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ON THE HYPERBOLICITY OF SMALL CANCELLATION GROUPS AND ONE-RELATOR GROUPS

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Abstract. In the article, a result relating to maps (= finite planar connected and simply connected 2-complexes) that satisfy a $C(p)\&T(q)$ condition (where (p,q) is one of $(3,6)$, $(4,4)$, $(6,3)$ which correspond to regular tessellations of the plane by triangles, squares, hexagons, respectively) is proven. On the base of this result a criterion for the Gromov hyperbolicity of finitely presented small cancellation groups satisfying non-metric $C(p)\&T(q)$-conditions is obtained and a complete (and explicit) description of hyperbolic groups in some classes of one-relator groups is given: All one-relator hyperbolic groups with > 0 and $\leq$ 3 occurrences of a letter are indicated; it is shown that a finitely generated one-relator group G whose reduced relator R is of the form $R \equiv aT_0aT_1 \dots aT_{n-1}$, where the words T_i are distinct and have no occurrences of the letter $a^{\pm 1}$, is not hyperbolic if and only if one has in the free group that (1) $n = 2$ and $T_0T_1^{-1}$ is a proper power; (2) $n = 3$ and for some i it is true (with subscripts mod 3) that $T_iT_{i+1}^{-1}T_iT_{i+2}^{-1} = 1$; (3) $n = 4$ and for some i it is true (with subscripts mod 4) that $T_iT_{i+1}^{-1}T_{i+2}T_{i+3}^{-1} = 1$.

0. Introduction

Let G be a group given by generators and defining relators:

$$G = \langle\, \mathcal{A} \parallel \mathcal{R} \,\rangle, \tag{0.1}$$

where $\mathcal{A}$ is the set of generators (alphabet) and $\mathcal{R}$ is the set of relators of G. The group G is called *finitely presented* if both $\mathcal{A}$ and $\mathcal{R}$ are finite.

Let $F(\mathcal{A})$ be the free group over the alphabet $\mathcal{A}$ and $\psi : F(\mathcal{A}) \to G$ the natural epimorphism. We will say that a word W equals 1 in G provided $\psi(W) = 1$ in G.

By the definition, a word W equals 1 in G ($W \overset{G}{=} 1$) if in the free group $F(\mathcal{A})$ we have

$$W = S_1 R_{i_1}^{\varepsilon_1} S_1^{-1} S_2 R_{i_2}^{\varepsilon_2} S_2^{-1} \dots S_d R_{i_d}^{\varepsilon_d} S_d^{-1}, \tag{0.2}$$

where $S_j \in F(\mathcal{A})$, $R_{i_j} \in \mathcal{R}$, and $\varepsilon_j = \pm 1$, $j = 1, \dots, d$.

The group G given by presentation (0.1) is said to satisfy a *linear isoperimetric inequality* if there is a constant $L \geq 0$ such that for every word W with $W \overset{G}{=} 1$ the minimal $d = d(W)$ in (0.2) does not exceed $L|W|$, where $|W|$ is the length of W.

Hyperbolic (or negatively curved) groups, introduced by Gromov [Gr], admit several equivalent definitions; see [Gr], [GH], [GS1] for further discussion. It is,

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perhaps, most natural to define these groups in terms of metric properties of their Cayley graphs, but the following definition is most convenient in the present article: A finitely presented group G is called *hyperbolic* provided G satisfies a linear isoperimetric inequality, that is, $d(W) \le L|W|$ for every $W \overset{G}{=} 1$. It is easy to see that the existence of such an L does not depend on the presentation of G, although its value does.

Among examples of hyperbolic groups are free groups (note $L = 0$ if and only if $\mathcal{R}$ is empty) and small cancellation groups satisfying a metric small cancellation condition $C'(\lambda)$ with $0 < \lambda \le 1/6$. In particular, the fundamental group of an oriented surface of genus $g > 1$ is hyperbolic. Every discontinuous cocompact group of isometries of a simply connected manifold of negative curvature is hyperbolic [Gr]. Clearly, finite groups are also hyperbolic. On the other hand, the intersection of the class of hyperbolic groups with the class of solvable (nilpotent) groups contains only cyclic-by-finite groups, for a free abelian group of rank 2 cannot be a subgroup of a hyperbolic group [Gr]. For the same reason a non-metric small cancellation condition $C(p)\&T(q)$, where p, q are positive integers with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, that is, (p, q) is one of (3,6), (4,4), (6,3)) does not guarantee the hyperbolicity of a finitely presented group with this condition (all definitions can be found in Sect. 1; see also [LS]). For example, a free abelian group A_2 of rank 2 itself admits a presentation $\langle\, a, b \parallel aba^{-1}b^{-1} \,\rangle$ that satisfies the $C(4)\&T(4)$-condition.

As was proved by Gersten and Short [GS2], any finitely presented group G satisfying the $C(p)\&T(q)$-condition, where p, q are as above, is automatic (see [ECHLPT] for an account of automatic groups). On the other hand, as was proven by the same authors [GS1], if one strengthens the standard $C(p)\&T(q)$-condition and considers the $C(p_1)\&T(q_1)$-condition, where p_1, q_1 are positive integers with $\frac{1}{p_1} + \frac{1}{q_1} < \frac{1}{2}$, then the corresponding finitely presented $C(p_1)\&T(q_1)$-groups *are* hyperbolic.

In this article, we will study the problem on hyperbolicity of finitely presented groups that satisfy the standard small cancellation $C(p)\&T(q)$-condition more closely. The result obtained in this direction is a criterion for the hyperbolicity of finitely presented $C(p)\&T(q)$-groups; see Theorem 2, the corollary in Sect. 2, and the dicsussion following the corollary.

The proof of Theorem 2 depends on Theorem 1 about (p, q)-maps, which is proved in Sect. 1 and was originally designed to help describe hyperbolic groups in some classes of one-relator groups. This description is given in Theorems 3 and 4, whose proofs occupy Sects. 4–5. It is worth mentioning that although it is more convenient (for briefness) to describe non-hyperbolic groups in Theorems 3–4, the most non-trivial and interesting part of proofs is to show that in all other cases the groups *are* hyperbolic.

We also note that Part (2) of Theorem 3 follows from results due to Bestvina, Feighn [BF], Kharlampovich, Myasnikov [KM], and Mikhajlovskii, Ol'shanskii [MO].

Theorem 3. *Let $G = \langle\, \mathcal{A} \parallel R \,\rangle$ be a one-relator group, R a cyclically reduced word over a finite alphabet $\mathcal{A}$. Suppose $a \in \mathcal{A}$ and the number of occurrences of a and a^{-1} in R, $|R|_a$, satisfies the inequalities $1 \le |R|_a \le 3$. Then G is not hyperbolic if and only if the relator R has one of the following forms up to cyclic permutations and taking inverses:*

(1) $|R|_a = 2$, $R = aBaC$ *and* BC^{-1} *is a proper power in the free group* $F(\mathcal{A}) = F$ *over* $\mathcal{A}$ *(i.e.* $BC^{-1} \overset{F}{=} X^\ell$, *with* $X \neq 1$ *and* $\ell > 0$*).*

(2) $|R|_a = 2$, $R = aBa^{-1}C$ *and either* (2*a*) *or* (2*b*) *is true:*

(2a) *B, C are conjugate in $F(\mathcal{A})$ to powers of a word X (i.e.* $B \overset{F}{=} S_1 X^{\ell_1} S_1^{-1}$, $C \overset{F}{=} S_2 X^{\ell_2} S_2^{-1}$ *with some integers* ℓ_1, ℓ_2 *and* $S_1, S_2 \in F$*).*

(2b) *Both B and C are proper powers in $F(\mathcal{A})$.*

(3) $|R|_a = 3$, $R = aBaCaD$, *the subgroup* $\langle CB^{-1},\ DB^{-1}\rangle$ *of* $F(\mathcal{A})$ *is cyclic, and if* $CB^{-1} = Z^{n_1}$, $DB^{-1} = Z^{n_2}$, *where Z is not a proper power, then one of* (3*a*)–(3*d*) *holds:*

(3a) $\min(|n_1|, |n_2|) = 0$ *and* $\max(|n_1|, |n_2|) > 1$.

(3b) $\min(|n_1|, |n_2|) > 0$ *and* $|n_1| = |n_2| \neq 1$.

(3c) $\min(|n_1|, |n_2|) > 0$ *and* $n_1 = -n_2$.

(3d) $\min(|n_1|, |n_2|) > 0$ *and* $n_1 = 2n_2$ *(or* $n_2 = 2n_1$*).*

(4) $|R|_a = 3$, $R = aBaCa^{-1}D$, *the subgroup* $\langle B^{-1}CB,\ D\rangle$ *of* $F(\mathcal{A})$ *is cyclic, and if* $B^{-1}CB = Z^{n_1}$, $D = Z^{n_2}$, *where Z is not a proper power, then either* (4*a*) *or* (4*b*) *holds:*

(4a) $|n_1| = |n_2|$.

(4b) $n_1 = -2n_2$ *(or* $n_2 = -2n_1$*).*

Theorem 4. *Let* $G = \langle\ \mathcal{A} \parallel R\ \rangle$ *be a one-relator group,* $a \in \mathcal{A}$, *and R be a cyclically reduced word over a finite alphabet $\mathcal{A}$ of the form*

$$R = aT_0aT_1 \dots aT_{n-1},$$

where the words T_i have no occurrences of $a^{\pm 1}$ *and* $T_i \neq T_j$ *for* $i \neq j$. *Then G is not hyperbolic if and only if R is of one of the following forms:*

(1) $n = 2$ *and* $T_0^{-1}T_1$ *is a proper power in the free group* $F(\mathcal{A})$ *over* $\mathcal{A}$.

(2) $n = 3$ *and for some* $i \in \{0, 1, 2\}$ *one has (with subscripts* $\bmod 3$*) that in* $F(\mathcal{A})$

$$T_i T_{i+1}^{-1} T_i T_{i+2}^{-1} = 1\ .$$

(3) $n = 4$ *and for some* $i \in \{0, 1, 2, 3\}$ *one has (with subscripts* $\bmod 4$*) that in* $F(\mathcal{A})$

$$T_i T_{i+1}^{-1} T_{i+2} T_{i+3}^{-1} = 1\ .$$

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1. (p, q)-Maps

By a *map* M we mean, as in [LS], [Ol], a finite planar connected and simply connected simplicial 2-complex. 0-, 1-, 2-cells of M are called *vertices, edges, faces* of M, respectively. The sets of all 0-, 1-, 2-cells of M are denoted by $M(0)$, $M(1)$, $M(2)$, respectively. The *degree* $d(v)$ of a vertex $v \in M(0)$ is the number of edges of M incident with v (loops are counted twice). The *degree* $d(\pi)$ of a face $\pi \in M(2)$ is the number of vertices of $v \in \partial\pi$ with $d(v) \geq 3$ (counting with their multiplicity). Clearly, $d(\pi) \leq |\partial\pi|$, where $|\partial\pi|$ is the length of the boundary $\partial\pi$ of π (i.e. the number of edges on $\partial\pi$). A vertex $v \in M(0)$ is called *interior* if $v \notin \partial M$, otherwise, v is *exterior*. A face $\pi \in M(2)$ is called *interior* provided $\partial\pi$ and ∂M have no (non-oriented) edges in common. Otherwise, π is an *exterior* face. If $s = e_1 \dots e_k$

is a path in M, where $e_1, \dots, e_k$ are edges, then s_- and s_+ denote the initial and terminal vertices of s, respectively.

Let (p,q) be one of $(3,6)$, $(4,4)$, $(6,3)$. M is referred to as a (p,q)-*map* provided the following conditions are satisfied:

(M1) $d(\pi) \ge p$ for every interior face $\pi \in M(2)$.
(M2) $d(v) \ge q$ for every interior vertex $v \in M(0)$ with $d(v) > 2$, and no interior vertex v of degree 1 exists in M.

A *submap* M' of a map M is a subcomplex of M that itself is a map. Clearly, any submap of a (p,q)-map is also a (p,q)-map.

A map M is called a *regular* (p,q)-map provided

(N1) $d(\pi) = p$ for every interior face $\pi \in M(2)$.
(N2) $d(v) = q$ for every interior vertex $v \in M(0)$ with $d(v) > 2$, and no interior vertex v of degree 1 exists in M.

It is easy to see that a regular (p,q)-map looks like a "piece" of a regular tessellation of the plane by triangles (when $(p,q) = (3,6)$), squares (when $(p,q) = (4,4)$), or hexagons (when $(p,q) = (6,3)$) provided interior vertices of degree 2 are disregarded

Let M be a regular (p,q)-map. The *radius* $\mathrm{rd}(M)$ of regular M is defined to be the maximum of distances $\mathrm{dist}(v, \partial M)$ over all $v \in M(0)$.

Our key technical result (that will be the base of arguments in Sects. 3–4) is the following

Theorem 1. *Let M be a (p,q)-map, and let the radius $\mathrm{rd}(M')$ of any regular (p,q)-submap of M be bounded by a constant K. Then there is a constant $L = L(p,q,K)$ (e.g. $L = 100q^K p$) such that the number of faces $|M(2)|$ in M satisfies*

$$|M(2)| \le L|\partial M|. \tag{1.1}$$

Proof. First let us show that we may assume M to have the following properties:

(P1) No proper subpath of ∂M bounds a submap of M.
(P2) There are no vertices of degree ≤ 2 in M.

If a path such as in (P1) existed, the map M would be the union of its two submaps, M_1 and M_2, with a single identified vertex. By apparent induction on $|\partial M|$,

$$|M_1(2)| \le L|\partial M_1|, \qquad |M_2(2)| \le L|\partial M_2|,$$

whence $|M(2)| \le L|\partial M|$ as well.

Next, turning all vertices of degree 2 of M into interior points of edges (and removing all exterior vertices of degree 1 along with the edges incident to them) produces a new (p,q)-map M' with $|\partial M'| \le |\partial M|$, $|M'(2)| = |M(2)|$ that has property (P2). Clearly, Theorem 1 will hold for M if it holds for M', and so properties (P1)–(P2) are proven.

Now we obtain another version of formulas (3.1)–(3.4) of [LS], Ch. V (see also (8)–(10) in Sect. 2.4 of the Appendix to [GH]) for (p,q)-maps. To simplify notation put

$$V = |M(0)|, \qquad E = |M(1)|, \qquad F = |M(2)|.$$

In view of (P2),

$$2E = \sum_{v \in M(0)} d(v), \qquad 2E = \sum_{\pi \in M(2)} d(\pi) + |\partial M|. \tag{1.2}$$

Applying Euler's formula to M, we also have

$$V - E + F = 1. \tag{1.3}$$

Consider a linear combination of the equalities in (1.2) and (1.3) with coefficients $\frac{p}{q}, 1, p$. Making use of the equality $2(\frac{p}{q}+1) = p$, we have

$$p(V+F) = \frac{p}{q}\sum_{v \in M(0)} d(v) + \sum_{\pi \in M(2)} d(\pi) + |\partial M| + p,$$

whence

$$\begin{aligned} p &= pV - \frac{p}{q}\sum_{v \in M(0)} d(v) + pF - \sum_{\pi \in M(2)} d(\pi) - |\partial M| \\ &= \frac{p}{q}\sum_{v \in M(0)} (q - d(v)) + \sum_{\pi \in M(2)} (p - d(\pi)) - |\partial M|. \end{aligned} \tag{1.4}$$

Let $M_E(0),\ M_I(0)$ $(M_E(2),\ M_I(2))$ denote the sets of exterior, interior vertices (faces) of M, respectively. Notice that, by (P1),

$$\frac{p}{q}\sum_{v \in M_E(0)} (q - d(v)) = \frac{p}{q}\sum_{v \in M_E(0)} (3 - d(v)) + \tfrac{p}{q}(q-3)|\partial M|. \tag{1.5}$$

In view of (1.5), we may rewrite (1.4) as follows:

$$\begin{aligned} p &= \frac{p}{q}\sum_{v \in M_E(0)} (q - d(v)) + \frac{p}{q}\sum_{v \in M_I(0)} (q - d(v)) \\ &\quad + \sum_{\pi \in M_E(2)} (p - d(\pi)) + \sum_{\pi \in M_I(2)} (p - d(\pi)) - |\partial M| \\ &= \frac{p}{q}\sum_{v \in M_E(0)} (3 - d(v)) + \frac{p}{q}\sum_{v \in M_I(0)} (q - d(v)) + \sum_{\pi \in M_E(2)} (p - d(\pi)) \\ &\quad + \sum_{\pi \in M_I(2)} (p - d(\pi)) + (\tfrac{p}{q}(q-3) - 1)|\partial M|. \end{aligned} \tag{1.6}$$

Note that $(\frac{p}{q}(q-3)-1)|\partial M| \le \frac{1}{2}|\partial M|$ when (p,q) is one of (3,6), (4,4), (6,3). Consequently, we finally have from (1.6) that

$$\begin{aligned} \tfrac{1}{2}|\partial M| &\ge \frac{p}{q}\sum_{v \in M_E(0)} (d(v) - 3) \\ &\quad + \frac{p}{q}\sum_{v \in M_I(0)} (d(v) - q) + \sum_{\pi \in M_E(2)} (d(\pi) - p) + \sum_{\pi \in M_I(2)} (d(\pi) - p). \end{aligned} \tag{1.7}$$

An interior face π will be called *regular* provided $d(\pi) = p$ and $d(v) = q$ for every vertex $v \in \partial\pi$. Otherwise, an interior face π is *irregular*.

Let $F_E,\ F_I,\ F_R,\ F_N$ denote the numbers of exterior, interior, regular, irregular faces, respectively, and let $M_R(2),\ M_N(2)$ be the sets of regular, irregular faces in M, respectively. Clearly, $F = F_E + F_I, \quad F_I = F_R + F_N$.

Now we are going to make use of the main assumption $\mathrm{rd}M' \le K$ of Theorem 1 to get an estimate for F_R (note this assumption is used only when proving Lemma 1.1).

Lemma 1.1. *In the above notation,*

$$F_R \le q^K \left(\sum_{\pi \in M_E(2)} d(\pi) + \sum_{\pi \in M_N(2)} d(\pi) + |\partial M| \right). \tag{1.8}$$

Proof. Consider a regular face $\pi \in M(2)$. Let t be a shortest path in M of the form $t = v - v'$, where $v = t_- \in \partial\pi$ and the vertex $v' = t_+$ is one of the following:

(1) $v' \in \partial M$;
(2) $v' \in \partial\pi'$ and π' is an exterior face in M;
(3) $v' \in \partial\pi'$ and π' is an irregular face in M.

It follows from the assumption of Theorem 1 that the length $|t|$ is not greater than K. Also, observe that we have $d(o) = q$ for every vertex $o \in t$ different from v', since every edge of t belongs to a regular face of M, by the choice of t.

Consider a function $f : \pi \to v'$, where π is a regular face and v' is a vertex of one of the forms (1)–(3) that arises from the above definition of t. The number of regular π's that are mapped by f into a vertex v' does not exceed q^K, because corresponding paths t^{-1} begin at v', go through vertices of degree q and are of length not greater than K.

On the other hand, the number of vertices v' with one of the properties (1)–(3) is not greater than

$$\sum_{\pi \in M_E(2)} d(\pi) + \sum_{\pi \in M_N(2)} d(\pi) + |\partial M|.$$

Now inequality (1.8) becomes obvious. □

Lemma 1.2. *The following are true*:

$$F_E \le |\partial M|, \qquad F_N \le 7(2p+1)|\partial M|. \tag{1.9}$$

Proof. The first inequality is obvious. Let us turn to the second. By an *angle* $\alpha = (e, f)$ of a face $\pi \in M(2)$ we mean two oriented edges e, f such that ef is a subpath of the counterclockwise oriented boundary $\partial\pi$. The vertex $e_+ = f_-$ will be called the *vertex* of an angle $\alpha = (e, f)$ and denoted by $\nu(\alpha)$. We will also say that α is *adjacent* to v.

Let $\alpha = (e, f)$ be an angle of a face $\pi \in M(2)$. The *weight* $\omega(\alpha)$ of α is defined to be $\frac{1}{7}$ provided π is irregular and $d(\nu(\alpha)) > q$. Otherwise, we set $\omega(\alpha) = 0$.

The *weight* $\omega(\pi)$ of an irregular face π is defined by the following formula:

$$\omega(\pi) = d(\pi) - p + S(\pi),$$

where $S(\pi)$ is the sum $\sum \omega(\alpha)$ over all angles α of π.

If π is not irregular, we put $\omega(\pi) = 0$.

It is immediate from the definition of an irregular face that $\omega(\pi) \ge \frac{1}{7}$ for every irregular π. Hence,

$$\tfrac{1}{7} F_N \le \sum_{\pi \in M(2)} \omega(\pi). \tag{1.10}$$

On the other hand, it follows from the definition of the weight function ω that

$$\sum_{\pi \in M(2)} \omega(\pi) = \sum_{\pi \in M_N(2)} (d(\pi) - p) + \sum_{v \in M_I(0)} S(v) + \sum_{v \in M_E(0)} S(v), \tag{1.11}$$

where $S(v)$ is the sum $\sum \omega(\alpha)$ over all angles α with $\nu(\alpha) = v$.

Let us estimate the two last terms in the right part of (1.11).

If v is interior and $d(v) = q$, then $S(v) = 0$. If v is interior and $d(v) \geq q$, then

$$S(v) \leq \tfrac{1}{7}d(v) \leq d(v) - q$$

as $q \leq 6$.

Assume v to be exterior. Clearly, at most $d(v) - 3$ angles α with $\nu(\alpha) = v$ can have positive weight. Hence, $S(v) \leq \frac{1}{7}(d(v) - 3)$.

Consequently, it follows from (1.11) that

(1.12)
$$\sum_{\pi \in M(2)} \omega(\pi) \leq \sum_{\pi \in M_N(2)} (d(\pi) - p) + \sum_{v \in M_I(0)} (d(v) - q) + \frac{1}{7} \sum_{v \in M_E(0)} (d(v) - 3),$$

and all sums on the right are non-negative by (M1)–(M2) and (P2).

Now note that

$$\sum_{\pi \in M_E(2)} (p - d(\pi)) \leq p|\partial M|$$

by the first inequality in (1.9). Hence, it follows from (1.7) that

$$(p + 0.5)|\partial M| \geq \frac{p}{q} \sum_{v \in M_E(0)} (d(v) - 3) + \frac{p}{q} \sum_{v \in M_I(0)} (d(v) - q) + \sum_{\pi \in M_I(2)} (d(\pi) - p).$$

Comparing this with (1.12), we see from

$$\sum_{\pi \in M_N(2)} (d(\pi) - p) = \sum_{\pi \in M_I(2)} (d(\pi) - p), \qquad \frac{p}{q} \geq \frac{1}{2},$$

that

$$\sum_{\pi \in M(2)} \omega(\pi) \leq 2(p + 0.5)|\partial M|.$$

It remains to refer to (1.10) to obtain that

$$F_N \leq 7(2p + 1)|\partial M| .$$

Lemma 1.2 is proven. □

Let us finish the proof of Theorem 1. It follows from Lemmas 1.1 and 1.2 that

$$\begin{aligned} F_R &\leq q^K \left(\sum_{\pi \in M_E(2)} (d(\pi) - p) + \sum_{\pi \in M_N(2)} (d(\pi) - p) + p(F_E + F_N) + |\partial M| \right) \\ &\leq q^K \left(\sum_{\pi \in M_E(2)} (d(\pi) - p) + \sum_{\pi \in M_I(2)} (d(\pi) - p) + (p(7(2p + 1) + 1) + 1)|\partial M| \right). \end{aligned}$$

Since the first two sums in (1.7) are non-negative by (P2) and (M1), one has from (1.7) that

$$0.5|\partial M| \geq \sum_{\pi \in M_E(2)} (d(\pi) - p) + \sum_{\pi \in M_I(2)} (d(\pi) - p).$$

Therefore, the estimate for F_R above implies that

$$F_R \leq q^K(p(7(2p+1)+1)+1.5)|\partial M|.$$

Finally, referring to Lemma 1.2 once again, we have

$$\begin{aligned} F = F_E + F_R + F_N &\leq \left(q^K(p(7(2p+1)+1)+1.5)+7(2p+1)+1\right)|\partial M| \\ &\leq 100q^K p|\partial M|. \end{aligned}$$

Theorem 1 is proven. □

2. Hyperbolicity of Small Cancellation Groups Satisfying the $C(p)\&T(q)$-condition

Let a group G be given by a presentation

$$G = \langle\, \mathcal{A} \parallel \mathcal{R}\, \rangle, \tag{2.1}$$

where $\mathcal{A}$ is a group alphabet ($\mathcal{A} = \mathcal{A}^{-1}$), $\mathcal{R}$ is the set of relators (which are cyclically reduced words over $\mathcal{A}$).

A (van Kampen) diagram Δ over G given by (2.1) is a map (see Sect. 1) that is equipped with a labeling function φ from the set of oriented edges of Δ to the alphabet $\mathcal{A}$ such that

- (L1) If $\varphi(e) = a$, then $\varphi(e^{-1}) = a^{-1}$.
- (L2) If Π is a face in Δ and $\partial\Pi = e_1 ... e_\ell$ is the boundary cycle of Π, where $e_1, \dots, e_\ell$ are oriented edges, then $\varphi(\partial\Pi) = \varphi(e_1)\dots\varphi(e_\ell)$ is a cyclic permutation of R^ε, where $\varepsilon = \pm 1$ and $R \in \mathcal{R}$.

It is convenient to fix the positive (counterclockwise) orientation for the boundary $\partial\Pi$ of a face Π in Δ and the negative (clockwise) orientation for the boundary $\partial\Delta$ of the diagram Δ. Following the terminology of [Ol], we will call faces of a diagram Δ *cells* and oriented above boundaries $\partial\Pi$, $\partial\Delta$ *contours* of Π, Δ, respectively.

Let Π_1, Π_2 be cells in a diagram Δ over (2.1), $\Pi_1 \neq \Pi_2$, and let v be a vertex such that $v \in \partial\Pi_1, \partial\Pi_2$. The cells Π_1, Π_2 are said to be a *reducible pair* provided the label $\varphi(\partial\Pi_1|_v)$ of the contour $\partial\Pi_1|_v$ starting at the vertex v is graphically (= letter-by-letter) equal to $\varphi(\partial\Pi_2|_v)^{-1}$, where $\partial\Pi_2|_v$ is the contour of Π_2 starting at v. Then we write $\varphi(\partial\Pi_1|_v) \equiv \varphi(\partial\Pi_2|_v)^{-1}$. A diagram Δ over (2.1) is said to be *reduced* provided Δ contains no reducible pairs of cells.

The following modification of van Kampen's lemma is almost obvious (see also [LS], [Ol]).

Lemma 2.1. *A cyclically reduced non-empty word W equals 1 in the group $G = \langle\, \mathcal{A} \parallel \mathcal{R}\, \rangle$ if and only if there is a reduced diagram Δ over G such that $\varphi(\partial\Delta) \equiv W$.*

We say that the group G given by (2.1) satisfies the small cancellation condition $C(p)\&T(q)$ (where (p,q) is one of (3,6), (4,4), (6,3) as in Sect.1) if any reduced diagram Δ over G is a (p,q)-map (see Sect.1). Note that this definition is equivalent to the standard one in [LS].

A diagram Δ over presentation (2.1) will be called *minimal* if for any diagram Δ' over (2.1) with the same label $\varphi(\partial\Delta')$ of $\partial\Delta'$ as that of $\partial\Delta$ the number of cells $|\Delta'(2)|$ in Δ' is not fewer than that in Δ. Clearly, every minimal diagram is reduced but the converse is not necessarily true.

The following theorem provides a criterion for the hyperbolicity of finitely presented groups satisfying one of small cancellation conditions $C(p)\&T(q)$.

Theorem 2. *A finitely presented group $G = \langle\, \mathcal{A} \parallel \mathcal{R}\, \rangle$ satisfying a small cancellation condition $C(p)\&T(q)$ ((p,q) is one of $(3,6)$, $(4,4)$, $(6,3)$) is hyperbolic if and only if there is a constant K such that for every minimal diagram Δ over G the radii of regular (p,q)-submaps of the map associated with Δ (by disregarding the labels of edges of Δ) do not exceed K.*

Proof. Let us prove the sufficiency of the condition $\mathrm{rd}(M) \le K$. Let W be a nonempty cyclically reduced word and $W = 1$ in G. By Lemma 2.1, there is a reduced diagram Δ with $\varphi(\partial\Delta) = W$. Disregarding the label function φ and considering Δ as a (p,q)-map, we have from Theorem 1 that the number of cells $|\Delta(2)|$ in Δ is estimated as follows:

$$|\Delta(2)| \le L|\partial\Delta| = L|W|,$$

where $L = L(K)$. Hence, G satisfies a linear isoperimetric inequality and, therefore, is hyperbolic.

Now assume G to be hyperbolic. Then there is a constant L such that

$$|\Gamma(2)| \le L|\partial\Gamma| = L|W|$$

for every minimal diagram Γ over G.

Consider a minimal diagram Δ over G and let Δ_0 be a subdiagram of Δ (that is, Δ_0 itself is a diagram) such that the map associated with Δ_0 is (p,q)-regular and has radius r. By the definition, there is a vertex $v \in \Delta_0$ with $d(v) \ge 3$ such that $\mathrm{dist}(v, \partial\Delta_0) = r$. Next, consider a maximal subdiagram Δ_1 in Δ_0 relative to the property that if Π is a cell in Δ_1 then $\mathrm{dist}(v, \partial\Pi) \le r$. Denote $m = \max\{|R| \mid R \in \mathcal{R}\}$. Then Δ_1 contains a subdiagram Δ_2 which (after disregarding the labels of edges and verices of degree 2) looks like a "piece" S of a regular (p,q)-tessellation of the plane such that S consists of all faces π of the (p,q)-tessellation with $\mathrm{dist}(\partial\pi, v) \le \frac{r}{2m}$. It is clear that there are positive real numbers κ_1, κ_2, κ_3, κ_4 such that

$$|S(2)| > \kappa_1 r^2 - \kappa_2, \qquad |\partial S| < \kappa_3 r + \kappa_4.$$

These inequalities imply that

$$|\Delta_2(2)| > \kappa_1 r^2 - \kappa_2, \qquad |\partial\Delta_2| < m(\kappa_3 r + \kappa_4). \tag{2.2}$$

On the other hand, Δ_2, being a subdiagram of Δ, is itself minimal. Therefore,

$$|\Delta_2(2)| \le L|\partial\Delta_2|.$$

Referring to (2.2), we have

$$\kappa_1 r^2 - \kappa_2 < Lm(\kappa_3 r + \kappa_4),$$

whence an estimate $r < K(L, m, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ becomes obvious.

Theorem 2 is proven. □

Let $G = \langle\, \mathcal{A} \parallel \mathcal{R}\, \rangle$ be defined as in Theorem 2 and Δ_k, $k = 1, 2, \dots$, be a sequence of diagrams over G such that every Δ_k is a fixed subdiagram of Δ_{k+1} with $\mathrm{dist}(\partial\Delta_k, \partial\Delta_{k+1}) > 0$, Δ_k is reduced, and the map associated as above with Δ_k is (p,q)-reqular. Call such a sequence Δ_k, $k = 1, 2, \dots$, a *flat* over G. (Less formally, a flat over G is an infinite reduced diagram over G that covers the entire plane and whose associated map is (p,q)-reqular.) A flat is *minimal* if each Δ_k is a minimal diagram. The following is immediate from Theorem 2 and its proof.

Corollary. *Let $G = \langle\, \mathcal{A} \,\|\, \mathcal{R} \,\rangle$ be defined as in Theorem 2. Then G is hyperbolic if and only if there are no minimal flats over G.*

Note that if all relators of G have length p, then the Cayley complex of G can easily be turned into a metric CAT(0)-space and the corollary can also be derived from Claim 4.1B of Gromov's work [Gr] (see also Bridson's article [B]). However, it seems unlikely that for every finitely presented $C(p)\&T(q)$-group G its Cayley complex can be regarded a metric CAT(0)-space. We would also like to ask whether the existence of a flat over G necessarily implies the existence of a minimal flat, and whether non-hyperbolicity of G implies that G contains an isomorphic copy of a free abelian group of rank two (the last problem is known for groups acting on metric CAT(0)-spaces; see [B]).

3. Some Reduction Lemmas

To prove Theorems 3 and 4 stated in the Introduction we will need several reduction lemmas (Lemmas 3.1–3.4 below) that we prove in this section.

Let

$$G = \langle\, \mathcal{A} \,\|\, R \,\rangle, \tag{3.1}$$

be a one-relator group, R a non-empty cyclically reduced word, $a \in \mathcal{A}$ a letter such that a and/or a^{-1} has occurrences in R, and

$$R \equiv a^{\varepsilon_0} B_0 a^{\varepsilon_1} B_1 \dots a^{\varepsilon_{k-1}} B_{k-1}, \tag{3.2}$$

where $k \geq 2$, $\varepsilon_i = \pm 1$, and the words B_i, $i = 0, 1, \dots, k-1$, have no occurrences of $a^{\pm 1}$.

Let us begin by giving some useful definitions.

Let Δ be a diagram over (3.1). An oriented edge e of Δ is called an *a-edge* provided $\varphi(e) = a^{\pm 1}$. Let e_0, f_0^{-1} be a-edges of $\partial\pi_0$, where π_0 is a cell of Δ (recall $\partial\pi$ is positively oriented for every π), such that the arc $e_0 v_0 f_0^{-1}$ of $\partial\pi_0$ has the property that v_0 contains no a-edges. Then by the *a-star* $\mathrm{St}(e_0, f_0)$ defined by the a-edges e_0, f_0 we mean the following sequence of a-edges. Assume e_0^{-1} belongs to $\partial\pi_1$ and consider the arc of $\partial\pi_1$ of the form $e_1 v_1 e_0^{-1}$, where e_1 is an a-edge, $e_1 \neq e_0^{-1}$, and v_1 contains no a-edges (this is possible due to the assumption that the number $k = |R|_a = |\partial\pi_1|_a$ of a-edges on $\partial\pi_1$ is at least 2; see (3.2)). Assuming $e_1^{-1} \in \partial\pi_2$, we again pick the arc of $\partial\pi_2$ of the form $e_2 v_2 e_1^{-1}$, where e_2 is an a-edge and v_2 has no a-edges. Keeping on doing so, we will get after $\ell - 1$ steps that either $e_{\ell-1} = f_0$ or $e_{\ell-1} \in \partial\Delta$ (see Fig. 3.1(a)–(b)). In the first case, the a-star $\mathrm{St}(e_0, f_0)$ consisting of the oriented a-edges $e_0, e_1, \dots, e_{\ell-1}$ is called *interior* and its *label* $\varphi(\mathrm{St}(e_0, f_0))$ is the word

$$\varphi(\mathrm{St}(e_0, f_0)) = \varphi(v_{\ell-1} v_{\ell-2} \dots v_1 v_0);$$

see Fig. 3.1(a).

In the second case, we extend our construction in the other direction: If $f_0 \in \partial\pi_{-1}$, then we pick the arc of $\partial\pi_{-1}$ of the form $f_0 v_{-1} f_{-1}^{-1}$, where f_{-1} is an a-edge and v_{-1} has no a-edges and so forth; see Fig. 3.1(b). After several steps, say $m-1$, we will get $f_{-(m-1)}^{-1} \in \partial\Delta$ (as $f_{-(m-1)}$ may not be e_0). Then the a-star $\mathrm{St}(e_0, f_0)$ consisting of the oriented a-edges $f_{-(m-1)}$, $f_{-(m-2)}, \dots,$ f_0, e_0, $e_1, \dots,$ $e_{\ell-1}$ is called *exterior* and its *label* $\varphi(\mathrm{St}(e_0, f_0))$ is the word

$$\varphi(\mathrm{St}(e_0, f_0)) = \varphi(v_{\ell-1} \dots v_1 v_0 v_{-1} \dots v_{-(m-1)});$$

see Fig. 3.1(b).

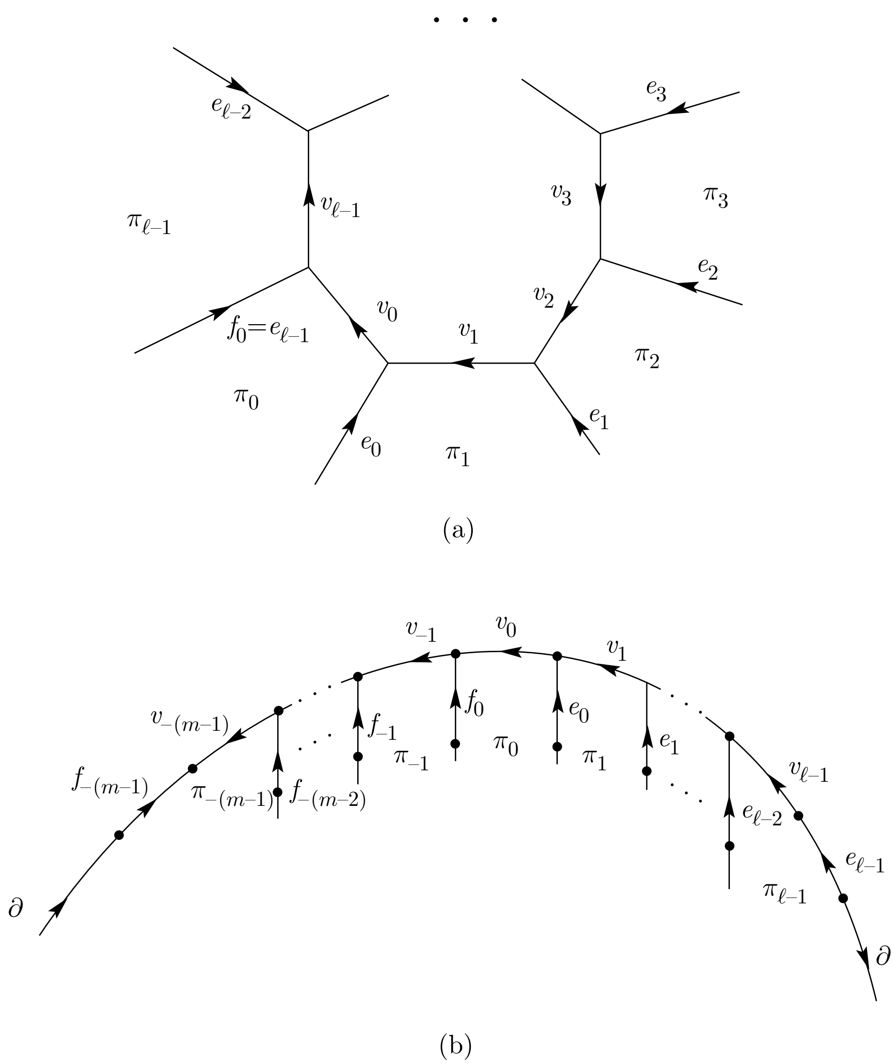


FIGURE 3.1

Now let $\mathcal{S} = \{\Gamma_1, \Gamma_2, \dots\}$ be a non-empty system of subdiagrams $\Gamma_1, \Gamma_2, \dots$ in Δ (as above, Δ is over (3.1)). The system $\mathcal{S}$ will be called *a-regular* provided it has the following three properties:

(A1) Every $\Gamma_i \in \mathcal{S}$ contains cells, distinct $\Gamma_i, \Gamma_j \in \mathcal{S}$ have no cells in common, and every cell $\pi \in \Delta$ is contained in some $\Gamma_i \in \mathcal{S}$.
(A2) For every $\Gamma_i \in \mathcal{S}$ the number of a-edges on $\partial\Gamma_i$, $|\partial\Gamma_i|_a$, is at least 2 and every (oriented) a-edge $e \in \partial\Gamma_i$ belongs to $\partial\Delta$.
(A3) For every $\Gamma_i \in \mathcal{S}$ the word $\varphi(\partial\Gamma_i)$ is cyclically reduced.

Let us define boundary a-stars in Δ with respect to an a-regular system $\mathcal{S} = \{\Gamma_1, \Gamma_2, \dots\}$. Let e_0, f_0, $e_0 \neq f_0$, be a-edges, $e_0, f_0 \in \partial\Gamma_{i_0}$, $\Gamma_{i_0} \in \mathcal{S}$ ($\partial\Gamma_{i_0}$ is negatively oriented), and let the subpath $e_0 u_0 f_0$ of $\partial\Gamma_{i_0}$ be such that u_0 contains

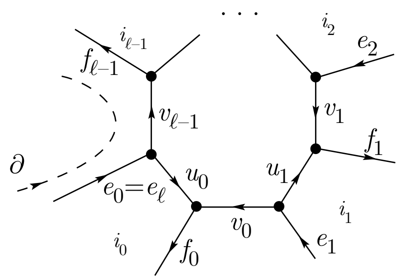


FIGURE 3.2

no a-edges. Then the *boundary a-star* $\mathrm{St}_{\mathcal{S}}(e_0, f_0)$ defined by the a-edges e_0, f_0 is the sequence of a-edges constructed as follows. Consider a subpath of $\partial\Delta$ of the form $e_1 v_0 f_0$, where e_1 is an a-edge, $e_1 \neq f_0$, and v_0 has no a-edges (this is possible because $|\partial\Delta|_a > 1$ by property (A2) of $\mathcal{S}$). Let $e_1 \in \partial\Gamma_{i_1}$, and let $e_1 u_1 f_1$ be a subpath of $\partial\Gamma_{i_1}$ such that f_1 is an a-edge and u_1 has no a-edges. Considering a subpath of $\partial\Delta$ of the form $e_2 v_1 f_1$, where e_2 is an a-edge and v_1 has no a-edges, we analogously define $\partial\Gamma_{i_2}$ and so on (see Fig. 3.2). Clearly, $e_\ell = e_0$ for some $\ell \geq 1$. Then the boundary a-star $\mathrm{St}_{\mathcal{S}}(e_0, f_0)$ consists of the a-edges $e_0, f_0, e_1, f_1, \dots, e_{\ell-1}, f_{\ell-1}$, and its *label* $\varphi(\mathrm{St}_{\mathcal{S}}(e_0, f_0))$ is the word

$$\varphi(\mathrm{St}_{\mathcal{S}}(e_0, f_0)) = \varphi(u_0 v_0^{-1} u_1 v_1^{-1} \dots u_{\ell-1} v_{\ell-1}^{-1}).$$

Clearly, $\varphi(\mathrm{St}_{\mathcal{S}}(e_0, f_0)) = 1$ in $F(\mathcal{A})$ for any boundary a-star, by (A1).

Coming back to the relator R of the form (3.2), let us state two additional properties of it (to be studied separately):

(P1) $k \geq 3$, at most one of $B_0, B_1, \dots, B_{k-1}$ is empty, and all non-empty words out of $B_0, B_1, \dots, B_{k-1}$ freely generate a free subgroup of the free group $F(\mathcal{A})$ over $\mathcal{A}$.

(P2) The subgroup $\langle B_0, B_1, \dots, B_{k-1} \rangle$ of $F(\mathcal{A})$ generated by $B_0, B_1, \dots, B_{k-1}$ is cyclic and is a subgroup of $\langle B \rangle$, where B is a non-empty cyclically reduced word that is not a proper power in $F(\mathcal{A})$.

First let us study property (P1):

Lemma 3.1. *The one-relator group $G = \langle\, \mathcal{A} \parallel R \,\rangle$ is hyperbolic provided its relator R has property (P1).*

Proof. Let Δ be a reduced diagram (for the definition see Sect. 2) over G. It is easy to see from (P1) that the existence of an interior a-star in Δ contradicts the assumption that Δ is reduced. Hence, all a-stars in Δ are exterior. But then it is evident from the inequality $k \geq 3$ that the number of a-edges $|\partial\Delta|_a$ on $\partial\Delta$ is at least $(k-2)|\Delta(2)|$. Hence, $|\partial\Delta| \geq |\partial\Delta|_a \geq (k-2)|\Delta(2)|$, and so G satisfies a linear isoperimetric inequality. □

To study groups whose relators have property (P2) we need some more preparations.

A word A over $\mathcal{A}$ will be called *simple* if A is non-empty, cyclically reduced, and is not a proper power.

Let Γ be a diagram over the free group $F(\mathcal{A})$ (so Γ looks like a tree), and let C be a simple word. If s is a subpath of $\partial\Gamma$ and $\varphi(s) = C^\ell$, then we will call s

C^ε-*periodic*, where $\varepsilon = 1$ if $\ell \geq 0$ and $\varepsilon = -1$ if $\ell < 0$. We will also say that $v \in s$ is a *phase vertex* of s provided the decomposition of s defined by v has the form $s = s_1 s_2$ with $|s_1|$, $|s_2|$ multiples of $|C|$. Let $t_1,\ t_2$ be C^{ε_1}-, C^{ε_2}-periodic subpaths of $\partial\Gamma$ with $\varepsilon_1, \varepsilon_2 = \pm 1$, respectively. The paths $t_1,\ t_2$ are said to be C-compatible provided $\varepsilon_1\varepsilon_2 = -1$ and $t_1,\ t_2$ have a common phase vertex.

Lemma 3.2. *Let A, B be simple words over $\mathcal{A}$ and either $A = B$ or A is not conjugate in $F(\mathcal{A})$ to B and B^{-1}. Furthermore, suppose Δ is a diagram over $F(\mathcal{A})$ such that $\partial\Delta = x_0 y_0 \dots x_{n-1} y_{n-1}$, where each of $x_0, \dots, x_{n-1}$ is an A- or B-periodic path,*

$$|x_0| + \dots + |x_{n-1}| > 7n(|A| + |B|),$$

and $\varphi(y_0), \dots, \varphi(y_{n-1})$ are reduced words. Then either

$$|y_0| + \dots + |y_{n-1}| > \tfrac{1}{7}(|x_0| + \dots + |x_{n-1}|) \tag{3.3}$$

or there are x_i, x_j, $\ i \neq j$, that are A- or B-compatible.

Proof. Assuming that there are no distinct x_i, x_j that are A- or B-compatible, we will prove inequality (3.3).

Let $u, v \in \{x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}\}$. We say that a path z with $|z| > 0$ is a *common arc* between u and v if z^ε, $z^{-\varepsilon}$ with $\varepsilon = \pm 1$ are subpaths of u, v, respectively. This z will be called a *maximal common arc* between u and v if z is not contained in a bigger common arc z' (with $|z'| > |z|$). Such maximal z will also be denoted be $z(u, v)$. Notice that if e is an edge, $e \in u$, then $e^{-1} \in u'$, where $u' \in \{x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}\}$, because $\varphi(x_0), \dots, \varphi(x_{n-1}), \varphi(y_0), \dots, \varphi(y_{n-1})$ are reduced words and Δ is over $F(\mathcal{A})$ (i.e. Δ has no cells). For the same reasons, up to orientation there is at most one maximal common arc between u and v and there is no common arc between u and u (for any u).

Consider a maximal common arc $z(x_i, x_j)$ between x_i and x_j. It is not difficult to see that if $|z(x_i, x_j)| \geq |A| + |B|$, then x_i and x_j are A- or B-compatible. Consequently,

$$|z(x_i, x_j)| < |A| + |B| \tag{3.4}$$

for any x_i, x_j.

To estimate the number of such $z(x_i, x_j)$, for every x_i we pick a point o_{x_i} close to x_i (in the sense that it is possible to connect all o_{x_i} with endpoints $(x_i)_-$, $(x_i)_+$ of x_i without intersections). Next, if there is a common arc $z(x_i, x_j)$ between x_i and x_j, then we connect o_{x_i} and o_{x_j} by a line L_{ij} going through $z(x_i, x_j)$. As a result, we obtain a planar graph Φ whose vertices are $o_{x_0}, \dots, o_{x_{n-1}}$ and whose edges are L_{ij}. As we saw above, Φ has no loops or multiple edges. Consequently, denoting the numbers of vertices, edges, faces in Φ by V, E, F, respectively, and applying Euler's formula, we have

$$V - E + F = 1, \qquad F \leq \frac{2E}{3},$$

whence $E \leq 3(V-1) = 3(n-1)$. Therefore, it follows from inequality (3.4) that

$$\sum |z(x_i, x_j)| < 6n(|A| + |B|). \tag{3.5}$$

It is obvious that

$$\sum |y_i| \geq \sum |z(x_i, y_j)| \geq \sum |x_i| - \sum |z(x_i, x_j)|.$$

Hence, it follows from the assumption $\sum |x_i| > 7n(|A|+|B|)$ and inequality (3.5) that

$$\sum |y_i| > \left(1 - \tfrac{6}{7}\right) \sum |x_i| = \tfrac{1}{7} \sum |x_i|,$$

as required. Lemma 3.2 is proven. □

Let us turn to property (P2). Assuming that R has property (P2), put $B_i = B^{\ell_i}$, $i = 0, \dots, k-1$, and consider a new word $\bar{R}$ obtained from R by replacing all B^{ℓ_i} by b^{ℓ_i}, where b is a new letter, $b \notin \mathcal{A}$. Define the group $\bar{G}$ as follows:

$$\bar{G} = \langle\, a^{\pm 1}, b^{\pm 1} \parallel \bar{R}\,\rangle. \tag{3.6}$$

Lemma 3.3. *Let a one-relator group G be given by presentation (3.1), let the relator R have property (P2), and let the group $\bar{G}$ be constructed as in (3.6). If $\bar{G}$ satisfies a linear isoperimetric inequality then so does G.*

Proof. Let W be a non-empty word over $\mathcal{A}$ such that $W = 1$ in G and no proper subword of W equals 1 in G. In particular, W is cyclically reduced. Clearly, to show that G satisfies a linear isoperimetric inequality it suffices to prove the existence of a constant L such that

$$|\Delta(2)| \le L|\partial\Delta| = L|W|, \tag{3.7}$$

where Δ is a diagram over G with $\varphi(\partial\Delta) = W$ minimal relative to $|\Delta(2)|$. Let Δ be such a diagram. A *tame component* Γ of Δ is defined to be a subdiagram of Δ with the following properties:

- (T1) The word $\varphi(\partial\Gamma)$ is non-empty and cyclically reduced (in particular, Γ contains cells).
- (T2) Let v_1, v_2 be endpoints of some a-edges in Γ, $s = v_1 - v_2$ a reduced path in Γ connecting v_1, v_2 (s is *reduced* if s contains no subpath of the form ee^{-1} with an edge e). Then $\varphi(s)$ is a word of the form $V(a^{\pm 1}, B^{\pm 1})$, where each subword $B^{\pm 1}$ of $V(a^{\pm 1}, B^{\pm 1})$ is the label of a path t whose endpoints t_-, t_+ are phase vertices of $B^{\pm 1}$-periodic subpaths of the contour $\partial\pi$ of a cell π in $\partial\Delta$ and t itself is a subpath of $\partial\pi^{\pm 1}$ (see (3.2) and (P2)).

A system $\mathcal{S} = \{\Gamma_1, \Gamma_2, \dots,\}$ of tame components $\Gamma_1, \Gamma_2, \dots$ in Δ will be called *complete* provided it has the following extra property

- (T3) Distinct $\Gamma_i, \Gamma_j \in \mathcal{S}$ have no cells in common, and every cell Π of Δ is contained in some $\Gamma_i \in \mathcal{S}$.

Notice that the existence of a complete system of tame components of Δ is obvious, as one may consider all cells of Δ as tame components.

Now let us introduce elementary transformations of Δ. Suppose v is a vertex in Δ and e_1, e_2 are distinct oriented edges with $\varphi(e_1) = \varphi(e_2)$ and $(e_1)_+ = (e_2)_+ = v$, $(e_1)_- \ne (e_2)_-$. Let us split v into v', v'', e_1 into e_1', e_1'', e_2 into e_2', e_2'', as shown in Fig. 3.3, creating thereby a hole in Δ (shaded in Fig. 3.3). Then we attach e_1' to e_2', e_1'' to e_2'', thus eliminating the hole. As a result, we obtain a new diagram Δ'. Such surgery will be called an *elementary transformation* of Δ applied to the edges e_1, e_2. It is clear that $\varphi(\partial\Delta') = \varphi(\partial\Delta)$ and $|\Delta'(2)| = |\Delta(2)|$. Therefore, we may consider Δ' instead of Δ to prove inequality (3.7).

Consider all complete systems $\mathcal{S}$ of tame components in all diagrams Δ' obtained from Δ by (finite) sequences of elementary transformations, and choose $\mathcal{S}, \Delta'$ so that the cardinality $|\mathcal{S}|$ is minimal. Renaming Δ' by Δ (recall $\varphi(\partial\Delta') = \varphi(\partial\Delta)$,

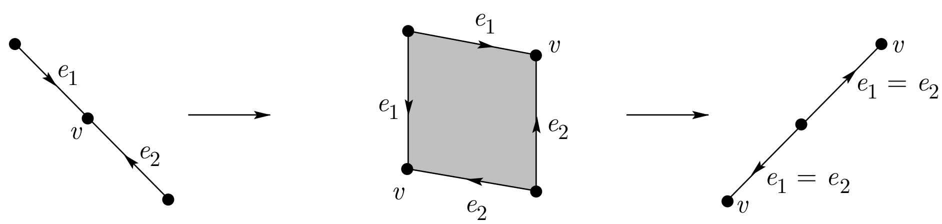


FIGURE 3.3

$|\Delta'(2)| = |\Delta(2)|$), let us show that such $\mathcal{S}$ has properties (A1)–(A3) of an a-regular system.

First, (A1) and (A3) follow from properties (T1), (T3).

To prove (A2), arguing on the contrary, assume $|\partial \Gamma_i|_a \le 1$. Then, by (T2), one has $\varphi(\partial \Gamma_i) = a^\varepsilon B^\ell$, where $\varepsilon = \pm 1$ and ℓ is an integer. But then a contradiction to $|R|_a \ge 2$ (see (3.2)) is easy to derive from the minimality of $|\Delta(2)|$ and Magnus's classical theorems on one-relator groups; see Theorems 4.10–4.12 in [MKS].

Now let $e \in \Gamma_i$ be an a-edge and $e \notin \partial\Delta$. This means that $e^{-1} \in \Gamma_j$. First assume $\Gamma_j \ne \Gamma_i$. Then it is not difficult to see that there are elementary transformations of Δ (applied, if necessary, to glue some paths of $\partial\Gamma_i$ and $\partial\Gamma_j^{-1}$) that do not affect properties (T1)–(T2) of each $\Gamma_{i'} \in \mathcal{S}$ and that make it possible to consider a subdiagram Γ, consisting of Γ_i, Γ_j, which is a tame component of a diagram Δ' obtained from Δ by these elementary transformations. This, however, contradicts the minimality of $|\mathcal{S}|$. Assume $\Gamma_j = \Gamma_i$. Then $\partial\Gamma_i$ contains a subpath of the form eue^{-1}. Picking such subpath minimal relative to $|u|$, one may claim that u has no a-edge f with $f^{-1} \in \partial\Gamma_i$. Hence, if u contains an a-edge f, then $f^{-1} \in \partial\Gamma_j$ with $j \ne i$. In this case, a contradiction follows as above. Thus, u has no a-edges and the subdiagram E of Δ given by $\partial\,\mathrm{E} = u$ has no cells by Theorem 4.10 of [MKS] and the minimality of $|\Delta(2)|$. But in this case, the subpath eue^{-1} of $\partial\Gamma_i$ is not reduced, contrary to (T1).

Thus every a-edge $e \in \partial\Gamma_i,\ \Gamma_i \in \mathcal{S}$, belongs to $\partial\Delta$, and it is proven that $\mathcal{S}$ is a-regular.

Since $\mathcal{S} = \{\Gamma_1, \Gamma_2, \dots,\}$ is a-regular, we may consider boundary a-stars in Δ. Let $\mathrm{St}_{\mathcal{S}}(e,f)$ be such an a-star. In the notation of its definition, let

$$\varphi(\mathrm{St}_{\mathcal{S}}(e,f)) = \varphi(u_0)\varphi(v_0)^{-1} \dots \varphi(u_{\ell-1})\varphi(v_{\ell-1})^{-1}. \tag{3.8}$$

Note that, by (T2), the words $\varphi(u_0), \dots, \varphi(u_{\ell-1})$ are $B^{\pm1}$-periodic, the words $\varphi(v_0)^{-1}, \dots, \varphi(v_{\ell-1})^{-1}$ are reduced (since they are subwords of $\varphi(\partial\Delta) = W$), and $\varphi(\mathrm{St}_{\mathcal{S}}(e,f)) = 1$ in $F(\mathcal{A})$. Hence it follows from Lemma 3.2 (where $A = B$) that either $\sum |u_i| \le 12\ell|B|$ or, otherwise, that either $\sum |v_i| > \frac{1}{7}\sum |u_i|$ or there are $u_i,\ u_j,\ i \ne j$, that are B-compatible.

Assume there is a boundary a-star in Δ such that $u_i, u_j,\ i \ne j$, are B-compatible, and let v be the common phase vertex of u_i, u_j (see the definition of B-compatibility). Let u_i, u_j be subpaths of $\partial\Gamma_{t_i},\ \partial\Gamma_{t_j}$, respectively. Clearly, $t_i \ne t_j$ for $i \ne j$, and so one may consider the subdiagram Γ of Δ, consisting of $\Gamma_{t_i},\ \Gamma_{t_j}$ "joined" by the path $v - v$ of length 0. In this case, however, as in proving property (A2) for $\mathcal{S}$, one can perform elementary transformations over Δ to merge $\Gamma_{t_i},\ \Gamma_{t_j}$ into a single tame component, contrary to the minimality of $|\mathcal{S}|$. Consequently, for every boundary

a-star in Δ one of the following two inequalities holds:

$$\sum |u_i| \le 12\ell|B|, \tag{3.9}$$

or

$$\sum |u_i| < 7\sum |v_i| \ . \tag{3.10}$$

Denote

$$\partial\Gamma_i = e_{i0}p_{i0}\dots e_{i\ell_i-1}p_{i\ell_i-1}, \tag{3.11}$$

where $e_{i0},\dots,e_{i\ell_i-1}$ are all a-edges of $\partial\Gamma_i$, and let

$$\partial\Delta = e_0q_0\dots e_{n-1}q_{n-1}, \tag{3.12}$$

where $e_0,\dots,e_{n-1}$ are all a-edges of $\partial\Delta$.

It is easy to see that every a-edge $e \in \partial\Delta$ occurs exactly twice in all boundary a-stars of Δ, each $\varphi(p_{ij})$, where p_{ij} is from (3.11), occurs as $\varphi(u_{i'})$ exactly once in labels of all boundary a-stars, and every $\varphi(q_i)^{-1}$, where q_i is from (3.12), occurs exactly once as $\varphi(v_{i'})^{-1}$ in labels of all boundary a-stars. These observations enable us to conclude that the sum Σ_1 of $\sum|u_i|$ over all boundary a-stars for which (3.9) holds can be estimated by

$$\Sigma_1 < 12|B|\sum \ell_i \le 24|B||\partial\Delta|_a$$

and that the sum Σ_2 of $\sum|u_i|$ over all boundary a-stars for which (3.10) holds can be estimated by

$$\Sigma_2 \le 7(|\partial\Delta| - |\partial\Delta|_a) \ .$$

Therefore,

$$\sum_{\Gamma_i\in\mathcal{S}} |\partial\Gamma_i| = \Sigma_1 + \Sigma_2 + |\partial\Delta|_a < (24|B|-6)|\partial\Delta|_a + 7|\partial\Delta| \le 31|B||\partial\Delta|, \tag{3.13}$$

as $|B| \ge 1$.

Now consider a diagram Γ over the group G given by (3.1) that is a tame component of itself, and let t be a path in Γ such that $\varphi(t) = B^\varepsilon$, $\varepsilon = \pm1$. Let us replace t by a single edge e and assign $\varphi(e) = b^\varepsilon$. It follows from (T1)–(T2) that after changing all such t's in Δ this way we will obtain a diagram $\bar\Gamma$ over the group $\bar G$ given by (3.6) such that if $\varphi(\partial\Gamma) = V(a^{\pm1}, B^{\pm1})$ then $\varphi(\partial\bar\Gamma) = V(a^{\pm1}, b^{\pm1})$. Conversely, considering a diagram E over (3.6) the label $\varphi(\partial\,\mathrm{E})$ of whose contour is non-empty and has the property that no proper subword of $\varphi(\partial\,\mathrm{E})$ equals 1 in $\bar G$, we define its image $\tilde{\mathrm{E}}$ over (3.1) by turning every edge e of E with $\varphi(e) = b^\varepsilon$ into a path t of length $|B|$ and assigning $\varphi(t) = B^\varepsilon$. Clearly, $\tilde{\bar\Gamma} = \Gamma$ and $\bar{\tilde{\mathrm{E}}} = \mathrm{E}$.

Coming back to tame components $\Gamma_1, \Gamma_2, \dots$ of Δ, consider $\bar\Gamma_i$. Suppose there is a diagram E_i over (3.6) such that $\varphi(\partial\,\mathrm{E}_i) = \varphi(\partial\bar\Gamma_i)$ and $|\,\mathrm{E}_i(2)| < |\bar\Gamma_i(2)|$. Then, obviously, $\tilde{\mathrm{E}}_i$ is a diagram over (3.1) such that $\varphi(\partial\tilde{\mathrm{E}}_i) = \varphi(\partial\Gamma_i)$ and $\tilde{\mathrm{E}}_i$ is a tame component of itself. Hence, we may take Γ_i out of Δ and fill in the resulting hole with $\tilde{\mathrm{E}}_i$. As a result, we obtain a diagram Δ_0 with $\varphi(\partial\Delta_0) = \varphi(\partial\Delta)$ and $|\Delta_0(2)| < |\Delta(2)|$, contrary to minimality of $|\Delta(2)|$. Thus, no such E_i exists, and so

$$|\bar\Gamma_i(2)| \le \bar L|\partial\bar\Gamma_i| \le \bar L|\partial\Gamma_i|$$

with a constant $\bar{L}$, as (3.6) satisfies a linear isoperimetric inequality and $|\partial\bar{\Gamma}_i| \le |\partial\Gamma_i|$. Finally, it follows from (3.13) that

$$|\Delta(2)| = \sum_{\Gamma_i \in \mathcal{S}} |\Gamma_i(2)| = \sum_{\Gamma_i \in \mathcal{S}} |\bar{\Gamma}_i(2)| \le \bar{L} \sum_{\Gamma_i \in \mathcal{S}} |\partial\Gamma_i| \le 31|B|\bar{L}|\partial\Delta| \, .$$

Lemma 3.3 is proven. □

Lemma 3.4. *Let $H = \langle\, a^{\pm 1}, b^{\pm 1} \parallel S \,\rangle$ be a one-relator group, where the relator $S = S(a^{\pm 1}, b^{\pm k})$ with an integer $k > 0$ is a cyclically reduced word over $\{a^{\pm 1}, b^{\pm 1}\}$, and $H_1 = \langle\, a^{\pm 1}, b^{\pm 1} \parallel S_1 \,\rangle$, where $S_1 = S(a^{\pm 1}, b^{\pm 1})$. Then H_1 is hyperbolic if H is.*

Proof. Let Δ be a diagram over H. By a *k-phase vertex* of a cell $\pi \in \Delta$ we mean a vertex o of any $b^{\pm 1}$-periodic subpath u of $\partial\pi$ such that if $u = u_1 u_2$ is the decomposition of u defined by o, then the lengths $|u_1|$, $|u_2|$ are multiples of k.

A diagram Δ over H is called *tame* provided non-empty $\varphi(\partial\Delta)$ is cyclically reduced and every reduced path t connecting endpoints of a-edges in Δ has a label of the form $\varphi(t) = V(a^{\pm 1}, b^{\pm k})$, where each distinguished subword $b^{\pm k}$ of $V(a^{\pm 1}, b^{\pm k})$ is the label of a subpath u' of $\partial\pi'$, where π' is a cell in Δ, whose endpoints u'_-, u'_+ are k-phase vertices.

It is easy to see that if Δ is a tame diagram over H, then replacement of every path t in Δ such that $\varphi(t) = b^{\varepsilon k}$ with $\varepsilon = \pm 1$ and t_-, t_+ are k-phase vertices by a single edge e with $\varphi(e) = b^{\varepsilon}$ results in a diagram Δ_1 over H_1 such that $|\Delta_1(2)| = |\Delta(2)|$ and $\varphi(\partial\Delta_1) = W(a^{\pm 1}, b^{\pm 1})$ provided $\varphi(\partial\Delta) = W(a^{\pm 1}, b^{\pm k})$. Conversely, if E_1 is a diagram over H_1 with non-empty cyclically reduced $\varphi(\partial\,\mathrm{E}_1)$, then turning each edge e of E_1 with $\varphi(e) = b^{\varepsilon}$, $\varepsilon = \pm 1$, into a path of length k and assigning $\varphi(t) = b^{\varepsilon k}$ produce a tame diagram E over H such that $|\,\mathrm{E}(2)| = |\,\mathrm{E}_1(2)|$ and $\varphi(\partial\,\mathrm{E}) = Y(a^{\pm 1}, b^{\pm k})$, provided $\varphi(\partial\,\mathrm{E}_1) = Y(a^{\pm 1}, b^{\pm 1})$.

It is easy to see that H_1 satisfies a linear isoperimetric inequality provided the following are true: (1) So does H; (2) Any diagram Δ over H such that the non-empty word $\varphi(\partial\Delta)$ is of the form $W(a^{\pm 1}, b^{\pm k})$ and no proper subword of $\varphi(\partial\Delta)$ equals 1 in H is a tame diagram. The first claim is true by the lemma's assumption, and the second is easy to establish by considering a-stars in Δ (with the initial observation that there are no a-edges e, e^{-1} on $\partial\Delta$).

Lemma 3.4 is proven. □

4. Proof of Theorem 3

To prove Theorems 3 and 4 stated in the Introduction we need three more lemmas, the first two of which are taken from Gromov's fundamental article [Gr].

Lemma 4.1 (8.2.A of [Gr]). *For any non-trivial element g of a torsion free hyperbolic group Γ there exist unique g_0, m, where $g_0 \in \Gamma$ is not a proper power in Γ and $m > 0$, such that $g = g_0^m$.*

Lemma 4.2 (3.1.A of [Gr]). *A free abelian group of rank 2 cannot be a subgroup of a hyperbolic group.*

Lemma 4.3. *Let x, y be elements of a torsion free hyperbolic group Γ, $y \ne 1$, and $xy^k x^{-1} = y^\ell$ with $k \ne 0$. Then $xyx^{-1} = y$; in particular, $k = \ell$.*

Proof. Consider unique y_0, m for y as in Lemma 4.1. Then it follows from Lemma 4.1 and the equation $(xy_0x^{-1})^{km} = y_0^{\ell m}$ that $xy_0x^{-1} = y_0^{\pm 1}$, whence $xyx^{-1} = y^{\pm 1}$.

If $xyx^{-1} = y^{-1}$, then $yx^2y^{-1} = x^2$. Applying Lemma 4.1 as above to the latter equation, one has $yxy^{-1} = x$. But then $xyx^{-1} = y^{-1} = y$, hence $y^2 = 1$ and $y = 1$ as Γ is torsion free. Thus $xyx^{-1} = y$, as claimed. $\square$

Let us turn to parts (1)–(4) of Theorem 3 directly.

Part (1). There is nothing to prove if $|R|_a = 1$, as G is a finitely generated free group and so hyperbolic. Let $|R|_a = 2$ and $R = aBaC$. Applying an obvious automorphism of $F = F(\mathcal{A})$ to the presentation $G = \langle\ \mathcal{A} \parallel R\ \rangle$, we get a new presentation for G of the form

$$G = \langle\ \mathcal{A} \parallel a^2D^n\ \rangle, \tag{4.1}$$

where D is cyclically reduced and not a proper power, $n > 0$ (otherwise, replace D by D^{-1}), and D^n is conjugate in $F(\mathcal{A})$ to $B^{-1}C$. If $|D| = 0$, then G is hyperbolic, being isomorphic to the free product of a free group and a group of order 2. Let $|D| > 0$ and $n = 1$. Then Lemma 3.3 applies to (4.1) and yields the hyperbolicity of G, by virtue of the hyperbolicity of the group $\langle\ a^{\pm 1}, d^{\pm 1} \parallel a^2d\ \rangle$.

Assume $|D| > 0$, $n > 1$, and G is hyperbolic. By Theorem 4.12 of [MKS], G is torsion free. Hence, by Lemma 4.3, it follows from the equation $Da^2D^{-1} = a^2$ in G (we denote this by the sign "$\overset{G}{=}$") that $DaD^{-1} \overset{G}{=} a$. But this is clearly false in the quotient group of G by the normal closure of a^2.

Coming back to the original presentation of G, we see that G is not hyperbolic if and only if $B^{-1}C$ is a proper power in $F(\mathcal{A})$, as required.

Part (2). Let $|R|_a = 2$ and $R = aBa^{-1}C$. Applying an obvious automorphism of $F(\mathcal{A})$ to the presentation $\langle\ \mathcal{A} \parallel R\ \rangle$ of G, one may assume that the words B, C are cyclically reduced and, if B, C are conjugate in $F(\mathcal{A})$ to elements of a cyclic subgroup $\langle D\rangle \subseteq F(\mathcal{A})$, where D is not a proper power, then $B = D^{n_1}$, $C = D^{n_2}$.

First, as in Case (2a), assume that both B and C *are* conjugate in $F(\mathcal{A})$ to powers of a word D. By Theorem 4.12 of [MKS], G is torsion free. If G were hyperbolic, then a contradiction to Lemma 4.2 would follow from Theorems 4.10–4.11 of [MKS] in view of Lemma 4.3.

So from now on we assume that B, C are not conjugate in $F(\mathcal{A})$ to powers of the same word. Denote $B = B_0^{n_1}$, $C = C_0^{n_2}$, where B_0, C_0 are not proper powers. Note B_0 is not conjugate in $F(\mathcal{A})$ to $C_0^{\pm 1}$ by our assumption.

Assume both B and C are proper powers, that is, $\min(|n_1|, |n_2|) > 1$, and G is hyperbolic. It follows from $aB_0^{n_1}a^{-1} \overset{G}{=} C_0^{-n_2}$ that $aB_0a^{-1}C_0^{n_2}aB_0^{-1}a^{-1} \overset{G}{=} C_0^{n_2}$. By Theorem 4.12 of [MKS], G is torsion free. By Lemma 4.3, the last equation implies that

$$aB_0a^{-1}C_0aB_0^{-1}a^{-1} \overset{G}{=} C_0.$$

Consider a reduced diagram Δ over G for the latter equation. It is obvious that Δ has no interior a-stars and that there must be exactly two exterior a-stars in Δ, each of which contains two a-edges on $\partial\Delta^{\pm 1}$. Note that the label of each exterior a-star is either $B_0^{n_1\ell}$ or $C_0^{n_2\ell}$ for some ℓ. On the other hand, since

$$\varphi(\partial\Delta) = aB_0a^{-1}C_0aB_0^{-1}a^{-1}C_0^{-1},$$

one of the words $B_0^{\pm 1}$, $C_0^{\pm 1}$ is the label of an exterior a-star. This obvious contradiction to the assumption $\min(|n_1|, |n_2|) > 1$ shows that in this case G is not hyperbolic.

It remains to prove the hyperbolicity of G provided $\min(|n_1|, |n_2|) = 1$. Without loss of generality we can put $n_1 = 1$. Then $R = aB_0a^{-1}C_0^{n_2}$, where, as was pointed out above, B_0 is not conjugate in $F(\mathcal{A})$ to $C_0^{\pm 1}$. Consider a non-empty word W such that $W \overset{G}{=} 1$ and no proper subword of W equals 1 in G. Let Δ be a diagram over G such that $\varphi(\partial\Delta) = W$ and $|\Delta(2)|$ is minimal (in particular, Δ is reduced). Let us show that there is a constant L such that

$$|\Delta(2)| \le L|\partial\Delta| = L|W|. \tag{4.2}$$

The proof of this is quite similar to the proof of inequality (3.7) in Lemma 3.3. First we introduce analogs of tame components of Δ by repeating (T1)–(T2) with a single correction: In (T2), $\varphi(s)$ is a word of the form $V(a^{\pm 1}, B_0^{\pm 1}, C_0^{\pm 1})$, where each subword $B_0^{\pm 1}$, $C_0^{\pm 1}$ of $V(a^{\pm 1}, B_0^{\pm 1}, C_0^{\pm 1})$ is the label of a path t whose endpoints t_-, t_+ are phase vertices of $B_0^{\pm 1}$-, $C_0^{\pm 1}$-periodic sections, respectively, of the contour $\partial\pi$ of a cell π in Δ and t itself is a subpath of $\partial\pi^{\pm 1}$. The definition of a complete system $\mathcal{S} = \{\Gamma_1, \Gamma_2, \dots\}$ of tame components $\Gamma_1, \Gamma_2, \dots$ in Δ is retained. As in Lemma 3.3, we consider all diagrams obtained from Δ by elementary transformations and all possible complete systems in such diagrams to pick Δ' and $\mathcal{S}$ in Δ' with the minimal $|\mathcal{S}|$. Renaming Δ' by Δ, we establish properties (A1)–(A3) for Δ as in proving Lemma 3.3 with a single correction: If $\Gamma_i \in \mathcal{S}$, then $|\partial\Gamma_i|_a \ge 2$ now follows from the form of the relator (that obviously forbids $|\partial\Gamma_i|_a = 1$) and Theorem 4.10 of [MKS]. Thus, $\mathcal{S}$ is an a-regular system, and it is still possible to consider boundary a-stars in Δ. Keeping the notation introduced in proving Lemma 3.3 (see (3.7)) and applying Lemma 3.2, we have from the minimality of $|\mathcal{S}|$ and $|\Delta(2)|$ that for every boundary a-star in Δ one of the following two inequalities holds:

$$\sum |u_i| \le 6\ell(|B_0| + |C_0|) < 31\ell \max(|B_0|, |C_0|),$$

$$\sum |u_i| < 7 \sum |v_i|.$$

Defining the sums Σ_1, Σ_2 and reasoning as in Lemma 3.3 to estimate them, we get

$$\begin{aligned} \sum_{\Gamma_i \in \mathcal{S}} |\partial\Gamma_i| &= \Sigma_1 + \Sigma_2 + |\partial\Delta|_a \\ &< (24\max(|B_0|, |C_0|) - 6)|\partial\Delta|_a + 7|\partial\Delta| \\ &< 31\max(|B_0|, |C_0|)|\partial\Delta| \, . \end{aligned} \tag{4.3}$$

Define the analog of presentation (3.6) as follows:

$$\bar{G} = \langle\, a^{\pm 1}, b^{\pm 1}, c^{\pm 1} \parallel aba^{-1}c^{n_2} \,\rangle \, .$$

In an obvious way, given a diagram Γ over G that is a tame component of itself, we construct a diagram $\bar{\Gamma}$ over $\bar{G}$. Conversely, given a diagram E over $\bar{G}$ the label of whose contour $\varphi(\partial\,\mathrm{E})$ is non-empty and has the property that no proper subword of $\varphi(\partial\,\mathrm{E})$ equals 1 in $\bar{G}$, we construct a diagram $\tilde{\mathrm{E}}$ over G which is a tame component of itself. Clearly, $\tilde{\bar{\Gamma}} = \Gamma$ and $\bar{\tilde{\mathrm{E}}} = \mathrm{E}$. As in Lemma 3.3, by minimality of $|\Delta(2)|$, no E_i exists such that $\varphi(\partial\,\mathrm{E}_i) = \varphi(\partial\bar{\Gamma}_i)$ and $|\,\mathrm{E}_i(2)| < |\bar{\Gamma}_i(2)|$. Hence, every $\bar{\Gamma}_i$ is reduced. But for any reduced diagram E over $\bar{G}$ we obviously have $|\,\mathrm{E}(2)| \le |\partial\,\mathrm{E}\,|_b \le |\partial\,\mathrm{E}\,|$.

Consequently, we have from (4.3) and $|\bar{\Gamma}_i(2)| \le |\partial\bar{\Gamma}_i|$ that

$$|\Delta(2)| = \sum_{\Gamma_i \in \mathcal{S}} |\Gamma_i(2)| = \sum_{\Gamma_i \in \mathcal{S}} |\bar{\Gamma}_i(2)| \le \sum_{\Gamma_i \in \mathcal{S}} |\partial\bar{\Gamma}_i| \le \sum_{\Gamma_i \in \mathcal{S}} |\partial\Gamma_i|$$
$$\le 31 \max(|B_0|, |C_0|)|\partial\Delta|$$

and (4.2) is proven. Now it becomes evident that G is hyperbolic.

Coming back to the original relator, we can conclude that G is not hyperbolic if and only if either B and C are conjugate to powers of the same word or both B and C are proper powers.

Part (2) is complete.

Part (3). Let $R = aBaCaD$. Applying an obvious automorphism of $F(\mathcal{A})$ to the presentation of G, we can assume $R = a^2CB^{-1}aDB^{-1}$. If the rank of the subgroup $\langle CB^{-1}, DB^{-1}\rangle$ is 2 or 0, then G is hyperbolic by Lemma 3.1. Suppose $\operatorname{rank}(\langle CB^{-1}, DB^{-1}\rangle) = 1$ and E is a word that is not a proper power, so that $CB^{-1} = E^{n_1}$, $DB^{-1} = E^{n_2}$. Without loss of generality assume E to be cyclically reduced.

Case (3a). Let $\min(|n_1|, |n_2|) = 0$. For definiteness assume $n_1 = 0$ (the case $n_2 = 0$ is analogous). Then $R = a^3E^{n_2}$. Arguing as in Part (2), it is easy to obtain that G is not hyperbolic if and only if $|n_2| > 1$, as stated in Case (3a).

Cases (3b)–(3d). From now on assume $\min(|n_1|, |n_2|) > 0$. Consider the following five cases:

1. $n_1 = n_2$.
2. $n_1 = -n_2$.
3. $n_1 = 2n_2$ or $n_2 = 2n_1$.
4. $n_1 = -2n_2$ or $n_2 = -2n_1$.
5. None of Cases 1–4 holds.

1. Applying the automorphism $a \to aE^{-n_1}$, $b \to b$, $b \neq a$, of $F(\mathcal{A})$, we see that G also has the presentation

$$G = \langle\, \mathcal{A} \parallel a^3E^{-n_1} \,\rangle.$$

Hence, as above, G is not hyperbolic if and only if $|n_1| > 1$.

2. Now $R = a^2E^{n_1}aE^{-n_1}$, $n_1 \neq 1$. Note G is torsion free by Theorem 4.12 of [MKS]. Hence, by Lemma 4.3, it would follow from $E^{n_1}aE^{-n_1} \overset{G}{=} a^{-2}$ that $E^{n_1}aE^{-n_1} \overset{G}{=} a$, and so $a^3 \overset{G}{=} 1$ provided G were hyperbolic. This contradiction proves that G is not hyperbolic.

3. Let $n_1 = 2n_2$ (the case $n_2 = 2n_1$ is analogous). Applying the automorphism $a \to aE^{-n_2}$ and $b \to b$, where $b \neq a$, of $F(\mathcal{A})$, we get

$$G = \langle\, \mathcal{A} \parallel a^2E^{-n_2}aE^{n_2} \,\rangle.$$

Hence, as in Case 2, G is not hyperbolic.

4. Let $n_1 = -2n_2$ (the case $n_2 = -2n_1$ is analogous). Applying the automorphism $a \to aE^{-n_2}$, $b \to b$, $b \neq a$ of $F(\mathcal{A})$, we get

$$G = \langle\, \mathcal{A} \parallel a^2E^{-n_2}aE^{-3n_2} \,\rangle.$$

Therefore, this case is reduced to Case 5.

5. By Lemma 3.3, to prove the hyperbolicity of G it suffices to show that the group

$$\bar{G} = \langle\, a^{\pm 1}, b^{\pm 1} \parallel a^2 b^{n_1} a b^{n_2} \,\rangle \tag{4.4}$$

is hyperbolic. To do it we consider a word W over the alphabet $\{a^{\pm 1}, b^{\pm 1}\}$ such that $W \overset{\bar{G}}{=} 1$ and no proper subword of W equals 1 in $\bar{G}$. Let Δ be a reduced diagram over $\bar{G}$ (given by (4.4)) with $\varphi(\partial\Delta) = W$. Let us construct a map M_Δ from Δ by contracting all edges e with $\varphi(e) = b^{\pm 1}$ into points (and disregarding labels of remaining a-edges). It is clear that $|\partial\pi| = 3$ for every face $\pi \in M_\Delta(2)$ and

$$|M_\Delta(2)| = |\Delta(2)|, \qquad |\partial M_\Delta| = |\partial\Delta|_a \le |\partial\Delta|\ . \tag{4.5}$$

Let v be an interior vertex of M_Δ, $d(v) = \ell$, and $e_0, e_1, \dots, e_{\ell-1}$ be all consecutive (in positive direction) oriented edges of M_Δ whose terminal vertices are v. It is clear that the pre-images of $e_0, e_1, \dots, e_{\ell-1}$ in Δ (to be also denoted by $e_0, e_1, \dots, e_{\ell-1}$) form an interior a-star $\mathrm{St}(e_0, e_1)$ in Δ. Keeping the notation of the definition of an a-star, we observe that v results from contraction of the closed path $p = v_{\ell-1} \dots v_1 v_0$ into a point. Notice that $\varphi(e_i) = a^{\varepsilon}$ implies that $\varphi(e_{i+1}) = a^{-\varepsilon}$ (where $\varepsilon = \pm 1$ and subscripts are mod ℓ) due to the form of the relator in (4.4). Hence, ℓ is even.

Let us show that $\ell \ge 6$. If $\ell = 2$, then $\varphi(v_1) = \varphi(v_0)^{-1}$ and so Δ is not reduced, contrary to its choice. Let $\ell = 4$. Since each of $\varphi(v_3), \varphi(v_2), \varphi(v_1), \varphi(v_0)$ is one of $1, b^{\pm n_1}, b^{\pm n_2}$ and $|n_1| \ne |n_2|$, we have that there are distinct v_i, v_j with $\varphi(v_i) = \varphi(v_j)^{\pm 1}$. Since Δ is reduced, we have $j = i+2 \pmod 4$, $\varphi(v_i) = \varphi(v_j)$, and $\varphi(v_{i+1}) \ne \varphi(v_i)^{-1}$, $\varphi(v_{i-1}) \ne \varphi(v_i)^{-1}$. Now we see from $\varphi(\mathrm{St}(e_0, e_1)) = 1$ in $F(\mathcal{A})$ that the existence of such v implies one of the following equations: $n_1 + n_2 = 0$, $2n_1 = 0$, $2n_2 = 0$ (when $\varphi(v_i) = 1$); $2n_1 = 0$, $2(n_1 - n_2) = 0$, $2n_1 - n_2 = 0$ (when $\varphi(v_i) = b^{\pm n_1}$); $2n_2 = 0$, $2(n_2 - n_1) = 0$, $2n_2 - n_1 = 0$ (when $\varphi(v_i) = b^{\pm n_2}$). None of them, however, is possible in Case 5. Thus, the inequality $d(v) \ge 6$ is proven for any interior vertex v and, therefore, M_Δ is a (3,6)-map.

Let us study an interior vertex v with $d(v) = 6$. First assume that among $\varphi(v_5), \dots, \varphi(v_0)$ there are exactly two 1's, two $b^{\pm n_1}$'s, and two $b^{\pm n_2}$'s. Then it follows from $\varphi(\mathrm{St}(e_0, e_1)) = 1$ in $F(\mathcal{A})$ that either one of the equations $2(n_1 \pm n_2) = 0$, $2n_1 = 0$, $2n_2 = 0$ holds, or $\varphi(v_{i_0}) = \varphi(v_{i_3})^{-1} = 1$, $\varphi(v_{i_1}) = \varphi(v_{i_4})^{-1} = b^{\pm n_1}$, $\varphi(v_{i_2}) = \varphi(v_{i_5})^{-1} = b^{\pm n_2}$, where $\{i_0, i_1, \dots, i_5\} = \{0, 1, \dots, 5\}$. Since Δ is reduced, we have $i_4 \ne i_1 \pm 1$. It is easy to see that the words $\varphi(v_{i_1+2})$, $\varphi(v_{i_1-2})$, being subwords of R, may not be b^{-n_1}. Hence, $i_4 \equiv i_1 + 3 \pmod 6$. Analogously, $i_5 \equiv i_2 + 3 \pmod 6$, whence $i_3 \equiv i_0 + 3 \pmod 6$. Such a vertex v of degree 6 will be called a *vertex of type* I.

Now assume that among $\varphi(v_5), \dots, \varphi(v_0)$ there are at least three $\varphi(v_i)$, $\varphi(v_j)$, $\varphi(v_k)$ with $\varphi(v_i) = \varphi(v_j)^{\pm 1} = \varphi(v_k)^{\pm 1}$. Since Δ is reduced, we have

$$\min(|i-j|, |j-k|, |k-i|) > 1.$$

Therefore, $j \equiv i + 2 \pmod 6$ and $k \equiv i + 4 \pmod 6$. If $\varphi(v_i) = 1$, then we get from $\varphi(\mathrm{St}(e_0, e_1)) = 1$ in $F(\mathcal{A})$ that $2n_1 + n_2 = 0$ or $2n_2 + n_1 = 0$, contrary to the assumption of Case 5. Hence, $\varphi(v_i)$ is either $b^{\pm n_1}$ or $b^{\pm n_2}$. Then, again by $\varphi(\mathrm{St}(e_0, e_1)) = 1$, we have one of the following equations: $3n_1 - n_2 = 0$, $3n_1 - 2n_2 = 0$ (when $\varphi(v_i) = b^{\pm n_1}$); or $3n_2 - n_1 = 0$, $3n_2 - 2n_1 = 0$ (when $\varphi(v_i) = b^{\pm n_2}$). Clearly, at most one of these 4 equations can hold. The vertex v (if any) corresponding to this equation will be referred to as a *vertex of type* II. Notice that there is one out of $1, b^{n_1}, b^{n_2}$ that appears exactly once in $\{\varphi(v_0)^{\pm 1}, \dots, \varphi(v_5)^{\pm 1}\}$ for any vertex v

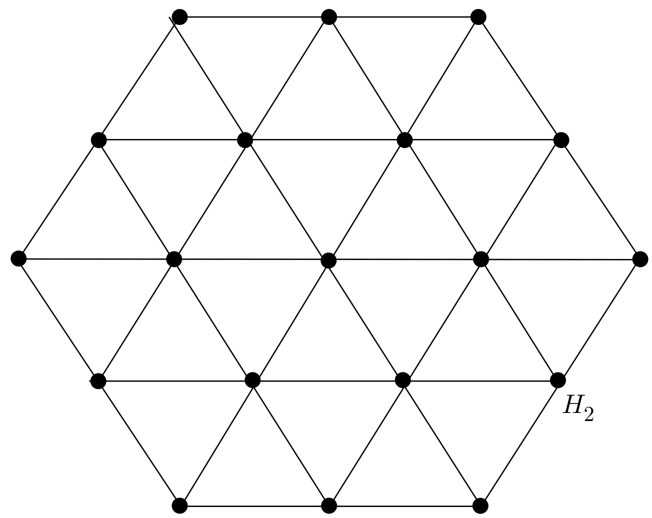


FIGURE 4.1

of type II, and that this one is the same for all such v's. Thus, it is proved that any vertex of degree 6 in M_Δ is of type I or II.

Let us assign labels to angles of faces in M_Δ (for the definition of an angle see Sect. 1). Let $\alpha = (e, f)$ be an angle of a face $\pi \in M_\Delta(2)$. Denote by e', f', π' the pre-images in Δ of e, f, π. Also, let $e'sf'$ be a subpath of $\partial\pi'$. Clearly, $\varphi(s)$ is one of $1, b^{\pm n_1}, b^{\pm n_2}$. Then the *label* $\psi(\alpha)$ of an angle $\alpha = (e, f)$ is defined as follows:

$$\psi(\alpha) = 0 \text{ provided } \varphi(s) = 1,$$
$$\psi(\alpha) = 1 \text{ provided } \varphi(s) = b^{\pm n_1},$$
$$\psi(\alpha) = 2 \text{ provided } \varphi(s) = b^{\pm n_2}.$$

Now assume that M_Δ contains a regular (3,6)-submap of radius $r \geq 2$. Then M_Δ also contains a regular (3,6)-submap that looks like a hexagon, H_r, whose sides have length r and that consists of $6r^2$ triangles (see Fig. 4.1, where $r = 2$).

Let π be an interior face in H_r whose vertices are interior in H_r and have type I. Also, denote $\partial\pi = e_0e_1e_2$, so that if $\alpha_i = (e_i, e_{i+1})$ is an angle, then $\psi(\alpha_i) = i$ (i is mod 3). Starting at the vertex $v = \nu(\alpha_0)$, let us try to assign labels to angles of faces of M_Δ that have a vertex in common with π. At the first step, assigning labels to angles of faces that contain v (and keeping in mind that v is of type I), we have two cases drawn in Figs. 4.2(a), (b), respectively.

After this the process of assignment becomes determined and quickly leads to a contradiction to the assumption that all 3 vertices of π are of type I (see Fig. 4.3(a)-(b)).

Consequently, at least one of vertices of π must be of type II. This observation enables us to conclude that if V is the number of all interior vertices in H_r and V_1, V_2 are the numbers of interior vertices of type I, II, respectively, then $V_2 \geq V/7$ (and, of course, $V_1 + V_2 = V$). Notice that the total number of vertices in H_r is $3r^2+3r+1$ and the numbers of exterior and interior vertices in H_r are $6r, 3r^2-3r+1$, respectively. By our remark above about vertices of type II, there is an $i \in \{0, 1, 2\}$

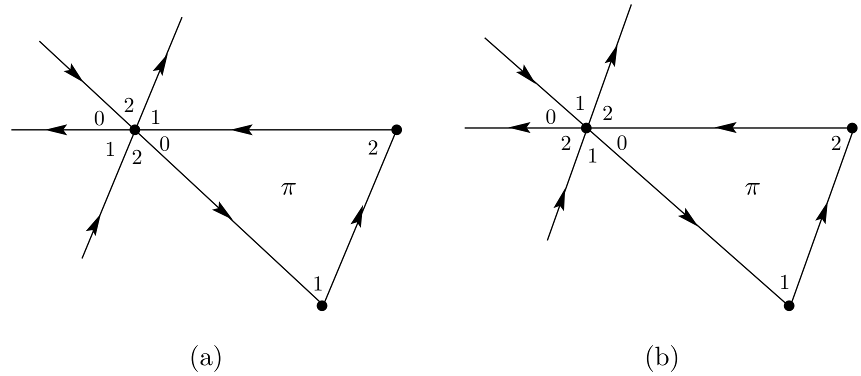


FIGURE 4.2

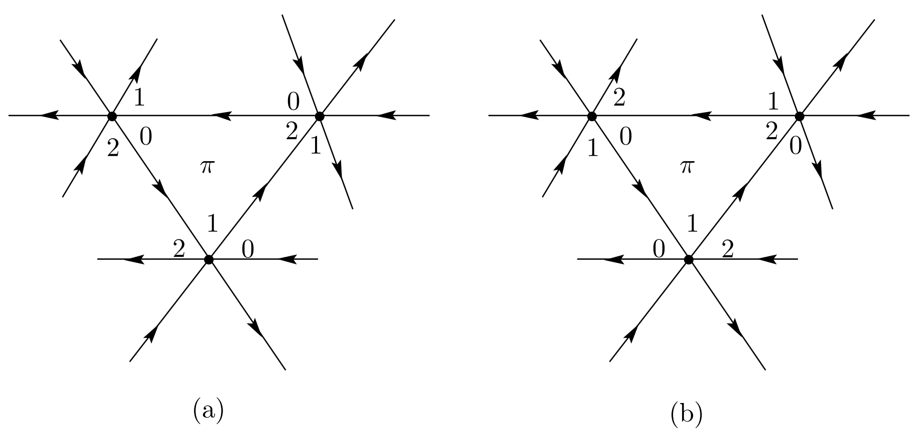


FIGURE 4.3

such that every vertex v of type II has exactly one angle α adjacent to v (i.e. $\nu(\alpha) = v$) with $\psi(\alpha) = i$. Next, it is easy to see that if v is of type I or v is an exterior vertex, then the number of angles α with $\psi(\alpha) = i$ is at most 2. Consequently, the total number, N_i, of angles α in H_r with $\psi(\alpha) = i$ can be estimated as follows:

$$\begin{aligned} N_i &\le 2(6r + V_1) + V_2 \le 12r + 2V - V_2 \le 12r + \tfrac{13}{7}V \\ &= 12r + \tfrac{13}{7}\left(3r^2 - 3r + 1\right) < \tfrac{39}{7}r^2 + 7r + 2. \end{aligned}$$

On the other hand, N_i obviously equals the total number of faces in H_r, that is, $6r^2$. Therefore, $6r^2 < \frac{39}{7}r^2 + 7r + 2$, whence $r \le 16$. Now we see that Theorem 1 applies to M_Δ (with $(p, q) = (3, 6)$ and $K = 16$) and yields that

$$|M_\Delta(2)| \le L|\partial M_\Delta|$$

with a constant L (e.g. $L = 300 \cdot 6^{17}$). The hyperbolicity of $\bar{G}$ now becomes obvious from (4.5). Case 5 is complete.

Thus, in Cases 1–3, G is not hyperbolic and, in Cases 4–5, G is hyperbolic.

Cases (3b)–(3d) and Part (3) of Theorem 3 are complete.

Part (4). Let $R = aBaCa^{-1}D$. Simplifying the presentation of G by an obvious automorphism of $F(\mathcal{A})$, assume $R = a^2Ba^{-1}C$ (where the new B, C are $B^{-1}CB$, D, respectively, in our original notation). If $\operatorname{rank}(\langle B, C\rangle) = 2$, the group G is hyperbolic by Lemma 3.1. Let $\operatorname{rank}(\langle B, C\rangle) = 1$ (the rank cannot be 0, for R is cyclically reduced) and let $\langle B, C\rangle$ be a subgroup of $\langle E\rangle$, where E is a nonempty cyclically reduced word which is not a proper power in $F(\mathcal{A})$ (if E were not cyclically reduced, then we could apply an inner automorphism of $F(\mathcal{A})$ to fix that), and $B = E^{n_1}$, $C = E^{n_2}$. Consider the following cases:

1. $n_1 = n_2$.
2. $n_1 = -n_2$.
3. $n_1 = -2n_2$ or $n_2 = -2n_1$.
4. $n_1 = 2n_2$ or $n_2 = 2n_1$.
5. None of Cases 1–4 holds.

1. It follows from $a^2E^{n_1}a^{-1}E^{n_1} \overset{G}{=} 1$ that $E^{n_1}a^3E^{-n_1}a^{-3} \overset{G}{=} 1$. Now the nonhyperbolicity of G follows from Theorems 4.11, 4.12 of [MKS] and our Lemma 4.3.

2. Assuming that G is hyperbolic, one quickly has a contradiction using Theorems 4.10, 4.12 of [MKS] and Lemma 4.3.

3. The second subcase $n_2 = -2n_1$ is reduced to the first one by renaming $a \to a^{-1}, E \to E^{-1}$. So let $R = a^2E^{-2n_2}a^{-1}E^{n_2}$. By Theorem 4.12 of [MKS], G is torsion free. Note that $aE^{2n_2}a^{-1} \overset{G}{=} E^{2n_2}a^2E^{-2n_2}$ (there is a diagram of this equation with 2 cells).

It is easy to see from Lemma 4.1 that $x^k = y^k$, $k \neq 0$, implies $x = y$ in a torsion free hyperbolic group. Hence, to prove that G is not hyperbolic it suffices to show that

$$aE^{n_2}a^{-1} \overset{G}{\neq} E^{2n_2}aE^{-2n_2}. \tag{4.6}$$

We will prove this by induction on the length $|E|$, using standard HNN-methods of studying one-relator groups (see Sect. 5 of Ch. IV in [LS]).

First assume that at least two different letters, say b and c, occur in E. Denote the sum of exponents on a letter d in a word W by $\sigma_d(W)$. If one of $\sigma_b(E)$, $\sigma_c(E)$ is 0, then we will present G as an HNN-extension of a one-relator group G_1 whose relator is $R_1 = a^2E_1^{-2n_2}a^{-1}E_1^{n_2}$ with the stable letter c (or b). Then (4.6) is immediate by the induction hypothesis, for $|E_1| = |E| - |E|_c < |E|$.

Suppose that $\sigma_b(E) = \alpha \neq 0$, $\sigma_c(E) = \beta \neq 0$. Introducing new letters x, y and applying the map

$$b \to yx^{-\beta},\ c \to x^{\alpha},\ d \to d,$$

where d is different from $b^{\pm1}, c^{\pm1}$, we have a homomorphism of G into the group G_0 whose relator, $R_0 = a^2E_0^{-2n_2}a^{-1}E_0^{n_2}$, results from rewriting the relator of G using the map above (E rewrites in E_0). It is clear that $\sigma_x(E_0) = 0$, and if we apply the reduction above to G_0, then the relator will become $a^2E_1^{-2n_2}a^{-1}E_1^{n_2}$ with $|E_1| = |E| - |E|_c < |E|$. By the induction hypothesis, (4.6) is proven provided E contains at least two different letters.

Now assume $E = b^k$, $k = \pm1$. Put $n = n_2k$. Then our presentation becomes

$$G = \langle\ a^{\pm1}, b^{\pm1} \parallel a^2b^{-2n}a^{-1}b^n\ \rangle$$

(for obvious reasons we suppose $\mathcal{A} = \{a^{\pm1}, b^{\pm1}\}$).

Applying the automorphism $a \to ab^n$, $b \to b$ of $F(\mathcal{A})$, we have another presentation for G:

$$G = \langle\ a^{\pm 1}, b^{\pm 1} \parallel ab^n ab^{-2n} a^{-1} b^n\ \rangle.$$

Renaming $a \to a_0$ and introducing new "HNN-generators" a_i and new "HNN-relations" $a_i = b^{-i} a_0 b^i$, $|i| \le |n|$, we also have that G is the HNN-extension

$$G = \langle\ b^{\pm 1}, a_i^{\pm 1} \parallel a_0 a_{-n} a_n^{-1}\ \ ,\ a_i = b^{-i} a_0 b^i\ ;\ |i| \le |n|\ \rangle\ .$$

Let us rewrite (4.6) in terms of the last presentation:

$$(a_0 b^n) b^n (a_0 b^n)^{-1} \overset{G}{\neq} b^{2n} a_0 b^{-2n},$$

whence

$$b^{-n} a_0 b^n a_0^{-1} b^n a_0^{-1} b^{-n} \overset{G}{\neq} 1, \text{ or } a_n a_0^{-1} a_{-n}^{-1} \overset{G}{\neq} 1.$$

Since $a_n \overset{G}{=} a_0 a_{-n}$, we now see that (4.6) is also equivalent to

$$a_0 a_{-n} a_0^{-1} a_{-n}^{-1} \overset{G}{\neq} 1.$$

But this is obviously true.

4. As in Case 3, the second subcase $n_2 = 2n_1$ is reduced to the first one by simple renaming $a \to a^{-1}, E \to E^{-1}$. So we let $n_1 = 2n_2$.

By Lemmas 3.3–3.4, the hyperbolicity of G would follow from the hyperbolicity of the group

$$G_1 = \langle\ a^{\pm 1}, b^{\pm 1} \parallel a^{6n_2} b^{2n_2} a^{-3n_2} b^{n_2}\ \rangle.$$

Let us show that this G_1 is indeed hyperbolic. Consider another presentation for G_1:

$$G_1 = \langle\ a^{\pm 1}, b^{\pm 1} \parallel a^{6n_2} (ba^{-1})^{2n_2} a^{-3n_2} (ba^{-1})^{n_2}\ \rangle.$$

Renaming $b \to b_0$ and introducing new "HNN-generators" b_i and "HNN-relations" $b_{i+1} = a^{-1} b_i a$, $i = 0, \pm 1, \pm 2, \dots$, we also have

$$\begin{aligned}(4.7)\quad G_1 = \langle\ a^{\pm 1}, b_i^{\pm 1} \parallel\ & a^{6n_2} (b_0 a^{-1})^{2n_2} a^{-3n_2} (b_0 a^{-1})^{n_2}, \\ & b_{i+1} = a^{-1} b_i a\ ;\ i = 0, \pm 1, \pm 2, \dots \rangle\ .\end{aligned}$$

Let G_0 be the subgroup of G_1 that consists of all words V such that the sum of exponents on a in V is 0. It is easy to see from our definitions that G_0, G_1 can be given by the following presentations:

$$\begin{aligned}G_1 = \langle\ a^{\pm 1}, b_i^{\pm 1} \parallel\ & b_0 b_1 \dots b_{2n_2 - 1} b_{5n_2} b_{5n_2+1} \dots b_{6n_2 - 1}, \\ & b_{i+1} = a^{-1} b_i a\ ;\ i = 0, \pm 1, \pm 2, \dots \rangle,\end{aligned}$$

$$\begin{aligned}(4.8)\quad G_0 = \langle\ b_i^{\pm 1} \parallel\ & b_i b_{i+1} \dots b_{i+2n_2-1} b_{i+5n_2} b_{i+5n_2+1} \dots b_{i+6n_2-1},\ ; \\ & i = 0, \pm 1, \pm 2, \dots \rangle\ .\end{aligned}$$

Note that presentation (4.7) satisfies a linear isoperimetric inequality provided (4.8) does. Indeed, in an obvious way one can establish a correspondance between diagrams Δ and Δ' (with the property that no proper subword of non-empty $\varphi(\partial\Delta), \varphi(\partial\Delta')$ equals 1 in G_1, G_0) over (4.7) and (4.8), respectively. Since this correspondance preserves the number of cells, one has that if Δ' over (4.8) corresponds in this way to Δ over (4.7), then $|\partial\Delta|_b = |\partial\Delta'|$. Now it is easy to see that

$$|\Delta(2)| \le L|\partial\Delta|_b \le L|\partial\Delta|,$$

since $|\Delta'(2)| \le L|\partial\Delta'|$.

Thus, to prove the hyperbolicity of G_1 it suffices to establish a linear isoperimetric inequality for the presentation (4.8) of G_0.

Let $W \overset{G_0}{=} 1$, $|W| > 0$, and suppose no proper subword of W equals 1 in G_0. Let Δ be a diagram over (4.8) with $\varphi(\partial\Delta) = W$ minimal relative to $|\Delta(2)|$ (in particular, Δ is reduced). Suppose π is a cell in Δ, $\partial\pi = e_0e_1\dots e_{3n_2-1}$, where $e_0, e_1, \dots, e_{3n_2-1}$ are edges. Denote the terminal vertex $(e_i)_+$ of e_i by o_i, $i = 0, 1, \dots, 3n_2 - 1$. Considering indices modulo $3n_2 - 1$, we will call o_i a *regular* vertex of π provided

$$\varphi(e_i) = b_{j_1}^{\varepsilon}, \quad \varphi(e_{i+1}) = b_{j_2}^{\varepsilon}$$

with $\varepsilon = \pm 1$ and $j_2 - j_1 = \varepsilon$.

Next, o_i is called a *positive pole* of π provided

$$\varphi(e_i) = b_{j_1}^{\varepsilon}, \quad \varphi(e_{i+1}) = b_{j_2}^{\varepsilon}$$

with $\varepsilon = \pm 1$ and $j_2 - j_1 = \varepsilon(3n_2 + 1)$, and a *negative pole* of π provided

$$\varphi(e_i) = b_{j_1}^{\varepsilon}, \quad \varphi(e_{i+1}) = b_{j_2}^{\varepsilon}$$

with $\varepsilon = \pm 1$ and $j_2 - j_1 = \varepsilon(-6n_2 + 1)$.

It is immediate from (4.8) that each vertex $o_i \in \partial\pi$ is either regular or a (positive or negative) pole of π and that every π of Δ has two poles: positive and negative ones.

Using the notation just introduced, we define labels of angles of cells in Δ as follows: Let $\alpha = (e_i, e_{i+1})$ be an angle of a cell π in Δ. Then the *label* $\chi(\alpha)$ of α is the integer $j_2 - j_1$. It is clear that $\chi(\alpha) \in \{\pm 1, \pm(3n_2 + 1), \pm(-6n_2 + 1)\}$ and the equations $\chi(\alpha) = \pm 1$, $\chi(\alpha) = \pm(3n_2 + 1)$, $\chi(\alpha) = \pm(-6n_2 + 1)$ are equivalent to the properties of the vertex $\nu(\alpha)$ of α of being regular, a positive pole, or a negative pole of π, respectively.

Let us make another useful observation. Suppose o is an interior vertex in Δ such that $d(o) = 4$, and if $o \in \partial\pi$ then o is a regular vertex of π. By e_1, e_2, e_3, e_4 denote the consecutive (in negative direction) edges whose terminal vertices are o. Define p_1, p_3 to be subpaths of maximal length $|p_1| = |p_3|$ with the following property: If $p_1 = f_1^1 \dots f_k^1$, $p_3 = f_1^3 \dots f_k^3$, where f_i^j are edges, then $f_k^1 = e_1$, $f_k^3 = e_3$, and the terminal vertices of f_i^1, f_i^3, $1 \le i < k$, are interior ones of degree 2 in Δ. Since Δ is reduced and the numbers $1, 3n_2 + 1, -6n_2 + 1$ are distinct, every vertex v out of the terminal vertices of f_i^1, f_i^3, $1 \le i < k$, is regular relative to either cell π with $v \in \partial\pi$. This observation enables us to conclude that $\varphi(p_1) = \varphi(p_3)$. Performing k obvious elementary transformations over Δ (which may be viewed as splitting p_1 into p_1', p_1'', p_3 into p_3', p_3'' with subsequent identification of p_1' and p_3', p_1'' and p_3''; see Fig. 4.4) results in a diagram that we denote by Δ'.

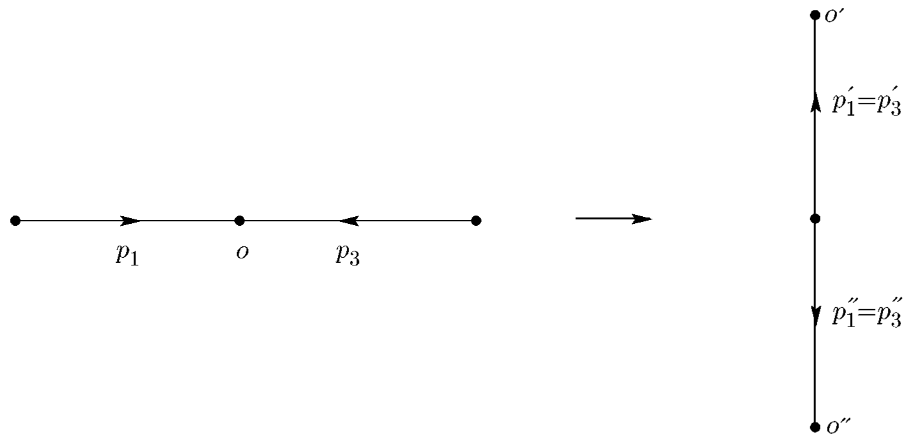


FIGURE 4.4

Notice that the choice of p_1, p_3 guarantees that the terminal vertex of p_1', p_1'' in Δ' does not have the properties of the vertex o in Δ. Hence, the described surgery on Δ diminishes the number of such o's in Δ by 1. Observing that Δ' is still reduced by the minimality of $|\Delta(2)| = |\Delta'(2)|$, we repeat our argument several times until we get a diagram $\Delta^{(\ell)}$ without interior vertex o such that $d(o) = 4$ and if $o \in \partial\pi$ then o is a regular vertex of π. Clearly, $\varphi(\partial\Delta^{(\ell)}) = \varphi(\partial\Delta)$ and $|\Delta^{(\ell)}(2)| = |\Delta(2)|$, so we rename $\Delta^{(\ell)}$ by Δ.

Now let us show that Δ (fixed in the above way) is a (3,6)-map. It is clear that the degree $d(o)$ of every interior vertex o is even. Assume $d(v) = 4$ and $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are angles of some cells $\pi_0, \pi_1, \pi_2, \pi_3$, respectively, whose vertex is o. Also assume that $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are counterclockwise consecutive (in the sense that $\alpha = (f_i, f_{i+1}^{-1})$, where f_i is an edge of $\partial\pi_i$ and $i \pmod 4$)). It is immediate that the inclusion $\alpha_i \in \{\varepsilon, \varepsilon(3n_2+1), \varepsilon(-6n_2+1)\}$ with $\varepsilon = \pm 1$ implies $\alpha_{i+1} \in \{-\varepsilon, -\varepsilon(3n_2+1), -\varepsilon(-6n_2+1)\}$ and that $\chi(\alpha_0) + \cdots + \chi(\alpha_3) = 0$. Since there is j such that o is not a regular vertex of π_j, we have $\chi(\alpha_j) \neq \pm 1$. It now becomes easy to see that there are no $\chi(\alpha_0), \ldots, \chi(\alpha_3) \in \{\pm 1, \pm(3n_2+1), \pm(-6n_2+1)\}$ with the properties pointed out above.

Thus the inequality $d(o) \geq 6$ for any interior vertex o of Δ with $d(v) > 2$ is proven.

Let us describe interior vertices of degree 6 in Δ. Let v be such a vertex, $\alpha_0, \ldots, \alpha_5$ consecutive angles of cells $\pi_0, \ldots, \pi_5$ whose vertex is v. If $\chi(\alpha_i) = \pm 1$ for each i, then we will call v to be a *vertex* of type III.

Assume v is not of type III. Denote $\Psi_\varepsilon = \{\varepsilon, \varepsilon(3n_2+1), \varepsilon(-6n_2+1)\}$, where $\varepsilon = \pm 1$. It is immediate that for some $\varepsilon = \pm 1$ it is true that $\alpha_0, \alpha_2, \alpha_4 \in \Psi_\varepsilon$ and that $\alpha_1, \alpha_3, \alpha_5 \in \Psi_{-\varepsilon}$. First suppose that the set $\pm\alpha_0, \ldots, \pm\alpha_5$ does not contain one of 1, $3n_2+1$, $-6n_2+1$. Using each time the fact that Δ is reduced, we easily obtain from $\sum \chi(\alpha_i) = 0$ one of the following contradictory equations:

$$(3n_2+1) - (-6n_2+1) = 0, \quad (-6n_2+1) - 1 = 0, \quad (3n_2+1) - 1 = 0.$$

Then each element of Ψ_1 is one of $\pm\chi(\alpha_i)$ for some i. Hence, applying a cyclic permutation of indices, we can put $\chi(\alpha_0) = \varepsilon(-6n_2+1)$ with $\varepsilon = \pm 1$. Since Δ is reduced, it follows from $\sum \chi(\alpha_i) = 0$ that either (a) $\chi(\alpha_3) = -\varepsilon(-6n_2+1)$, or (b) $\chi(\alpha_2) = \chi(\alpha_4) = \varepsilon(3n_2+1)$.

In case (a), we have again from $\sum \chi(\alpha_i) = 0$ that either $\chi(\alpha_1) = -\chi(\alpha_4) = -\varepsilon(3n_2+1)$ and $\chi(\alpha_5) = -\chi(\alpha_2) = -\varepsilon$, or $\chi(\alpha_1) = -\chi(\alpha_4) = -\varepsilon(3n_2+1)$ and $\chi(\alpha_5) = -\chi(\alpha_2) = -\varepsilon$. Such v will be referred to as a *vertex of type* I.

In case (b), we have $\chi(\alpha_1) + \chi(\alpha_3) + \chi(\alpha_5) = -3\varepsilon$, whence $\chi(\alpha_1) = \chi(\alpha_3) = \chi(\alpha_5) = -\varepsilon$. Such v will be referred to as a *vertex of type* II.

Thus it is proven that any interior vertex v of degree 6 in Δ has one of the described types I, II, III.

Let π be a cell in Δ with no edge $e \in \partial\pi$ such $e^{-1} \in \partial\Delta$. Since Δ is reduced, we have $d(o^+), d(o^-) > 2$, where o^+, o^- are positive and negative poles of π. Let $\partial\pi^\varepsilon = u_1u_2$, where $\varepsilon = \pm 1$ is such that $(u_1)_- = o^+$, $(u_2)_- = o^-$. Then

$$\varphi(u_1) = b_{j+5n_2}b_{j+5n_2+1}\dots b_{j+6n_2-1}, \quad \varphi(u_2) = b_jb_{j+1}\dots b_{j+2n_2-1}.$$

It follows from $|u_1| = n_2$, $|u_2| = 2n_2$ that the path u_2 must contain a vertex o different from the endpoints of u_2 with $d(o) > 2$.

Obviously, Δ has no interior vertices of degree 1 (as the relators are cyclically reduced).

Now it is proven that the map associated with Δ is a (3,6)-map.

Consider another map, M'_Δ, obtained from Δ by disregarding interior vertices of degree 2 in Δ (and labels of edges of Δ). Define a labeling function, ψ, on the set of angles of faces in M'_Δ as follows: Let α' be an angle of a face π' in M'_Δ, $\alpha' = (e', f')$, and let the edges e', f' be the images in M'_Δ of subpaths $e = e_1\dots e_{\ell_1}$, $f = f_1\dots f_{\ell_2}$ of π, where π is the pre-image in Δ of π'. Consider the angle $\alpha = (e_{\ell_1}, f_1)$ of π in Δ. Then we put $\psi(\alpha') = 0$ provided $\chi(\alpha) = \pm 1$, $\psi(\alpha') = 1$ provided $\chi(\alpha) = \pm(3n_2+1)$, and $\psi(\alpha') = 2$ provided $\chi(\alpha) = \pm 1(-6n_2+1)$.

The same argument we used in proving the inequality $d(\partial\pi) \geq 3$ for any interior face π of Δ enables us to claim that if π' is a face in M'_Δ with $d(\pi') = 3$ and α'_0, α'_1, α'_2 are angles of π', then the sequence $\psi(\alpha'_0)$, $\psi(\alpha'_1)$, $\psi(\alpha'_2)$ is a cyclic permutation of either $0, 1, 2$ or $2, 1, 0$. Also, notice that the classification of interior vertices in Δ obtained above yields a similar classification of vertices of degree 6 in M'_Δ. In particular, if o is a vertex of degree 6 in M'_Δ and $\alpha_0, \dots, \alpha_5$ are the consecutive angles whose vertex is o, then o is of type III provided

$$\psi(\alpha_0) = \dots = \psi(\alpha_5) = 1.$$

Next, o is of type I provided for some i either

$$\psi(\alpha_i) = \psi(\alpha_{i+3}) = 0, \quad \psi(\alpha_{i+1}) = \psi(\alpha_{i+4}) = 1, \quad \psi(\alpha_{i+2}) = \psi(\alpha_{i+5}) = 2,$$

or

$$\psi(\alpha_i) = \psi(\alpha_{i+3}) = 0, \quad \psi(\alpha_{i+1}) = \psi(\alpha_{i+4}) = 2, \quad \psi(\alpha_{i+2}) = \psi(\alpha_{i+5}) = 1.$$

Finally, o is of type II if for some i

$$\psi(\alpha_i) = \psi(\alpha_{i+2}) = \psi(\alpha_{i+4}) = 0$$

and the sequence $\psi(\alpha_{i+1}),\ \psi(\alpha_{i+3}),\ \psi(\alpha_{i+5})$ contains two 1's and one 2.

Now assume that M'_Δ contains a regular (3,6)-submap of radius $r \geq 2$. Then M'_Δ also contains a regular (3,6)-submap that looks like a hexagon, H_{r-1}, whose sides have length $r-1$ (see Fig. 4.1 with $r-1 = 2$) such that each triangle (= face) in H_{r-1} is an interior face of degree 3 in M'_Δ.

Keeping in mind the construction of M'_Δ, it is easy to show that H_{r-1} has no interior (relative to H_{r-1}) vertices of type III (for otherwise, Δ would not be reduced). Now we can literally repeat our reasoning in Case 5 of Part (3) to obtain, as there, that $r-1 \le 16$. By Theorem 1 (with $K = 17$), there is a constant L (e.g. $L = 300 \cdot 6^{17}$) such that

$$|\Delta(2)| = |M'_\Delta(2)| \le L|\partial M'_\Delta| = L|\partial\Delta|.$$

Thus presentation (4.8) does satisfy a linear isoperimetric inequality, and Case 4 is complete.

5. By Lemma 3.3, to prove that G is hyperbolic it suffices to show that the group

$$\bar{G} = \langle\, a^{\pm 1}, b^{\pm 1} \parallel a^2 b^{n_1} a^{-1} b^{n_2} \,\rangle. \tag{4.9}$$

is hyperbolic.

Consider a non-empty word W over $\{a^{\pm 1}, b^{\pm 1}\}$ such that $W \overset{\bar{G}}{=} 1$ and no proper subword of W equals 1 in $\bar{G}$. Let Δ be a diagram over $\bar{G}$ with $\varphi(\partial\Delta) = W$ and minimal $|\Delta(2)|$. As in Case 5 of Part (3), we construct a map M_Δ from Δ by contracting all edges e with $\varphi(e) = b^{\pm 1}$ into points. Let v be an interior vertex in M_Δ and $e_0, \dots, e_{\ell-1}$ be all consecutive (in the positive direction) oriented edges whose terminal vertex is v. Denoting the pre-images in Δ of $e_0, \dots, e_{\ell-1}$ also by $e_0, \dots, e_{\ell-1}$, we see that $e_0, \dots, e_{\ell-1}$ form an interior a-star $\mathrm{St}(e_0, e_1)$ in Δ. Since Δ is reduced and $\varphi(\mathrm{St}(e_0, e_1)) = 1$ in the free group $F(a^{\pm 1}, b^{\pm 1})$, we have that $\ell \ge 4$. Analogously, the assumption $\ell = 4$ implies one of the two equations $n_1 \pm n_2 = 0$, which are impossible in Case 5. The assumption $\ell = 5$ implies one of the four equations $n_1 \pm 2n_2 = 0$, $n_2 \pm 2n_1 = 0$, also not possible. Thus, M_Δ is a (3,6)-map.

Keeping the above notation, assume $\ell = 6$. Then one of the four equations $n_1 \pm 3n_2 = 0$, $n_2 \pm 3n_1 = 0$ holds. Let $n_1 = 3n_2$ (the other three cases are analogous). As in Case 5 of Part (3) (see the notation introduced there), we assign labels to angles of a face π in M_Δ by putting $\psi(\alpha) = 0$ provided $\varphi(s) = 1$, $\psi(\alpha) = 1$ provided $\varphi(s) = b^{\pm n_1}$, and $\psi(\alpha) = 2$ provided $\varphi(s) = b^{\pm n_2}$. Clearly, if $\alpha_0, \dots, \alpha_5$ are the angles adjacent to the vertex v of degree 6, then in the sequence $\psi(\alpha_0), \dots, \psi(\alpha_5)$ there are exactly two 0's, one 1, and three 2's.

Now assume M_Δ contains a regular (3,6)-submap of radius r. Then M_Δ also contains H_r. Denote the number of angles in α's with $\psi(\alpha) = 1$ by N_1 and the number of interior vertices in H_r by V. Then, arguing as in Case 5 of Part (3), we have

$$N_1 = 6r^2 \le 3|\partial H_r| + V = 18r + (3r^2 - 3r + 1).$$

Consequently, $r^2 \le 5r + 1/3$, whence $r \le 5$. Now a reference to Theorem 1 completes Case 5, as in Case 5 of Part (3).

Part (4) is also complete.

It remains to make the easy observation that, up to cyclic permutations and taking inverses, a non-empty cyclically reduced word R with $2 \le |R|_a \le 3$ has one of the forms that have been discussed above.

Theorem 3 is proven. □

5. Proof of Theorem 4

Our claims for $n \le 3$ follow directly from Theorem 3. So we let $n \ge 4$. Let Δ be a reduced diagram over G, $\mathrm{St}(e_0, e_1) = \{e_0, \dots, e_{\ell-1}\}$ an interior a-star in Δ.

Clearly, ℓ is even and $\ell > 2$, by our assumptions on the relator R. Consequently, the map M_Δ, obtained from Δ by contracting all but a-edges as in Case 5 of Parts 3–4 of the proof of Theorem 3, is a (4,4)-map. Assume M_Δ contains a regular (4,4)-submap of radius $2r+1 > 0$. Then M_Δ also contains a "large" square S_r tessellated into r^2 "small" faces.

Clearly, $2r+1 < 2$ provided $n \geq 5$ (because $d(\pi) > 4$ for any face π in M_Δ). Then, as in Case 5 of Parts 3–4 of the proof of Theorem 3, G is hyperbolic by Theorem 1.

Let $n = 4$ and consider the following two basic cases:

A. For some $i \in \{0,1,2,3\}$ it is true in $F(\mathcal{A})$ with subscripts mod 4 that

$$T_i T_{i+1}^{-1} T_{i+2} T_{i+3}^{-1} = 1.$$

B. For every $i \in \{0,1,2,3\}$ it is true in $F(\mathcal{A})$ with subscripts mod 4 that

$$T_i T_{i+1}^{-1} T_{i+2} T_{i+3}^{-1} \neq 1.$$

Case A. Let

$$W = a^{\varepsilon_0} V_0 a^{\varepsilon_1} V_1 \dots a^{\varepsilon_{k-1}} V_{k-1}$$

be a cyclically reduced word over $\mathcal{A}$, $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{k-1} = \pm 1$, and let the words $V_0, \dots, V_{k-1}$ contain no occurrences of $a^{\pm 1}$. Then by the *a-skeleton* $\mathrm{Sk}_a(W)$ of W we mean the cyclic word $\varepsilon_0\varepsilon_1 \dots \varepsilon_{k-1}$ over the alphabet $\{1,-1\}$. A subword $\varepsilon_i\varepsilon_{i+1}$ (subscripts mod k) of the word $\varepsilon_0\varepsilon_1 \dots \varepsilon_{k-1}$ with $k > 1$ is called *regular* provided the product $\varepsilon_i\varepsilon_{i+1}$ equals -1. Otherwise, the subword $\varepsilon_i\varepsilon_{i+1}$ is *irregular*. The *defect*, $\mathrm{df}(\varepsilon_0\varepsilon_1 \dots \varepsilon_{k-1})$, of $\mathrm{Sk}_a(W) = \varepsilon_0\varepsilon_1 \dots \varepsilon_{k-1}$ with $k > 1$ is defined to be the number of all irregular subwords of $\varepsilon_0\varepsilon_1 \dots \varepsilon_{k-1}$. For example, $\mathrm{df}(11) = 2$, $\mathrm{df}(-11-1) = 1$.

Lemma 5.1. *Let W be a non-empty cyclically reduced word and $W = 1$ in G. Then $|W|_a > 1$ and $\mathrm{df}(\mathrm{Sk}_a(W)) \geq 4$.*

Proof. Set $\mathrm{Sk}_a(W) = \varepsilon_0\varepsilon_1 \dots \varepsilon_{k-1}$. Since $|W|_a$ is even (for $\mathrm{Sk}_a(R) = 1111$), we see from Theorem 4.10 of [MKS] that $|W|_a > 1$.

Arguing on the contrary, assume the existence of a word W such that $W = 1$ in G and $\mathrm{df}(\mathrm{Sk}_a(W)) < 4$. Let us pick such a W so that $|W|_a$ is minimal and, if $|W|_a$ is fixed, $|W|$ is minimal. It is easy to see that the word W contains no subword V such that $0 < |V|_a < |W|_a$ and $V = V'$ in G with $|V'|_a = 0$. Consider a reduced diagram Δ over G with $\varphi(\partial\Delta) = W$. Clearly, the map M_Δ (constructed as above from Δ) has no proper subpath p of ∂M_Δ such that $p_- = p_+$.

Set $\partial M_\Delta = e_0 e_1 \dots e_{k-1}$ and consider the pre-image in Δ of e_i, which we will also denote by e_i (we may assume the indices of e_i match those of ε_i, that is, $\varphi(e_i) = a^{\varepsilon_i}$). Notice that if $\varepsilon_i\varepsilon_{i+1} = -1$, then the exterior vertex $(e_i)_+$ has degree ≥ 3, because of the absence of edges $e \in \partial M_\Delta$ with $e^{-1} \in \partial M_\Delta$. Consequently, if $v = (e_j)_+$ is an exterior vertex in M_Δ and $d(v) = 2$, then $\varepsilon_j\varepsilon_{j+1} = 1$. Therefore, the number of exterior vertices v with $d(v) = 2$ does not exceed $\mathrm{df}(\mathrm{Sk}_a(W)) < 4$.

As was pointed out above, the map M_Δ (for any reduced Δ) is a $(4,4)$-map and there is no proper subpath p of ∂M_Δ with $p_- = p_+$. Consequently, equation (1.6)

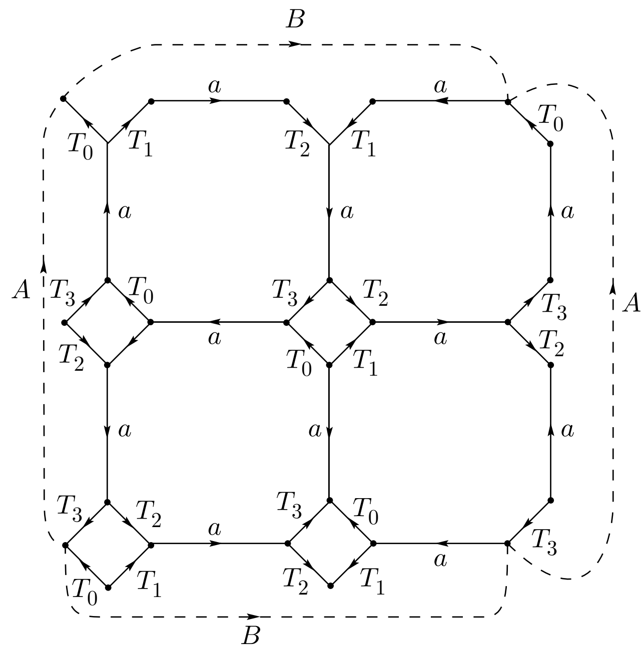


FIGURE 5.1

applies to M_Δ (where $(p,q)=(4,4)$ and hence $(\frac{p}{q}(q-3)-1)|\partial M|=0$), and yields

$$4=\sum_{v\in M_{\Delta E}(0)}(3-d(v))+\sum_{v\in M_{\Delta I}(0)}(4-d(v))$$
$$+\sum_{\pi\in M_{\Delta E}(2)}(4-d(\pi))+\sum_{\pi\in M_{\Delta I}(2)}(4-d(\pi)).$$

Clearly, positive terms in the right part of this equation may only occur in the first sum. Hence, the right part is less than or equal to $\mathrm{df}(\mathrm{Sk}_a(W))<4$. This contradiction completes the proof of Lemma 5.1. □

By the assumption of Case A, for some i we have $T_iT_{i+1}^{-1}T_{i+2}T_{i+3}^{-1}=1$ in $F(\mathcal{A})$. For definiteness put $i=0$, that is, $T_0T_1^{-1}T_2T_3^{-1}=1$ in $F(\mathcal{A})$. Then it follows from

$$aT_0aT_1aT_2aT_3\overset{G}{=}1$$

that

$$[A,B]=[T_3^{-1}a^{-1}T_2^{-1}T_3aT_0,\ T_0^{-1}T_1aT_2T_1^{-1}a^{-1}]\overset{G}{=}1,$$

where $[A,B]=ABA^{-1}B^{-1}$ denotes the commutator of the words

$$A=T_3^{-1}a^{-1}T_2^{-1}T_3aT_0,\qquad B=T_0^{-1}T_1aT_2T_1^{-1}a^{-1}$$

(see Fig. 5.1 for a diagram of this equality consisting of 4 cells).

Assume G to be hyperbolic. By Theorem 4.12 of [MKS], G is torsion free. Then, by Lemma 4.3, it follows from $[A,B]\overset{G}{=}1$ that $A^k\overset{G}{=}B^\ell$ with $\max(|k|,|\ell|)>0$.

Assume $\min(|k|,|\ell|)=0$. Since G is torsion free, we have $\max(|k|,|\ell|)=1$. Let, say, $k=1$. Then $A=T_3^{-1}a^{-1}T_2^{-1}T_3aT_0\overset{G}{=}1$. However, by Lemma 5.1, this equation is impossible for $\mathrm{df}(\mathrm{Sk}_a(A)) = 0$. Analogously, $B \overset{G}{\neq} 1$. Thus $\min(|k|, |\ell|) > 0$ and one can write

$$A^k B^{-\ell} \overset{G}{=} 1$$

with $k\ell \neq 0$. Consider a reduced diagram Δ with $\varphi(\partial\Delta) = V$, where V is a cyclically reduced word conjugate in $F(\mathcal{A})$ to $A^k B^{-\ell}$. It is easy to see that $|V|_a = |A^k B^{-\ell}|_a$, and so $\mathrm{Sk}_a(V) = \mathrm{Sk}_a(A^k B^{-\ell}) = (-11)^{|k|}(1-1)^{|\ell|}$. Consequently, $\mathrm{df}(\mathrm{Sk}_a(V)) = 2$.

Let W be a non-empty subword of V such that $W \overset{G}{=} 1$, and let W be minimal relative to $|W|$. It is easy to check that no proper subword of the cyclic word W equals 1 in G and that $\mathrm{df}(\mathrm{Sk}_a(W)) \leq 3$. But the existence of such W contradicts Lemma 5.1.

Thus $k = \ell = 0$, and G is not hyperbolic by Lemma 4.2.

Case A1 is complete.

Case B. Assume that $r \geq 6$ (recall M_Δ contains a "big square" S_r). Let us further subdivide this case into:

B1. $\mathrm{rank}(\langle T_i T_0^{-1} | i = 0, 1, 2, 3\rangle) = 1$.
B2. $\mathrm{rank}(\langle T_i T_0^{-1} | i = 0, 1, 2, 3\rangle) = 2$.
B3. $\mathrm{rank}(\langle T_i T_0^{-1} | i = 0, 1, 2, 3\rangle) = 3$.

B1. Without loss of generality we may obviously suppose that $\langle T_0, T_1, T_2, T_3\rangle \subseteq \langle D\rangle$, where D is a non-empty cyclically reduced word. So we put $T_i = D^{n_i}$, $i = 0, 1, 2, 3$, and $R = aD^{n_0}aD^{n_1}aD^{n_2}aD^{n_3}$. Then the assumption $T_0 T_1^{-1} T_2 T_3^{-1} \neq 1$ is equivalent to

$$n_0 + n_2 \neq n_1 + n_3, \tag{5.1}$$

where n_0, n_1, n_2, n_3 are some distinct integers.

Let us come back to the diagram Δ and maps M_Δ, S_r defined in the beginning of this section. Keeping the notation of the definition of the labeling function $\varphi(\mathrm{St}(e_0, e_1))$, we assign ψ-labels to angles of faces in M_Δ as follows: $\psi(\alpha) = i$ provided $\varphi(v_i) = T_i^{\pm 1}$ $(= D^{n_i})$.

To proceed we need to introduce more special cases:

B1.1. There is a vertex $o \in S_r$ with $\mathrm{dist}(o, \partial S_r) \geq 2$ such that if $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are the angles adjacent to o (i.e. their vertex is o), then $\{\psi(\alpha_0), \dots, \psi(\alpha_3)\} = \{0, 1, 2, 3\}$.
B1.2. Case B1.1 does not hold.

Case B1.1. Let the angles $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ be consecutive, i.e. $\alpha_i = (e_i, e_{i+1}^{-1})$, where $e_0, \dots, e_3$ are oriented edges whose terminal vertex is o.

Renaming if necessary, one may assume $\psi(\alpha_0) = 0$.

Consider the following six cases (see Fig. 5.2, in which the arrows denote directions of the a-edges e in Δ with $\varphi(e) = a$):

C1. $\psi(\alpha_1) = 1$, $\psi(\alpha_2) = 2$, $\psi(\alpha_3) = 3$.
C2. $\psi(\alpha_1) = 1$, $\psi(\alpha_2) = 3$, $\psi(\alpha_3) = 2$.
C3. $\psi(\alpha_1) = 2$, $\psi(\alpha_2) = 1$, $\psi(\alpha_3) = 3$.
C4. $\psi(\alpha_1) = 2$, $\psi(\alpha_2) = 3$, $\psi(\alpha_3) = 1$.
C5. $\psi(\alpha_1) = 3$, $\psi(\alpha_2) = 1$, $\psi(\alpha_3) = 2$.
C6. $\psi(\alpha_1) = 3$, $\psi(\alpha_2) = 2$, $\psi(\alpha_3) = 1$.

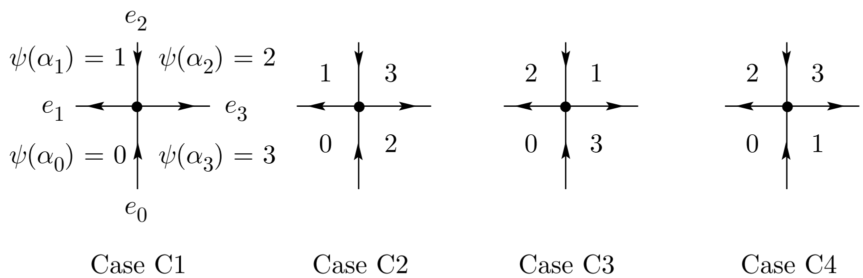


FIGURE 5.2

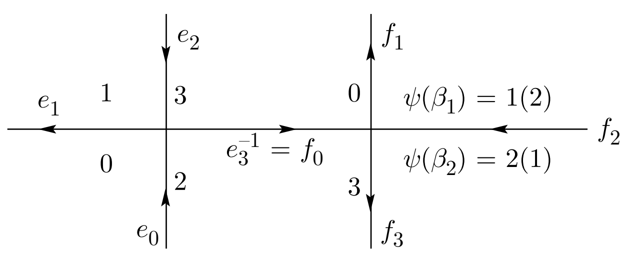


FIGURE 5.3

Case C1. Reading off the label of the interior a-star in Δ whose image in M_Δ is $\{e_0, \dots, e_3\}$, we get $D^{n_0 - n_1 + n_2 - n_3} = 1$ in $F(\mathcal{A})$, whence $n_0 - n_1 + n_2 - n_3 = 0$, contrary to (5.1). Thus Case C1 is impossible.

Case C2. Consider two more subcases:

C2.1. There is an $i \in \{1, 3\}$ such that if $\beta_0, \dots, \beta_3$ are the angles adjacent to $(e_i)_-$ (recall that $\alpha_i = (e_i, e_{i+1}^{-1})$), then $\{\psi(\beta_0), \dots, \psi(\beta_3)\} = \{0, 1, 2, 3\}$.

C2.2. Case C2.1 does not hold.

Case C2.1. Note the case $i = 1$ is reduced to $i = 3$ by shifting indices by $2 \pmod 4$. So we assume $i = 3$ and let f_j, $j = 0, 1, 2, 3$, be the edges such that $(f_j)_+ = (e_3)_-$, $\beta_j = (f_j, f_{j+1}^{-1})$ are the angles adjacent to $(e_3)_-$, and $f_0 = e_3^{-1}$. Then, by construction of M_Δ, we have $\psi(\beta_0) = 0$ (for $\psi(\alpha_2) = 3$) and $\psi(\beta_3) = 3$ (for $\psi(\alpha_3) = 2$). See Fig. 5.3.

Clearly, either $\psi(\beta_1) = 1$ and $\psi(\beta_2) = 2$, or $\psi(\beta_1) = 2$ and $\psi(\beta_2) = 1$. Considering the labels of interior a-stars $\{e_0, \dots, e_3\}$, $\{f_0, \dots, f_3\}$ in Δ, we obtain one of

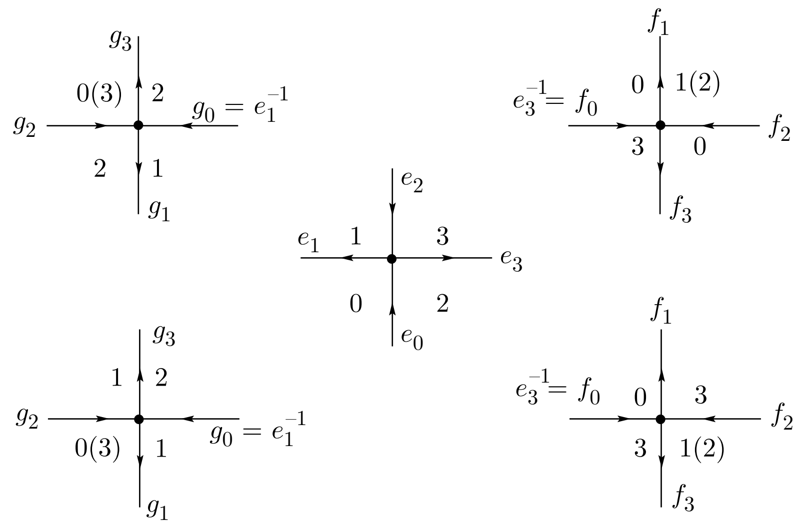


FIGURE 5.4

the following 2 systems of 2 equations (see Fig. 5.3):

$$\begin{gathered} n_1 + n_2 = n_0 + n_3, \\ n_0 + n_2 = n_3 + n_1, \text{ or } n_0 + n_1 = n_3 + n_2. \end{gathered} \tag{5.2}$$

The second equation of the first system contradicts (5.1). The second system implies the impossible equation $n_1 = n_3$.

Thus Case C2.1 is impossible.

Case C2.2. Keeping the notation introduced in Case C2.1, we also introduce analogous notation relating to the vertex $(e_1)_-$: Let g_j, $j = 0, 1, 2, 3$, be the edges such that $(g_j)_+ = (e_1)_-$, $g_0 = e_1^{-1}$, and $\gamma_j = (g_j, g_{j+1}^{-1})$ the angles adjacent to $(e_1)_-$.

As in Case C2.1, it is clear that $\psi(\beta_0) = 0$, $\psi(\beta_3) = 3$ because $\psi(\alpha_2) = 3$, $\psi(\alpha_3) = 2$, respectively, and that $\psi(\gamma_0) = 1$, $\psi(\gamma_3) = 2$ because $\psi(\alpha_0) = 0$, $\psi(\alpha_1) = 1$, respectively.

Since Δ is reduced and all n_i's are distinct, there is no interior vertex o in M_Δ such that the set of labels of the angles $\delta_0, \dots, \delta_{\ell-1}$ adjacent to o consists of ≤ 2 elements. Making use of this observation and the fact that $\psi(\beta_j) \neq \psi(\beta_{j+1})$, $\psi(\gamma_j) \neq \psi(\gamma_{j+1})$ for each j (mod 4), it is easy to see that there are 16 ways to assign labels to the angles $\beta_1, \beta_2, \gamma_1, \gamma_2$. See Fig. 5.4.

Reading off the labels of the interior a-stars $\{e_0, \dots, e_3\}$, $\{f_0, \dots, f_3\}$, and $\{g_0, \dots, g_3\}$ in Δ, we have one of the following 16 systems of 3 equations (see Fig. 5.4):

(5.3)

$$\begin{gathered} n_0 + n_3 = n_1 + n_2, \\ 2n_0 = n_3 + n_1, \text{ or } 2n_0 = n_3 + n_2, \text{ or } 2n_3 = n_0 + n_1, \text{ or } 2n_3 = n_0 + n_2, \\ 2n_1 = n_2 + n_0, \text{ or } 2n_1 = n_2 + n_3, \text{ or } 2n_2 = n_1 + n_0, \text{ or } 2n_2 = n_1 + n_3. \end{gathered}$$

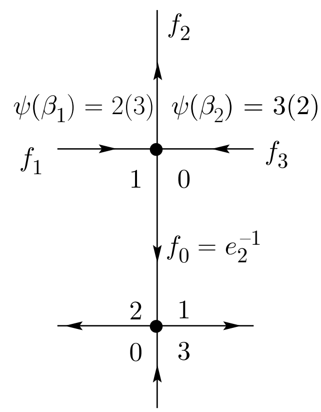


FIGURE 5.5

Let $x_i = n_i - n_0$, $i = 0, 1, 2, 3$. Then the systems (5.3) can be rewritten as follows:

$$\begin{gathered} x_3 = x_1 + x_2, \\ x_3 + x_1 = 0, \text{ or } x_3 + x_2 = 0, \text{ or } 2x_3 = x_1, \text{ or } 2x_3 = x_2, \\ 2x_1 = x_2, \text{ or } 2x_1 = x_2 + x_3, \text{ or } 2x_2 = x_1, \text{ or } 2x_2 = x_1 + x_3. \end{gathered}$$

Eliminating x_3, we have

$$\begin{gathered} 2x_1 + x_2 = 0, \text{ or } x_1 + 2x_2 = 0, \text{ or } x_1 + 2x_2 = 0, \text{ or } 2x_1 + x_2 = 0, \\ 2x_1 - x_2 = 0, \text{ or } 2x_2 - x_1 = 0, \text{ or } 2x_2 - x_1 = 0, \text{ or } 2x_1 - x_2 = 0. \end{gathered}$$

Now it becomes evident that any of these 16 systems has only the trivial solution, whence $n_0 = n_1 = n_2 = n_3$.

This contradiction proves that Cases C2.2 and C2 are not possible either.

Case C3. Consider two more subcases:

C3.1. There is an $i \in \{0, 2\}$ such that if $\beta_0, \dots, \beta_3$ are the angles adjacent to $(e_i)_-$ (recall that $\alpha_i = (e_i, e_{i+1}^{-1})$), then $\{\psi(\beta_0), \dots, \psi(\beta_3)\} = \{0, 1, 2, 3\}$.

C3.2. Case C3.1 does not hold.

Case C3.1. Note that the case $i = 0$ is reduced to $i = 2$ by shifting indexes by 2 (mod 4). So we assume $i = 2$ and let f_j, $j = 0, 1, 2, 3$, be the edges such that $(f_j)_+ = (e_2)_-$, $\beta_j = (f_j, f_{j+1}^{-1})$ are the angles adjacent to $(e_2)_-$, and $f_0 = e_2^{-1}$. Then, by construction of M_Δ, we have $\psi(\beta_0) = 1$ (for $\psi(\alpha_1) = 2$) and $\psi(\beta_3) = 0$ (for $\psi(\alpha_2) = 1$); see Fig. 5.5.

It is clear that either $\psi(\beta_1) = 2$ and $\psi(\beta_2) = 3$, or $\psi(\beta_1) = 3$ and $\psi(\beta_2) = 2$. Considering the labels of the interior a-stars $\{e_0, \dots, e_3\}$, $\{f_0, \dots, f_3\}$ in Δ, we obtain one of the following 2 systems of 2 equations:

$$\begin{gathered} n_0 + n_1 = n_2 + n_3, \\ n_1 + n_3 = n_0 + n_2, \quad \text{or} \quad n_1 + n_2 = n_0 + n_3, \end{gathered} \tag{5.4}$$

which all contradict either (5.1) or $n_1 \neq n_3$.

Hence Case C3.1 is impossible.

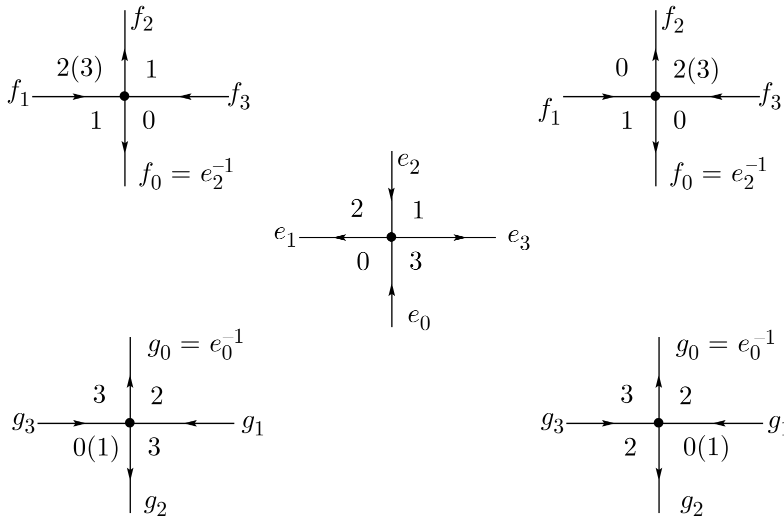


FIGURE 5.6

Case C3.2. Keeping the notation introduced in Case C3.1, we also let g_j, $j = 0, 1, 2, 3$, be the edges such that $(g_j)_+ = (e_0)_-$, $g_0 = e_0^{-1}$, and $\gamma_j = (g_j, g_{j+1}^{-1})$ are the angles adjacent to $(e_0)_-$.

Clearly, $\psi(\beta_0) = 1$, $\psi(\beta_3) = 0$, because $\psi(\alpha_1) = 2$, $\psi(\alpha_2) = 1$, respectively, and $\psi(\gamma_0) = 2$, $\psi(\gamma_3) = 3$, because $\psi(\alpha_3) = 3$, $\psi(\alpha_0) = 0$, respectively.

As in Case C2.2, since Δ is reduced, we see that there are 16 ways to assign labels to angles β_1, β_2, γ_1, γ_2, as shown in Fig. 5.6.

Referring to the labels of the interior a-stars $\{e_0, \dots, e_3\}$, $\{f_0, \dots, f_3\}$, and $\{g_0, \dots, g_3\}$ in Δ, we have one of the following 16 systems of 3 equations (see Fig. 5.6):

(5.5)

$$n_0 + n_1 = n_2 + n_3,$$
$$2n_1 = n_0 + n_2, \text{ or } 2n_1 = n_0 + n_3, \text{ or } 2n_0 = n_1 + n_2, \text{ or } 2n_0 = n_1 + n_3,$$
$$2n_3 = n_2 + n_0, \text{ or } 2n_3 = n_2 + n_1, \text{ or } 2n_2 = n_3 + n_0, \text{ or } 2n_2 = n_3 + n_1.$$

Setting $x_i = n_i - n_0$, $i = 0, 1, 2, 3$, rewrite (5.5) as follows:

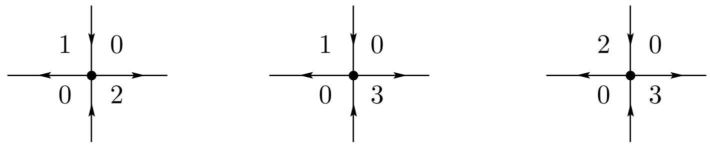
$$x_1 = x_2 + x_3,$$
$$2x_1 = x_2, \text{ or } 2x_1 = x_3, \text{ or } x_1 + x_2 = 0, \text{ or } x_1 + x_3 = 0,$$
$$2x_3 = x_2, \text{ or } 2x_3 = x_2 + x_1, \text{ or } 2x_2 = x_3, \text{ or } 2x_2 = x_3 + x_1.$$

Eliminating x_1, we have

$$2x_3 + x_2 = 0, \text{ or } 2x_2 + x_3 = 0, \text{ or } 2x_2 + x_3 = 0, \text{ or } 2x_3 + x_2 = 0,$$
$$2x_3 - x_2 = 0, \text{ or } 2x_2 - x_3 = 0, \text{ or } 2x_2 - x_3 = 0, \text{ or } 2x_3 - x_2 = 0.$$

Hence, any of these 16 systems implies that $x_1 = x_2 = x_3 = 0$ and so $n_0 = n_1 = n_2 = n_3$.

This contradiction shows that Cases C3.2 and C3 are not possible either.



Case D1 Case D2 Case D3

FIGURE 5.7

Cases C4–6. Cases C4, C5, C6 are reduced to Cases C3, C2, C1, respectively, by changing the indices $i \to i+1 \pmod 4$.

Case B1.1 is proven to be impossible.

Case B1.2. In this case, it is obvious that S_r contains a submap of the form S_{r-2} such that if $\alpha_0, \dots, \alpha_3$ are the angles adjacent to any interior vertex o in S_{r-2}, then the set $\{\psi(\alpha_0), \dots, \psi(\alpha_3)\}$ consists of 3 elements.

Since $r-2 \geq 4$, we can pick a vertex o in S_{r-2} with $\operatorname{dist}(o, \partial S_{r-1}) \geq 2$.

Let $\alpha_0, \dots, \alpha_3$ be the angles adjacent to the vertex o in S_{r-2}, $\alpha_i = (e_i, e_{i+1}^{-1})$, where e_i, $i = 0,1,2,3$, are the edges with $(e_i)_+ = o$. Without loss of generality, one may assume $\psi(\alpha_0) = \psi(\alpha_2) = 0$. Since Δ is reduced, up to apparent symmetry one of the following 3 cases holds (see Fig. 5.7 where an arrow again denotes the orientation of an edge e so that $\varphi(e) = a$ in Δ):

D1. $\psi(\alpha_1) = 1$ and $\psi(\alpha_3) = 2$.
D2. $\psi(\alpha_1) = 1$ and $\psi(\alpha_3) = 3$.
D3. $\psi(\alpha_1) = 2$ and $\psi(\alpha_3) = 3$.

Before getting down to Cases D1–D3 let us introduce some more notation.

Let $f_j,\ g_j,\ h_j,\ k_j$, $j = 0,1,2,3$, be the edges given by

$$(f_j)_+ = (e_0)_- = v_0,\ (g_j)_+ = (e_1)_- = v_1,\ (h_j)_+ = (e_2)_- = v_2,\ (k_j)_+ = (e_3)_- = v_3$$

and let $\beta_j,\ \gamma_j,\ \delta_j,\ \varepsilon_j$, $j = 0,1,2,3$, be the angles adjacent to v_0, v_1, v_2, v_3, respectively, such that $f_0 = e_0^{-1}$, $g_0 = e_1^{-1}$, $h_0 = e_2^{-1}$, $k_0 = e_3^{-1}$, and

$$\beta_j = (f_j, f_{j+1}^{-1}),\ \gamma_j = (g_j, g_{j+1}^{-1}),\ \delta_j = (h_j, h_{j+1}^{-1}),\ \varepsilon_j = (k_j, k_{j+1}^{-1})$$

(see Fig. 5.8).

Case D1. It is clear that

$$\psi(\beta_0) = 1,\ \psi(\beta_3) = 3,\ \psi(\gamma_0) = 1,\ \psi(\gamma_3) = 2,\ \psi(\delta_0) = 0,\ \psi(\delta_0) = 3,$$

and that there are 256 ways to assign labels to $\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2$, all shown in Fig. 5.9 (recall that the set of labels of the angles adjacent to any of v_1, v_2, v_3, v_4 consists of 3 elements).

Reading off the labels of $\{e_0, \dots, e_3\}$, $\{f_0, \dots, f_3\}$, $\{g_0, \dots, g_3\}$, $\{h_0, \dots, h_3\}$ in Δ, we have one of the following 64 systems of 4 equations (see Fig. 5.9):

$$\begin{aligned}
&2n_0 = n_1 + n_2,\\
&2n_1 = n_3 + n_0, \text{ or } 2n_1 = n_3 + n_2, \text{ or } 2n_3 = n_1 + n_0, \text{ or } 2n_3 = n_1 + n_2,\\
&2n_1 = n_2 + n_0, \text{ or } 2n_1 = n_2 + n_3, \text{ or } 2n_2 = n_1 + n_0, \text{ or } 2n_2 = n_1 + n_3,\\
&2n_0 = n_3 + n_1, \text{ or } 2n_0 = n_3 + n_2, \text{ or } 2n_3 = n_0 + n_1, \text{ or } 2n_3 = n_0 + n_2.
\end{aligned} \tag{5.6}$$

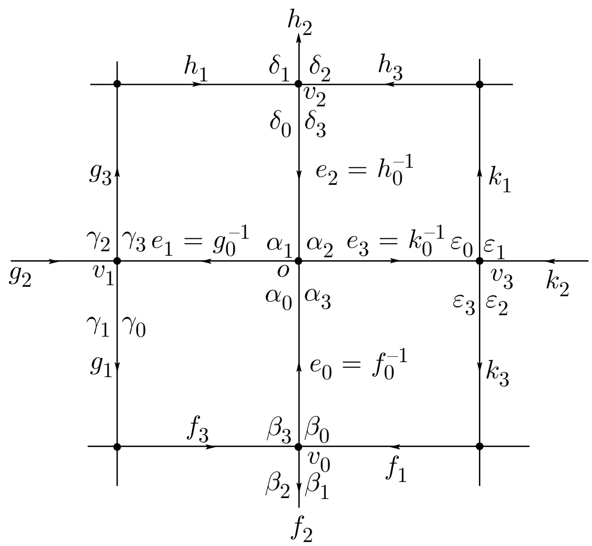


FIGURE 5.8

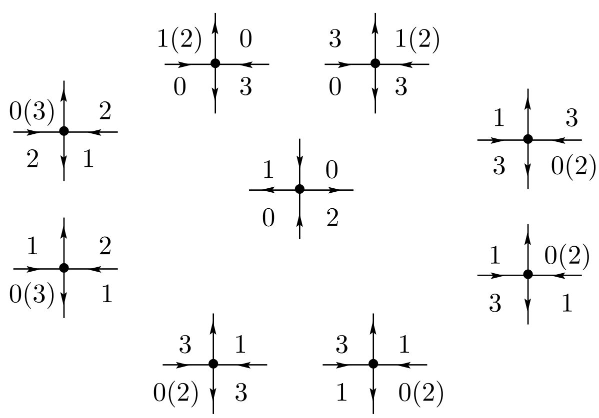


FIGURE 5.9

Setting $x_i = n_i - n_0$, $i = 0, 1, 2, 3$, we have

$$
\begin{aligned}
&x_1 + x_2 = 0,\\
&\quad 2x_1 = x_3, \text{ or } 2x_1 = x_3 + x_2, \text{ or } 2x_3 = x_1, \text{ or } 2x_3 = x_1 + x_2,\\
&\quad 2x_1 = x_2, \text{ or } 2x_1 = x_2 + x_3, \text{ or } 2x_2 = x_1, \text{ or } 2x_2 = x_1 + x_3,\\
&x_1 + x_3 = 0, \text{ or } x_3 + x_2 = 0, \text{ or } 2x_3 = x_1, \text{ or } 2x_3 = x_2.
\end{aligned}
$$

Eliminating x_1, we further have

$$
\begin{aligned}
-2x_2 = x_3, &\text{ or } -3x_2 = x_3, \text{ or } 2x_3 = -x_2, \text{ or } 2x_3 = 0,\\
3x_2 = 0, &\text{ or } -3x_2 = x_3, \text{ or } 3x_2 = 0, \text{ or } 3x_2 = x_3,\\
x_2 = x_3, &\text{ or } x_2 = -x_3, \text{ or } 2x_3 = -x_2, \text{ or } 2x_3 = x_2.
\end{aligned}
$$

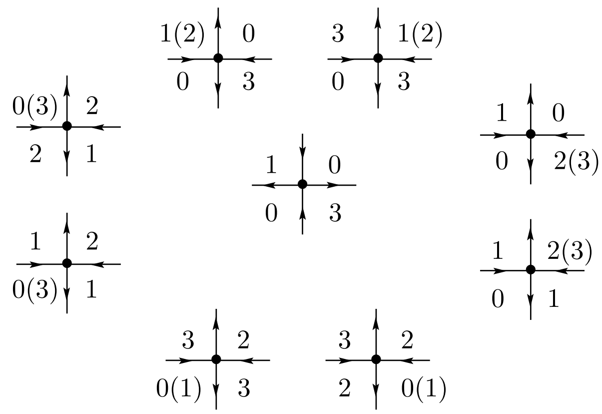


Figure 5.10

Now it becomes easy to see that any of these 64 systems has only the trivial solution. Hence $n_0 = n_1 = n_2 = n_3$.

This contradiction shows that Case D1 is impossible.

Case D2. Quite analogously to Case D1, we have 256 ways to assign labels to the angles β_1, β_2, γ_1, γ_2, δ_1, δ_2, $\varepsilon_1, \varepsilon_2$ that are all shown in Fig. 5.10.

Considering the labels of $\{e_0, \dots, e_3\}$, $\{f_0, \dots, f_3\}$, $\{g_0, \dots, g_3\}$, $\{h_0, \dots, h_3\}$ in Δ, we have one of the following 64 systems of 4 equations (see Fig. 5.10):

$$
\begin{aligned}
&2n_0 = n_1 + n_3,\\
&2n_2 = n_3 + n_0, \text{ or } 2n_2 = n_3 + n_1, \text{ or } 2n_3 = n_2 + n_0, \text{ or } 2n_3 = n_2 + n_1,\\
&2n_1 = n_2 + n_0, \text{ or } 2n_1 = n_2 + n_3, \text{ or } 2n_2 = n_1 + n_0, \text{ or } 2n_2 = n_1 + n_3,\\
&2n_0 = n_3 + n_1, \text{ or } 2n_0 = n_3 + n_2, \text{ or } 2n_3 = n_0 + n_1, \text{ or } 2n_3 = n_0 + n_2.
\end{aligned} \tag{5.7}
$$

Setting $x_i = n_i - n_0$, $i = 0, 1, 2, 3$, we have

$$
\begin{aligned}
&x_1 + x_3 = 0,\\
&\quad 2x_2 = x_3, \text{ or } 2x_2 = x_3 + x_1, \text{ or } 2x_3 = x_2, \text{ or } 2x_3 = x_2 + x_1,\\
&\quad 2x_1 = x_2, \text{ or } 2x_1 = x_2 + x_3, \text{ or } 2x_2 = x_1, \text{ or } 2x_2 = x_1 + x_3,\\
&x_3 + x_1 = 0, \text{ or } x_3 + x_2 = 0, \text{ or } 2x_3 = x_1, \text{ or } 2x_3 = x_2.
\end{aligned}
$$

Let us eliminate x_1:

$$
\begin{gathered}
2x_2 = x_3, \text{ or } 2x_2 = 0, \text{ or } 2x_3 = x_2, \text{ or } 3x_3 = x_2,\\
-2x_3 = x_2, \text{ or } -3x_3 = x_2, \text{ or } 2x_2 = -x_3, \text{ or } 2x_2 = 0,\\
0 = 0, \text{ or } x_3 = -x_2, \text{ or } 3x_3 = 0, \text{ or } 2x_3 = x_2.
\end{gathered}
$$

Any of these 64 systems implies that $x_2 = 0$, whence $n_0 = n_2$, contrary to $n_i \neq n_j$.

Case D2 is impossible.

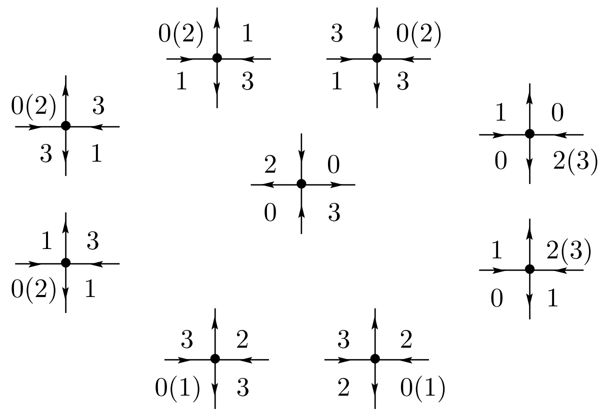


FIGURE 5.11

Case D3. Reading off the labels of the interior a-stars $\{e_0, \dots, e_3\}$, $\{f_0, \dots, f_3\}$, $\{g_0, \dots, g_3\}$, $\{k_0, \dots, k_3\}$ in Δ, we have one of the following 64 systems of 4 equations (see Fig. 5.11):

$$\begin{aligned} &2n_0 = n_2 + n_3, \\ &2n_2 = n_3 + n_0, \text{ or } 2n_2 = n_3 + n_1, \text{ or } 2n_3 = n_2 + n_0, \text{ or } 2n_3 = n_2 + n_1, \\ &2n_1 = n_3 + n_0, \text{ or } 2n_1 = n_3 + n_2, \text{ or } 2n_3 = n_1 + n_0, \text{ or } 2n_3 = n_1 + n_2, \\ &2n_0 = n_1 + n_2, \text{ or } 2n_0 = n_1 + n_3, \text{ or } 2n_1 = n_0 + n_2, \text{ or } 2n_1 = n_0 + n_3. \end{aligned} \tag{5.8}$$

Setting $x_i = n_i - n_0$, $i = 0, 1, 2, 3$, and eliminating x_3, we have

$$\begin{aligned} &3x_2 = 0, \text{ or } 3x_2 = x_1, \text{ or } 3x_2 = 0, \text{ or } -3x_2 = x_1, \\ &2x_1 = -x_2, \text{ or } 2x_1 = 0, \text{ or } -2x_2 = x_1, \text{ or } -3x_2 = x_1, \\ &x_1 = -x_2, \text{ or } x_1 = x_2, \text{ or } 2x_1 = x_2, \text{ or } 2x_1 = -x_2. \end{aligned}$$

Now it is clear that any of these 64 systems has only the trivial solution. Hence $n_0 = n_1 = n_2 = n_3$.

This contradiction shows that Case D3 is not possible either.

Cases B1.2 and B1 are thus proven to be impossible.

Case B2. Our approach is basically the same as in Case B1. But instead of solving systems of linear equations we are going to solve the systems of equations in a free group that are obtained the same way as in Case B1.

Without loss of generality we may obviously assume that

$$T_0 T_1^{-1} T_2 T_3^{-1} \neq 1 \tag{5.9}$$

in $F(\mathcal{A})$.

The definitions of analogs of Cases B1.1–B1.2, C1–C6, C2.1–C2.2, C3.1–C3.2, D1–D3 are literally retained.

Case C1. As in Case C1 above, we now have $T_0 T_1^{-1} T_2 T_3^{-1} = 1$, contrary to (5.9).

Case C2.1. Instead of (5.2), we now have one of the following 2 systems of 2 equations in $F(\mathcal{A})$:

$$T_0T_1^{-1}T_3T_2^{-1} = 1,$$
$$T_0T_1^{-1}T_2T_3^{-1} = 1, \text{ or } T_0T_2^{-1}T_1T_3^{-1} = 1.$$

Let $X_i = T_iT_0^{-1}$, $i = 1, 2, 3$. Then our systems turn into

$$X_1^{-1}X_3X_2^{-1} = 1,$$
$$X_1^{-1}X_2X_3^{-1} = 1, \text{ or } X_2^{-1}X_1X_3^{-1} = 1.$$

Since $\operatorname{rank}(\langle X_1, X_2, X_3\rangle) = 2$, it follows from the first equation of the systems above that $\operatorname{rank}(\langle X_i, X_j \mid i \neq j\rangle) = 2$. Eliminating X_1 yields that

$$X_2X_3^{-1}X_2X_3^{-1} = 1, \text{ or } X_2^{-1}X_3X_2^{-1}X_3^{-1} = 1,$$

contrary to $\operatorname{rank}(\langle X_2, X_3\rangle) = 2$.

Case C2.2. Similarly to (5.3), we have one of the following 16 systems of 3 equations in $F(\mathcal{A})$ (we do not explicitly write down the last line):

$$T_0T_1^{-1}T_3T_2^{-1} = 1,$$
$$T_0T_1^{-1}T_0T_3^{-1} = 1, \text{ or } T_0T_2^{-1}T_0T_3^{-1} = 1, \text{ or } T_3T_0^{-1}T_3T_1^{-1} = 1, \text{ or } T_3T_0^{-1}T_3T_2^{-1} = 1,$$
$$\ldots = \ldots.$$

Let $X_i = T_iT_0^{-1}$, $i = 1, 2, 3$. Then we can rewrite the above systems as follows:

$$X_1^{-1}X_3X_2^{-1} = 1,$$
$$X_1^{-1}X_3^{-1} = 1, \text{ or } X_2^{-1}X_3^{-1} = 1, \text{ or } X_3^2X_1^{-1} = 1, \text{ or } X_3^2X_2^{-1} = 1,$$
$$\ldots = \ldots.$$

Since $\operatorname{rank}(\langle X_1, X_2, X_3\rangle) = 2$, it follows from the first equation of the systems above that $\operatorname{rank}(\langle X_i, X_j \mid i \neq j\rangle) = 2$. This, however, is impossible in view of the second equation of any system.

Case C3.1. Similarly to (5.4), we have one of the following 2 systems of 2 equations in $F(\mathcal{A})$:

$$T_0T_2^{-1}T_1T_3^{-1} = 1,$$
$$T_0T_1^{-1}T_2T_3^{-1} = 1, \text{ or } T_0T_1^{-1}T_3T_2^{-1} = 1.$$

Let $X_i = T_iT_0^{-1}$, $i = 1, 2, 3$. Then our systems become

$$X_2^{-1}X_1X_3^{-1} = 1,$$
$$X_1^{-1}X_2X_3^{-1} = 1, \text{ or } X_1^{-1}X_3X_2^{-1} = 1.$$

Since $\operatorname{rank}(\langle X_1, X_2, X_3\rangle) = 2$, it follows from the first equation of the systems above that $\operatorname{rank}(\langle X_i, X_j \mid i \neq j\rangle) = 2$. Eliminating $X_1 = X_2X_3$, we have

$$X_3^{-2} = 1, \text{ or } X_3^{-1}X_2^{-1}X_3X_2^{-1} = 1.$$

This obvious contradiction to $\operatorname{rank}(\langle X_2, X_3\rangle) = 2$ completes Case C3.1.

Case C3.2. Similarly to (5.5), we have one of the following 16 systems of 3 equations in $F(\mathcal{A})$ (we do not write down explicitly the last line):

$$\begin{aligned}
&T_0T_2^{-1}T_1T_3^{-1} = 1,\\
&T_1T_0^{-1}T_1T_2^{-1} = 1 \text{ or } T_1T_0^{-1}T_1T_3^{-1} = 1 \text{ or } T_0T_1^{-1}T_0T_2^{-1} = 1 \text{ or } T_0T_1^{-1}T_0T_3^{-1} = 1,\\
&\qquad \ldots = \ldots .
\end{aligned}$$

Let $X_i = T_iT_0^{-1}$, $i = 1, 2, 3$. Then we can rewrite the above systems as follows:

$$\begin{aligned}
&X_2^{-1}X_1X_3^{-1} = 1,\\
&\quad X_1^2X_2^{-1} = 1, \text{ or } X_1^2X_3^{-1} = 1, \text{ or } X_1^{-1}X_2^{-1} = 1, \text{ or } X_1^{-1}X_3^{-1} = 1,\\
&\qquad \ldots = \ldots .
\end{aligned}$$

Since $\operatorname{rank}(\langle X_1, X_2, X_3\rangle) = 2$, it follows from the first equation of the systems above that $\operatorname{rank}(\langle X_i, X_j \mid i \neq j\rangle) = 2$. This, however, is impossible in view of the second equation of any system.

Case D1. Similarly to (5.6), we have one of the following 64 systems of 4 equations in $F(\mathcal{A})$ (we skip the last two lines):

$$\begin{aligned}
&T_0T_1^{-1}T_0T_2^{-1} = 1,\\
&T_1T_3^{-1}T_1T_0^{-1} = 1, \text{ or } T_1T_3^{-1}T_1T_2^{-1} = 1, \text{ or } T_3T_1^{-1}T_3T_0^{-1} = 1, \text{ or } T_3T_1^{-1}T_3T_2^{-1} = 1,\\
&\qquad \ldots = \ldots ,\\
&\qquad \ldots = \ldots .
\end{aligned}$$

Let $X_i = T_iT_0^{-1}$, $i = 1, 2, 3$. Then the above systems turn into

$$\begin{aligned}
&X_1^{-1}X_2^{-1} = 1,\\
&X_1X_3^{-1}X_1 = 1, \text{ or } X_1X_3^{-1}X_1X_2^{-1} = 1, \text{ or } X_3X_1^{-1}X_3 = 1, \text{ or } X_3X_1^{-1}X_3X_2^{-1} = 1,\\
&\qquad \ldots = \ldots ,\\
&\qquad \ldots = \ldots .
\end{aligned}$$

Since $\operatorname{rank}(\langle X_1, X_2, X_3\rangle) = 2$, it follows from the first equation of the systems above that $\operatorname{rank}(\langle X_1, X_3\rangle) = \operatorname{rank}(\langle X_2, X_3\rangle) = 2$. This, however, is impossible in view of the second equation of any system.

Case D2. Similarly to (5.7), we have one of the following 64 systems of 4 equations in $F(\mathcal{A})$ (we skip the last two lines):

$$\begin{aligned}
&T_0T_3^{-1}T_0T_1^{-1} = 1,\\
&T_2T_3^{-1}T_2T_0^{-1} = 1, \text{ or } T_2T_3^{-1}T_2T_1^{-1} = 1, \text{ or } T_3T_2^{-1}T_3T_0^{-1} = 1, \text{ or } T_3T_2^{-1}T_3T_1^{-1} = 1,\\
&\qquad \ldots = \ldots ,\\
&\qquad \ldots = \ldots .
\end{aligned}$$

Let $X_i = T_i T_0^{-1}$, $i = 1, 2, 3$. Then the above systems become

$$
\begin{aligned}
&X_3^{-1} X_1^{-1} = 1,\\
&X_2 X_3^{-1} X_2 = 1, \text{ or } X_2 X_3^{-1} X_2 X_1^{-1} = 1, \text{ or } X_3 X_2^{-1} X_3 = 1, \text{ or } X_3 X_2^{-1} X_3 X_1^{-1} = 1,\\
&\qquad \ldots = \ldots,\\
&\qquad \ldots = \ldots.
\end{aligned}
$$

Since $\operatorname{rank}(\langle X_1, X_2, X_3\rangle) = 2$, it follows from the first equation of the systems above that $\operatorname{rank}(\langle X_1, X_2\rangle) = \operatorname{rank}(\langle X_2, X_3\rangle) = 2$. This, however, is impossible in view of the second equation of any system.

Case D3. Similarly to (5.8), we have one of the following 64 systems of 4 equations in $F(\mathcal{A})$ (we skip the two middle lines):

$$
\begin{aligned}
&T_0 T_2^{-1} T_0 T_3^{-1} = 1,\\
&\qquad \ldots = \ldots,\\
&\qquad \ldots = \ldots,\\
&T_0 T_1^{-1} T_0 T_2^{-1} = 1 \text{ or } T_0 T_1^{-1} T_0 T_3^{-1} = 1 \text{ or } T_1 T_0^{-1} T_1 T_2^{-1} = 1 \text{ or } T_1 T_0^{-1} T_1 T_3^{-1} = 1.
\end{aligned}
$$

Let $X_i = T_i T_0^{-1}$, $i = 1, 2, 3$. Then the above systems become

$$
\begin{aligned}
&X_2^{-1} X_3^{-1} = 1,\\
&\qquad \ldots = \ldots,\\
&\qquad \ldots = \ldots,\\
&X_1^{-1} X_2 = 1, \text{ or } X_1^{-1} X_3 = 1, \text{ or } X_1^2 X_2^{-1} = 1, \text{ or } X_1^2 X_3^{-1} = 1.
\end{aligned}
$$

Since $\operatorname{rank}(\langle X_1, X_2, X_3\rangle) = 2$, it follows from the first equation of the systems above that $\operatorname{rank}(\langle X_1, X_2\rangle) = \operatorname{rank}(\langle X_1, X_3\rangle) = 2$. This, however, is impossible in view of the second equation of any system.

Thus Case B2 is proven to be impossible.

Case B3. In this case, the map M_Δ obviously has no interior vertices, whence a contradiction to $r \ge 6$ is immediate.

Thus, in Case B (consisting of Cases B1–B3), a contradiction to the inequality $r \ge 6$ is obtained. Consequently $r < 8$, and so radii of regular (4,4)-submaps of M_Δ are bounded by $2r + 1 \le 11$. It remains to refer to Theorem 1 to get that the group G satisfies a linear isoperimetric inequality with $L = 100 \cdot 4^{12}$.

Hence G is hyperbolic, and the proof of Theorem 4 is complete. $\square$

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