THE $L_2$-LOCALIZATION OF $W(n)$

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Abstract. In this paper we analyze the localization of $W(n)$, the fiber of the double suspension map $S^{2n-1} \to \Omega^2 S^{2n+1}$, with respect to $E(2)$. If four cells at the bottom of $D_p M^{2np-1}$, the $p$th extended power spectrum of the Moore spectrum, are collapsed to a point, then one obtains a spectrum $C$. Let $QM^{2np-1} \to QC$ be the James-Hopf map followed by the collapse map. Then we show that the secondary suspension map $BW(n) \to QM^{2np-1}$ has a lifting to the fiber of $QM^{2np-1} \to QC$ and this lifting is shown to be a $v_2$-periodic equivalence, hence an $E(2)$-equivalence.

1. Introduction

We begin by recalling the following construction from [24]. Consider the fiber sequence

$$
F \longrightarrow QS^{2n+1} \xrightarrow{j_p} QD_p S^{2n+1}
$$

where $j_p$ is the James-Hopf map and $D_p S^{2n+1}$ is the $p$th extended power construction on the sphere. The stabilization map $S^{2n+1} \to QS^{2n+1}$ lifts to a map $S^{2n+1} \to F$, and in [24] it is shown that this lifting induces an isomorphism in complex $K$-theory. It follows that there is an equivalence $L_1 S^{2n+1} \simeq L_1 F$ where $L_1$ stands for Bousfield localization with respect to $K$-theory on the category of spaces. This result enables one to get a handle on $L_1 S^{2n+1}$ since the functor $L_1$ is reasonably well behaved on fiber sequences, $L_1$ of an infinite loop space is something very close to the localization of the corresponding spectrum, and $K$-theory localization stably is well understood.

The aim of this paper is to explore an analogous construction for $L_2 W(n)$. $L_2$ refers to Bousfield localization with respect to the $p$-local homology theory $E(2)$ with coefficients $E(2)_* = Z(p)[v_1, v_2, v_2^{-1}]$ (for example see [28]). $W(n)$ is the homotopy fiber of the double suspension map $S^{2n-1} \to \Omega^2 S^{2n+1}$, localized at a prime $p$. For technical reasons which probably have to do with our method of proof more than anything else, we will assume $p \geq 5$. The analogue of the stabilization map is a ‘secondary suspension map’, which is a map $W(n) \to QM^{2np-2}$ that is degree one on the bottom Moore space. Here $M^k$ denotes a mod $p$ Moore space with top cell in dimension $k$. There are various constructions of maps such as this, for example see [8]. It will be more convenient to start with a delooped version of the secondary suspension. In [12] it is shown that there exists a delooping of $W(n)$,
denoted $BW(n)$. It follows from the construction of $BW(n)$ that there is a map $BW(n) \to QM^{2np-1}$ which has degree one on the bottom cell. See [12] for details.

Consider the James-Hopf map

$$QM^{2np-1} \xrightarrow{j_p} QD_p M^{2np-1}.$$ 

The left side of Figure 1 gives a cell diagram for $D_p M^{2np-1}$. The short and long lines represent the actions of the Milnor primitives $Q_0$ and $Q_1$ respectively. Note the four cells near the bottom in dimensions $2np^2 - 1, 2np^2 - 2, 2np^2 - 2p + 1$, and $2np^2 - 2p$. Denote this 4-cell complex by $X$. Since $p$ is odd, $X$ can be collapsed to a point. Let $C$ denote the complex $D_p M^{2np-1}/X$ which is pictured on the right side of Figure 1, and consider the fiber sequence

$$(1.1) \quad G^n \xrightarrow{i} QM^{2np-1} \longrightarrow QC,$$

where the second map is the James-Hopf map $j_p$ composed with $Q(\pi)$ where $\pi$ is the collapse map.

In the case of the sphere $S^{2n+1}$, the lifting of the stabilization map exists for purely dimensional reasons. Since $BW(n)$ is not finite dimensional, a secondary suspension map does not lift for such a simple reason.

Our first result is the following:
**Theorem 1.2.** Assume \( p \geq 5 \) and \( n \geq 1 \). There exists a map
\[
\sigma_1 : \text{BW}(n) \to \Omega^{2p}\text{BW}(n + 1)
\]
which is degree one on the bottom Moore space. The mapping telescope of the diagram
\[
\text{BW}(n) \to \Omega^{2p}\text{BW}(n + 1) \to \Omega^{4p}\text{BW}(n + 2) \to \ldots
\]
is \( \text{QM}^{2np-1} \). If we let \( \sigma : \text{BW}(n) \to \text{QM}^{2np-1} \) denote the inclusion into the telescope, then there exists a map \( \lambda : \text{BW}(n) \to \text{G}^n \) such that \( i \circ \lambda = \sigma \).

This will be proved in section 2 by analyzing some properties of the James-Hopf maps. The hypothesis that \( p \geq 5 \) is required in order to use certain properties of Gray’s delooping of \( W(n) \) ([12]).

Our main result is the following:

**Theorem 1.3.** Assume \( p \geq 5 \) and \( 2np - 2 - k \) is sufficiently large. Then
\[
\lambda : \Omega^k\text{BW}(n) \to \Omega^k\text{G}^n
\]
induces an isomorphism in \( E(2)_* \), hence
\[
L_2\Omega^k\text{BW}(n) \simeq L_2\Omega^k\text{G}^n.
\]

Just how large \( 2np - 2 - k \) must be for the theorem to hold is discussed below.

In [24] the \( K \)-theory isomorphism induced by the map \( S^{2n+1} \to F \) is established by direct calculation of \( K_*(F) \) relying on, among other things, the results of [27]. Techniques for calculating the \( E(2) \)-homology of spaces such as \( \Omega^k\text{BW}(n) \) and \( \Omega^k\text{G}^n \) are not in place yet, so Theorem 1.3 will be deduced from Theorem 1.5 stated below, via the following theorem of A. K. Bousfield [3]. In order to state this we recall some definitions.

For each \( m \geq 1 \), let \( V_{m-1} \) denote some finite cell complex which has type \( m \), i.e. \( K(i)_{V_{m-1}} = 0 \) if \( i < m \) and \( K(m)_{V_{m-1}} \neq 0 \), where \( K(i) \) is the \( i \)th Morava \( K \)-theory spectrum (see [28]). Let \( v : \Sigma^1V_{m-1} \to V_{m-1} \) be a \( v_m \) self map, i.e. a map inducing an isomorphism in \( K(m)_* \), and inducing the zero map in \( K(i)_* \) if \( i \neq m \). Define the homotopy groups of a space \( Y \) with coefficients in \( V_{m-1} \) by
\[
\pi_t(Y; V_{m-1}) = [\Sigma^1V_{m-1}, Y]
\]
and define the \( v_m \)-periodic homotopy groups of \( Y \), which we will denote by
\[
v_m^{-1}\pi_t(Y; V_{m-1}),
\]
as the colimit of the sequence
\[
\pi_t(Y; V_{m-1}) \xrightarrow{v_*} \pi_{t+d}(Y; V_{m-1}) \xrightarrow{v_*} \ldots
\]
It can be shown that these periodic groups do not depend on the choice of \( v \). They do depend on the choice of \( V_{m-1} \), however if a map induces an isomorphism in \( v_m^{-1}\pi_t(V_{m-1}) \) with one choice of \( V_{m-1} \), then it also will with any other choice (Corollary 11.11, [3]). So for purposes of making statements about \( v_m \)-periodic isomorphisms, we are free to choose \( V_{m-1} \) as we like.

For each \( n \), Bousfield defines an integer \( c(n) \). The precise value of \( c(n) \) is not known. Very roughly, \( c(n) \) is bounded above by the dimension of the bottom cell of a minimally connected type \( n \) complex \( V_{n-1} \) which is a suspension. Also, \( c(n) \) is bounded below by \( n + 1 \). It is known that \( c(0) = 1 \) and \( c(1) = 2 \). Define a
functor \( \tilde{\Omega} \), going from the category of \( c(n) \)-connected spaces to itself, as the \( c(n) \)-connected cover of the loop space functor \( \Omega \). Let \( E_* \) be a homology theory. We say a map \( f : X \to Y \) in the homotopy category of \( c(n) \)-connected spaces is a durable \( E_* \)-equivalence if \( \tilde{\Omega}^k f : \tilde{\Omega}^k X \to \tilde{\Omega}^k Y \) is an \( E_* \)-equivalence for all \( k \geq 0 \).

The following is distilled from Bousfield [3].

**Theorem 1.4** (Bousfield, 13.3 and 13.15 of [3]). Let \( f : X \to Y \) be a map in the homotopy category of \( c(n) \)-connected spaces. Then \( f \) induces an isomorphism in \( v^{-1}_m \pi_t(V^{m-1}) \) for all \( 0 \leq m \leq n \) if and only if \( f \) is a durable \( E_* \)-equivalence for all spectra \( E \) such that \( E^*(V_n) = 0 \).

Such an equivalence is called a \( v_n \)-periodic equivalence. In particular, a \( v_n \)-periodic equivalence is always an \( E(n)_* \)-isomorphism.

The condition on \( n \) and \( k \) in Theorem 1.3 can be stated more precisely now: \( 2np - 2 - k \) is sufficiently large if \( \Omega^k BW(n) \) is \( c(2) \)-connected.

Thus by using Bousfield’s theorem we see that Theorem 1.3 follows from the following:

**Theorem 1.5.** Assume that \( p \geq 5 \) and \( n \geq 1 \). The map \( \lambda : BW(n) \to G^n \) induces an isomorphism in unstable \( v_m \)-periodic homotopy groups for \( 0 \leq m \leq 2 \), i.e \( \lambda \) is a \( v_2 \)-periodic equivalence.

Theorem 1.5 will be proved in section 3. The proof is an adaptation to the present situation of the methods employed in [25], [23], [22], and [30]. In particular, Theorem 1.5 could be viewed as an odd primary analogue of the main result [25] which deals with the case \( p = 2 \). However there are two significant differences. The first is that in [25], we do not know if there is a map analogous to \( \lambda \) of Theorem 1.2. This means that the statement concerning \( v_2 \)-periodic homotopy groups does not obviously translate into a result concerning homological localization. The second is that the lambda algebra calculations of [25] for \( p = 2 \) do not readily carry over to the odd primary case.

We deal with this second point by using the results of B. Gray concerning the odd primary lambda algebra [13] and [14]. Thus Theorem 1.5 is concerned with the application of the machinery of [13] and [14] to the unstable Adams spectral sequence. This was part of the original motivation for studying such subquotients of the lambda algebra. See [21], [15], [22], and [30].

**Remark 1.6.** If we localize with respect to \( K(2) \) instead of \( E(2) \) then we can say more. In [10] it is shown that Bousfield localization with respect to the Morava \( K \)-theory spectrum \( K(n) \) preserves fiber sequences which are double loops except possibly in dimensions \( n-1 \), \( n \), and \( n+1 \). Combining this with Theorem 1.3 yields the following corollary:

**Corollary 1.7.** Let \( p \geq 5 \) and \( 2np - 4 > c(2) \). Then there is a map from \( L_{K(2)} \Omega W(n) \) to the homotopy fiber of

\[ L_{K(2)} QM^{2np-3} \to L_{K(2)} Q \Sigma^{-2} C \]

which induces an isomorphism in homotopy groups except possibly in dimensions 1, 2, and 3.

Furthermore, in [2] Bousfield proves that the localization of any infinite loop space \( \Omega^\infty Z \) with respect to any spectrum \( E \) is again an infinite loop space. There is a certain localization functor associated to \( E \) on the category of \((-1)\)-connected
spectra, called the $E_* \Omega^\infty$ localization, and in [2] it is shown that the $E$-localization of the space $\Omega^\infty Z$ is $\Omega^\infty$ applied to the spectrum $E_* \Omega^\infty Z$. Thus Corollary 1.7 shows that the homotopy groups of $L_{K(2)} \Omega W(n)$ could in principle be computed from the LES associated to the $K(2)$-localization of (1.1), if one had explicit information about the $K(2)_* \Omega^\infty$ localization functor on connective spectra.

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2. James-Hopf maps

Theorem 1.2 follows from some basic properties of James-Hopf maps in conjunction with some properties of Gray’s construction of $BW(n)$. We recall James-Hopf maps:

For nonnegative integers $k$ and $q$ (or $k$ infinite) and each space $X$ there are James-Hopf maps

$$j_q : \Omega^k \Sigma^k X \to QD_{k,q} X,$$

natural in $X$, where $D_{k,q} X$ is the extended power space $C_k(q)^+ \wedge \Sigma^q X[n]$. Here $C_k(q)$ is the space of ordered $q$-tuples of little cubes disjointly embedded in $I^k$. If $k$ is infinite, we simply write $D_q X$. The maps $j_q$ are defined in [6]. In [4] an important Cartan formula is proved for the James-Hopf maps, and in [18] various compatibility relations between the James-Hopf maps are established which are extremely useful.

Taking the wedge sum of the adjoints of the James-Hopf maps yields a map of spectra

$$J : \Sigma^\infty \Omega^k \Sigma^k X \to \bigvee_{q \geq 1} \Sigma^\infty D_{k,q} X$$

which is a stable equivalence. Such a stable splitting was first established in [17] for $k = \infty$ and [29] for finite $k$ and then generalized in [6] and [4]. Such a splitting is not unique of course. Throughout this paper $j_q$ will always refer to the James-Hopf maps of [6], [4], and the stable splitting of $\Omega^k \Sigma^k X$ will be the one in (2.1) induced by the maps $j_q$ unless otherwise noted.

In [12] Gray shows that $W(n)$ is a loop space. More precisely, he shows that there exists a space $BW(n)$, together with a map $\Omega^2 S^{2n+1} \to BW(n)$ such that the homotopy fiber of $\nu$ is $S^{2n-1}$. For $p$ odd, $BW(n)$ is shown to be an H-space, and for $p \geq 5$, $\nu$ is an H-map. In what follows we need $\nu$ to be an H-map, hence the hypothesis in Theorem 1.2 that $p \geq 5$. Furthermore, in Proposition 7 of [12], it is shown that there is a splitting

$$\Sigma^2 \Omega^2 S^{2n+1} \cong \Sigma^2 (S^{2n-1} \times BW(n))$$

$$\cong \Sigma^2 (S^{2n-1} \vee BW(n) \vee \Sigma^{2n-1} BW(n)).$$

In [8], it is shown that the James-Hopf map admits a factorization

$$\Omega^2 S^{2n+1} \to \Omega^2 \Sigma^2 \nu M^{2np-1} \to QM^{2np-1} = QD_{2,p}(S^{2n-1}).$$
Definition 2.3. Let \( s : \Sigma^2 BW(n) \to \Sigma^2 \Omega^2 S^{2n+1} \) be the right inverse of \( \Sigma^2 \nu \) corresponding to (2.2). Let \( \sigma_{1'} : BW(n) \to \Omega^{2p} \Sigma^{2p} M^{2np-1} \) be the adjoint of the composite

\[
\Sigma^{2p} BW(n) \xrightarrow{\Sigma^{2p-2}} \Sigma^{2p} \Omega^2 S^{2n+1} \xrightarrow{J_p} \Sigma^{2p} M^{2np-1}.
\]

Finally, let \( \sigma_1 : BW(n) \to \Omega^{2p} BW(n+1) \) be the composite

\[
BW(n) \xrightarrow{\sigma_{1'}} \Omega^{2p} \Sigma^{2p} M^{2np-1} \to \Omega^{2p} BW(n+1)
\]

where the second map is \( \Omega^{2p} \) on the inclusion of the bottom cell.

The proof that \( Q M^{2np-1} \) is the mapping telescope of \( \sigma_1 \) is the same as that in [8]. Note that the map \( BW(n) \xrightarrow{\alpha} Q M^{2np-1} \) is just

\[
BW(n) \xrightarrow{\sigma_{1'}} \Omega^{2p} \Sigma^{2p} M^{2np-1} \to Q M^{2np-1}
\]

where the second map is the inclusion.

For the last statement in Theorem 1.2 we need several lemmas.

The following lemma is a variation of Lemma 3.6 of [20]. The difference is that the secondary suspension map \( \alpha \) defined in Lemma 3.6 of [20] is not \( a \ priori \) the same as the map \( \sigma \) defined here. One can conclude after the fact that \( \alpha \) and \( \sigma \) are the same since \( BW(n) \) splits off of \( \Omega^2 S^{2n+1} \) stably.

Lemma 2.4. There exists a factorization up to homotopy of the James-Hopf map:

\[
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \xrightarrow{\nu} & BW(n) \\
\downarrow & & \downarrow \sigma \\
\Omega^2 S^{2n+1} & \xrightarrow{J_p} & QM^{2np-1}
\end{array}
\]

Proof. The mod \( p \) homology algebra of \( \Omega^2 S^{2n+1} \) for \( p \) odd is given by ([5])

\[
E(t, Q_1t, Q_1^2t, \ldots) \otimes P(\beta Q_1t, \beta Q_1^2t, \ldots).
\]

If we assign weights to the monomials by \( wt(Q_1^j t) = wt(\beta Q_1^j t) = p^j \) and \( wt(xy) = wt(x) + wt(y) \) then the homology of \( D_{2,j} S^{2n-1} \) is the vector space of monomials of weight \( j \). It follows that \( \Omega^2 S^{2n+1} \), localized at \( p \), splits stably into a wedge \( \bigvee_{j=1}^\infty D_{2,j} S^{2n-1} \) where \( j \equiv 0 \) or \( 1 \) (mod \( p \)). Let \( J^{-1} \) stand for the homotopy equivalence which is inverse to the stable splitting of (2.1) given by the James-Hopf maps.

It can be verified by an easy calculation in homology that the composite

(2.5) \[
\Sigma^\infty BW(n) \xrightarrow{\Sigma^\infty s} \Sigma^\infty \Omega^2 S^{2n+1} \xrightarrow{\Sigma^\infty J} \bigvee_{j=0}^{\infty} \Sigma^\infty D_{2,j} S^{2n-1} \quad j \equiv 0 \pmod{p}
\]

is a homotopy equivalence. Thus we have a stable splitting

\[
\Sigma^\infty BW(n) \vee (\bigvee_{j=1}^{\infty} \Sigma^\infty D_{2,j} S^{2n-1}) \xrightarrow{\Sigma^\infty s \vee J^{-1}} \Sigma^\infty \Omega^2 S^{2n+1} \quad \equiv
\]

Consider the adjoint of the diagram in Lemma 2.4. It is immediate that the adjoint diagram commutes when restricted to the piece \( \Sigma^\infty BW(n) \). To show that the diagram commutes on the other piece first note that \( J_p : \Sigma^\infty \Omega^2 S^{2n+1} \to \Sigma^\infty M^{2np-1} \), is null homotopic on the pieces of the splitting where \( j \equiv 1 \) (mod \( p \)).

Thus the proof of 2.4 is completed by the following lemma. \( \square \)
**Lemma 2.6.** The composite map

\[ \bigvee_{j \equiv 1 \pmod{p}} D_{2,j} S^{2n-1} \xrightarrow{J^{-1}} \Sigma^\infty \Omega^2 S^{2n+1} \xrightarrow{\Sigma^\infty j} \Sigma^\infty BW(n) \]

is null homotopic.

**Proof.** This makes use of the Cartan formula for James-Hopf maps given in [4] and the fact that \( BW(n) \) is an H-space [12]. There are pairings \( D_{k,j} \times D_{k,j} \rightarrow D_{k,j} \) induced by the inclusion \( \Sigma_j \times \Sigma_r \subset \Sigma_{j+r} \) and the Cartan formula for James-Hopf maps says that these pairings are compatible, via the stable splitting, with the stabilization of the H-space multiplication on \( \Omega^k \Sigma^k X \). In the following diagram we will abbreviate \( D_{2,j} S^{2n-1} \to D_j \). We will suppress the symbol \( \Sigma^\infty \) but the diagram is to be understood as being stable.

\[
\begin{array}{cccccc}
S^{2n-1} \wedge D_{pk} & \longrightarrow & S^{2n-1} \wedge \Omega^2 S^{2n+1} & \longrightarrow & * \\
\downarrow & & \downarrow & & \\
D_1 \wedge D_{pk} & \xrightarrow{J^{-1}} & \Omega^2 S^{2n+1} \wedge \Omega^2 S^{2n+1} & \xrightarrow{\nu \wedge \nu} & BW(n) \wedge BW(n) \\
\downarrow & & \downarrow m & & \downarrow m \\
D_{pk+1} & \xrightarrow{J^{-1}} & \Omega^2 S^{2n+1} & \xrightarrow{\nu} & BW(n)
\end{array}
\]

The upper left vertical map is an equivalence because \( D_1 = S^{2n-1} \). The lower left vertical map induces an isomorphism in homology hence is an equivalence. The lower middle vertical map is the Hopf construction on the H-space multiplication on \( \Omega^2 S^{2n+1} \). The right hand lower vertical map is the Hopf construction on the H-space multiplication on \( BW(n) \). Since \( \nu \) is an H-map, the lower right hand square commutes. The upper right square commutes since \( S^{2n-1} \to \Omega^2 S^{2n+1} \wedge \nu \), \( BW(n) \) is null. This completes the proof of 2.6. \( \square \)

Before completing the proof of Theorem 1.2 we recall a result from [19] concerning the composite of two James-Hopf maps:

**Theorem 2.7** (part of 5.2 of [19]). For \( k, n, r, q \geq 1 \) let \( f_{r,q}^n \) be the composite map

\[ \Sigma^\infty D_{k,n} X \leftarrow \Sigma^\infty \Omega^k \Sigma^k X \xrightarrow{\Sigma^\infty j_q} \Sigma^\infty QD_{k,q} X \to \Sigma^\infty D_r D_{k,r} X \]

Then \( f_{r,q}^n \) is null homotopic if \( n > rq \).

**Proof.** (of Theorem 1.2)

In order to get a lifting \( BW(n) \xrightarrow{\Delta} G \) we need to know that the composite \( BW(n) \xrightarrow{\sigma} QM^{2np-1} \to QC \) is null homotopic. Gray’s map \( \Omega^2 S^{2n+1} \xrightarrow{\nu} BW(n) \) has a right inverse stably (Theorem 8(e) [12]), so by Lemma 2.4 it suffices to show that \( \Omega^2 S^{2n+1} \xrightarrow{J_p} QM^{2np-1} \to QC \) is null homotopic. This is equivalent to a factorization of \( j_p \circ j_p \) through \( QX \), where \( X \) is the four cell complex at the bottom of \( D_p M^{2np-1} \) defined in Section 1. See the diagram below. Notice that \( X \) is homotopy equivalent to \( D_{2,p^2} (S^{2n-1}) \), so Theorem 1.2 is proved once we know that
the following square commutes up to homotopy:
\[ \begin{array}{c}
QM^{2np-1} \\
\downarrow j_p
\end{array} \xrightarrow{j_p} \begin{array}{c}
QD_p M^{2np-1} \\
\uparrow
\end{array} \]
\[ \Omega^2 S^{2n+1} \xrightarrow{j_p^2} QD_{2,p^2}(S^{2n-1}) \xrightarrow{\sim} QX \]

Equivalently, we consider the adjoint diagram and check that it commutes on each piece of the stable splitting of \( \Omega^2 S^{2n+1} \):

\[ \begin{array}{c}
\Sigma^\infty QM^{2np-1} \\
\uparrow \Sigma^\infty j_p
\end{array} \xrightarrow{j_p} \begin{array}{c}
\Sigma^\infty D_p M^{2np-1} \\
\uparrow
\end{array} \]
\[ \Sigma^\infty \Omega^2 S^{2n+1} \xrightarrow{j_p^2} \Sigma^\infty D_{2,p^2}(S^{2n-1}) \]

The right hand vertical map is a sort of transfer, defined as the composite

\[ \Sigma^\infty D_{2,p^2}(S^{2n-1}) \hookrightarrow \Sigma^\infty \Omega^2 S^{2n+1} \xrightarrow{\Sigma^\infty j_p}, \Sigma^\infty QM^{2np-1} \xrightarrow{j_p}, \Sigma^\infty D_p M^{2np-1}. \]

Thus the square (2.8) commutes on the \( p^2 \) piece of the splitting by definition. The bottom horizontal map is null on \( \Sigma^\infty D_{2,m}(S^{2n-1}) \) for each \( m \neq p^2 \). The composite \( j_p \circ \Sigma^\infty j_p \) is null on \( \Sigma^\infty D_{2,m}(S^{2n-1}) \) for \( m < p^2 \) for purely dimensional reasons. Finally, \( j_p \circ \Sigma^\infty j_p \) is null on \( \Sigma^\infty D_{2,m}(S^{2n-1}) \) for \( m > p^2 \) by Theorem 2.7.

\[ \square \]

3. Unstable \( v_2 \)-periodic homotopy groups

In this section we will prove Theorem 1.5. To start, we have

**Lemma 3.1.** \( \lambda : BW(n) \to G^n \) induces an isomorphism in \( v_0^{-1} \pi_*(\ ) \) and \( v_1^{-1} \pi_*(\ ) \).

**Proof.** \( v_0^{-1} \pi_*(\ ) \) is just rational homotopy and both spaces are torsion. The map \( \sigma : BW(n) \to QM^{2np-1} \) induces an isomorphism in \( v_1^{-1} \pi_*(\ ) \) by [30]. To see that \( v_1^{-1} \pi_*(QC) = 0 \), use the fact that \( C \) has a filtration with subquotients \( V(1) \), and so \( K(1), V(1) = 0 \). By the telescope theorem for \( n = 1 \) (Theorem 4.11 of [1]), we have that stably \( v_1^{-1} \pi_*(V(1)) = 0 \).

We will define unstable \( v_2 \)-periodic homotopy groups by taking \( V_1 \) to be the Smith-Toda complex \( V(1) \), which we will denote simply by \( V \). Since \( p \geq 5 \), \( V \) has a \( v_2 \)-self map \( v_2 : \Sigma^{v_2} V \to V \). Using a \( p \)-local version of the Freudenthal suspension theorem (see [11]) we see that this \( v_2 \)-self map is defined unstably as long as \( V \) is at least \( d - 1 \)-connected, where \( d = \frac{2p^2 + 1}{p-1} + 3 \).

Consider the map of pairs

\[ (QM^{2np-1}, BW(n)) \to (QM^{2np-1}, G^n) \to (QC, \ast). \]

It suffices to show that this induces an isomorphism

\[ v_2^{-1} \pi_*(QM^{2np-1}, BW(n); V) \xrightarrow{\sim} v_2^{-1} \pi_*(QC; V). \]

The proof of this is based on the modified unstable Adams spectral sequence techniques of [22], [23], [25], [30]. This machinery takes as input certain calculations involving subquotients of the lambda algebra. See [15] and [21]. In the present case,
the relevant lambda algebra calculations are provided by [13] and [14] so we will use that framework. We recall the construction.

In [13] it is shown that there are spaces \( \{W^n(0)\}_{n \geq 0} \) and maps

\[
\begin{align*}
(3.3) \quad & \Omega W^{2n-1}_n(0) \to \Omega^2 W^{2n+1}_n(0) \to \Omega^3 W^{2n+3}_n(0) \to \Omega^7 W^{2n+5}_n(0) \to \ldots
\end{align*}
\]

The two cell complex at the bottom of \( \Omega^{2k+1} W^{2(n+k)-1}_n(0) \) is \( M^ {2n-2} \) and each of the above maps is degree one on this bottom Moore space. The homotopy colimit of this sequence is \( QM^{2n-2} \). The spaces \( W^{2n-1}_n(0) \) are defined as follows:

\[
W^{2n-1}_n(0) = \text{fiber}(\pi_n : \Omega^2 S^{2n+1} \to S^{2n-1})
\]

where

\[
(3.4) \quad \pi_n = \begin{cases} 
\pi_n & \text{from [7] if } (n, p) = 1, \\
\phi_m & \text{from [12] if } n = pm.
\end{cases}
\]

Thus \( \Omega W^{2n-1}_n = W(n) \).

We need to prove that there is an isomorphism

\[
(3.5) \quad v_2^{-1} \pi_* (QM^{2n-2}_n, \Omega W^{2n-1}_n(0); V) \cong v_2^{-1} \pi_* (Q \Sigma^{-1} C; V).
\]

Even though the map \( \Omega W^{2n-1}_n(0) \to QM^{2n-2}_n \) defined by (3.3) is not necessarily the same as \( \Omega \sigma : W(n) \to QM^{2n-2}_n \), we will nevertheless see that the proof of (3.5) leads to the proof of (3.2).

In [14] certain subquotients of \( \Lambda \), the odd primary lambda algebra, are defined.

These are denoted by \( \Lambda_{(m)}(n) \), \( m \geq -1, n \geq 0 \). There are SES's

\[
0 \to \Lambda_{(m)}(2n-1) \to \Lambda_{(m)}(2n) \to \Lambda_{(m)}(2np - 1) \to 0
\]

and

\[
0 \to \Lambda_{(m)}(2n) \to \Lambda_{(m)}(2n+1) \to \Lambda_{(m)}(2np + 2p^{m+1} - 1) \to 0
\]

which yield EHP sequences in homology and a SES

\[
(3.6) \quad 0 \to \Lambda_{(m)}(2n-1) \to \Lambda_{(m)}(2n+1) \to \Lambda_{(m+1)}(2np - 1) \to 0
\]

which yields the double suspension sequence.

We have

\[
\Lambda_{(m)} = \bigcup_{n=1}^{\infty} \Lambda_{(m)}(n) = E(\tau_0, \ldots, \tau_m) \otimes \Lambda
\]

where \( E(\tau_0, \ldots, \tau_m) \) is the exterior subalgebra of the dual Steenrod algebra \( A_* \). In those cases where \( V(m) \) exists we have \( E(\tau_0, \ldots, \tau_m) = H_* V(m) \) and

\[
H_*(E(\tau_0, \ldots, \tau_m) \otimes \Lambda) = \text{Ext}_{A_*}(H_* V(m)).
\]

The chain complex \( \Lambda_{(m+1)}(k) \) has a splitting given by the SES's

\[
0 \to \Lambda_{(m)}(2n+1) \to \Lambda_{(m+1)}(2n+1) \to \Lambda_{(m)}(2n + 2p^{m+1} + 1) \to 0
\]

and

\[
0 \to \Lambda_{(m)}(2n + 1) \to \Lambda_{(m+1)}(2n) \to \Lambda_{(m)}(2n + 2p^{m+1} - 1) \to 0
\]

There are \( v_m \)-self maps

\[
v_m : \Lambda_{(m-1)}(2n + 2p^m - 1) \to \Lambda_{(m-1)}(2n - 1)
\]
and isomorphisms
\[ v_m^{-1} \Lambda_{m-1}(2n-1) \cong v_m^{-1} \Lambda_{m-1}(2n+1) \cong v_m^{-1}(E(\tau_0, \ldots, \tau_{m-1}) \otimes \Lambda). \]

Recall from [22] and [30] that a resolution of a space \( X \) is a tower of fibrations,
\[
\begin{array}{ccc}
F_0 & \longrightarrow & F_1 \longrightarrow F_2 \\
\downarrow & & \downarrow \\
X_0 & \longrightarrow & X_1 \longrightarrow X_2 \longrightarrow \ldots
\end{array}
\]
with each fiber \( F_s \) being a GEM, and compatible maps \( f_s : X \to X_s \), with \( f_\infty : X \to X_\infty \) being the \( \mu \)-completion. Given a resolution of a space, there is the usual homotopy spectral sequence.

**Lemma 3.7.**
1. There is a resolution of \( W^{2n-1}_{(0)} \) with
\[ E_2^{s,t} \cong H_*(\Lambda(0)(2n-1)). \]
2. The map \( \Omega W^{2n-1}_{(0)} \to \Omega^2 W^{2n+1}_{(0)} \) is covered by a map of resolutions, and the induced map of \( E_2 \)-terms is \( H_*(\Lambda(0)(2n-1)) \to H_*(\Lambda(0)(2n+1)) \) from (3.6).
3. Let
\[ \Omega^2 W_{(0)}^{2np-1} \to \Omega W_{(0)}^{2n-1} \to \Omega^3 W_{(0)}^{2n+1} \]
be the homotopy fiber sequence of [13]. Then there is a resolution of \( \Omega^2 W_{(0)}^{2np-1} \)
with \( E_2^{s,t} \cong H_*(\Lambda(1)(2np-1)). \)

**Proof.** Proposition 6.3 of [25] states that if we are given a map of spaces \( f : X \to Y \), and resolutions of \( X \) and \( Y \), then there is a map of resolutions covering \( f \) if the largest dimensional homotopy class in \( \pi_i F_s \), for the target space \( Y \), is in the range through which \( f_s \), for the source space \( X \), is surjective in cohomology. This was used in [22] (Proposition 4.10) and [30] (Theorem 2.27) to produce a map of resolutions covering a secondary suspension map \( W(n) \to \Omega^2 p W(n+1) \). The proof of Proposition 6.3 of [25] is the same as the proof of Proposition 4.10 of [22]. If we replace the resolution of the target space \( Y \) by the same tower starting in degree \( i \), then we have the result that there is a filtration \( i \) map of resolutions covering \( f \) if the largest dimensional homotopy class in \( \pi_i F_{s+i} \), for the target, is in the range through which \( f_s \), for the source, is surjective in cohomology.

We apply this to the map \( \pi_n : \Omega^2 S^{2n+1} \to S^{2n-1} \) of (3.4). As usual, take the Adams resolution for \( S^{2n-1} \) with \( \Lambda(2n-1) \) as \( E_1 \)-term, and for \( \Omega^2 S^{2n+1} \) take double loops on the Adams resolution for \( S^{2n+1} \). The map of resolutions needs to be a filtration one map. As in [30], the dimension of a class in \( \pi_n F_{s+1} \), for \( S^{2n-1} \), is at most
\[ q(n-1)[1 + p + \cdots + p^q] + 2n - 1. \]
This is less than \( (2n-1)p^s+1 + (p-2)p^s \), which is the range through which \( f_s^* : H^* \Omega^2 S^{2n+1} \to H^* X_s \) is onto in the resolution of \( \Omega^2 S^{2n+1} \).

Proposition 3.3 of [22] (see also 2.20 of [30]) states that if we are given a map of resolutions covering a given map \( f \), then there is a resolution of the fiber of \( f \), and a long exact sequence of \( E_2 \)-terms. It is implicit in [22] that one of the maps in the LES is induced by the map \( f \). This last fact is proved explicitly in [23].

For our resolution of \( W^{2n-1}_{(0)} \) we take the resolution of the fiber corresponding to the map of resolutions covering \( \pi_n : \Omega^2 S^{2n+1} \to S^{2n-1} \) constructed above.
The statement regarding the $E_2$-term follows from the LES of $E_2$-terms, once we check that the map of resolutions induces the homomorphism $v_0 : \Lambda_0(2n + 1) \to \Lambda(2n - 1)$ at least on $E_2$. Following the proof of Proposition 2.32 and Lemma 2.29 of [30], let $P'$ be a chain complex of free unstable $A$-modules corresponding to the resolution of $\Omega^2 S^{2n+1}$, and $P''$ a chain complex of free unstable $A$-modules for the resolution of $S^{2n-1}$. Let $\epsilon : P'' \to P'$ denote the difference between the chain map induced by the map of resolutions constructed above, and the given map $v_0$. Since $P''$ is acyclic (the resolution of $S^{2n-1}$ is an Adams resolution), the composite $P'' \xrightarrow{\epsilon} P' \xrightarrow{v} P''$ is chain homotopically trivial, where $\sigma$ is double suspension. Thus there is a lifting $P'' \xrightarrow{v} P''$ of $\sigma$. Now ker $\sigma$ is a chain complex of free unstable $A$-modules corresponding to a resolution of $W(n)$. This lifting is zero since ker $\sigma$ is acyclic in the range of dimensions in which $\text{Hom}_A(P'', Z/p)$ is nonzero, which is easy to check by the calculations of section two of [30].

Part 3) follows immediately from part 2) by using the resolution of the fiber. Part 2) uses the same argument as Theorem 2.27 and Lemma 2.29 of [30]. Actually no new calculations are needed as the estimates given in [30] yield part 2) directly.

The 4-cell complex at the bottom of $\Omega^2 W_{(1)}^{2np^2 - 1}$ is $V = V(1)$ with the bottom cell in dimension $2np^2 - 4$. Checking the $p$-local Freudenthal suspension condition, we see that as long as $n \geq 1$ this $V$ at the bottom is the target of the self map $v_2$.

By [16] there is an exponent $k$ such that $v_2^k \wedge 1$ is the same as $1 \wedge v_2^k$ as a stable self map of $V \wedge V$. As in [25] we consider the following diagram of pointed mapping spaces. For brevity, denote $\Omega^2 W_{(1)}^{2np^2 - 1}$ by $W$, $v_2^k$ by $v$, and set $j = |v_2^k|$. 

\[
\begin{array}{cccccc}
\text{map}_*(V, W) & \to & \text{map}_*(\Sigma V, W) & \to & \text{map}_*(\Sigma^2 V, W) & \to & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \\
\text{map}_*(V, V) & \to & \text{map}_*(\Sigma V, V) & \to & \text{map}_*(\Sigma^2 V, V) & \to & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{map}_*(\Sigma V, \Sigma V) & \to & \text{map}_*(\Sigma^2 V, \Sigma V) & \to & \ldots \\
\downarrow & & \downarrow & & \\
\text{map}_*(\Sigma^2 V, \Sigma^2 V) & \to & \ldots
\end{array}
\]

As in [25], this yields a commutative diagram of abelian groups after applying $\pi_*$. This produces a homomorphism

$$\pi_*^S(V; V) \to v_2^{-1} \pi_*(W; V)$$

which extends to give a homomorphism

$$v_2^{-1} \pi_*^S(V; V) \xrightarrow{\phi} v_2^{-1} \pi_*(W; V).$$

**Theorem 3.8.** The homomorphism $\phi$ is an isomorphism.

**Proof.** As in [25], we have a corresponding diagram of $E_2$-terms

\[
\begin{array}{cccccc}
E_2^{s,*}(W; V) & \to & E_2^{s+1,*}(W; V) & \to & E_2^{s+2,*}(W; V) & \to & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \\
E_2^{s,*}(V; V) & \to & E_2^{s+1,*}(V; V) & \to & E_2^{s+2,*}(V; V) & \to & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
E_2^{s,*}(\Sigma V; V) & \to & E_2^{s+1,*}(\Sigma V; V) & \to & \ldots \\
\downarrow & & \downarrow & & \\
E_2^{s,*}(\Sigma^2 V; V) & \to & \ldots
\end{array}
\]
which gives a homomorphism
\[ \text{Ext}^{s,t}_A(H_* V, H_* V) \to v_2^{-1} E^{s,t}_2(W; V) \]
which extends to a homomorphism
\[ v_2^{-1} \text{Ext}^{s,t}_A(H_* V, H_* V) \xrightarrow{\psi} v_2^{-1} E^{s,t}_2(W; V). \]
In [14] it shown that there is an isomorphism
\[ v_2^{-1} E^{s,t}_2(W) \xrightarrow{\theta} v_2^{-1} \text{Ext}^{s,t}_A(H_* V). \]
Reducing mod \( V \), we get an isomorphism
\[ v_2^{-1} E^{s,t}_2(W; V) \xrightarrow{\theta} v_2^{-1} \text{Ext}^{s,t}_A(H_* V, H_* V). \]
Now the argument of [9], Theorem 3.10, shows that the composite
\[ v_2^{-1} \text{Ext}^{s,t}_A(H_* V, H_* V) \xrightarrow{\psi} v_2^{-1} E^{s,t}_2(W; V) \xrightarrow{\theta} v_2^{-1} \text{Ext}^{s,t}_A(H_* V, H_* V) \]
is an isomorphism, and this proves Theorem 3.8.

Returning to the proof of the isomorphism in (3.5), consider the tower of fibrations
\[
\begin{array}{cccc}
\ast & \to & (\Omega^3 W_{(0)}^{2n+1}, \Omega W_{(0)}^{2n-1}) & \to & (\Omega^5 W_{(0)}^{2n+3}, \Omega W_{(0)}^{2n-1}) & \to & \ldots \\
& & \uparrow & & \uparrow & & \\
\Omega^2 W_{(1)}^{2n^2-1} & \to & \Omega^4 W_{(1)}^{2n^2+2p-1} & \to & \Omega^6 W_{(1)}^{2n^2+4p-1}
\end{array}
\]
The homotopy colimit of this tower is the pair \((QM^{2n-2}, \Omega W_{(0)}^{2n-1})\). By applying the functor \( v_2^{-1} \pi_* \) to this tower we get a spectral sequence which converges to \( v_2^{-1} \pi_* (QM^{2n-2}, \Omega W_{(0)}^{2n-1}; V) \).

The complex \( C \) has a filtration with subquotients copies of \( V \), (see Figure 1), and this filtration is compatible with the above tower. This gives a map of spectral sequences, with \( v_2 \) inverted. Theorem 3.8 says this map of spectral sequences is an isomorphism on \( E_2 \)-terms and (3.5) follows.

Now consider (3.2). First note that if we “speed up the filtration” of the pair \((QM^{2n-2}, \Omega W_{(0)}^{2n-1})\) we get
\[ \ast \to (\Omega^{2p+1} W_{(0)}^{2(n+1)p-1}, \Omega W_{(0)}^{2n-1}) \to (\Omega^{4p+1} W_{(0)}^{2(n+2)p-1}, \Omega W_{(0)}^{2n-1}) \to \ldots \]
The fiber at each stage is a space \( F \) whose bottom \( 4p \) cells is a complex \( A_1 \), whose cohomology is \( A(1) \), the subalgebra of the Steenrod algebra generated by \( \beta \) and \( P^1 \). Note that \( A_1 \) consists of \( p \) copies of \( V(1) \) attached together.

In [26] it shown that there is a \( v_2 \)-self map \( \Sigma^{v_2} A_1 \to A_1 \). Again, by the \( p \)-local Freudenthal suspension condition, this map desuspends to a map of spaces, as long as \( A_1 \) is at least \( d-1 \)-connected, where \( d = \frac{2q^2+1}{p-1} + 2p + 1 \). The dimension of the bottom cell of the first \( A_1 \) is \( 2np^2 - 4 \), and so it is the target of the self map \( v_2 \).

Lemma 2.27 of [30] shows that the map
\[ W(n) \xrightarrow{\Omega^{v_2} \pi_*} \Omega^3 W(n + 1) \]
is covered by a map of resolutions and Lemma 2.29 of [30] shows that the induced map of \( E_2 \)-terms is the same as that of the argument above. Thus we have an isomorphism
\[ v_2^{-1} E^{s,t}_2(F; V) \xrightarrow{\theta} v_2^{-1} \text{Ext}^{s,t}_A (A_1, H_* V) \]
and the $v_2$-periodic homotopy of $F$ is the stable $v_2$-periodic homotopy of $A_1$.
Now the proof of (3.2) proceeds exactly as above with $W$ replaced by $F$ and $\text{map}_*(V,V)$ replaced by $\text{map}_*(V,A_1)$.

References


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