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THE IRRATIONALITY OF $\log(1+1/q)\log(1-1/q)$

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Abstract. We shall show that the numbers $1, \log(1+1/q), \log(1-1/q)$ and $\log(1+1/q)\log(1-1/q)$ are linearly independent over $\mathbf{Q}$ for any natural number $q \geq 54$. The key is to construct explicit Padé-type approximations using Legendre-type polynomials.

0. Introduction

Galochkin [3] obtained the lower estimates of polynomials with integral coefficients in the values of certain Siegel's G-functions at algebraic points of a special form. He used Padé-type approximations of the first kind in Mahler's classification, and all the constants in his estimates are effectively computable. As a simple and instructive application of his general results, which concerns neither a polynomial in one logarithm nor a linear form in several logarithms, the irrationality of the number

$$\log\left(1+\frac{1}{q}\right)\log\left(1-\frac{1}{q}\right) \tag{0.1}$$

is shown for any natural number $q > e^{795}$. Galochkin [4] also pointed out that the bound e^{795} can be improved to e^{170} if one applies Chudnovsky and Chudnovsky's result [2], which was obtained by Padé-type approximations of the second kind. The authors in both papers used the so-called Siegel's lemma coming from Dirichlet's box principle.

Galochkin's result on the number (0.1) can now be improved remarkably as follows:

Theorem 0.1. *The number* (0.1) *is irrational for any natural number* $q \geq 54$.

To see this we will use Padé-type approximations in the same way as in our previous study [7] on the values of the dilogarithm. More precisely, we will construct explicitly a polynomial $P(w) \in \mathbf{Z}[w]$ of degree n satisfying

$$P(z^2)F(z) - Q(z^2) = O\left(z^{2(n+1+[\lambda n])}\right), \tag{0.2}$$

where $Q(w)$ is some polynomial of degree at most n, $\lambda = 1/4$ and

$$F(z) = -\log(1+z)\log(1-z) = \sum_{k=1}^{\infty}\left(\sum_{\substack{r+s=2k\\ r,s\geq 1}} \frac{(-1)^{r+1}}{rs}\right) z^{2k}.$$

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It is still open to find explicit Padé approximations (the case $\lambda = 1$) or Padé-type approximations for some $\lambda \in (1/4, 1)$ to $F(z)$.

Put $P^*(w) = w^n P(w^{-1})$ for brevity. By noticing that

$$F(z) = \int_0^1 \frac{1}{z^{-2} - x} \log\left(1 + \frac{1}{\sqrt{x}}\right) dx$$

for $|z| < 1$, we get

$$\begin{aligned} P(z^2)F(z) - z^{2n} \int_0^1 \frac{P^*(z^{-2}) - P^*(x)}{z^{-2} - x} \log\left(1 + \frac{1}{\sqrt{x}}\right) dx \\ = z^{2n+2} \int_0^1 \frac{P^*(x)}{1 - z^2 x} \log\left(1 + \frac{1}{\sqrt{x}}\right) dx \\ = \sum_{k=0}^{\infty} z^{2n+2+2k} \int_0^1 x^k P^*(x) \log\left(1 + \frac{1}{\sqrt{x}}\right) dx. \end{aligned}$$

Since the second term of the left-hand side is a polynomial in z^2 with rational coefficients of degree at most n, it follows that $P(z^2)$ satisfies (0.2) if and only if

$$\int_0^1 x^k P^*(x) \log\left(1 + \frac{1}{\sqrt{x}}\right) dx = 0 \quad \text{for} \quad 0 \leq k < [\lambda n]. \tag{0.3}$$

However, it seems to be difficult to solve (0.3) directly. Instead of (0.3) we will consider the following orthogonality property:

$$\int_0^1 x^m P^*(x^2) \log\left(1 + \frac{1}{x}\right) dx = 0 \quad \text{for} \quad 0 \leq m < [2\lambda n], \tag{0.4}$$

which is easier to handle. Indeed, we can construct explicitly the polynomial $P^*(w)$ which satisfies (0.4) for $\lambda = 1/4$ by somewhat modifying the Legendre polynomial $(x^n(1-x)^n)^{(n)}/n!$ in Section 1.

The specific family of linear fractional transformations with real coefficients

$$\mathcal{T} = \left\{ \tau_c(x) = \frac{1 - x}{1 - cx}; c < 1 \right\}$$

plays an important role in this paper. Note that $\tau_c : [0,1] \to [0,1]$ is an orientation-reserving homeomorphism and satisfies $\tau_c \equiv \tau_c^{-1}$ for any $\tau_c \in \mathcal{T}$. It is also easily seen that $d\tau_c/(1 - c\tau_c) = -dx/(1 - cx)$. We call τ_c a *nice transformation*. Some kinds of definite integrals over $[0,1]^\ell$, $\ell \in \mathbf{N}$, change into simpler and more useful ones by nice transformations. Such an example can be found in Beukers' paper [1], in which he used the nice transformation $\tau_{1-xy}(z)$ on some triple integral, so that the asymptotic behavior of the integral was easily obtained. As another example, Rhin and Viola [9] used some birational transformations involving nice transformation $\tau_y(x)$ in order to choose relevant polynomial factors. In this paper nice transformations will appear in Sections 1 and 5.

In Section 1 we will investigate several properties of our Padé-type approximant $P(x^2)$. Then a upper estimate of the remainder term in (0.2) will be given in Sections 4 and 5. Our main result on linear independence measures of the numbers $1, \log(1 + 1/q),\ \log(1 - 1/q)$ and $\log(1 + 1/q)\log(1 - 1/q)$ will be proved in Section 6.

1. Preliminaries

Lemma 1.1. *For any $x \in (0,1)$ and $0 \le j < n$, we have*

$$\frac{1}{n!}\left(x^j \log\left(1+\frac{1}{x}\right)\right)^{(n)} = \frac{(-1)^{n-j}}{(n-j)\binom{n}{j}} \sum_{k=0}^{n-j-1} \frac{\binom{n}{k}}{x^{n-j-k}(1+x)^n}. \tag{1.1}$$

Proof. In the region $\mathbf{C}_0$ obtained by omitting the segment $[-1,0]$ from the complex plane $\mathbf{C}$, the principal branch of $\log(1+1/z)$ is single-valued and hence analytic. The left-hand side of (1.1) is equal to

$$\frac{1}{2\pi i}\int_C \frac{\zeta^j}{(\zeta-x)^{n+1}} \log\left(1+\frac{1}{\zeta}\right) d\zeta \qquad \text{(call it } J\text{)},$$

where C is a small circle in $\mathbf{C}_0$ centered at x. Since the integrand is estimated by $O(|\zeta|^{-2})$ when $|\zeta|$ is sufficiently large, the contour C can be changed to the (degenerate) curve C' adhering to the branch cut $[-1,0]$. Hence

$$\begin{aligned} J &= \frac{(-1)^{n-j+1}}{2\pi i}\int_0^1 \frac{t^j}{(t+x)^{n+1}}\left[\left\{\log\left(\frac{1}{t}-1\right)-\pi i\right\} - \left\{\log\left(\frac{1}{t}-1\right)+\pi i\right\}\right] dt \\ &= (-1)^{n-j}\int_0^1 \frac{t^j}{(t+x)^{n+1}} dt. \end{aligned}$$

We now use the nice transformation $\tau_{-1/x}(t) \in \mathcal{T}$. Substituting $\tau = \tau_{-1/x}(t)$, we get

$$\begin{aligned} J &= \frac{(-1)^{n-j}}{x^{n-j}(1+x)^n}\int_0^1 (1-\tau)^j(\tau+x)^{n-j-1} d\tau \\ &= \frac{(-1)^{n-j}}{x^{n-j}(1+x)^n}\sum_{k=0}^{n-j-1}\binom{n-j-1}{k} x^k \int_0^1 \tau^{n-j-k-1}(1-\tau)^j d\tau \\ &= \frac{(-1)^{n-j}}{(n-j)\binom{n}{j}}\sum_{k=0}^{n-j-1}\frac{\binom{n}{k}}{x^{n-j-k}(1+x)^n}. \qquad \square \end{aligned}$$

It follows from Lemma 1.1 that

$$U_{j,n}(x) = x^n(1+x)^n\left(x^j\log\left(1+\frac{1}{x}\right)\right)^{(n)}$$

is a polynomial satisfying $\deg(U_{j,n}) = n-1$ and $\mathrm{ord}_{x=0}(U_{j,n}) = j$ for $0 \le j < n$. It seems to be hard to obtain $\deg(U_{j,n}) = n-1$ from the usual differential calculation by Leibniz' formula.

Let $\mathcal{A}(I)$ be the set of all real-analytic and integrable functions defined on an open interval I. For any $f(x) \in \mathcal{A}(I)$ let $\nu_I(f) \in [0,\infty]$ be the number of zero points of $f(x)$ in I without counting the multiplicities. (Note that $\nu_I(f) = \infty$ if $f(x)$ is identically zero in I.) The following basic lemma can be easily shown by a simple application of Rolle's theorem.

Lemma 1.2. $\nu_I(f') \ge \nu_I(f) - 1$ *for any* $f(x) \in \mathcal{A}(I)$. *Put* $I = (a,b)$ *and suppose further that* $\lim_{x\to a+} f(x) = \lim_{x\to b-} f(x) = 0$. *Then* $\nu_I(f') \ge \nu_I(f) + 1$.

As a corollary, we have

Corollary 1.3. *Suppose that $f(x) \in \mathcal{A}(I)$ satisfies $\int_I x^j f(x)\,dx = 0$ for $0 \le j < n$. Then $\nu_I(f) \ge n$.*

Proof. Put $I = (a,b)$ and $f_0(x) \equiv f(x)$. We define $f_k(x) = \int_a^x f_{k-1}(t)\,dt$ inductively for $x \in \bar{I}$ and $k \in \mathbf{N}$. Obviously $f_j(x) \in \mathcal{A}(I)$ and $f_j(a) = 0$ for $j \ge 1$. Moreover we can show that $f_j(b) = 0$ for $1 \le j \le n$ by induction. Therefore it follows from Lemma 1.2 that $\nu_I(f) = \nu_I(f_0) \ge \nu_I(f_1) + 1 \ge \cdots \ge \nu_I(f_n) + n \ge n$. □

Lemma 1.4. *Let K be the open unit interval $(0,1)$. Suppose that $R(x)$ is a real polynomial in x^2 and*

$$\int_0^1 x^j R(x)\,dx = 0 = \int_0^1 x^j R(x) \log\left(1 + \frac{1}{x}\right) dx$$

for $0 \le j < n$. Then $\nu_K(R) \ge 2n$. Suppose further that $\int_0^1 R(x) \log x\,dx = 0$. Then $\nu_K(R) \ge 2n+1$.

Proof. Put $R_0(x) \equiv R(x)$ and define $R_k(x) = \int_0^x R_{k-1}(t)\,dt$ inductively for $k \in \mathbf{N}$. Then one can show that $R_j(0) = R_j(1) = 0$ for $1 \le j \le n$ by induction; hence $R_n(x) = x^n(1-x)^n S(x)$ for some polynomial $S(x)$. Since $R_n(x) \equiv (-1)^n R_n(-x)$, we get $(1-x)^n S(x) \equiv (1+x)^n S(-x)$; therefore $S(x) = (1+x)^n T(x)$ for some polynomial $T(x)$.

On the other hand, by n-fold partial integration, we have

$$\begin{aligned} 0 = \int_0^1 \left(x^j \log\left(1 + \frac{1}{x}\right)\right)^{(n)} R_n(x)\,dx &= \int_0^1 U_{j,n}(x)(1-x)^n T(x)\,dx \\ &= \int_0^1 U_{j,n}(x)\tilde{T}(x)\,dx \end{aligned}$$

for $0 \le j < n$, where $\tilde{T}(x) = (1-x)^n T(x)$ is a polynomial. Since $\deg(U_{j,n}) = n-1$ and $\mathrm{ord}_{x=0}(U_{j,n}) = j$, it is easily seen that

$$\int_0^1 x^k \tilde{T}(x)\,dx = 0 \quad \text{for} \quad 0 \le k < n.$$

Hence $\nu_K(R_n) = \nu_K(S) = \nu_K(T) = \nu_K(\tilde{T}) \ge n$ by Corollary 1.3 and $\nu_K(R) = \nu_K(R_0) \ge \nu_K(R_1) + 1 \ge \cdots \ge \nu_K(R_n) + n \ge 2n$, as required.

Moreover, if $\int_0^1 R(x)\log x\,dx = 0$ in addition, then $\int_0^1 (\log x)^{(n)} R_n(x)\,dx = 0$; hence

$$0 = \int_0^1 x^{-n} R_n(x)\,dx = \int_0^1 (1-x^2)^n T(x)\,dx = \int_0^1 (1+x)^n \tilde{T}(x)\,dx.$$

Therefore $\int_0^1 x^n \tilde{T}(x)\,dx = 0$ and $\nu_K(R_n) = \nu_K(\tilde{T}) \ge n+1$ by Corollary 1.3. We thus have $\nu_K(R) \ge \nu_K(R_n) + n \ge 2n+1$, which completes the proof. □

We now introduce the following Legendre-type polynomial:

$$\begin{aligned} L_{2n}(x) &= \frac{1}{n![n/2]!}\left(x^{[n/2]}(1-x^2)^{[n/2]}\left(x^n(1-x^2)^{[(n+1)/2]}\right)^{(n)}\right)^{([n/2])} \\ &= \sum_{k=0}^{[n/2]} \sum_{\ell=0}^{[(n+1)/2]} (-1)^{k+\ell} \binom{[n/2]}{k}\binom{[(n+1)/2]}{\ell}\binom{2\ell+n}{n}\binom{2k+2\ell+[n/2]}{[n/2]} x^{2(k+\ell)} \end{aligned}$$

for $n \in \mathbf{N}$. Since $L_{2n}(x)$ is a polynomial in x^2, one can put $L_{2n}(x) \equiv A_n(x^2)$, so that $A_n(z)$ is a polynomial of degree $[n/2]+[(n+1)/2] = n$ with integral coefficients. Concerning the orthogonality of $A_n(x^2)$, we have

Lemma 1.5. *For $0 \le j < [n/2]$,*

$$\int_0^1 x^j A_n(x^2)\,dx = 0 = \int_0^1 x^j A_n(x^2) \log\left(1+\frac{1}{x}\right) dx.$$

Furthermore

$$\int_0^1 A_1(x^2)\,dx = 0 \quad \textit{and} \quad \int_0^1 A_n(x^2) \log x\,dx = 0$$

for every odd integer $n \ge 3$.

Proof. Obviously $\int_0^1 x^j A_n(x^2)\,dx = 0$ for $0 \le j < [n/2]$, and $\int_0^1 A_1(x^2)\,dx = 0$. By $[n/2]$-fold partial integration,

$$\begin{aligned}
&\int_0^1 x^j A_n(x^2) \log\left(1+\frac{1}{x}\right) dx \\
&\quad = \text{const.} \int_0^1 U_{j,[n/2]}(x)(1-x)^{[n/2]} \left(x^n(1-x^2)^{[(n+1)/2]}\right)^{(n)} dx \\
&\quad = 0
\end{aligned}$$

for $0 \le j < [n/2]$, since $\deg(U_{j,[n/2]}) = [n/2]-1$. Moreover, if $n \ge 3$ is odd,

$$\int_0^1 A_n(x^2)\log x\,dx = \text{const}'. \int_0^1 (1-x^2)^{[n/2]} \left(x^n(1-x^2)^{[(n+1)/2]}\right)^{(n)} dx = 0,$$

since $2[n/2] < n$. □

Therefore our polynomial $A_n(x^2)$ gives a solution to (0.4) for $\lambda = 1/4$.

Lemma 1.6. *All zero points of $A_n(z)$ are simple and lie on $(0,1)$.*

Proof. Since $A_1(z) = 1-3z$, we can assume that $n \ge 2$. It follows from Lemma 1.4 that $\nu_K(L_{2n}) \ge 2[n/2] = n$ if n is even and that $\nu_K(L_{2n}) \ge 2[n/2]+1 = n$ if n is odd. Therefore $n \ge \nu_K(A_n) = \nu_K(L_{2n}) \ge n$; hence $\nu_K(A_n) = n$. This completes the proof. □

Lemma 1.7. *For an arbitrarily fixed $z \in \mathbf{C}$ the sequence $X_n \equiv A_n(z^2)$ satisfies the linear recurrence*

$$X_{n+1} = (\alpha_n z^2 + \beta_n)X_n + \sum_{k=1}^{5} \gamma_{k,n} X_{n-k} \tag{1.2}$$

for $n \ge 6$, where α_n, β_n and $\gamma_{k,n}$ $(1 \le k \le 5)$ are rational constants depending only on n. Moreover, for an arbitrarily fixed $z \in \mathbf{C}\backslash[-1,1]$, all the sequences

$$I_n^\delta(z^2) = \int_0^1 \frac{A_n(x^2)}{z^2-x^2} x^\delta dx \quad \textit{and} \quad J_n^\delta(z^2) = \int_0^1 \frac{A_n(x^2)}{z^2-x^2} x^\delta \log\left(1+\frac{1}{x}\right) dx$$

$(\delta = 0, 1)$ satisfy the same linear recurrence (1.2) for $n \ge 6$ as well.

Proof. We define $m_n = 4$ if n is even and $m_n = 5$ if n is odd. Put

$$\tilde{L}(z) \equiv \tilde{A}(z^2) = A_{n+1}(z^2) - \left(\alpha_n z^2 + \beta_n\right) A_n(z^2) - \sum_{k=1}^{m_n} \gamma_{k,n} A_{n-k}(z^2)$$

for $n \geq 6$, where $\alpha_n,\ \beta_n$ and $\gamma_{k,n}$ $(1 \leq k \leq m_n)$ are rational numbers chosen so that the degree of $\tilde{A}(z^2)$ is less than $2(n-m_n)$. Then $\int_0^1 x^j \tilde{A}(x^2)\, dx = 0$ for $0 \leq j < m$, where

$$m = \min\left\{\left[\frac{n+1}{2}\right], \left[\frac{n}{2}\right] - 2, \left[\frac{n-1}{2}\right], ..., \left[\frac{n-m_n}{2}\right]\right\} = \left[\frac{n-m_n}{2}\right].$$

Similarly we have $\int_0^1 x^j \tilde{A}(x^2)\log(1+1/x)\, dx = 0$ for $0 \leq j < [(n-m_n)/2]$. Since $n - m_n$ is an even integer, it follows from Lemma 1.4 that

$$\nu_K(\tilde{A}) = \nu_K(\tilde{L}) \geq 2\left[\frac{n-m_n}{2}\right] = n - m_n > \deg(\tilde{A});$$

therefore $\tilde{A}(z)$ must be identically zero. Hence $X_n = A_n(z^2)$ satisfies (1.2), if we put $\gamma_{5,n} = 0$ for every even $n \geq 6$. Moreover, for any $z \in \mathbf{C}\backslash[-1,1]$, it is easily seen that

$$I_{n+1}^{\delta}(z^2) - \left(\alpha_n z^2 + \beta_n\right) I_n^{\delta}(z^2) - \sum_{k=1}^{5} \gamma_{k,n} I_{n-k}^{\delta}(z^2) = -\alpha_n \int_0^1 x^{\delta} A_n(x^2)\, dx = 0$$

for $\delta = 0, 1$, since $[n/2] \geq 3$. Similarly $X_n = J_n^{\delta}(z^2)$ satisfies (1.2). This completes the proof. $\square$

Finally, we need the following:

Lemma 1.8. *Suppose that $g(x)$ is a real-valued integrable function on $(0,1)$ and satisfies*

$$\int_0^1 A_k(x^2) g(x)\, dx = 0$$

for all $k \geq 2n$. Then $g(x) = V(x) + W(x)\log(1+1/x)$ almost everywhere for some polynomials $V(x)$ and $W(x)$ of degrees less than n.

Proof. For any vector $\mathbf{v} = (a_0, a_1, ..., a_{n-1}, b_0, b_1, ..., b_{n-1}) \in \mathbf{R}^{2n}$, we put

$$\phi_{\mathbf{v}}(x) = \sum_{j=0}^{n-1} a_j x^j + \sum_{j=0}^{n-1} b_j x^j \log\left(1 + \frac{1}{x}\right).$$

Obviously $\phi_{\mathbf{v}} \in \mathcal{A}(0,1)$. We next define the linear mapping $\Phi : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ by

$$\Phi(\mathbf{v}) = \left(\int_0^1 \phi_{\mathbf{v}}(x)\, dx, \int_0^1 x^2 \phi_{\mathbf{v}}(x)\, dx, ..., \int_0^1 x^{2(2n-1)} \phi_{\mathbf{v}}(x)\, dx\right).$$

We first show that Φ is a homeomorphism. To see this, it suffices to show that Φ is one-to-one. Suppose, on the contrary, that $\Phi(\mathbf{v}) = \mathbf{0}$ for some $\mathbf{v} \neq \mathbf{0}$; that is,

$$0 = \int_0^1 x^{2j} \phi_{\mathbf{v}}(x)\, dx = \frac{1}{2}\int_0^1 t^j \frac{\phi_{\mathbf{v}}(\sqrt{t})}{\sqrt{t}}\, dt$$

for $0 \leq j < 2n$. Since $\varphi(x) \equiv \phi_{\mathbf{v}}(\sqrt{x})/\sqrt{x} \in \mathcal{A}(0,1)$, we get $\nu_K(\phi_{\mathbf{v}}) = \nu_K(\varphi) \geq 2n$ by Corollary 1.3. Thus it follows from Lemma 1.2 that $\nu_K(\phi_{\mathbf{v}}^{(n)}) \geq \nu_K(\phi_{\mathbf{v}}) - n \geq n$.

Therefore, putting $\tilde{U}(x) \equiv x^n(1+x)^n\phi_{\mathbf{v}}^{(n)}(x)$, we get $\nu_K(\tilde{U}) = \nu_K(\phi_{\mathbf{v}}^{(n)}) \geq n$. Hence we have $\tilde{U}(x) \equiv 0$, since

$$\tilde{U}(x) = \sum_{j=0}^{n-1} b_j x^n (1+x)^n \left(x^j \log\left(1+\frac{1}{x}\right)\right)^{(n)} = \sum_{j=0}^{n-1} b_j U_{j,n}(x)$$

is a polynomial of degree less than n. Thus $\phi_{\mathbf{v}}(x)$ is some polynomial of degree less than n; hence $\phi_{\mathbf{v}}(x) \equiv 0$, since $\nu_K(\phi_{\mathbf{v}}) \geq 2n$. We thus have

$$\phi_{\mathbf{v}}(x) = V_0(x) + W_0(x)\log\left(1+\frac{1}{x}\right) \equiv 0$$

for $x \in (0,1)$, where $V_0(x) = \sum_{j=0}^{n-1} a_j x^j$ and $W_0(x) = \sum_{j=0}^{n-1} b_j x^j$. Then it is easily seen that this occurs if and only if $V_0(x) = W_0(x) \equiv 0$; hence $\mathbf{v} = \mathbf{0}$. This contradiction implies that $\Phi : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is a homeomorphism, as required.

Since Φ is a homeomorphism, there exists a unique vector $\mathbf{w} \in \mathbf{R}^{2n}$ such that

$$\int_0^1 x^{2j}\left(g(x) - \phi_{\mathbf{w}}(x)\right)\,dx = 0 \quad \text{for} \quad 0 \leq j < 2n. \tag{1.3}$$

We now have

$$\int_0^1 A_m(x^2)\phi_{\mathbf{w}}(x)\,dx = 0 \quad \text{for all} \quad m \geq 2n$$

by Lemma 1.5; hence

$$\int_0^1 A_m(x^2)\left(g(x) - \phi_{\mathbf{w}}(x)\right)\,dx = 0 \quad \text{for all} \quad m \geq 2n. \tag{1.4}$$

Combining (1.3) and (1.4), we conclude that

$$\int_0^1 x^{2j}\left(g(x) - \phi_{\mathbf{w}}(x)\right)\,dx = 0 \quad \text{for all} \quad j \geq 0.$$

Then it can be seen that $g(x) = \phi_{\mathbf{w}}(x)$ almost everywhere. This completes the proof. $\square$

2. Arithmetical properties of the coefficients

For any $n \in \mathbf{N}$ and $\delta = 0, 1$, we define

$$B_n^\delta(z^2) = \int_0^1 \frac{A_n(z^2) - A_n(x^2)}{z^2 - x^2} x^\delta\,dx,$$
$$C_n^\delta(z^2) = \int_0^1 \frac{A_n(z^2) - A_n(x^2)}{z^2 - x^2} x^\delta \log\left(1+\frac{1}{x}\right)dx.$$

Obviously $B_n^\delta(w) \in \mathbf{Q}[w]$. Since it can be seen that

$$\int_0^1 x^{2k+1}\log\left(1+\frac{1}{x}\right)dx = \frac{1}{2(k+1)}\sum_{j=1}^{2k+1}\frac{(-1)^{j+1}}{j}$$

and

$$\int_0^1 x^{2k}\log\left(1+\frac{1}{x}\right)dx = \frac{1}{2k+1}\left(2\log 2 + \sum_{j=1}^{2k}\frac{(-1)^j}{j}\right)$$

for any integer $k \geq 0$, we have $C_n^1(w), C_n^0(w) - 2\log 2 B_n^0(w) \in \mathbf{Q}[w]$. In this section we will investigate some arithmetical properties of the coefficients of $B_n^\delta(w), C_n^1(w)$ and $C_n^0(w) - 2\log 2 B_n^0(w)$.

Put $A_n(w) = \sum_{j=0}^n c_{j,n} w^j$; that is,

$$c_{j,n} = (-1)^j \binom{2j+[n/2]}{[n/2]} \sum_{\substack{k+\ell=j \\ 0\leq k\leq [n/2] \\ 0\leq \ell \leq [(n+1)/2]}} \binom{[n/2]}{k} \binom{[(n+1)/2]}{\ell} \binom{2\ell+n}{n}.$$

In our previous study [5], [7] on rational approximation to numbers like $\log 2, \pi/\sqrt{3}$, $\zeta(2), \zeta(3)$ and the values of the dilogarithm at rational points of a special form, the Legendre-type polynomials play an important role in each case, since the coefficients have a large common divisor when the degree is sufficiently large. However, in this paper, the coefficients of $A_n(w)$ do not possess any common divisor greater than 1, because $c_{0,n} \equiv 1$. Nevertheless the coefficients of $A_n(w)$ have the following interesting property, thanks to the binomial coefficient $\binom{2\ell+n}{n}$.

Lemma 2.1. *For any positive integers j, n and any prime number p satisfying $j \leq n$ and $2[(n+1)/2] < p < 2j$, we have $p | c_{j,n}$.*

Proof. Suppose, on the contrary, that some prime number $p \in (2[(n+1)/2], 2j)$ is not a divisor of $c_{j,n}$. Then there exist $k \in [0, [n/2]]$ and $\ell \in [0, [(n+1)/2]]$ such that $k + \ell = j$ and p is not a divisor of the binomial coefficient $\binom{2\ell+n}{n}$. Hence we have

$$\left[\frac{2\ell+n}{p}\right] = \left[\frac{2\ell}{p}\right] + \left[\frac{n}{p}\right]$$

since $p > 2[(n+1)/2] \geq \sqrt{2\ell+n}$. Put $\omega = \{n/p\}$ and $\eta = \{\ell/p\}$ for brevity, where $\{x\}$ denotes the fractional part of x. Then $[\omega + 2\eta] = [2\eta]$. Since $\ell \leq [(n+1)/2] < p/2$, we have $\eta = \ell/p < 1/2$; hence $[\omega + 2\eta] = [2\eta] = 0$.

On the other hand, it follows that

$$\ell = j - k \geq j - \left[\frac{n}{2}\right] \geq \frac{p+1}{2} - \left[\frac{n}{2}\right] > \frac{p+1}{2} - \frac{n}{2} - 1 = \frac{p-n-1}{2};$$

therefore $\ell \geq (p-n-1)/2 + 1/2 = (p-n)/2$. Thus $\ell/p \geq 1/2 - n/(2p)$. Since $n \leq 2[(n+1)/2] < p < 2j \leq 2n$, we have $1/2 < n/p < 1$; hence $\omega = n/p$ and $\eta \geq 1/2 - \omega/2$. Therefore $[\omega + 2\eta] \geq 1$. This contradiction completes the proof. $\square$

Lemma 2.2. *Let D_n be the least common multiple of $1, 2, ..., n$. Then, for any integers $1 \leq \ell < m \leq 2n$,*

$$\frac{1}{\ell m} \in \frac{\mathbf{Z}}{D_n D_{2n}}.$$

Proof. We distinguish two cases: (a) $\ell \leq n$ and (b) $\ell > n$. In case (a) the statement is clear. In case (b) we also have

$$\frac{1}{\ell m} = \frac{1}{m-\ell}\left(\frac{1}{\ell} - \frac{1}{m}\right) \in \frac{\mathbf{Z}}{D_n D_{2n}},$$

since $m - \ell < 2n - n = n$. $\square$

Theorem 2.3. *The polynomials* $B_n^\delta(w)$ $(\delta=0,1)$, $C_n^1(w)$, $C_n^0(w)-2\log 2B_n^0(w)$ *all belong to the set* $\mathbf{Z}[w]/M_n$, *where*

$$M_n=\frac{D_nD_{2n}}{\prod_{\substack{p:\text{prime}\\ n<p<2n}} p}.$$

Proof. We first consider the polynomial

$$C_n^1(w)=\sum_{j=1}^{n}\sum_{r=1}^{j}\sum_{s=1}^{2r-1}\frac{(-1)^{s+1}c_{j,n}w^{j-r}}{2rs}.$$

Since $1\le s<2r\le 2n$, it follows from Lemma 2.2 that $D_nD_{2n}C_n^1(w)\in\mathbf{Z}[w]$. Suppose now that some denominator $2rs$ has a prime factor $p\in(n,2n)$. Note that $p>2[(n+1)/2]$. We then have $s=p$, because $r\le n<p$ and $s<2n<2p$. Hence $p=s\le 2r-1\le 2j-1$; that is, $p<2j$. Therefore we have $p|c_{j,n}$ by Lemma 2.1. This implies that

$$D_nD_{2n}C_n^1(w)\in\left(\prod_{\substack{p:\text{prime}\\ n<p<2n}} p\right)\mathbf{Z}[w],$$

as required.

The similar argument can be applied to the polynomials

$$B_n^\delta(w)=\sum_{j=1}^{n}\sum_{r=1}^{j}\frac{c_{j,n}w^{j-r}}{2r-1+\delta}\qquad(\delta=0,1)$$

and

$$C_n^0(w)-2\log 2B_n^0(w)=\sum_{j=1}^{n}\sum_{r=1}^{j}\sum_{s=1}^{2r-2}\frac{(-1)^{s}c_{j,n}w^{j-r}}{(2r-1)s}.$$

□

3. Simultaneous rational approximations

We recall that

$$I_n^\delta(z^2)=\int_0^1\frac{A_n(x^2)}{z^2-x^2}x^\delta\,dx\quad\text{and}\quad J_n^\delta(z^2)=\int_0^1\frac{A_n(x^2)}{z^2-x^2}x^\delta\log\left(1+\frac{1}{x}\right)dx$$

for any $n\in\mathbf{N}$, $z\in\mathbf{C}\backslash[-1,1]$ and $\delta=0,1$. Taking $z=\sqrt{k}$ for any integer $k\ge 2$, it is easily seen that

$$A_n(k)\sqrt{k}\log\frac{\sqrt{k}+1}{\sqrt{k}-1}-2kB_n^0(k)=2kI_n^0(k),\tag{3.1}$$

$$A_n(k)\log\left(1-\frac{1}{k}\right)+2B_n^1(k)=-2I_n^1(k),\tag{3.2}$$

$$A_n(k)\log\left(1+\frac{1}{\sqrt{k}}\right)\log\left(1-\frac{1}{\sqrt{k}}\right)+2C_n^1(k)=-2J_n^1(k)\tag{3.3}$$

and

$$A_n(k)\sqrt{k}\Lambda\left(\frac{1}{\sqrt{k}}\right) - 4k\left(C_n^0(k) - 2\log 2B_n^0(k)\right) = 4k\left(J_n^0(k) - 2\log 2I_n^0(k)\right), \tag{3.4}$$

where

$$\Lambda(x) = \sum_{r=1}^{\infty} \frac{2}{2r+1}\left(\sum_{s=1}^{2r}\frac{(-1)^s}{s}\right)x^{2r+1} = -\int_{-x}^{x}\frac{\log(1-t)}{1+t}dt.$$

Let $\mathrm{Li}_2(x) = \sum_{r=1}^{\infty} x^r/r^2$ be the dilogarithm. (For the dilogarithm see Lewin's book [8, Chapter 1].) Then it can be seen that

$$\Lambda(x) = \mathrm{Li}_2\left(\frac{1+x}{2}\right) - \mathrm{Li}_2\left(\frac{1-x}{2}\right) - \log 2\log\frac{1+x}{1-x}. \tag{3.5}$$

Thus (3.1)-(3.4) give a system of simultaneous rational approximations to the numbers including $\log(1+1/\sqrt{k})\log(1-1/\sqrt{k})$.

We now give an upper bound of $|A_n(k)|$. It follows that

$$A_n(k) = L_{2n}\left(\sqrt{k}\right) = \left(\frac{1}{2\pi i}\right)^2 \int_{C_0}\int_{C_1} \frac{\left(\zeta(1-\zeta^2)\right)^{[n/2]}}{(\zeta-\sqrt{k})^{[n/2]+1}} \cdot \frac{w^n(1-w^2)^{[(n+1)/2]}}{(w-\zeta)^{n+1}} dw\, d\zeta,$$

where C_0 and C_1 are the circles centered at $\zeta = \sqrt{k}$ and $w = \zeta$ with radii $\sqrt{k}/4$ and $5\sqrt{k}/4$ respectively. Then it can be seen that

$$|A_n(k)| \le \left(\frac{125k}{16} - 5\right)^{[n/2]} 2^n \left(\frac{25k}{4} - 1\right)^{[(n+1)/2]} < \left(\frac{5^{5/2}k}{4}\right)^n. \tag{3.6}$$

We next consider the remainder term $J_n^\delta(k)$. It follows from $[n/2]$-fold partial integration that

$$J_n^\delta(k) = (-1)^{[n/2]}\int_0^1 \left(x(1-x^2)\right)^{[n/2]} \frac{1}{n!}\left(x^n(1-x^2)^{[(n+1)/2]}\right)^{(n)} E_{k,n}^\delta(x)\, dx,$$

where

$$E_{k,n}^\delta(x) = \frac{1}{[n/2]!}\left(\frac{x^\delta}{k-x^2}\log\left(1+\frac{1}{x}\right)\right)^{([n/2])}.$$

Then one has

$$\begin{aligned} E_{k,n}^\delta(x) &= \frac{1}{2\pi i}\int_{C_2} \frac{1}{(\zeta-x)^{[n/2]+1}} \cdot \frac{\zeta^\delta}{k-\zeta^2}\log\left(1+\frac{1}{\zeta}\right) d\zeta \\ &\equiv -\frac{1}{2\pi i}\int_{C_2} G_{k,n}^\delta(x,\zeta)\, d\zeta, \qquad \text{say,} \end{aligned}$$

where C_2 is a small circle in $\mathbf{C}_0 = \mathbf{C}\backslash[-1,0]$ centered at x. Since $G_{k,n}^\delta(x,\zeta) = O(|\zeta|^{-2})$ when $|\zeta|$ is sufficiently large, the contour C_2 can be changed to the same curve C' as in the proof of Lemma 1.1 by taking account of the residues of $G_{k,n}^\delta(x,\zeta)$ at the poles $\zeta = \pm\sqrt{k}$. Therefore we get

$$E_{k,n}^\delta(x) = \mathrm{Res}_{\zeta=\sqrt{k}}\, G_{k,n}^\delta(x,\zeta) + \mathrm{Res}_{\zeta=-\sqrt{k}}\, G_{k,n}^\delta(x,\zeta) - \frac{1}{2\pi i}\int_{C'} G_{k,n}^\delta(x,\zeta)\, d\zeta.$$

Then it can be seen that

$$\mathrm{Res}_{\zeta=\pm\sqrt{k}}\, G_{k,n}^\delta(x,\zeta) = \frac{(-1)^{\kappa^\pm}}{2}k^{(\delta-1)/2}\log\left(1\pm\frac{1}{\sqrt{k}}\right)\frac{1}{(\sqrt{k}\mp x)^{[n/2]+1}}$$

respectively, where $\kappa^+ = 0$ and $\kappa^- = \delta + [n/2]$, and that

$$-\frac{1}{2\pi i}\int_{C'} G^{\delta}_{k,n}(x,\zeta)\, d\zeta = (-1)^{\kappa^-}\int_0^1 \frac{1}{(t+x)^{[n/2]+1}}\cdot\frac{t^\delta}{k-t^2}dt.$$

So it would be convenient to introduce the following:

$$\varepsilon_n^{\pm}(k) = \int_0^1 \frac{1}{n!}\left(x^n(1-x^2)^{[(n+1)/2]}\right)^{(n)}\frac{\big(x(1-x^2)\big)^{[n/2]}}{(\sqrt{k}\pm x)^{[n/2]+1}}dx$$

respectively, and

$$\mu_n^{\delta}(k) = \int_0^1\int_0^1 \frac{1}{n!}\left(x^n(1-x^2)^{[(n+1)/2]}\right)^{(n)}\frac{\big(x(1-x^2)\big)^{[n/2]}}{(t+x)^{[n/2]+1}}\cdot\frac{t^\delta}{k-t^2}dt\, dx$$

for $\delta = 0, 1$. We then have

$$\begin{aligned} J_n^{\delta}(k) = {}& \frac{(-1)^\delta}{2}k^{(\delta-1)/2}\log\left(1-\frac{1}{\sqrt{k}}\right)\varepsilon_n^{+}(k) \\ &+ \frac{(-1)^{[n/2]}}{2}k^{(\delta-1)/2}\log\left(1+\frac{1}{\sqrt{k}}\right)\varepsilon_n^{-}(k) + (-1)^\delta\mu_n^\delta(k). \end{aligned} \tag{3.7}$$

Similarly it is easily seen that

$$I_n^\delta(k) = \frac{k^{(\delta-1)/2}}{2}\left((-1)^\delta\varepsilon_n^+(k) + (-1)^{[n/2]}\varepsilon_n^-(k)\right). \tag{3.8}$$

Upper estimates of the terms $\varepsilon_n^{\pm}(k)$ and $\mu_n^\delta(k)$ will be discussed in Sections 4 and 5, respectively.

4. Upper estimates of $|\varepsilon_n^{\pm}(k)|$

It follows from n-fold partial integration that

$$\begin{aligned}\varepsilon_n^{\pm}(k) &= (-1)^n\int_0^1 x^n(1-x^2)^{[(n+1)/2]}\frac{1}{n!}\left(\frac{\big(x(1-x^2)\big)^{[n/2]}}{(\sqrt{k}\pm x)^{[n/2]+1}}\right)^{(n)} dx \\ &= \frac{(-1)^n}{2\pi i}\int_0^1 x^n(1-x^2)^{[(n+1)/2]}\int_{C_x}\frac{1}{(\zeta-x)^{n+1}}\cdot\frac{\big(\zeta(1-\zeta^2)\big)^{[n/2]}}{(\sqrt{k}\pm\zeta)^{[n/2]+1}}d\zeta\, dx,\end{aligned}$$

where C_x is the unit circle centered at x. Then, for any integer $k \geq 9$, we have

$$|\varepsilon_n^{\pm}(k)| \leq \int_0^1 \frac{x^n(1-x^2)^{[(n+1)/2]}}{(\sqrt{k}-1\pm x)^{[n/2]+1}}\left(\Omega(x)\right)^{[n/2]}dx \leq \left(\max_{0\leq x\leq 1}\frac{x^2(1-x^2)\Omega(x)}{\sqrt{k}-1-x}\right)^{n/2}, \tag{4.1}$$

where $\Omega(x) = \max_{\zeta\in C_x}|\zeta(1-\zeta^2)|$. Note that $\Omega(x) \geq 2$ for $x \in [0,1]$. We need a slightly sharp upper estimate of $\Omega(x)$ as follows:

Lemma 4.1. *Let* $\rho = (\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} - 2)/3 = 0.83928...$ *(*ρ *is the unique root in* $(0,1)$ *of* $x^3+2x^2-2=0$*). Then*

$$\Omega(x) \leq \begin{cases} \dfrac{(1+x)(2-x^2)}{\sqrt{1-x^2}} & \text{for } 0 \leq x \leq \rho, \\ x(1+x)(2+x) & \text{for } \rho < x \leq 1. \end{cases}$$

Proof. Putting $\zeta = x + e^{i\theta}$, we have $|1-\zeta^2|^2 = Y(\cos\theta)$, where $Y(w) = 4(x^2-1)w^2 + 4x^3 w + x^4 + 4$; hence

$$\Omega(x) \le (1+x)\sqrt{\max_{|t|\le 1} Y(t)}.$$

The maximum of $Y(t)$ as t varies in $[-1,1]$ is attained at $t = x^3/(2(1-x^2))$ or $t=1$ according as $x \in [0,\rho]$ or $x \in (\rho, 1]$ respectively. This completes the proof. □

It thus follows from (4.1) and Lemma 4.1 that

$$\begin{aligned} |\varepsilon_n^{\pm}(k)|^{2/n} &\le \max\left\{ \max_{0\le x\le\rho} \frac{x^2(1-x^2)\Omega(x)}{\sqrt{k}-1-x}, \max_{\rho\le x\le 1} \frac{x^2(1-x^2)\Omega(x)}{\sqrt{k}-1-x} \right\} \\ &\le \max\left\{ \frac{1}{\sqrt{k}-1-\rho} \max_{0\le x\le\rho} x^2\sqrt{1-x^2}(1+x)(2-x^2)\ , \right. \\ &\qquad \left. \frac{1}{\sqrt{k}-2} \max_{\rho\le x\le 1} x^3(1-x^2)(1+x)(2+x) \right\}. \end{aligned} \tag{4.2}$$

Since

$$\begin{aligned} &\max_{0\le x\le\rho} x^2\sqrt{1-x^2}(1+x)(2-x^2) \\ &\quad < \max_{0\le u\le 1} u\sqrt{1-u}(2-u) + \max_{0\le v\le 1} v^{3/2}\sqrt{1-v}(2-v) \\ &\quad = \frac{4}{5^{5/4}} + \frac{2^{7/2}}{27} < 0.9541 \end{aligned}$$

and

$$\begin{aligned} &\max_{\rho\le x\le 1} x^3(1-x^2)(1+x)(2+x) \\ &\quad < 2\max_{\rho^2\le u\le 1} u^{3/2}(1-u) + 3\max_{\rho^2\le v\le 1} v^2(1-v) + \max_{0\le w\le 1} w^{5/2}(1-w) \\ &\quad = 2\rho^3(1-\rho^2) + 3\rho^4(1-\rho^2) + \frac{2\cdot 5^{5/2}}{7^{7/2}} < 0.9128, \end{aligned}$$

it follows from (3.8) and (4.2) that

$$|I_n^{\delta}(k)| \le \max\left\{ |\varepsilon_n^{+}(k)|, |\varepsilon_n^{-}(k)| \right\} < \left(\frac{0.9541}{\sqrt{k}-1-\rho} \right)^{n/2} \tag{4.3}$$

for $\delta = 0, 1$ and for any integer $k \ge 31$.

5. Upper estimates of $|\mu_n^{\delta}(k)|$

We use the same nice transformation $\tau_{-1/x}(t) \in \mathcal{T}$ as in the proof of Lemma 1.1. Substituting $\tau = \tau_{-1/x}(t)$, we get

$$\int_0^1 \frac{1}{(t+x)^{[n/2]+1}} \cdot \frac{t^{\delta}}{k-t^2} dt = x^{\delta-[n/2]}(1+x)^{-[n/2]} \int_0^1 \frac{(1-\tau)^{\delta}(\tau+x)^{[n/2]+1-\delta}}{k(\tau+x)^2 - x^2(1-\tau)^2} d\tau;$$

therefore, by n-fold partial integration,

$$\begin{aligned} \mu_n^{\delta}(k) = \int_0^1 \int_0^1 x^n(1-x^2)^{[(n+1)/2]} \frac{1}{n!} \\ \times \left(\frac{(1-\tau)^{\delta} x^{\delta}(1-x)^{[n/2]}(\tau+x)^{[n/2]+1-\delta}}{k(\tau+x)^2 - x^2(1-\tau)^2} \right)^{(n)} d\tau\, dx. \end{aligned}$$

The rational function in x in the big parentheses in the right-hand side can be written in the form

$$P_0(x)+\frac{1}{2\sqrt{k}\tau}\Big(\frac{A^\delta(1-A)^{[n/2]}(\tau+A)^{[n/2]+1-\delta}}{x-A} \\ -\frac{B^\delta(1-B)^{[n/2]}(\tau+B)^{[n/2]+1-\delta}}{x-B}\Big),$$

where $A=-\sqrt{k}\tau/(\sqrt{k}-1+\tau)$, $B=-\sqrt{k}\tau/(\sqrt{k}+1-\tau)$ and $P_0(x)$ is some polynomial in x of degree less than n. Hence we have

$$\mu_n^\delta(k)=\frac{(-1)^{n+1}}{2}k^{(\delta-1)/2}\left((-1)^\delta\xi_{\delta,n}^+(k)+(-1)^{[n/2]}\xi_{\delta,n}^-(k)\right) \tag{5.1}$$

where

$$\xi_{\delta,n}^\pm(k)=\int_0^1\int_0^1 x^n(1-x^2)^{[(n+1)/2]}\tau^{[n/2]}(1-\tau)^{[n/2]+1-\delta} \\ \times\frac{\left(\sqrt{k}\pm(1-\tau)\right)^{n-2[n/2]}\left(\sqrt{k}\tau+\sqrt{k}\pm(1-\tau)\right)^{[n/2]}}{\left((\sqrt{k}\pm(1-\tau))x+\sqrt{k}\tau\right)^{n+1}}d\tau\,dx$$

respectively. We now use the transformations $T^\pm:[0,1]\to[0,1]$ defined by

$$T^\pm(\tau)=\frac{\sqrt{k}\tau}{\sqrt{k}\pm(1-\tau)} \tag{5.2}$$

respectively. Substituting $T=T^\pm(\tau)$, we get

$$\xi_{\delta,n}^\pm(k)=k^{(\delta-1)/2}\int_0^1\int_0^1\frac{x^n(1-x^2)^{[(n+1)/2]}}{(x+T)^{n+1}} \\ \times\frac{T^{[n/2]}(1-T)^{[n/2]+1-\delta}(1+T)^{[n/2]}}{(\sqrt{k}\pm T)^{[n/2]+2-\delta}}dT\,dx;$$

hence

$$|\xi_{\delta,n}^\pm(k)|\le\left(\max_{0\le x,T\le 1}\frac{x^2(1-x^2)}{(x+T)^2}\cdot\frac{T(1-T^2)}{\sqrt{k}\pm T}\right)^{[n/2]}$$

for any $k\ge 4$. Using the inequality $x+T\ge 2\sqrt{xT}$, the maximum in the big parentheses in the right-hand side is estimated above by

$$\frac14\max_{0\le x\le 1}x(1-x^2)\cdot\max_{0\le T\le 1}\frac{1-T^2}{\sqrt{k}\pm T}<\frac{1}{6\sqrt{3}(\sqrt{k}-1)}<\frac{0.1}{\sqrt{k}-1}.$$

Thus it follows from (5.1) that

$$|\mu_n^\delta(k)|\le\max\left\{|\xi_{\delta,n}^+(k)|,|\xi_{\delta,n}^-(k)|\right\}<\left(\frac{0.1}{\sqrt{k}-1}\right)^{[n/2]};$$

hence, from (3.7) and (4.3), we obtain

$$|J_n^\delta(k)|\le\max\left\{|\varepsilon_n^+(k)|,|\varepsilon_n^-(k)|\right\}+|\mu_n^\delta(k)|\le 2\left(\frac{0.9541}{\sqrt{k}-1-\rho}\right)^{[n/2]}. \tag{5.3}$$

Remark 5.1. The transformations $T^{\pm}(\tau)$ defined in (5.2) are not nice transformations, since they are orientation-preserving homeomorphisms. However it is easily seen that both $T^{\pm} \circ \tau_c$ and $\tau_c \circ T^{\pm}$ belong to $\mathcal{T}$ for any $c < 1$. In particular, $T^{\pm} \in \mathcal{T}^2 = \{\tau_c \circ \tau_{c'}; c, c' < 1\}$. Note that $\mathcal{T}^3 = \{\tau_c \circ \tau_{c'} \circ \tau_{c''}; c, c', c'' < 1\}$ coincides with $\mathcal{T}$.

6. Main results

Although we gave the upper estimates for the remainder terms in the system of simultaneous rational approximations (3.1)-(3.4) in the previous sections, it seems difficult to give their exact asymptotic behaviors as n tends to infinity. This is the reason why we need the various lemmas concerning our Legendre-type polynomials in Section 1. To derive linear independence measures from the system (3.1)-(3.4), we need the following lemma, which is a generalization of [5, Lemma 3.2].

Lemma 6.1. *Let $M \in \mathbf{N}$, and let $\gamma_1, \gamma_2, ..., \gamma_M$ be given real numbers. Let d be a fixed positive number. For given sequences $\{q_n\}_{n\geq 1}$ and $\{p_{m,n}\}_{n\geq 1}$ in $\mathbf{Z} + id\mathbf{Z}$ satisfying $q_n\gamma_m - p_{m,n} = \varepsilon_{m,n}$ $(1 \leq m \leq M)$, suppose that $q_n \neq 0$ for all $n \in \mathbf{N}$, and*

$$\limsup_{n\to\infty} \frac{1}{n} \log |q_n| \leq \sigma, \qquad \max_{1\leq m\leq M} \limsup_{n\to\infty} \frac{1}{n} \log |\varepsilon_{m,n}| \leq -\tau$$

for some positive numbers σ, τ. Suppose further that there exists a positive integer N satisfying

$$\sum_{j=0}^{N} \left| n_0 q_{n+j} + \sum_{m=1}^{M} n_m p_{m,n+j} \right| > 0 \tag{6.1}$$

for all $n \in \mathbf{N}$ and any $(n_0, n_1, ..., n_M) \in \mathbf{Z}^{M+1} \backslash \{(0, 0, ..., 0)\}$. Then, for any $\varepsilon > 0$, there exists an effectively computable constant $H_0 \equiv H_0(\varepsilon)$ such that

$$\left| n_0 + \sum_{m=1}^{M} n_m \gamma_m \right| \geq H^{-\sigma/\tau - \varepsilon}$$

for any $(n_1, n_2, ..., n_M) \in \mathbf{Z}^M$ with $H = \max_{1\leq m\leq M} |n_m| \geq H_0$.

Proof. We put $\Theta(n_0, n_1, ..., n_M) = n_0 + \sum_{m=1}^{M} n_m\gamma_m$ for brevity. Then

$$q_n\Theta(n_0, n_1, ..., n_M) = \left(n_0 q_n + \sum_{m=1}^{M} n_m p_{m,n} \right) + \sum_{m=1}^{M} n_m \varepsilon_{m,n} \equiv S_n + \omega_n, \quad \text{say,} \tag{6.2}$$

for all $n \in \mathbf{N}$. The condition (6.1) implies that there exists an integer $r(n) \in [n, n + N]$ satisfying $S_{r(n)} \neq 0$ for all $n \in \mathbf{N}$; hence $|S_{r(n)}| \geq \kappa_d$ for some constant $\kappa_d \in (0, 1]$ depending only on d, because $S_{r(n)} \in \mathbf{Z} + id\mathbf{Z}$.

For any $\varepsilon > 0$, we can define a sufficiently small $\varepsilon' \in (0, \tau)$ satisfying $\sigma/\tau + \varepsilon/2 > (\sigma + \varepsilon')/(\tau + \varepsilon')$. Then there exists an integer $n^* \equiv n^*(\varepsilon)$ such that $|q_n| \leq e^{(\sigma+\varepsilon')n}$ and $|\varepsilon_{m,n}| \leq e^{-(\tau-\varepsilon')n}$ for $1 \leq m \leq M$ and any $n \geq n^*$. Let $H_0 \equiv H_0(\varepsilon)$ be the least positive integer satisfying

$$2H_0 M e^{-(\tau-\varepsilon')n^*} \geq \kappa_d \qquad \text{and} \qquad H_0^{\varepsilon/2} \geq \left(\frac{2M}{\kappa_d} \right)^{(\sigma+\tau)/(\tau-\varepsilon')} e^{(\sigma+\varepsilon')(N+1)}.$$

Then, for any $(n_1, n_2, ..., n_M) \in \mathbf{Z}^M$ with $H = \max_{1\le m\le M} |n_m| \ge H_0$, let $\tilde{n}$ be the least positive integer satisfying $2HMe^{-(\tau-\varepsilon')\tilde{n}} < \kappa_d$. Obviously $n^* < \tilde{n} \le r(\tilde{n})$.

We now take $n = r(\tilde{n})$ in (6.2). Then

$$|\Theta(n_0, n_1, ..., n_M)| \ge \frac{|S_{r(\tilde{n})}| - |\omega_{r(\tilde{n})}|}{|q_{r(\tilde{n})}|} \ge \frac{\kappa_d - HMe^{-(\tau-\varepsilon')\tilde{n}}}{e^{(\sigma+\varepsilon')(\tilde{n}+N)}} \ge \frac{\kappa_d}{2e^{(\sigma+\varepsilon')(\tilde{n}+N)}}.$$

Since $2HMe^{-(\tau-\varepsilon')(\tilde{n}-1)} \ge \kappa_d$, we get

$$\begin{aligned}|\Theta(n_0, n_1, ..., n_M)| &\ge \frac{\kappa_d}{2e^{(\sigma+\varepsilon')(N+1)}} \cdot \left(\frac{\kappa_d}{2MH}\right)^{(\sigma+\varepsilon')/(\tau-\varepsilon')} \\ &\ge H_0^{-\varepsilon/2} H^{-(\sigma+\varepsilon')/(\tau-\varepsilon')} \ge H^{-((\sigma+\varepsilon')/(\tau-\varepsilon')+\varepsilon/2)} > H^{-\sigma/\tau-\varepsilon}.\end{aligned}$$

This completes the proof. $\square$

For an arbitrarily fixed integer $k \ge 2$, we now put $q_n = M_n A_n(k) \in \mathbf{Z}$. Then $q_n \ne 0$ for all $n \in \mathbf{N}$ by Lemma 1.6. Since $\lim_{n\to\infty}(\log M_n)/n = 2$ by the prime number theorem, it follows from (3.6) that

$$\limsup_{n\to\infty} \frac{1}{n} \log|q_n| \le \log\left(\frac{5^{5/2}k}{4}\right) + 2 = \sigma(k), \quad \text{say.} \tag{6.3}$$

Putting $p_{1+\delta,n} = (-1)^\delta 2k^{1-\delta} M_n B_n^\delta(k) \;\; (\delta = 0, 1)$, $p_{3,n} = -2M_n C_n^1(k)$ and $p_{4,n} = 4kM_n(C_n^0(k) - 2\log 2 B_n^0(k))$, we have $p_{m,n} \in \mathbf{Z}$ for $1 \le m \le 4$ and $n \in \mathbf{N}$ by Theorem 2.3. We also put $\varepsilon_{1+\delta,n} = (-1)^\delta 2k^{1-\delta} M_n I_n^\delta(k) \;\; (\delta = 0, 1)$, $\varepsilon_{3,n} = -2M_n J_n^1(k)$ and $\varepsilon_{4,n} = 4kM_n(J_n^0(k) - 2\log 2 I_n^0(k))$. Then it follows from (4.3) and (5.3) that

$$\max_{1\le m\le 4} \limsup_{n\to\infty} \frac{1}{n} \log|\varepsilon_{m,n}| \le \frac{1}{2}\log\left(\frac{0.9541}{\sqrt{k} - 1 - \rho}\right) + 2 = -\tau(k), \quad \text{say,} \tag{6.4}$$

for any integer $k \ge 31$. We have $\tau(k) > 0$ for any integer $k \ge 2909$.

We next show that our sequences $\{q_n\}$ and $\{p_{m,n}\}$ $(1 \le m \le 4)$ defined above satisfy the condition (6.1) for $N = 5$. Suppose, on the contrary, that

$$S_{r+j} = r_0 q_{r+j} + \sum_{m=1}^{4} r_m p_{m,r+j} = 0 \quad (0 \le j \le 5)$$

for some $r \in \mathbf{N}$ and some $(r_0, r_1, ..., r_4) \in \mathbf{Z}^5 \backslash \{(0, 0, ..., 0)\}$. Since the sequence $X_n = S_n/M_n$ can be expressed as a linear combination of $A_n(k), I_n^\delta(k)$ and $J_n^\delta(k)$ $(\delta = 0, 1)$ with coefficients independent of n, it follows from Lemma 1.7 that $\{X_n\}$ also satisfies the recurrence (1.2). Therefore $S_n = 0$; hence $q_n\Theta(r_0, r_1, ..., r_4) = \omega_n$ for all $n \ge r$. If $k \ge 2909$, then $\omega_n \to 0$ as $n \to \infty$; thus we get $\Theta(r_0, r_1, ..., r_4) = 0$ and $\omega_n = 0$ for all $n \ge r$. Since

$$\begin{aligned}0 = \sum_{m=1}^{4} r_m \varepsilon_{m,n} &= k(r_1 - 4r_4\log 2) I_n^0(k) - r_2 I_n^1(k) + 2kr_4 J_n^0(k) - r_3 J_n^1(k) \\ &= \int_0^1 \frac{k(r_1 - 4r_4\log 2) - r_2 x + (2kr_4 - r_3 x)\log(1 + 1/x)}{k - x^2} A_n(x^2)dx\end{aligned}$$

for all $n \ge r$, it follows from Lemma 1.8 that

$$\frac{k(r_1 - 4r_4\log 2) - r_2 x}{k - x^2} - U(x) + \left(\frac{2kr_4 - r_3 x}{k - x^2} - W(x)\right)\log\left(1 + \frac{1}{x}\right) \equiv 0$$

for $0 < x < 1$ and for some polynomials $U(x)$ and $W(x)$. Therefore we have $k(r_1 - 4r_4 \log 2) - r_2 x \equiv (k - x^2)U(x)$ and $2kr_4 - r_3 x \equiv (k - x^2)W(x)$; hence $U(x) = W(x) \equiv 0$ and $r_1 = r_2 = r_3 = r_4 = 0$. We thus have $r_0 = 0$, because $0 = S_r = r_0 q_r$. This contradiction implies that our sequences satisfy the condition (6.1), as required.

Thus Lemma 6.1 can be applied to the system (3.1)-(3.4), so that we have

Theorem 6.2. *Let $k \geq 2909$ be an integer. Then, for any $\varepsilon > 0$, there exists an effectively computable constant $H_0 \equiv H_0(\varepsilon, k)$ such that*

$$\Big| n_0 + n_1 \sqrt{k} \log \frac{\sqrt{k}+1}{\sqrt{k}-1} + n_2 \log\left(1 - \frac{1}{k}\right) + n_3 \log\left(1 + \frac{1}{\sqrt{k}}\right) \log\left(1 - \frac{1}{\sqrt{k}}\right) + n_4 \sqrt{k} \Lambda\left(\frac{1}{\sqrt{k}}\right) \Big| \geq H^{-\sigma(k)/\tau(k) - \varepsilon}$$

for any $(n_0, n_1, ..., n_4) \in \mathbf{Z}^5$ satisfying $H = \max_{1 \leq m \leq 4} |n_m| \geq H_0$. (For the definitions of $\sigma(k), \tau(k)$ and the function $\Lambda(x)$, see (6.3), (6.4) *and* (3.5) *respectively.)*

In particular, taking $k = q^2$, we have

Corollary 6.3. *Let $q \geq 54$ be an integer. Then, for any $\varepsilon > 0$, there exists an effectively computable constant $H_1 \equiv H_1(\varepsilon, q)$ such that*

$$\Big| n_0 + n_1 \log\left(1 + \frac{1}{q}\right) + n_2 \log\left(1 - \frac{1}{q}\right) + n_3 \log\left(1 + \frac{1}{q}\right) \log\left(1 - \frac{1}{q}\right) + n_4 \Lambda\left(\frac{1}{q}\right) \Big| \geq H^{-\sigma(q^2)/\tau(q^2) - \varepsilon}$$

for any $(n_0, n_1, ..., n_4) \in \mathbf{Z}^5$ satisfying $H = \max_{1 \leq m \leq 4} |n_m| \geq H_1$.

Theorem 0.1 in the Introduction follows immediately from this corollary.

The linear independence measure $\sigma(k)/\tau(k)$ in Theorem 6.2 is fairly large when k is not so large. For example, one has $\sigma(2909)/\tau(2909) = 349075.6....$ However it is easily seen that $\sigma(k)/\tau(k)$ tends to 4 as $k \to \infty$.

Linear independence results for other sets of four numbers will be obtained if we put $z = i\sqrt{k}$ or $z = e^{\pi i/3}\sqrt{k}$ instead of $z = \sqrt{k}$. To such cases we will be able to apply Lemma 6.1 for $d = 1$ or $d = \sqrt{3}$ respectively.

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