq 0}\mathcal{M}_S$ and $Mod - QS(R^{op})$ is the category of right modules over the differential graded algebra $QS(R^{op})$.

9. Linearization and Functors of Spaces

As indicated in the introduction, we have written this paper to provide an introduction to the algebraic version of Goodwillie's calculus of homotopy functors. Though this paper has been motivated by results in topology, we have taken care to develop ideas algebraically, independent of topological results. As a consequence, we have thus far described nothing more than a formal analogy between Dold-Puppe stabilization and the stabilization of a functor of spaces. But, in fact, the Dold-Puppe stabilization is a special case of the stabilization of a functor of spaces, and in this section we explain this relationship. This section is written for topologists, and assumes some familiarity with basic homotopy theory and the terminology of Goodwillie's calculus (in particular, section 1 of [G1]).

Let $\mathcal{F}_R$ be the category of free R-modules. Given any functor $F : \mathcal{F}_R \to \mathcal{M}_S$, one can produce from it a functor, $\widehat{F}$, of topological spaces in the following way. (By a topological space we will mean a basepointed space with the homotopy type of a finite CW-complex.) Let $\widetilde{R}[-]$ be the reduced free functor from finite basepointed sets to R-modules: for a set X with basepoint $*$, $\widetilde{R}[X] = R[X]/R[*]$. For a space Y, let $Y.$ denote its simplicial replacement. Applying $\widetilde{R}[-]$ degreewise to $Y.$ produces a simplicial R-module, to which F can also be applied degreewise to obtain a simplicial S-module. Forgetting the S-module structure, we have a simplicial set whose realization gives us a space. In other words, let $\widehat{F}$ be the functor of spaces defined by

$$\widehat{F}(Y) = |F(\widetilde{R}[Y.])|.$$

Recall from [G1] that a functor of spaces G is linear at a point if the following three conditions hold:

1) G preserves weak homotopy equivalences.
2) $G(*) \simeq *$ for a contractible space $*$.
3) G is excisive, i.e., it takes every co-Cartesian square of spaces to a Cartesian square.

Proposition 9.1. *If $F : \mathcal{F}_R \to \mathcal{M}_S$ is additive, then $\widehat{F}$ is linear as a functor of spaces.*

Proof. Recall that the prolongation of F is linear as a functor from $Ch_{\geq 0}\mathcal{P}_R$ to $Ch_{\geq 0}\mathcal{M}_S$ by corollary 5.5. It is straightforward to check that the first two conditions of linearity hold for $\widehat{F}$. We will concentrate on the third. Consider the

simplicial replacement of a co-Cartesian diagram of spaces in which each map is a cofibration:

$$\begin{array}{ccc} X_0 & \to & X_1 \\ \downarrow & & \downarrow \\ X_2 & \to & X_{12}. \end{array}$$

(Any co-Cartesian square is equivalent to such a diagram.) Being co-Cartesian in this setting is equivalent to the map from $cofiber(X_0 \to X_1)$ to $cofiber(X_2 \to X_{12})$ being a simplicial homotopy equivalence. Since cofibrations of simplicial sets are inclusions, it follows that the functor $\widetilde{R}[-]$ preserves cofibers in addition to preserving the equivalence. Then, since F is linear, we obtain a map of quasi-exact sequences

$$\begin{array}{ccccc} F\widetilde{R}[X_0] & \to & F\widetilde{R}[X_1] & \to & F\widetilde{R}[cofiber(X_0 \to X_1)] \\ \downarrow & & \downarrow & & \downarrow \\ F\widetilde{R}[X_2] & \to & F\widetilde{[}X_{12}] & \to & F\widetilde{R}[cofiber(X_2 \to X_{12})] \end{array}$$

and so, by the five lemma, $F\widetilde{R}[cofiber(X_0 \to X_1)] \to F\widetilde{R}[cofiber(X_2 \to X_{12})]$ is a quasi-isomorphism. Using the five lemma again, we obtain quasi-isomorphisms

$$\begin{array}{ccc} cofiber(F\widetilde{R}[X_0] \to F\widetilde{R}[X_1]) & \to & cofiber(F\widetilde{R}[X_2] \to F\widetilde{R}[X_{12}]) \\ \cong\downarrow & & \downarrow\cong \\ F\widetilde{R}(cofiber(X_0 \to X_1)) & \xrightarrow{\cong} & F\widetilde{R}[cofiber(X_2 \to X_{12})] \end{array}$$

and so $cofiber(F\widetilde{R}[X_0] \to F\widetilde{R}[X_1])$ is quasi-isomorphic to $cofiber(F\widetilde{R}[X_2] \to F\widetilde{R}[X_{12}])$. Thus, the square

$$\begin{array}{ccc} \widehat{F}(X_0) & \to & \widehat{F}(X_1) \\ \downarrow & & \downarrow \\ \widehat{F}(X_2) & \to & \widehat{F}(X_{12}) \end{array}$$

is Cartesian, and $\widehat{F}$ is linear as a functor of spaces.

Under this correspondence between functors of modules and functors of spaces, one can also show that the Dold-Puppe stabilization of F corresponds to the linearization of $\widehat{F}$. For any functor of spaces G, let D_1G denote its linearization.

Proposition 9.2. *If $F : \mathcal{F}_R \to \mathcal{M}_S$ is a reduced functor, then $\widehat{D_1F}$ and $D_1\widehat{F}$ are weakly equivalent as functors of spaces.*

Proof. Recall that the stabilization of a functor of spaces, G, is equivalent to $hocolim_n \Omega^n G\Sigma^n$ for some appropriate choice of model for $\Omega^n G\Sigma^n$. That is, D_1G is equivalent to the homotopy colimit (and, in this case, the strict colimit) of a diagram of the form

$$G \to T_1G \to T_2G \to \ldots T_nG \to \ldots,$$

where $T_nG \simeq \Omega^n G\Sigma^n$ for each $n \geq 1$.

On the other hand, we have seen that for $F : \mathcal{M}_R \to \mathcal{M}_S$,

$$D_1F = \varinjlim_n sh_{-n}\mathbf{F}sh_n$$

and hence

$$\widehat{D_1F} \simeq \varinjlim_n sh\widehat{_{-n}Fsh}_n,$$

so that it suffices to show that $\widehat{sh_{-n}Fsh_n}$ and $\Omega^n\widehat{F}\Sigma^n$ are equivalent as functors of spaces. Consider $\widehat{F}\Sigma^n$ and $\widehat{Fsh_n}$. For a space X (considered as a simplicial set) and a simplicial model for S^n,

$$\widehat{F}\Sigma^n X \simeq |F(\widetilde{R}[S^n \wedge X])|,$$

where F is applied degreewise to $\widetilde{R}[S^n \wedge X]$. Using the bar construction (definition A.1) and lemma A.3, one can show that

$$\widehat{Fsh_n}X \simeq F\Delta B^n \widetilde{R}[X],$$

where F is again applied degreewise. However, one can show that $\widetilde{R}[S^n \wedge X]$ and $\Delta B^n \widetilde{R}[X]$ are isomorphic as simplicial R-modules if one uses $\Delta^1/\partial\Delta^1$ as a simplicial model for S^1. To see this, first note that $\widetilde{R}[S^n]$ and $\Delta B^n R$ are isomorphic as simplicial R-modules. Degreewise, X, S^n and $S^n \wedge X$ are just finite sets, and in degree k we have

$$\begin{aligned}
\widetilde{R}[(S^n \wedge X)_k] &= \widetilde{R}[\bigvee_{|X_k|-1} S^n_k] \\
&= \bigoplus_{(|X_k|-1)(|S^n_k|-1)} R \\
&\cong \bigoplus_{|X_k|-1} \bigoplus_{|S^n_k|-1} R \\
&\cong \bigoplus_{|X_k|-1} \widetilde{R}[S^n_k].
\end{aligned}$$

Thus,

$$\begin{aligned}
\widetilde{R}[S^n \wedge X_k] &\cong \bigoplus_{|X_k|-1} \widetilde{R}[S^n] \\
&\simeq \bigoplus_{|X_k|-1} \Delta B^n R \\
&\cong \Delta B^n (\bigoplus_{|X_k|-1} R) \\
&\cong \Delta B^n \widetilde{R}[X_k].
\end{aligned}$$

Hence $\Delta B^n \widetilde{R}[X] \simeq \widetilde{R}[\Sigma^n X]$, and so $\widehat{Fsh_n} \simeq \widehat{F}\Sigma^n$. Since Ω^n shifts the homotopy of a simplicial set down by n and sh_{-n} does the same for the homology of a chain complex, the result follows.

Appendix

In this appendix we will prove the equivalence of functors needed in the proof of 3.17. To obtain the result, one essentially needs a way of moving the functor sh_n past the functor Γ. But, sh_n is a functor of chain complexes and Γ is a functor of simplicial modules, so what one really needs is a simplicial version of sh_n. This is given in the definition of B^n below. The relationship between sh_n and B^n is described in the first two lemmas.

Definition A.1. Let M be an R-module. The simplicial R module BM is defined by

$$(BM)_n = \begin{cases} 0 & \text{if } n = 0, \\ M^{\oplus n} & \text{if } n > 0. \end{cases}$$

with face and degeneracy maps given by

$$d_i(m_1, m_2, \ldots, m_n) = \begin{cases} (m_2, \ldots, m_n) & \text{if } i = 0, \\ (m_1, \ldots, m_{i-1}, m_i + m_{i+1}, m_{i+2}, \ldots, m_n) & \text{if } 0 < i < n, \\ (m_1, m_2, \ldots, m_{n-1}) & \text{if } i = n, \end{cases}$$

and

$$\sigma_i(m_1, m_2, \ldots, m_n) = (m_1, m_2, \ldots, m_i, 0, m_{i+1}, \ldots, m_n).$$

This is the bar construction, familiar to topologists. BM has the property that its geometric realization, $|BM|$, is the Eilenberg-Mac Lane space $K(M, 1)$ (see [Ma], chapter 3 or [We], p. 257). We will iterate the bar construction, to produce an n-simplicial object $B^n A$ from an object A.

Lemma A.2. *If we let Δ be the diagonal simplicial module of an n-simplicial module, then $Tot(N\Delta B^n)$ and sh_n are naturally chain homotopic as functors from $Ch_{\geq 0}\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_R$.*

Proof. For an R-module A it follows directly from the definitions of N, B, and sh_1 that $N(BA)$ and $sh_1 A$ are chain isomorphic. In addition, it is easy to verify that $sh_n A$ is equal to $Tot(sh_1 \circ sh_1 \circ \cdots \circ sh_1 A)$, where $sh_1 \circ sh_1 \circ \cdots \circ sh_1 A$ is the n-dimensional chain complex obtained by shifting A (considered as a chain complex concentrated in degree 0) by one degree in n directions. Then

$$\begin{aligned} sh_n A &= Tot(sh_1 \circ \cdots \circ sh_1 A) \\ &\cong Tot(N(B) \circ \cdots \circ N(B))A \\ &\simeq N(\Delta B^n)A, \end{aligned}$$

where the last equivalence is due to the n-dimensional version of the Eilenberg-Zilber theorem. Since all of the above equivalences are natural in A, it follows immediately that sh_n and $Tot(N\Delta B^n)$ are naturally chain homotopy equivalent when considered as functors from $Ch_{\geq 0}\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_R$.

Lemma A.3. *Γsh_n and $\Delta(B^n\Gamma)$ are naturally simplicially homotopic as functors from $Ch_{\geq 0}\mathcal{M}_R$ to $Simp_{\mathcal{M}_R}$.*

Proof. By the previous lemma, we know that $sh_n \simeq Tot(N\Delta B^n)$. Since $N\Gamma \cong id$, $N \simeq C$, and N and C convert simplicial homotopies to chain homotopies, we see that

$$\begin{aligned} sh_n &\simeq Tot(N\Delta B^n) \\ &\simeq Tot(N\Delta B^n N\Gamma) \\ &\simeq Tot(N\Delta B^n C\Gamma). \end{aligned}$$

Since $C(X)_k = X_k$ for any simplicial object $X.$ and ΔB^n is being applied degreewise to an object, it is clear that $\Delta B^n C \cong C\Delta B^n$. Thus,

$$\begin{aligned} sh_n &\simeq Tot(N\Delta B^n C\Gamma) \\ &\simeq Tot(NC\Delta B^n\Gamma) \\ &\simeq Tot(NN\Delta B^n\Gamma). \end{aligned}$$

Applying Γ to both sides (and noting that it takes chain homotopies to simplicial homotopies) and then applying the Eilenberg-Zilber theorem, we have

$$\begin{aligned} \Gamma(Tot[N(\Delta B^n\Gamma)]) &\simeq \Gamma N\Delta((\Delta B^n)\Gamma) \\ &\cong \Gamma N\Delta(B^n\Gamma) \\ &\cong \Delta(B^n\Gamma). \end{aligned}$$

Since each chain homotopy above is natural, we obtain the result.

Lemma A.4. *Let F be a functor from $\mathcal{M}_R$ to $\mathcal{M}_S$ and let $G = \mathbf{F}sh_n$, considered as a functor from $\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$. Then, $\mathbf{G}$ and $\mathbf{F} \circ sh_n$ are naturally chain homotopic as functors from $Ch_{\geq 0}\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$.*

Proof. Given a chain complex P, we have, by definition,

$$\begin{aligned} \mathbf{G} &= TotNG\Gamma(P) \\ &= TotN\mathbf{F}sh_n\Gamma(P) \\ &= TotNNF\Gamma sh_n\Gamma(P). \end{aligned}$$

The key here is to recognize that G (and hence $\mathbf{F}sh_n$) is being applied degreewise to $\Gamma(P)$. This means that sh_n is applied to $\Gamma(P)$ in each simplicial degree to yield a simplicial chain complex, and $\Gamma sh_n\Gamma(P)$ is the bisimplicial R-module obtained from $sh_n\Gamma(P)$ by applying Γ to the chain complex in each simplicial degree. Then, F is applied degreewise to $\Gamma sh_n\Gamma(P)$ to yield a bisimplicial S-module. Finally, $Tot(NNF\Gamma sh_n\Gamma(P))$ represents the total complex of the bicomplex produced by normalizing $F\Gamma sh_n\Gamma(P)$ in both simplicial directions. By the Eilenberg-Zilber theorem we have

$$Tot(NNF\Gamma sh_n\Gamma(P)) \cong N\Delta(F\Gamma sh_n\Gamma(P)).$$

Now, since sh_n is being applied degreewise to $\Gamma(P)$, we know from the proof of lemma A.2 that $sh_n\Gamma(P)$ is equivalent to $(N\Delta B^n)(\Gamma P)$. Using the fact that $\Gamma N \cong id$ gives us

$$\begin{aligned} N\Delta(F\Gamma sh_n\Gamma(P)) &\cong N\Delta(F\Gamma(N\Delta B^n)(\Gamma P)) \\ &\cong N\Delta F((\Delta B^n)(\Gamma P)). \end{aligned}$$

Finally, since F was applied degreewise, it commutes with Δ, and so

$$N\Delta F((\Delta B^n)(\Gamma P)) = NF(\Delta(B^n\Gamma)(P)).$$

By lemma A.3,

$$NF(\Delta(B^n\Gamma)(P)) \simeq NF(\Gamma sh_n P) = \mathbf{F}sh_n P.$$

Therefore, $\mathbf{G}$ and $\mathbf{F}sh_n$ are naturally chain homotopic as functors from $Ch_{\geq 0}\mathcal{M}_R$ to $Ch_{\geq 0}\mathcal{M}_S$.

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Department of Mathematics, Union College, Schenectady, New York 12308
E-mail address: johnsonb@union.edu

Department of Mathematics, University of Illinois, Urbana, Illinois 61801
E-mail address: randy@math.uiuc.edu