TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 350, Number 5, May 1998, Pages 1811–1822
S 0002-9947(98)01735-8

INTEGRAL REPRESENTATION OF CONTINUOUS COMONOTONICALLY ADDITIVE FUNCTIONALS

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Abstract. In this paper, I first prove an integral representation theorem: Every quasi-integral on a Stone lattice can be represented by a unique upper-continuous capacity. I then apply this representation theorem to study the topological structure of the space of all upper-continuous capacities on a compact space, and to prove the existence of an upper-continuous capacity on the product space of infinitely many compact Hausdorff spaces with a collection of consistent finite marginals.

1. Introduction

The notion of capacity was first introduced into mathematics literature by Choquet [1]. Though originally motivated by problems arising from potential theory in physics, it has since found many applications in various fields. In particular, owing to Schmeidler's work [7, 8] in the 1980's, capacity and Choquet integral have become important tools in describing individuals' behavior under uncertainty. In this paper, I will investigate some properties of capacities that arise from decision theory and game theory applications. Readers interested in these applications are referred to some recent papers by Epstein and Wang [4] and by the author [9].

Specifically, I study upper-continuous capacities from the duality viewpoint. The main result I prove in Section 2 is a representation theorem for upper-continuous capacities. The distinction between my result and other previous representation theorems for capacities (Greco [5], Schmeidler [7]) is the emphasis my result places on the upper-continuity of capacities. Though various regular conditions on capacities have been considered by many authors since Choquet [1], most existing representation results do not involve capacities with proper, if any, regularity conditions. My representation theorem reveals that upper-continuity is actually one of the more natural regularity conditions on capacities from the duality viewpoint. In Sections 3 and 4, I consider two applications of this representation theorem. I first show that the space of upper-continuous capacities on any compact space, endowed

Received by the editors August 16, 1995 and, in revised form, October 30, 1995.
1991 *Mathematics Subject Classification.* Primary 28A12, 28C05, 28C15; Secondary 60A05, 60A15, 90A05.
Key words and phrases. Upper-continuous capacities, regular capacities, Choquet integrals, Stone lattices, comonotonically additive functionals, monotonic functionals, continuous functionals, the weak topology, Kolmogorov's theorem, consistent marginals.
I want to thank L. Epstein, D. Schmeidler, and in particular, M. Marinacci, as well as an anonymous referee, for their helpful comments. The revision was done while I was visiting the Economics Department of the Hong Kong University of Science and Technology, whose hospitality I deeply appreciated.

with the weak topology, is compact and Hausdorff. It is further a compact metric space if the base space is also a compact metric space. Second, I prove the existence of an upper-continuous capacity on the product space of infinitely many compact Hausdorff spaces with a collection of consistent finite marginals.

2. A representation theorem

In this section I prove a representation theorem in a rather general setting. Let X be an arbitrary set and Σ a collection of subsets of X. A real-valued function μ on Σ is called a capacity if it satisfies: $\mu(\varnothing) = 0$; $\mu(S) = 1$; and $\mu(A) \leq \mu(B)$ for all $A \subseteq B$, $A, B \in \Sigma$. A capacity μ is upper-continuous if $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_{n=1}^{\infty} A_n)$ for any non-increasing sequence of sets $\{A_n\}$ ($A_n \in \Sigma$ for all n).

Remark 1. Although most authors require that a capacity μ be defined on an algebra (or σ-algebra) Σ, I will not insist on this requirement. Of course, one can always extend a capacity μ on Σ to any algebra that contains Σ (say by an outer capacity operation). Such an extension, however, might be irrelevant as far as representation is concerned. Moreover, such an extension might not inherit some properties that μ possesses on Σ. For instance, the next representation theorem claims that for any functional I satisfying certain properties (yet to be specified) there exists a unique upper-continuous capacity μ on some Σ, which is not necessarily an algebra. But neither the unicity nor the upper-continuity can be claimed for extensions of μ on the algebra generated by Σ. Hence, our treatment allows us to make sharper statements about μ on Σ that would have been lost had we insisted that μ be defined on an algebra.

Remark 2. Upper-continuity of a capacity is a generalization of countable additivity of an additive measure. If Σ is a σ-algebra and μ additive, upper-continuity is indeed the same as countable additivity. Some previous authors have used upper-continuity, or other closely related properties, as a part of their definitions of a regular capacity, but none seem to have used it alone in their discussions.

Let us now consider functionals on some spaces of real-valued functions on X. Let L be a collection of bounded real-valued functions on X. L is called a Stone vector lattice if: (i) L is a vector space; (ii) L is a lattice, i.e., $\max(a, b)$, $\min(a, b) \in L$ for all $a, b \in L$; and (iii) L contains all constant functions on X.

Let I be a (real-valued) function from L to R. I is called a quasi-integral if:

I is comonotonically additive, i.e., $I(a+b) = I(a)+I(b)$ for all functions $a, b \in L$ that satisfy $(a(x) - a(x'))(b(x) - b(x')) \geq 0$ for all $x, x' \in X$;[1]

I is monotonic, i.e., $I(a) \geq I(b)$ for all $a, b \in L$ that satisfy $a(x) \geq b(x)$ for all x; and

I is continuous in the sense that $\lim_{n\to\infty} I(a_n) = I(a)$ for all sequences $\{a_n\}$ and a in L such that $\{a_n(x)\}$ is non-increasing and $\lim_{n\to\infty} a_n(x) = a(x)$ for all x.

Notice that a quasi-integral I is always homogeneous, i.e., $I(\alpha a) = \alpha I(a)$ for all $\alpha \geq 0$ and $a \in L$, since homogeneity is implied by comonotonic additivity and monotonicity [7].

The first main result of the paper establishes a one-to-one correspondence between upper-continuous capacities and quasi-integrals. For a given capacity μ on

[1] Any pair of such functions are called comonotonic.

Σ and a bounded Σ-measurable function a on X,[2] the Choquet integral of a with respect to μ is defined as follows (Choquet [1]):

$$\int_X a\,d\mu = \int_0^\infty \mu(a \geq t)dt + \int_{-\infty}^0 (\mu(a \geq t) - 1)dt,$$

where integrals on the right hand side are in Riemann's sense.

Theorem 1. *Assume that I is a quasi-integral on a Stone lattice L and $I(\mathbf{1}) = 1$ $(\mathbf{1}(x) \equiv 1)$. Then there exists a unique upper-continuous capacity μ on Σ, which is the collection of all upper contour sets of all functions in L, such that, for all $a \in L$,*

$$I(a) = \int_X a\,d\mu = \int_0^\infty \mu(a \geq t)dt + \int_{-\infty}^0 (\mu(a \geq t) - 1)dt. \tag{1}$$

On the other hand, for any upper-continuous capacity μ, the functional defined on L by (1) is a quasi-integral.

Before I prove Theorem 1, I present a lemma that relates continuity of quasi-integrals and upper-continuity of capacities to some seemingly weaker, albeit equivalent, properties.[3]

Lemma 1 (Marinacci)**.** (A) *For any functional I on a lattice L that is comonotonically additive and monotonic, the following two properties are equivalent*:

(Ai) $\lim_{n\to\infty} I(a_n) = I(a)$ *for all sequences $\{a_n\}$ and a in L such that $\{a_n(x)\}$ is non-increasing and $\lim_{n\to\infty} a_n(x) = a(x)$ for all x;*

(Aii) $\lim_{n\to\infty} I(a_n) \leq I(a)$ *for all sequences $\{a_n\}$ and a in L such that $\{a_n(x)\}$ is non-increasing and, for all x, there is an n_x with $a_n(x) \leq a(x)$ for all $n \geq n_x$.*

(B) *For any capacity μ on Σ, which is closed under the union operation, the following two properties are equivalent*:

(Bi) $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ *for any non-increasing sequence of sets $\{A_n\}$ and $A \in \Sigma$ such that $\bigcap_{n=1}^\infty A_n = A$;*

(Bii) $\lim_{n\to\infty} \mu(A_n) \leq \mu(A)$ *for any non-increasing sequence of sets $\{A_n\}$ and $A \in \Sigma$ such that $\bigcap_{n=1}^\infty A_n \subseteq A$.*

Proof. (Ai)$\Rightarrow$(Aii). Take any sequence $\{a_n\}$ and a in L such that $\{a_n\}$ is non-increasing and, for all x, there is an n_x with $a_n(x) \leq a(x)$ for all $n \geq n_x$. Since $\{\max(a_n, a)\}$ is also a non-increasing sequence in L and $\lim_{n\to\infty} \max(a_n(x), a(x)) = a(x)$ for all x, (Ai) implies $\lim_{n\to\infty} I(\max(a_n, a)) = I(a)$. By monotonicity, $\lim_{n\to\infty} I(a_n) \leq \lim_{n\to\infty} I(\max(a_n, a)) = I(a)$.

(Aii)$\Rightarrow$(Ai). Take any sequence $\{a_n\}$ and a in L such that $\{a_n\}$ is non-increasing and $\lim_{n\to\infty} a_n(x) = a(x)$ for all x. Fix any $\varepsilon > 0$. Since, for all $x \in S$, there is an n_x with $a_n(x) \leq a(x) + \varepsilon$ for all $n \geq n_x$, (Aii) implies $\lim_{n\to\infty} I(a_n) \leq I(a + \varepsilon\mathbf{1}) = I(a) + \varepsilon I(\mathbf{1})$. Let ε go to zero, we have $\lim_{n\to\infty} I(a_n) \leq I(a)$. On the other hand, $\lim_{n\to\infty} I(a_n) \geq I(a)$ by monotonicity. Hence, $\lim_{n\to\infty} I(a_n) = I(a)$.

It is similar to prove (Bi)$\Leftrightarrow$(Bii). $\square$

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[2] A function a on X is Σ-measurable if all of its upper contour sets $(a \geq t) = \{x \mid a(x) \geq t\} \in \Sigma$.

[3] In an earlier version of this paper, I used the seemingly weaker continuity properties throughout. In a private correspondence, however, M. Marinacci showed me the equivalence.

From now on, I will use (Ai) and (Aii), or (Bi) and (Bii) interchangeably, whichever is more convenient.

Proof of Theorem 1. Suppose that I is a quasi-integral on L. Let us first construct a capacity μ on Σ. For any set $A \in \Sigma$, there is some $a \in L$ and some real number t with $A = (a \geq t)$. Consider a sequence of functions $\{a_n^A\}$, which are defined by

$$a_n^A(x) = 1 - \min(1,\ n\max(0, t - a(x))), \quad \text{for all } x \in X. \tag{2}$$

Since L is a Stone lattice, each a_n^A belongs to L. It is obvious that $\{a_n^A\}$ satisfies

$$a_n^A(x) \begin{cases} = 1, & a(x) \geq t, \text{ i.e., } x \in A, \\ \in (0,1), & t - n^{-1} < a(x) < t, \\ = 0, & a(x) \leq t - n^{-1}, \end{cases} \tag{3}$$

and that $\{a_n^A(x)\}$ is a non-increasing sequence of real numbers that converge to $\chi^A(x)$ for all $x \in X$, where χ^A is the indicator function of A. Since I is monotonic, $\{I(a_n^A)\}$ is non-increasing. Since the sequence is also bounded below by zero, it will converge to a nonnegative number. Let us denote

$$\mu(A) = \lim_{n\to\infty} I(a_n^A). \tag{4}$$

To show that μ is well-defined by (4) as a capacity on Σ, however, we have to first verify that $\mu(A)$ is independent of any representation of A. Suppose that $A = (b \geq s)$ for some other $b \in L$ and s. We then can form a sequence of functions $\{b_n^A\}$ according to (2). Fix an integer m, consider the sequence $\{b_n^A\}$ and a_m^A. Since $\{b_n^A(x)\}$ also converges to $\chi^A(x)$, it is easy to see from (2) that $\{b_n^A\}$ and a_m^A satisfy the condition in (Aii); thus

$$\lim_{n\to\infty} I(b_n^A) \leq I(a_m^A), \tag{5}$$

for I is continuous. We then take the limit on the right hand side of (5)

$$\lim_{n\to\infty} I(b_n^A) \leq \lim_{n\to\infty} I(a_n^A). \tag{6}$$

Similarly, we can show $\lim\limits_{n\to\infty} I(b_n^A) \geq \lim\limits_{n\to\infty} I(a_n^A)$. So μ is well-defined by (4).

It is easy to check that μ is indeed a capacity: $\mu(\varnothing) = 0$ and $\mu(S) = 1$ are trivial; we can also prove $\mu(A) \leq \mu(B)$ for all $A \subseteq B$ by an argument similar to the argument that derives (6). Let us now show that μ is upper-continuous.

Take any non-increasing sequence of sets $\{A_n\}$ and $A \in \Sigma$ with $\bigcap_{n=1}^{\infty} A_n \subseteq A$. By definition, there are a sequence of functions $\{a_n\}$ and $a \in L$ such that $A_n = (a_n \geq t_n)$ and $A = (a \geq t)$. Since L is a Stone lattice, we may assume, without loss of generality, that $\{a_n\}$ is a non-increasing sequence, that $a_n(x) \leq 1$ and $a(x) \leq 1$ for all x, and that $A_n = (a_n = 1)$ and $A = (a = 1)$. Take any arbitrary $\varepsilon > 0$. By the definition of μ, there is some m such that

$$|\mu(A) - I(a_m^A)| < \varepsilon, \tag{7}$$

and there is also, without loss of generality, an increasing sequence of integers $\{m_n\}$ such that

$$|\mu(A_n) - I(a_{m_n}^{A_n})| < \varepsilon, \quad \text{for all } n. \tag{8}$$

We claim that the sequence $\{a_{m_n}^{A_n}\}$ and a_m^A satisfy the condition in (Aii). First, for every $x \in X$, $\{a_{m_n}^{A_n}(x)\}$ is a non-increasing of real numbers because $\{a_n(x)\}$ is a non-increasing sequence and $\{m_n\}$ is an increasing sequence of integers. Second, if $a_m^A(x) = 1$, then $a_{m_n}^{A_n}(x) \le a(x)$ for all n; and if $a_n^A(x) < 1$, then $x \notin A$. Since $\bigcap_{n=1}^{\infty} A_n \subseteq A$, there is a large n_x with $x \notin A_n$ for all $n \ge n_x$. Fix some $k \ge n_x$. By the construction of $a_l^{A_k}$, there exists a large L such that $a_l^{A_k}(x) = 0$ for all $l \ge L$. Since $\{m_n\}$ is an increasing sequence, there is a large $\tilde{n}_x$ such that $m_n \ge L$ for all $n \ge \tilde{n}_x$. Then $a_{m_n}^{A_n}(x) \le a_L^{A_k}(x) = 0 \le a_m^A(x)$ for all $n \ge \max\{k, \tilde{n}_x\}$.

The continuity of I, applied to $\{a_{m_n}^{A_n}\}$ and a_m^A, leads to $\lim_{n\to\infty} I(a_{m_n}^{A_n}) \le I(a_m^A)$. Then, there is a large N such that

$$I(a_{m_n}^{A_n}) \le I(a_m^A) + \varepsilon, \quad \text{for all } n \ge N. \tag{9}$$

Combining (7), (8), and (9) together, we have $\lim_{n\to\infty} \mu(A_n) \le \mu(A) + 3\varepsilon$. We then let ε go to zero to obtain

$$\lim_{n\to\infty} \mu(A_n) \le \mu(A).$$

Thus, the capacity μ defined by (4) is indeed upper-continuous.

We now verify (1). As shown in [7], we only need to do it for every non-negative function $a \in L$. Take an arbitrary $\varepsilon > 0$.

Since $\mu(a \ge t)$ is a non-increasing function of t, it is Riemann integrable. Let K be an upper bound of a. Divide $[0, K]$ into m subintervals of equal length: $0 = t_0 \le t_1 \le \cdots \le t_m$. By the definition of the Riemann integral, we can choose a large m such that

$$\left| \int_0^K \mu(a \ge t)dt - \sum_{i=0}^{m-1} \mu(a \ge t_i)(t_{i+1} - t_i) \right| \le \varepsilon, \quad \text{and} \tag{10}$$
$$m^{-1}K < \varepsilon.$$

For every i from 0 to $m-1$, let $C_i = (a \ge t_i)$. By the definition of μ, there exists an integer n large enough so that:

$$|\mu(a \ge t_i) - I(a_n^{C_i})| \le K^{-1}\varepsilon, \quad \text{for all } i, \text{ and} \tag{11}$$

$$n^{-1} < m^{-1}K. \tag{12}$$

Because of (3) and (12), $a_n^{C_i}(t_{i+1} - t_i)$ and $\sum_{j=i+1}^{m-1} a_n^{C_j}(t_{j+1} - t_j)$ are comonotonic for all i; hence, we can apply comonotonic additivity (and homogeneity) m times to obtain

$$I\left(\sum_{i=0}^{m-1} a_n^{C_i}(t_{i+1} - t_i)\right) = \sum_{i=0}^{m-1} I(a_n^{C_i})(t_{i+1} - t_i).$$

Therefore, we have

$$\begin{aligned} &\left| \sum_{i=0}^{m-1} \mu(a \ge t_i)(t_{i+1} - t_i) - I\left(\sum_{i=0}^{m-1} a_n^{C_i}(t_{i+1} - t_i)\right) \right| \\ &\quad = \left| \sum_{i=0}^{m-1} \mu(a \ge t_i)(t_{i+1} - t_i) - \sum_{i=0}^{m-1} I(a_n^{C_i})(t_{i+1} - t_i) \right| \le \varepsilon, \end{aligned} \tag{13}$$

with the last inequality due to (11).

It is straightforward to verify that

$$a \le \sum_{i=0}^{m-1} a_n^{C_i}(t_{i+1} - t_i) \le a + 2\varepsilon.$$

Thus, because I is monotonic,

$$\left| I(a) - I\left(\sum_{j=i+1}^{m-1} a_n^{C_j}(t_{j+1} - t_j) \right) \right| \leq 2\varepsilon. \tag{14}$$

Finally, we combine (10), (13), and (14) to obtain

$$\left| \int_0^K \mu(a \geq t)dt - I(a) \right| \leq 4\varepsilon.$$

Given that ε is arbitrary, it must be true that $I(a) = \int_0^K \mu(a \geq t)dt$. The uniqueness of μ is obvious from the above proof.

On the other hand, suppose that a functional I is defined by (1) for an upper-continuous capacity μ. It is clear that I is monotonic. It is also known that I is comonotonically additive ([3], [7]). So we only have to prove that I is continuous. Suppose that there exist a sequence $\{a_n\}$ and a such that, for every $x \in X$, $\{a_n(x)\}$ is a non-increasing sequence of real numbers, and there exists an n_x with $a_n(x) \leq a(x)$ for all $n \geq n_x$. By assumption, we have $\bigcap_{n=1}^{\infty} (a_n \geq t) \subseteq (a \geq t)$ for all real numbers t. Because μ is upper-continuous, we have, for the integrand in (1),

$$\lim_{n\to\infty} \mu(a_n \geq t) \leq \mu(a \geq t), \quad \text{for all } t.$$

Then we can apply the monotone convergence theorem to conclude $\lim_{n\to\infty} I(a_n) \leq I(a)$. This completes the proof of Theorem 1. □

Remark 3. Theorem 1 is closely related to an earlier result of Greco [5][4]. In a comparable setting, Greco proved that a functional I on a Stone lattice L is comonotonically additive and monotonic if and only if it has an integral representation by a capacity μ. But her result does not address continuity properties of I and μ, nor the unicity of μ. Both issues will be important for our discussion in the next two sections.

Remark 4. Theorem 1 can also be viewed as an extension of the classical Stone-Daniell theorem. The S-D theorem deals with linear functionals on a Stone lattice and it establishes an integral representation for a linear functional by a countably additive measure. While linearity in the S-D theorem implies comonotonic additivity in Theorem 1, the monotonicity and continuity conditions in both are the same when applied to linear functionals. Together with the following lemma, Theorem 1 provides an alternative proof of the S-D theorem.

Lemma 2. *If a functional I in Theorem 1 is linear, then the derived capacity μ can be extended to a countably additive measure on $\sigma(\Sigma)$, the σ-algebra generated by Σ.*

Proof. I will provide just an outline here. First consider $\alpha(\Sigma)$, the algebra generated by Σ. Since Σ is a lattice in the sense that $A \cup B$, $A \cap B \in \Sigma$ for all $A, B \in \Sigma$,

[4]I was unaware of Greco's work when I began to work on this paper. Although Greco's original work was written in Italian, an exposition of her major contributions in English is now included in Chapter 13 of "Non-Additive Measure and Integral" by Denneberg [3], a book I was referred to by M. Marinacci and S. Grant.

$\alpha(\Sigma)$ consists of all sets of the form $\bigcup_{i=1}^{k}(A_i \backslash B_i)$, where $A_i, B_i \in \Sigma$, $A_i \supseteq B_i$, and $A_i \backslash B_i$ are disjoint. We can extend μ to $\alpha(\Sigma)$ by

$$\mu\left(\bigcup_{i=1}^{k}(A_i \backslash B_i)\right) = \sum_{i=1}^{k} \mu(A_i \backslash B_i).$$

Given that I is linear, we can show that μ is well-defined on $\alpha(\Sigma)$, using an argument similar to the one used in the above proof showing that μ is well-defined on Σ. Then, by definition, μ is finitely additive. Moreover, we can show that μ vanishes at the empty set, using an argument similar to the one used in the proof showing that μ is upper-continuous. Hence, μ is countably additive on $\alpha(\Sigma)$; therefore, it can be extended to a countably additive measure on $\sigma(\Sigma)$. □

3. Upper-continuous capacities on compact spaces

Throughout this section we assume that (X, τ) is a compact space. Let $C(X)$ denote the space of all continuous functions on X, and $\Sigma(X, \tau)$, or simply Σ, the collection of all upper contour sets of all functions in $C(X)$. Finally, let $M(X)$ be the space of all upper-continuous capacities on Σ. We use Theorem 1 to study the topological structure of $M(X)$.

Remark 5. Every set in $\Sigma(X, \tau)$ is obviously closed, but not every closed set belongs to $\Sigma(X, \tau)$. The difference being similar to that between a Baire set and a Borel set, we call a set in $\Sigma(X, \tau)$ a Baire closed set. When (X, τ) is a metric space, however, each closed set is also a Baire closed set because $F = \{x \in X \mid -\operatorname{dist}(x, F) \geq 0\}$ for every closed set F.

Since all assumptions of Theorem 1 are satisfied by $C(X)$ and Σ, we can identify each $\mu \in M(X)$ with a functional I on $C(X)$ that is comonotonically additive, monotonic, and continuous. Given that X is compact, this identification can be further simplified.

Lemma 3. *Assume that X is a compact topological space. If a functional I on $C(X)$ is comonotonically additive and monotonic, then it is also continuous.*

Proof. Take any non-increasing sequence $\{a_n\}$ in $C(X)$ with $\lim_{n\to\infty} a_n = a$ for some $a \in C(X)$. Since I is monotonic, $\lim_{n\to\infty} I(a_n) \geq I(a)$. On the other hand, fix any arbitrary $\varepsilon > 0$. Consider $b = a + \varepsilon\mathbf{1}$. Then, $\lim_{n\to\infty} a_n(x) = a(x) < b(x)$ for every $x \in X$. Since X is compact and $\{a_n\}$ is non-increasing sequence of continuous functions, by Dini's theorem, there is an integer N, such that

$$a_n(x) < b(x), \quad \text{for all } x \in X \text{ and all } n \geq N.$$

Then, since I is comonotonically additive and monotonic (hence homogeneous as well),

$$I(a_n) \leq I(b) = I(a + \varepsilon\mathbf{1}) = I(a) + \varepsilon I(\mathbf{1}), \quad \text{for all } n \geq N. \tag{15}$$

Taking limits twice (first n, and second ε) in (15), we obtain $\lim_{n\to\infty} I(a_n) \leq I(a)$. Therefore, $\lim_{n\to\infty} I(a_n) = I(a)$. This proves that I is continuous. □

We are ready to study the topological structure of $M(X)$. We say that a sequence $\{\mu_n\}$ weakly converges to μ if

$$\int_X a\, d\mu \to \int_X a\, d\mu, \quad \text{for all } a \in C(X).$$

It is clear that this topology on $M(X)$ subsumes the weak topology on $\Delta(X)$, the space of all countably additive measures. I now prove

Theorem 2. *Assume that X is compact. Then $M(X)$, endowed with the weak topology, is compact and Hausdorff.*

Proof. Given Theorem 1 and Lemma 3, we can identify $M(X)$ with Φ, the set of all functionals on $C(X)$ that are comonotonically additive and monotonic. To prove Theorem 2, it suffices to show that Φ is compact and Hausdorff for the corresponding topology. For each $a \in C(X)$, consider the interval $i_a = [-\|a\|_\infty, \|a\|_\infty]$. Since each i_a is compact, the product space $\prod_{a\in C(X)} i_a$ is compact in the product topology κ by the Tychonoff Theorem. It is obvious that the topology induced by κ on Φ is exactly the weak topology. Hence we only have to show that Φ is closed for κ. Suppose that I is a limit point of Φ in κ. For any pair of $a, b \in C(X)$ that are comonotonic, consider an open set U around I defined by

$$\left\{ J \in \prod_{a\in C(X)} i_a \mid |J(a) - I(a)| < \varepsilon,\ |J(b) - I(b)| < \varepsilon,\ |J(a+b) - I(a+b)| < \varepsilon \right\}.$$

Since I is a limit point of Φ, there is a $J \in \Phi \cap U$. Because $J(a+b) = J(a) + J(b)$, we have

$$|I(a+b) - I(a) - I(b)| < 3\varepsilon.$$

But the ε is arbitrary, $I(a+b) = I(a) + I(b)$. This shows that I is comonotonically additive. We can also show that I is monotonic. Hence, $I \in \Phi$. This means that Φ is closed. □

When X is a compact metric space, the weak topology on $M(X)$ is also metrizable.

Theorem 3. *Assume that (X, d) is a compact metric space. Then $M(X)$ is also a compact metric space, i.e., there is a metric e on $M(X)$ that induces the weak topology.*

Proof. When (X, d) is a compact metric space, $C(X)$ is a separable Banach space under the supreme norm. Find a countable set D that is dense in $C(X)$. It is clear that for any sequence $\{\mu_n\}$ and μ in $M(X)$,

$$\int_X a\, d\mu_n \to \int_X a\, d\mu, \quad \text{for all } a \in C(X),$$

if and only if

$$\int_X a\, d\mu_n \to \int_X a\, d\mu, \quad \text{for all } a \in D.$$

Hence, the following metric e induces the weak topology on $M(X)$:

$$e(\mu, \mu') = \sum_{a_n \in D} 2^{-n} \|a_n\|_\infty^{-1} \left| \int_X a_n\, d\mu - \int_X a_n\, d\mu' \right|. \quad \square$$

Remark 6. Theorems 2 and 3 indicate that when X is a compact space, in particular, a compact metric space, the topological structure of $M(X)$ is similar to that of $\Delta(X)$. This is again a reflection of the parallel between countable additivity for additive measures and upper-continuity for capacities.

Remark 7. Some earlier authors have considered spaces of capacities with other regularity conditions. Since there are often differences in basic settings, it is not easy to make general comparisons. When (X, d) is a compact metric space, an upper-continuous capacity on Σ is, however, virtually equivalent to a regular capacity on $\sigma(X)$, the Borel (Baire) algebra of X, of a standard version. Following [6], a capacity μ^* on $\sigma(X)$ is regular if

R.1. $\mu^*(A) = \sup\{\mu^*(K) \mid K \subseteq A \text{ closed}\}$ for all measurable sets A;

R.2. $\mu^*(K) = \inf\{\mu^*(O) \mid O \supseteq K \text{ open}\}$ for all closed sets K.

Lemma 4. (i) *The restriction of any regular capacity μ^* on Σ is upper-continuous*; (ii) *Any upper-continuous capacity μ can be extended* (*uniquely*) *to a regular capacity on $\sigma(X)$.*

Proof. The proof of (i) follows immediately from R.1. For any non-increasing sequence of closed sets $\{A_n\}$ and any open set O that contains $\bigcap_{n=1}^{\infty} A_n$, it must be true that A_n will be contained in O for all large n. Hence, $\lim_{n\to\infty} \mu^*(A_n) \le \mu^*(O)$. Then R.2 implies

$$\lim_{n\to\infty} \mu^*(A_n) = \mu^*\left(\bigcap_{n=1}^{\infty} A_n\right).$$

The proof of (ii) is also direct. For any upper-continuous capacity μ on Σ, we can extend it to a capacity μ^* on $\sigma(X)$ by the formula in R.1.Then, of course, μ^* satisfies R.1. Next we verify that μ^* satisfies R.2. For any closed set K, consider a sequence of open sets $\{O_n\}$ defined by $O_n = \{x \in X \mid d(x, K) < 1/n\}$. Obviously, $\mu^*(O_n) \le \mu^*(K_n)$ where $K_n = \{x \in X \mid d(x, K) \le 1/n\}$. But $\lim_{n\to\infty} \mu^*(K_n) = \mu^*(K)$ since μ^* agrees with μ on closed sets and μ is upper-continuous on Σ. Therefore, $\lim_{n\to\infty} \mu^*(O_n) = \mu^*(K)$. Hence, μ^* satisfies R.2. The uniqueness of μ^* is due to R.1. □

Because of Lemma 4, upper-continuous capacities μ on Σ can be identified with regular capacities μ^* on $\sigma(X)$ for a compact metric space (X, d). Our results, specialized to this case, have also been proved in a recent article by Epstein and Wang [4]. They approach the problem using the so-called vague topology that, as they show, coincides with the weak topology on $M(X)$ for a compact metric space (X, d).

4. A Kolmogorov-type theorem for upper-continuous capacities

In this section we establish a result similar to Kolmogorov's theorem concerning the existence of a stochastic process with given consistent finite marginals. Suppose that there are infinitely many compact Hausdorff spaces $(X_\lambda, \tau_\lambda)$ $(\lambda \in \Lambda)$. For each finite subset B of Λ, there is an upper-continuous capacity μ_B on $\prod_{\lambda\in B} X_\lambda$. When does there exist an upper-continuous capacity μ on $\prod_{\lambda\in\Lambda} X_\lambda$ such that μ_B is the marginal capacity of μ for every finite subset B of Λ? Obviously, it is necessary that all μ_B's are consistent, i.e., the marginal of μ_B on $\prod_{\lambda\in A} X_\lambda$ is μ_A for any pair of A, B with $A \subseteq B$. I now show that this condition is also sufficient.

Theorem 4. *Suppose that there are infinitely many compact Hausdorff spaces $(X_\lambda, \tau_\lambda)$ $(\lambda \in \Lambda)$ and that the product space $X = \prod_{\lambda\in\Lambda} X_\lambda$ is endowed with the product topology τ. For every finite subsets B of Λ, there is an upper-continuous capacity μ_B on $\prod_{\lambda\in B} X_\lambda$. If μ_B's are consistent, then there exists an upper-continuous capacity μ on $\prod_{\lambda\in\Lambda} X_\lambda$ such that μ_B is the marginal capacity of μ for every finite subset B.*

Proof. Given Theorem 1 and Lemma 3, we only have to prove that there is a functional I on $C(X)$ that is comonotonically additive, monotonic, and induces on $C(\prod_{\lambda\in B} X_\lambda)$ the same functional μ_B induces for every finite subset B of Λ.

First, let us define the functional I. Take any $a \in C(X)$, and fix an arbitrary $x^* \in X$. Since each $(X_\lambda, \tau_\lambda)$ is compact and Hausdorff, so is the product space $X = \prod_{\lambda\in\Lambda} X_\lambda$. Let D be the subspace of $C(X)$ that contains all functions which depend on only finitely many x_λ. By the Weierstrass theorem, D is dense in $C(X)$. Hence, for any integer n, there is a finite subset B_n of Λ such that there is a continuous function b that depends on only x_λ for $\lambda \in B_n$ and

$$|a(x) - b(x)| \le \frac{1}{2n}, \quad \text{for all } x \in X.$$

Now for any finite subset A of Λ that contains B_n, we have

$$\begin{aligned} |a(x) - a(x_A, x^*_{-A})| &\le |a(x) - b(x)| + |b(x) - a(x_A, x^*_{-A})| \\ &\le \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}, \quad \text{for all } x \in X. \end{aligned}$$

Once we have done this for every n, we obtain a sequence of finite subsets $\{B_n\}$ of Λ such that for any finite subset A of Λ that contains B_n,

$$|a(x) - a(x_A, x^*_{-A})| \le \frac{1}{n}, \quad \text{for all } x \in X. \tag{16}$$

We can now use this sequence to define $I(a)$ as follows:

$$I(a) = \lim_{n\to\infty} \int_X a(x_{B_n}, x^*_{-B_n}) d\mu_{B_n}. \tag{17}$$

To show that I is well-defined by (17), however, we need to verify that $I(a)$ does not depend on any particularly chosen sequence $\{B_n\}$. Suppose that $\{A_n\}$ is another sequence that satisfies $B_n \subseteq A_n$ for all n. Apply (16) to both B_n and A_n,

$$\begin{aligned} |a(x_{B_n}, x^*_{B_n}) - a(x)| &\le \frac{1}{n}, \quad \text{and} \\ |a(x_{A_n}, x^*_{-A_n}) - a(x)| &\le \frac{1}{n}, \quad \text{for all } x \in X. \end{aligned}$$

Hence,

$$\begin{cases} a(x_{B_n}, x^*_{-B_n}) \le a(x_{A_n}, x^*_{-A_n}) + \frac{2}{n}, \\ a(x_{A_n}, x^*_{-A_n}) \le a(x_{B_n}, x^*_{-B_n}) + \frac{2}{n}. \end{cases} \tag{18}$$

Given that μ_{B_n} is the marginal of μ_{A_n}, and that μ_{A_n} defines a comonotonically additive and monotonic functional, (18) leads to

$$\left| \int_X a(x_{B_n}, x^*_{-B_n}) d\mu_{B_n} - \int_X a(x_{A_n}, x^*_{-A_n}) d\mu_{A_n} \right| \le \frac{2}{n}.$$

Therefore,

$$\lim_{n\to\infty}\int_X a(x_{B_n}, x^*_{-B_n})d\mu_{B_n} = \lim_{n\to\infty}\int_X a(x_{A_n}, x^*_{-A_n})d\mu_{A_n}. \tag{19}$$

For a general sequence $\{C_n\}$ that satisfies (16) but not necessarily $B_n \subseteq C_n$ for all n, we now may apply (19) twice to obtain

$$\begin{aligned}\lim_{n\to\infty}\int_X a(x_{B_n}, x^*_{-B_n})d\mu_{B_n} &= \lim_{n\to\infty}\int_X a(x_{B_n\cup C_n}, x^*_{-B_n\cup C_n})d\mu_{B_n\cup C_n}\\ &= \lim_{n\to\infty}\int_X a(x_{C_n}, x^*_{-C_n})d\mu_{C_n}.\end{aligned}$$

Thus, I is well-defined by (17) for any sequence $\{B_n\}$ that satisfies (16).

Next we prove that I is comonotonically additive and monotonic on $C(X)$. Suppose that two functions $a, b \in C(X)$ are comonotonic so that $(a(x) - a(x'))(b(x) - b(x')) \geq 0$ for all x, $x' \in X$. Let $\{B_n\}$ and $\{A_n\}$ be two sequences that define $I(a)$ and $I(b)$. According to (19), we may replace both by $\{B_n \cup A_n\}$ so we can assume $B_n = A_n$. Moreover, we also assume without loss of generality that the same sequence defines $I(a+b)$. Since $a(x)$ and $b(x)$ are comonotonic in $x, a(x_{B_n}, x^*_{-B_n})$ and $b(x_{B_n}, x^*_{-B_n})$ are comonotonic in x_{B_n}. Therefore,

$$\begin{aligned}I(a+b) &= \lim_{n\to\infty}\int_X (a(x_{B_n}, x^*_{-B_n}) + b(x_{B_n}, x^*_{-B_n}))d\mu_{B_n}\\ &= \lim_{n\to\infty}\int_X a(x_{B_n}, x^*_{-B_n})d\mu_{B_n} + \lim_{n\to\infty}\int_X b(x_{B_n}, x^*_{-B_n}))d\mu_{B_n}\\ &= I(a) + I(b).\end{aligned}$$

This shows that I is comonotonically additive. In a similar fashion, we can prove that I is monotonic. □

Remark 8. If there are only countably many X_λ and every X_λ is a compact metric space, the product space $X = \prod_{\lambda\in\Lambda} X_\lambda$ is also a compact metric space. Hence, according to Lemma 4, upper-continuity in Theorem 4 can be replaced by regularity. This particular result has also been proved by Epstein and Wang [4].

Remark 9. It is the requirement that capacities be upper-continuous that makes Theorem 4 interesting. When this requirement is dropped, the result is trivial. This is similar to the case of the original Kolmogorov theorem, which would have also been trivial had countable additivity of measures been weakened to finite additivity.

5. Conclusion

In this paper we have studied upper-continuous capacities from the duality viewpoint by way of the main representation theorem proved in Section 2. The power of this approach is illustrated by results proved in Sections 3 and 4. It is interesting to know whether results in Sections 3 and 4 can also be proved directly without any duality argument. It is even more intriguing to know when these results can be extended in the case of non-compact spaces.

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