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WEAK*-CLOSEDNESS OF SUBSPACES OF FOURIER-STIELTJES ALGEBRAS AND WEAK*-CONTINUITY OF THE RESTRICTION MAP

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Dedicated to Professor Elmar Thoma on the occasion of his seventieth birthday

ABSTRACT. Let G be a locally compact group and $B(G)$ the Fourier-Stieltjes algebra of G. We study the problem of how weak*-closedness of some translation invariant subspaces of $B(G)$ is related to the structure of G. Moreover, we prove that for a closed subgroup H of G, the restriction map from $B(G)$ to $B(H)$ is weak*-continuous only when H is open in G.

INTRODUCTION

Let G be a locally compact group, and let $B(G)$ be the Fourier-Stieltjes algebra of G as defined by Eymard [8]. Recall that $B(G)$ is the linear span of all continuous positive definite functions on G and can be identified with the Banach space dual of $C^*(G)$, the group C^*-algebra of G. The space $B(G)$, with the norm as dual of $C^*(G)$, is a commutative Banach *-algebra with pointwise multiplication and complex conjugation. The Fourier algebra $A(G)$ of G is the closed *-subalgebra of $B(G)$ generated by the functions in $B(G)$ with compact support. In particular, $A(G)$ is contained in $C_0(G)$, the algebra of complex valued continuous functions on G vanishing at infinity. As is well known $A(G)$ is weak*-dense in $B(G)$ if and only if G is amenable. In [3] translation invariant *-subalgebras A of $B(G)$ were studied, and it was shown that if such A is weak*-closed and point separating, then it must contain $A(G)$. However, apart from this, very little seems to be known about weak*-closed subspaces of $B(G)$.

The first purpose of this paper is to investigate the relation between weak*-closedness of certain interesting norm-closed translation invariant subspaces of $B(G)$ and the structure of G. Secondly, we solve the problem of when, for a closed subgroup H of G, the restriction map from $B(G)$ to $B(H)$ is weak*-continuous.

A brief outline of the paper is as follows. In Section 2 we establish for almost connected locally compact groups G the relation between weak*-closedness of $B_0(G) = B(G) \cap C_0(G)$ in $B(G)$ and the structure of G (Theorem 2.10). The key result is that for a connected Lie group G, $B_0(G)$ is weak*-closed in $B(G)$ if and only if G is a reductive Lie group with compact centre and Kazhdan's property (T).

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If G is a compact group, then for any unitary representation π of G, the Fourier space $A_\pi(G)$ associated to π (see [1] for the definition and properties of $A_\pi(G)$) is weak*-closed in $B(G)$. Note that $A(G) = A_{\lambda_G}(G)$, where λ_G denotes the left regular representation of G. In Theorem 3.6 we shall show that, conversely, if G contains an almost connected open normal subgroup and $A(G)$ is weak*-closed in $B(G)$, then G is compact. We also give a characterization of compactness of G in terms of the weak* and the norm topologies on the unit sphere of $B(G)$ (Theorem 3.9).

Besides the left regular representation, one of the most interesting representations of a locally compact group G is the conjugation representation γ_G of G on $L^2(G)$. In contrast to $A(G), A_{\gamma_G}(G)$ need not be a subalgebra and it can at best determine the structure of $G/Z(G)$, where $Z(G)$ denotes the centre of G. We prove that if G is a Lie group with countably many connected components and $A_{\gamma_G}(G)$ is weak*-closed in $B(G)$, then $G/Z(G)$ is compact (Theorem 4.8).

Let H be a closed subgroup of an arbitrary locally compact group G. Clearly, the restriction map $B(G) \to B(H)$ is continuous for the weak*-topologies whenever H is open. In the final section 5 we succeed in showing that conversely weak*-continuity of the restriction map forces H to be open in G.

1. Preliminaries

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure dx and modular function Δ, $L^1(G)$ the convolution algebra of integrable functions on G and $C^*(G)$ the group C^*-algebra of G. The Fourier-Stieltjes algebra $B(G)$ is the Banach space dual of $C^*(G)$ and as such carries the weak*-topology (w^*-topology, for short) $\sigma(B(G), C^*(G))$. The basic reference on Fourier and Fourier-Stieltjes algebras is [8].

Next, we have to introduce some notation from representation theory. We use the same letter, for example π, for a unitary representation of G and for the corresponding *-representation of $C^*(G)$. $\mathcal{H}_\pi$ will always denote the Hilbert space of π and $\ker\pi$ the C^*-kernel of π. If S and T are sets of unitary representations of G, then S is weakly contained in T ($S \prec T$) if $\bigcap_{\sigma\in S}\ker\sigma \supseteq \bigcap_{\tau\in T}\ker\tau$ or, equivalently, if every positive definite function associated to S can be uniformly approximated on compact subsets of G by sums of positive definite functions associated to T. Also S and T are weakly equivalent ($S \sim T$) if $S \prec T$ and $T \prec S$.

The dual space $\widehat{G}$ of G is the set of equivalence classes of irreducible representations of G, endowed with the Jacobson topology. For a representation π of G, the support of π is the closed subset $\operatorname{supp}\pi = \{\rho \in \widehat{G} : \rho \prec \pi\}$ of $\widehat{G}$. In particular, the support of the left regular representation λ_G is the reduced dual $\widehat{G}_r$, and $\lambda_G(C^*(G))$ is the so-called reduced group C^*-algebra of G which is denoted by $C^*_r(G)$. If N is a closed normal subgroup of G, then every representation of G/N can be lifted to a representation of G, and in this sense, $(G/N)\widehat{\ }$ will always be regarded as a subset of $\widehat{G}$. For general references to representation theory and dual spaces we mention [5] and [10].

G is called amenable if there exists a non-zero positive linear functional m on the space $C^b(G)$ of bounded continuous complex valued functions on G such that $m({}_xf) = m(f)$ for all $f \in C^b(G)$ and $x \in G$, where ${}_xf(y) = f(x^{-1}y)$. Recall that amenability is equivalent to a number of different conditions: $C^*_r(G) = C^*(G)$,

$\widehat{G}_r = \widehat{G}$ or $1_G \prec \lambda_G$ where 1_G is the trivial one-dimensional representation of G. Concerning the theory of amenable groups, we refer the reader to [29] and [30].

Let H be a closed subgroup of G, and suppose that σ and π are representations of H and G respectively. The representation of G induced by σ is denoted $\operatorname{ind}_H^G \sigma$. Then the tensor product $\pi \otimes \operatorname{ind}_H^G \sigma$ is equivalent to $\operatorname{ind}_H^G(\pi|H \otimes \sigma)$. Furthermore, by the theorem on induction in stages, $\operatorname{ind}_H^G \sigma = \operatorname{ind}_K^G(\operatorname{ind}_H^K \sigma)$ for every closed subgroup K of G containing H. Finally, we will frequently use that $\pi \prec \operatorname{ind}_H^G(\pi|H)$ if G is amenable [12, Theorem 5.1].

Let A be a C^*-algebra and $\widehat{A}$ its dual space, i.e. the set of equivalence classes of non-degenerated irreducible *-representations of A. We will several times use the fact that if the Banach space dual A^* of A is separable in the norm topology, then $\widehat{A}$ is countable (see [15, Theorem 3.1] or [23, Lemma 4.12]).

2. When is $B_0(G)$ w^*-closed in $B(G)$?

For a locally compact group G, let $B_0(G) = B(G) \cap C_0(G)$ denote the norm-closed and translation invariant subalgebra of $B(G)$ consisting of all functions in $B(G)$ that vanish at infinity. In this section we shall study the problem of when $B_0(G)$ is closed in $B(G)$ with respect to the w^*-topology on $B(G)$.

It turns out to be appropriate to reformulate this condition in terms of convergence of positive definite functions. The following lemma, which is a consequence of [8, (2.1) and (2.12)], will be used frequently.

Lemma 2.1. *Let $P(G)$ denote the set of all normalized continuous positive definite functions on G. The following are equivalent*:

(i) *$B_0(G)$ is w^*-closed in $B(G)$.*

(ii) *If (φ_ι) is a net in $P(G) \cap C_0(G)$ converging to some $\varphi \in P(G)$ uniformly on compact subsets of G, then $\varphi \in C_0(G)$.*

Lemma 2.2. *Suppose that $B_0(G)$ is w^*-closed in $B(G)$. Then every amenable closed normal subgroup of G is compact. In particular, for an amenable group G, $B_0(G)$ is w^*-closed in $B(G)$ only when G is compact.*

Proof. Let N be an amenable closed normal subgroup of G. Then the trivial one-dimensional representation 1_N of N is weakly contained in the left regular representation λ_N of N. Hence, by continuity of inducing, $\operatorname{ind}_N^G 1_N$ is weakly contained in $\operatorname{ind}_N^G \lambda_N$, which is equivalent to λ_G.

Let $q : G \to G/N$ denote the quotient homomorphism. Then $\operatorname{ind}_N^G 1_N = \lambda_{G/N} \circ q$. Thus every positive definite function φ associated to $\lambda_{G/N} \circ q$ is a uniform on compacta limit of functions in $A(G) \cap P(G) \subseteq B_0(G)$. Hence, by hypothesis, $\varphi \in C_0(G)$. Since such a φ is constant on cosets of N, N must be compact. □

We continue with two inheritance properties.

Lemma 2.3. *Let H be an open subgroup of the locally compact group G. If $B_0(G)$ is w^*-closed in $B(G)$, then $B_0(H)$ is w^*-closed in $B(H)$.*

Proof. Let (φ_ι) be a net in $P(H) \cap C_0(H)$ such that $\varphi_\iota \to \varphi \in P(H)$ uniformly on compact subsets of H. Let $\widetilde{\varphi_\iota}$ and $\tilde{\varphi}$ denote the trivial extensions of φ_ι and φ to G, that is, $\tilde{\varphi}(x) = \widetilde{\varphi_\iota}(x) = 0$ for $x \in G \setminus H$. Clearly, then $\widetilde{\varphi_\iota} \in P(G) \cap C_0(G)$, $\tilde{\varphi} \in P(G)$ and $\widetilde{\varphi_\iota} \to \tilde{\varphi}$ uniformly on compact subsets of G. By hypothesis, $\tilde{\varphi} \in C_0(G)$ and hence $\varphi \in C_0(H)$. □

Lemma 2.4. *Let G be a locally compact group and K a compact normal subgroup of G. Then $B_0(G)$ is w^*-closed in $B(G)$ if and only if $B_0(G/K)$ is w^*-closed in $B(G/K)$.*

Proof. Suppose that $B_0(G)$ is w^*-closed in $B(G)$, and let (φ_ι) be a net in $P(G/K)\cap C_0(G/K)$ converging to some $\varphi \in P(G/K)$ uniformly on compact subsets of G/K. Then, with $q : G \to G/K$ the quotient homomorphism, $\varphi_\iota \circ q \to \varphi \circ q$ uniformly on compact subsets of G and $\varphi_\iota \circ q \in C_0(G)$ since K is comapct. Hence, by hypothesis, $\varphi \circ q \in C_0(G)$ and so $\varphi \in C_0(G/K)$.

Conversely, suppose that $B_0(G/K)$ is w^*-closed in $B(G/K)$, and let $\varphi \in P(G)$ and $(\varphi_\iota) \subseteq P(G) \cap C_0(G)$ such that $\varphi_\iota \to \varphi$ uniformly on compact subsets of G. Define ψ_ι and ψ on G/K by

$$\psi_\iota(xK) = \int_K |\varphi_\iota(xk)|^2 dk \text{ and } \psi(xK) = \int_K |\varphi(xk)|^2 dk,$$

$x \in G$ (dk being the normalized Haar measure on K). Then

$$\psi \in P(G/K) \text{ and } \psi_\iota \in P(G/K) \cap C_0(G/K),$$

and $\psi_\iota \to \psi$ uniformly on compact subsets of G/K. Hence $\psi \in C_0(G/K)$.

For $\delta \in \widehat{K}$, let χ_δ denote the corresponding minimal idempotent in $L^1(K)$. Then by the Cauchy-Schwarz inequality,

$$|\varphi * \chi_\delta(x)| \leq |\psi(xK)|^{1/2}$$

for every $x \in G$. Since K is compact, this implies $\varphi * \chi_\delta \in C_0(G)$ for each $\delta \in \widehat{K}$. Now the linear span of $\{\chi_\delta : \delta \in \widehat{K}\}$ is dense in $Z(L^1(K))$, the centre of $L^1(K)$. It follows that for any $f \in Z(L^1(K))$, $\varphi * f$ is a uniform limit on G of finite linear combinations of functions $\varphi * \chi_\delta$, $\delta \in \widehat{K}$. Hence $\varphi * f \in C_0(G)$ for every $f \in Z(L^1(K))$. Finally, taking for f functions in $Z(L^1(K))$ with support shrinking to $\{e\}$, we easily conclude that $\varphi \in C_0(G)$. This completes the proof. □

We now turn to connected Lie groups. Theorem 2.7 below is the key result in this section.

Lemma 2.5. *Let G be a connected Lie group and N a connected closed normal subgroup. If $B_0(G)$ is w^*-closed in $B(G)$, then the centre $Z(N)$ of N is compact and $N/Z(N)$ is semisimple.*

Proof. Let R denote the radical of N. Then R and $Z(N)$ are amenable normal subgroups of G, and therefore both must be compact by Lemma 2.2. Since R is solvable and connected Lie, it is isomorphic to a torus $\mathbb{T}^n$. Hence $\mathrm{Aut}(R)$, the automorphism group of R, is discrete. Now, G acts by conjugation on R, and this defines a continuous homomorphism from G into $\mathrm{Aut}(R)$. G being connected, this homomorphism has to be trivial. This shows that R is contained in the centre of G. So $R \subseteq Z(N)$ and hence $N/Z(N)$ is semisimple. □

We remind the reader that a locally compact group G is said to have Kazhdan's property (T) if the trivial representation 1_G is an isolated point in the dual space $\widehat{G}$. An amenable group satisfies (T) if and only if it is compact. On the other hand, many connected semisimple Lie groups and many discrete groups share property (T). A comprehensive account on groups with property (T) has been given in [14].

Lemma 2.6. *Let G be a connected Lie group such that $B_0(G)$ is w^*-closed in $B(G)$. Then G has property (T).*

Proof. By Lemma 2.5, G is reductive with compact centre. Let

$$\widetilde{G} = \mathbb{R}^n \times G_1 \times \cdots \times G_m$$

be the universal covering group of G, where $G_1, \ldots, G_m$ are simply connected Lie groups. Denote by Z_i the (discrete) centre of G_i, $i = 1, \ldots, m$. Then $G = \widetilde{G}/\Gamma$ for some discrete subgroup Γ of the centre $Z(\widetilde{G}) = \mathbb{R}^n \times Z_1 \times \cdots \times Z_m$ of $\widetilde{G}$. Moreover, $Z(G) = Z(\widetilde{G})/\Gamma$. Hence

$$G/Z(G) = \widetilde{G}/Z(\widetilde{G}) = G_1/Z_1 \times \cdots \times G_m/Z_m.$$

Since $Z(G)$ is compact, $B_0(G/Z(G))$ is w^*-closed in $B(G/Z(G))$ (Lemma 2.4). Assume, towards a contradiction, that G does not have property (T). Then $G/Z(G)$ does not have property (T). Hence some factor, say G_1/Z_1, fails to have property (T) (see [35, Lemma 7.4.1]). Now, recall the following result due to Howe and Moore [35, Theorem 2.2.20]. If π is a unitary representation of a simple Lie group with finite centre and if there are no non-zero π-invariant vectors, then all the matrix coefficients of π vanish at infinity. Therefore, there exists a sequence

$$(\varphi_n^{(1)}) \subseteq P(G_1/Z_1) \cap C_0(G_1/Z_1)$$

converging to 1 uniformly on compact subsets of G_1/Z_1. Observe that G_1/Z_1 is not compact. Now choose arbitrary

$$\varphi^{(k)} \in P(G_k/Z_k) \cap C_0(G_k/Z_k),$$

$k = 2, \ldots, m$, and set

$$\varphi_n = \varphi_n^{(1)} \times \varphi^{(2)} \times \cdots \times \varphi^{(m)}.$$

Clearly, (φ_n) is a sequence of continuous positive definite functions on $G/Z(G)$ that vanish at infinity, and

$$\varphi_n \to 1 \times \varphi^{(2)} \times \cdots \times \varphi^{(m)}$$

uniformly on compact subsets of $G/Z(G)$. Since $1 \times \varphi^{(2)} \times \cdots \times \varphi^{(m)}$ does not vanish at infinity, we have reached a contradiction. Thus G has property (T). □

Theorem 2.7. *Let G be a connected Lie group. Then $B_0(G)$ is w^*-closed in $B(G)$ if and only if G is a reductive Lie group with compact centre and Kazhdan's property (T).*

Proof. From Lemma 2.5, applied to $N = G$, and Lemma 2.6 we know that w^*-closedness of $B_0(G)$ in $B(G)$ implies the stated conditions on G.

Suppose now that G is a connected reductive Lie group with compact centre $Z(G)$ and property (T). According to Lemma 2.4 it suffices to show that $B_0(G/Z(G))$ is w^*-closed in $B(G/Z(G))$. Recall from the proof of Lemma 2.6 that $G/Z(G)$ has a decomposition

$$G/Z(G) = G_1/Z_1 \times \cdots \times G_m/Z_m,$$

where $G_1, \ldots, G_m$ are connected simple Lie groups with centres $Z_1, \ldots, Z_m$, respectively. Set $H = G/Z(G)$ and $H_i = G_i/Z_i$ for $i = 1, \ldots, m$, and observe that every H_i has property (T). Of course, according to Lemma 2.4 we can assume that none of the H_i is compact.

Let $\varphi_n \in P(H) \cap C_0(H)$, $n \in \mathbb{N}$, such that $\varphi_n \to \varphi$ uniformly on compact subsets of G for some $\varphi \in P(H)$. Let π_n and π denote the representations of H associated to φ_n and φ through the GNS-construction. Then π is weakly contained in the direct sum $\bigoplus_{n=1}^{\infty} \pi_n$.

We claim that the restriction $\pi|H_i$ of π to H_i does not contain the trivial representation 1_{H_i}. Indeed, otherwise for some $n \in \mathbb{N}$, $\pi_n|H_i$ contains 1_{H_i} since H_i has property (T). However, since $\varphi_n \in C_0(H)$, all the matrix coefficients of π_n vanish at infinity.

We have thus verified that H satisfies the hypotheses of the Howe-Moore theorem [35, Theorem 2.2.20]. It follows that all the matrix coefficients of π vanish at infinity. This proves that $\varphi \in C_0(H)$. □

In order to deal with almost connected groups we need one more lemma.

Lemma 2.8. *Let G be a locally compact group and H a closed subgroup such that G/H is compact. If $B_0(H)$ is w^*-closed in $B(H)$, then $B_0(G)$ is w^*-closed in $B(G)$.*

Proof. For any function ϕ on G and $x, y \in G$, let

$${}_x\phi(y) = \phi(x^{-1}y), \phi_x(y) = \phi(yx) \text{ and } \phi^x(y) = \phi(x^{-1}yx).$$

Notice first that if ψ is a positive definite function on G, then $\psi^x + \psi + {}_x\psi + \psi_x$ and $\psi^x + \psi + i({}_x\psi - \psi_x)$ are also positive definite for every $x \in G$. Indeed, for all $f \in L^1(G)$

$$\langle \psi^x + \psi + {}_x\psi + \psi_x, f\rangle = \langle \psi, (\delta_x + \delta_e)^* * f * (\delta_x + \delta_e)\rangle$$

and

$$\langle \psi^x + \psi + i({}_x\psi - \psi_x), f\rangle = \langle \psi, (\delta_e - i\delta_x)^* * f * (\delta_e - i\delta_x)\rangle.$$

Let (φ_ι) be a net in $P(G) \cap C_0(G)$ converging to some $\varphi \in P(G)$ uniformly on compact subsets of G. Then, uniformly on compact subsets of G, $\varphi_\iota^x \to \varphi^x$,

$$\varphi_\iota^x + \varphi_\iota + {}_x(\varphi_\iota) + (\varphi_\iota)_x \to \varphi^x + \varphi + {}_x\varphi + \varphi_x$$

and

$$\varphi_\iota^x + \varphi_\iota + i({}_x\varphi_\iota - (\varphi_\iota)_x) \to \varphi^x + \varphi + i({}_x\varphi - \varphi_x)$$

for every $x \in G$. Thus, since $B_0(H)$ is w^*-closed in $B(H)$,

$$\varphi|H, \varphi^x|H, (\varphi^x + \varphi + {}_x\varphi + \varphi_x)|H \text{ and } (\varphi^x + \varphi + i({}_x\varphi - \varphi_x))|H$$

vanish at infinity on H. It follows that ${}_x\varphi|H \in C_0(H)$ for each $x \in G$. Since G/H is compact, employing the uniform continuity of φ, it is easily verified that $\varphi \in C_0(G)$. □

The converse to Lemma 2.8 does not hold in general. That is, if $B_0(G)$ is w^*-closed in $B(G)$ and H is a closed cocompact subgroup of G, then $B_0(H)$ need not be w^*-closed in $B(H)$. As an example, take for G a simply connected Lie group with finite centre and property (T) and for H a minimal parabolic subgroup. Then by the Howe-Moore result referred to in the proofs of Lemma 2.6 and Theorem 2.7, $B_0(G)$ is w^*-closed in $B(G)$, while $B_0(H)$ fails to be w^*-closed in $B(H)$ since H is non-compact and amenable.

Corollary 2.9. *Let G be a connected Lie group and N a connected closed normal subgroup of G such that G/N is compact. Then $B_0(G)$ is w^*-closed in $B(G)$ if and only if $B_0(N)$ is w^*-closed in $B(N)$.*

Proof. Suppose that $B_0(G)$ is w^*-closed in $B(G)$. By Lemma 2.5, $Z(N)$ is compact and $N/Z(N)$ is semisimple. Also, since G has property (T) by Lemma 2.6 and G/N is compact, N has property (T) [34, Theorem 3.7]. By Theorem 2.7 this implies that $B_0(N)$ is w^*-closed in $B(N)$.

The converse is a special case of Lemma 2.8. □

Theorem 2.10. *Let G be an almost connected locally compact group. Then $B_0(G)$ is w^*-closed in $B(G)$ if and only if the connected component G_0 of G is a projective limit of reductive Lie groups with property (T) and compact centres.*

Proof. Suppose first that G_0 has the indicated structure. Choose a compact normal subgroup K of G_0 such that G_0/K is a reductive Lie group with property (T) and compact centre. By Theorem 2.7, $B_0(G_0/K)$ is w^*-closed in $B(G_0/K)$. Since K and G/G_0 are compact, an application of Lemmas 2.4 and 2.8 yields that $B_0(G)$ is w^*-closed in $B(G)$.

Conversely, suppose that $B_0(G)$ is w^*-closed in $B(G)$. G being almost connected it is a projective limit of Lie groups G/K_ι. Thus there are closed normal subgroups H_ι of finite index in G such that $K_\iota \subseteq H_\iota$ and $H_\iota/K_\iota = (G/K_\iota)_0$. Then G_0 is the projective limit of the groups $G_0/G_0 \cap K_\iota$, and the $G_0/G_0 \cap K_\iota$ are connected Lie groups since

$$G_0/G_0 \cap K_\iota = G_0K_\iota/K_\iota,$$

a closed connected subgroup of G/K_ι. By Theorem 2.7 it suffices to show that $B_0(G_0/G_0 \cap K_\iota)$ is w^*-closed in $B(G_0/G_0 \cap K_\iota)$.

Now, since $B_0(G)$ is w^*-closed in $B(G)$, $B_0(G/K_\iota)$ is w^*-closed in $B(G/K_\iota)$ by Lemma 2.4, and hence $B_0(H_\iota/K_\iota)$ is w^*-closed in $B(H_\iota/K_\iota)$ by Lemma 2.3. Moreover, G_0K_ι/K_ι is a cocompact connected normal subgroup of the connected Lie group H_ι/K_ι. Thus, by Corollary 2.9, $B_0(G_0K_\iota/K_\iota)$ is w^*-closed in $B(G_0K_\iota/K_\iota)$. This proves that $B_0(G_0/G_0 \cap K_\iota)$ is w^*-closed in $B(G_0/G_0 \cap K_\iota)$. □

We conclude this section with some remarks.

Remarks 2.11. (i) The connected reductive Lie groups with property (T) and compact centres are precisely the groups of the form $G = (\mathbb{R}^n \times G_1 \times \cdots \times G_m)/\Gamma$, where $G_1, \ldots, G_m$ are simple Lie groups not locally isomorphic to $\mathrm{SO}(k,1)$, $k \geq 2$, or $\mathrm{SU}(k,1)$, $k \geq 1$, and Γ is a discrete cocompact subgroup of $\mathbb{R}^n \times Z_1 \times \cdots \times Z_m$, the centre of $\mathbb{R}^n \times G_1 \times \cdots \times G_m$.

Indeed, if G is of this form, then $G/Z(G) = G_1 \times \cdots \times G_m$, where $Z(G)$ denotes the centre of G, has property (T) (see [14, Chap. 2, 13. Corollaire, 9. Remarque and Chap. 9]). As $Z(G)$ is compact, G has property (T) [14, Chap. 1, 9. Proposition].

Conversely, let G be a connected reductive Lie group with property (T) and compact centre. Let $\widetilde{G} = \mathbb{R}^n \times G_1 \times \cdots \times G_m$ be its universal covering group, where $G_1, \ldots, G_m$ are simple Lie groups with centres $Z_1, \ldots, Z_m$. The arguments used in the proof of Lemma 2.6 show that $G_1, \ldots, G_m$ have property (T) and hence are not locally isomorphic to $\mathrm{SO}(k,1)$ or $\mathrm{SU}(k,1)$ (see [14, Chap. 6, 23. Corollaire]).

(ii) Let G be a discrete group such that $B_0(G)$ is w^*-closed in $B(G)$. Then every element in G has finite order. Indeed, this follows immediately by applying Lemma 2.2 to the cyclic subgroups of G.

(iii) Let G be a linear group (that is, a subgroup $\mathrm{GL}(n,K)$ for some field K) with the discrete topology. If G is infinite, then $B_0(G)$ is not w^*-closed in $B(G)$. In fact, this is clear from (ii) if G has an element of infinite order. On the other hand, if G is a torsion group, then it is well known to be locally finite [6, Theorem 9.2] and hence amenable, so that the claim follows from Lemma 2.2.

3. Characterizations of compact groups

If G is a compact group then $A(G) = B(G)$. It seems likely that the converse is also true; i.e. w^*-closedness of $A(G)$ in $B(G)$ already forces G to be compact. We have been able to show this for groups containing an almost connected open normal subgroup (Theorem 3.6). The case that remains open is that of a totally disconnected group.

We start with a lemma which will be generalized in Section 4 (Lemma 4.2).

Lemma 3.1. *If $A(G)$ is w^*-closed in $B(G)$ and H is an open subgroup of G, then $A(H)$ is w^*-closed in $B(H)$.*

Proof. It suffices to show that the unit ball of $A(H)$ is w^*-closed in the unit ball of $B(H)$. Thus, let (φ_ι) be a net in $A(H)$ and $\varphi \in B(H)$ such that

$$\|\varphi_\iota\| \leq 1, \|\varphi\| \leq 1 \text{ and } \varphi_\iota \to \varphi$$

in the w^*-topology. Let $\tilde{\varphi}_\iota$ and $\tilde{\varphi}$ denote the trivial extensions of φ_ι and φ to G. Then $\tilde{\varphi}_\iota \in A(G)$, $\|\tilde{\varphi}_\iota\| \leq 1$ and, for each $f \in L^1(G)$,

$$\begin{aligned} \int_G \tilde{\varphi}_\iota(x) f(x) dx &= \int_H \varphi_\iota(h) f(h) dh \\ &\to \int_H \varphi(h) f(h) dh = \int_G \tilde{\varphi}(x) f(x) dx. \end{aligned}$$

Hence $\tilde{\varphi}_\iota \to \tilde{\varphi}$ in the $\sigma(B(G), L^1(G))$-topology. Since $(\tilde{\varphi}_\iota)$ is a bounded net, it follows that $(\tilde{\varphi}_\iota)$ is w^*-convergent to $\tilde{\varphi}$. By hypothesis, $\tilde{\varphi} \in A(G)$ and so $\varphi \in A(H)$. □

Lemma 3.2. *Let K be a compact normal subgroup of G. If $A(G)$ is w^*-closed in $B(G)$, then $A(G/K)$ is w^*-closed in $B(G/K)$.*

Proof. Consider the map $T_K : f \to T_K f$ from $L^1(G)$ onto $L^1(G/K)$ given by

$$T_K f(xK) = \int_K f(xk) dk.$$

This map extends to a *-homomorphism from $C^*(G)$ onto $C^*(G/K)$ with dual map $T_K^* : B(G/K) \to B(G)$. Furthermore, $T_K^*(B(G/K))$ consists precisely of those functions in $B(G)$ that are constant on cosets of K [8, (2.26)]. Also, since K is compact,

$$T_K^*(A(G/K)) = A(G) \cap T_K^*(B(G/K)).$$

Now, let $\varphi_\iota \in A(G/K)$ such that $\varphi_\iota \to \varphi$ in the w^*-topology for some $\varphi \in B(G/K)$. Then

$$\langle T_K^*(\varphi_\iota), f\rangle = \langle \varphi_\iota, T_K(f)\rangle \to \langle \varphi, T_K(f)\rangle = \langle T_K^*(\varphi), f\rangle$$

for each $f \in C^*(G)$. Thus, by hypothesis, $T_K^*(\varphi) \in A(G)$ and so

$$T_K^*(\varphi) \in A(G) \cap T_K^*(B(G/K)),$$

whence $\varphi \in A(G/K)$. □

Lemma 3.3. *Let G be an almost connected locally compact group. If $A(G)$ is w^*-closed in $B(G)$, then G is compact.*

Proof. Since an almost connected group is a projective limit of Lie groups and $A(G/K)$ is w^*-closed in $B(G/K)$ for every compact normal subgroup K of G (Lemma 3.2), we can assume that G is a Lie group. Being a compactly generated Lie group, G is second countable and hence $A(G)$ is a separable Banach space. By hypothesis,

$$A(G) = B_\lambda(G) = C_r^*(G)^*.$$

Now, a C^*-algebra A with separable dual Banach space has a countable dual $\widehat{A}$ (see Section 1). It follows that $\widehat{G}_r$, the reduced dual of G, is countable. Finally, by [2, Theorem 2.5] a separable Lie group with countable reduced dual is compact. This shows that G is compact. □

Corollary 3.4. *Let G be any locally compact group and suppose that $A(G)$ is w^*-closed in $B(G)$. Then G contains a compact open subgroup.*

Proof. Since G/G_0 is totally disconnected, there exists an open subgroup H of G so that H/G_0 is compact. By Lemma 3.1, $A(H)$ is w^*-closed and hence H is compact by Lemma 3.3. □

Lemma 3.5. *If G is a discrete group and $A(G)$ is w^*-closed in $B(G)$, then G is finite.*

Proof. Assume that G is infinite. Then G has a countable infinite subgroup H. By Lemma 3.1, $A(H)$ is w^*-closed in $B(H)$. As in the proof of Lemma 3.3 we now conclude that $\widehat{H}_r$ is countable. Applying Baggett's result again, it follows that H is finite, a contradiction. □

Theorem 3.6. *Suppose that G contains an almost connected open normal subgroup. Then $A(G)$ is w^*-closed in $B(G)$ if and only if G is compact.*

Proof. Let N be an almost connected open normal subgroup of G. Then $A(N)$ is w^*-closed in $B(N)$ by Lemma 3.1, and Lemma 3.3 implies that N is compact. By Lemma 3.2, $A(G/N)$ is w^*-closed in $B(G/N)$. Since N is open, Lemma 3.5 gives that G/N is finite. Thus G is compact. □

We now turn to a second characterization of compact groups in terms of certain properties of the w^*-topology on $B(G)$. If G is a compact group, then the w^*-topology and the norm topology agree on the unit sphere of $B(G) = A(G)$ [11, Corollary 2]. We are going to establish the converse to this (see [22, Theorem 5] for the amenable case). Actually, we prove a stronger result in that we replace the unit sphere of $B(G)$ by the smaller set $P_\lambda(G) = B_\lambda(G) \cap P(G)$ of all normalized positive definite functions on G associated to representations that are weakly contained in the left regular representation. Note that this property implies the Radon-Nikodym property for $B(G)$ but not conversely (see [11] and [33]).

For any locally compact group G, let $P_\lambda(G) = B_\lambda(G) \cap P(G)$, the set of all normalized continuous positive definite functions on G associated to representations that are weakly contained in the left regular representation. $P_\lambda(G)$ is a w^*-compact convex subset of $B(G)$. We denote by $\mathrm{ex}(P_\lambda(G))$ the set of extreme points of the w^*-compact convex subset $P_\lambda(G)$ of $B(G)$.

Lemma 3.7. *Let G be a locally compact group and $\varphi \in \mathrm{ex}(P_\lambda(G))$. Suppose that φ is a point of continuity of the identity map*

$$(\mathrm{ex}(P_\lambda(G)), w^*) \to (\mathrm{ex}(P_\lambda(G)), \|\cdot\|).$$

Then π_φ is an isolated point in $\widehat{G}_r$.

Proof. Notice first that $\mathrm{ex}(P_\lambda(G)) \subseteq \mathrm{ex}(P(G))$ because if $\varphi \in P_\lambda(G)$ and $\psi \in P(G)$ are such that $c\varphi - \psi$ is positive definite for some $c \geq 0$, then $\psi \in P_\lambda(G)$. By [5, 2.12.1], if $\varphi_1, \varphi_2 \in \mathrm{ex}(P(G))$ and π_{φ_1} and π_{φ_2} are not equivalent, then $\|\varphi_1 - \varphi_2\| \geq 2$. By assumption there exists a w^*-open subset U of $\mathrm{ex}(P_\lambda(G))$ such that

$$U \subseteq \{\psi \in \mathrm{ex}(P_\lambda(G)) : \|\psi - \varphi\| < 2\}.$$

It follows that $\pi_\psi = \pi_\varphi$ for all $\psi \in U$. Now, by [5, Theorem 3.4.11], the map $q : \psi \to \pi_\psi$ from $\mathrm{ex}(P_\lambda(G))$ onto $\widehat{G}_r$ is open. Thus $\{\pi_\varphi\} = q(U)$ is open in $\widehat{G}_r$. □

Lemma 3.8. *Let H be an open subgroup of G. If the identity map from $(P_\lambda(G), w^*)$ to $(P_\lambda(H), \|\cdot\|)$ is continuous, then the identity map from $(P_\lambda(H), w^*)$ to $(P_\lambda(H), \|\cdot\|)$ is continuous.*

Proof. For any $\varphi \in P_\lambda(H)$, the trivial extension $\tilde{\varphi}$ belongs to $P_\lambda(G)$. Indeed, $\tilde{\varphi}$ is a positive definite function associated to the induced representation $\mathrm{ind}_H^G \pi_\varphi$, and $\pi_\varphi \prec \lambda_H$ implies

$$\mathrm{ind}_H^G \pi_\varphi \prec \mathrm{ind}_H^G \lambda_H = \lambda_G.$$

Let (φ_α) be a net in $P_\lambda(H)$ converging to $\varphi \in P_\lambda(H)$ in the w^*-topology. Then $\tilde{\varphi}_\alpha \to \tilde{\varphi}$ in the w^*-topology on $P_\lambda(G)$ (compare the proof of Lemma 3.1). By hypothesis, $\|\tilde{\varphi}_\alpha - \tilde{\varphi}\| \to 0$ and hence $\|\varphi_\alpha - \varphi\| \to 0$. □

Theorem 3.9. *For any locally compact group G the following conditions are equivalent.*

(i) *G is compact.*
(ii) *The w^*-topology and the norm topology agree on the unit sphere of $B(G)$.*
(iii) *The w^*-topology and the norm topology agree on $P_\lambda(G)$.*

Proof. As mentioned above, (i)⇒(ii) is due to Granirer and Leinert [11]. Since (ii)⇒(iii) is trivial, it only remains to prove (iii)⇒(i).

Assume that G fails to be compact. Then G contains a non-compact, σ-compact, open subgroup H. By Lemma 3.8, the w^*-topology and the norm topology coincide on $P_\lambda(H)$. It follows from Lemma 3.7 that $\widehat{H}_r$ is discrete. Since H is σ-compact Theorem 7.6 of [34] now shows that H is compact, a contradiction. □

4. When is $A_\gamma(G)$ w^*-closed in $B(G)$?

For a locally compact group G and any unitary representation π of G, the Fourier space $A_\pi(G)$ associated to π is defined to be the norm-closed linear subspace of $B(G)$ generated by all the coordinate functions of π [1], that is, the functions of the form $x \to \langle \pi(x)\xi, \eta\rangle$, $\xi, \eta \in \mathcal{H}_\pi$.

The conjugation representation γ_G (or simply γ, if no confusion can arise) on $L^2(G)$ is defined by

$$\gamma_G(x)f(y) = \Delta(x)^{1/2} f(x^{-1}yx),$$

$f \in L^2(G)$, $x, y \in G$. The purpose of this section is to investigate the question of when $A_\gamma(G)$ is w^*-closed in $B(G)$. It will turn out that this is closely related to problems on the support of γ as studied in [21]. We start with two simple facts on $A_\pi(G)$ for general representations π.

Lemma 4.1. *If G is a compact group, then $A_\pi(G)$ is w^*-closed in $B(G)$ for every representation π of G.*

Proof. The w^*-closure $\overline{A_\pi(G)}^{w^*}$ of $A_\pi(G)$ is the dual space of the C^*-algebra $\pi(C^*(G))$, which is a quotient of $C^*(G)$. Hence each $\varphi \in \overline{A_\pi(G)}^{w^*}$ is a linear combination of positive definite functions in $\overline{A_\pi(G)}^{w^*}$. Therefore, it suffices to prove that every positive definite $\varphi \in \overline{A_\pi(G)}^{w^*}$ actually is in $A_\pi(G)$.

For that, notice that there is a net (φ_α) in $A_\pi(G)$ such that $\varphi_\alpha \to \varphi$ in the w^*-topology and $\|\varphi_\alpha\| \to \|\varphi\|$ (compare [10, p. 565]). By [11, Theorem A] it follows that $\|\varphi_\alpha\psi - \varphi\psi\| \to 0$ for every $\psi \in A(G)$. In particular, $\|\varphi_\alpha - \varphi\| \to 0$ by setting $\psi = 1 \in B(G) = A(G)$. This shows that $\varphi \in A_\pi(G)$. □

Lemma 4.2. *Suppose that π is a representation of G such that $A_\pi(G)$ is w^*-closed in $B(G)$. Then $A_{\pi|H}(H)$ is w^*-closed in $B(H)$ for every open subgroup H of G.*

Proof. Recall that, by [1, Theorem 2.2], $\varphi \in B(G)$ belongs to $A_\pi(G)$ if and only if φ can be written as

$$\varphi = \sum_{n=1}^{\infty} \langle \pi(\cdot)\xi_n, \eta_n \rangle$$

where $\xi_n, \eta_n \in \mathcal{H}_\pi$ and $\sum_{n=1}^{\infty} \|\xi_n\| \cdot \|\eta_n\| < \infty$. In particular, $A_{\pi|H}(H) = A_\pi(G)|H$. It suffices to show that the unit ball of $A_{\pi|H}(H)$ is w^*-closed in the unit ball of $B(H)$. Thus, let $\varphi_i \in A_{\pi|H}(H)$, $i \in I$, and $\varphi \in B(H)$ such that $\|\varphi_i\| \le 1$, $\|\varphi\| \le 1$ and $\varphi_i \to \varphi$ in the w^*-topology. Choose representations

$$\varphi_i = \sum_{n=1}^{\infty} \langle \pi(\cdot)\xi_{in}, \eta_{in} \rangle$$

such that $\sum_{n=1}^{\infty} \|\xi_{in}\| \cdot \|\eta_{in}\| \le 2$ (see [1, Proposition 2.9]). Define

$$\psi_i(x) = \sum_{n=1}^{\infty} \langle \pi(x)\xi_{in}, \eta_{in} \rangle$$

for all $x \in G$ and $i \in I$. Then $\psi_i \in A_\pi(G)$ and $\|\psi_i\| \le 2$. Since the unit ball in $B(G)$ is w^*-compact, we can assume that $\psi_i \to \psi$ in the w^*-topology for some $\psi \in B(G)$. Now, $A_\pi(G)$ is w^*-closed in $B(G)$, so that $\psi \in A_\pi(G)$ and hence $\psi|H \in A_{\pi|H}(H)$. On the other hand, the restriction map $B(G) \to B(H)$ is w^*-continuous as H is open (see Section 5). Indeed, this follows from the fact that $C^*(H)$ is a subalgebra of $C^*(G)$ whenever H is open in G. Thus

$$\varphi_i = \psi_i|H \to \psi|H \text{ and } \varphi_i \to \varphi$$

in the w^*-topology. This proves $\varphi = \psi|H \in A_{\pi|H}(H)$. □

We now apply the preceding lemmas to the conjugation representation. The following corollary will be used several times in the sequel.

Corollary 4.3. *Suppose that G is second countable and $A_{\gamma_G}(G)$ is w^*-closed in $B(G)$. Then, for every open subgroup H of G, $\operatorname{supp}\gamma_H$ is countable.*

Proof. Since G is second countable, $A_\gamma(G)$ is norm separable. Since the restriction map from $B(G)$ to $B(H)$ is norm continuous, $A_{\gamma_G|H}(H) = A_{\gamma_G}(G)|H$ is norm separable. Now, since γ_H is a subrepresentation of $\gamma_G|H$ and $A_{\gamma_G|H}(H)$ is w^*-closed in $B(H)$ by Lemma 4.2,

$$\overline{A_{\gamma_H}(H)}^{w^*} \subseteq A_{\gamma_G|H}(H),$$

so that $\overline{A_{\gamma_H}(H)}^{w^*}$ is norm separable. Thus $\gamma_H(C^*(H))$ has a norm separable dual Banach space, $\overline{A_{\gamma_H}(H)}^{w^*}$, and hence

$$\operatorname{supp}\gamma_H = \gamma_H(C^*(H))^\frown$$

is countable. □

Corollary 4.4. *Let $Z(G)$ denote the centre of G. If $G/Z(G)$ is compact, then $A_{\gamma_G}(G)$ is w^*-closed in $B(G)$.*

Proof. For $z \in Z(G)$, $\gamma_G(z)$ is the identity on $L^2(G)$. Thus $\pi(xZ(G)) = \gamma_G(x)$, $x \in G$, defines a representation of $G/Z(G)$, and therefore $A_\pi(G/Z(G))$ is w^*-closed in $B(G/Z(G))$ by Lemma 4.1. Denoting by $q : G \to G/Z(G)$ the quotient homomorphism, we have

$$A_{\gamma_G}(G) = A_\pi(G/Z(G)) \circ q.$$

By [1, (2.10)], $A_{\gamma_G}(G)$ is w^*-closed in $B(G)$. □

Our goal is to establish the converse to Corollary 4.4 for Lie groups with countably many connected components (Theorem 4.8). Apart from using various results from [21], a major step in proving the theorem will be the next lemma.

We remind the reader that a group G is called an FC-group if all its conjugacy classes are finite. Such a group, more generally every locally compact group all of whose conjugacy classes are relatively compact, is amenable.

Lemma 4.5. *Let G be a countable discrete* FC*-group. If* $\operatorname{supp}\gamma_G$ *is countable, then G has a finite commutator subgroup.*

Proof. Let $S = \operatorname{supp}\gamma_G$ and notice first that points in S are closed in $\widehat{G}$. Indeed, the primitive ideal space of any FC-group is a T_1 space (see [28, Theorem 5.2]) and $C^*_\gamma(G)$, being a separable C^*-algebra with countable dual, is of type I. Thus the points of S are closed in $\widehat{G}$. Since $C^*(G)$ is unital, it follows that every $\sigma \in S$ is finite dimensional.

Next, employing the facts that points in S are closed, that S is countable and that duals of C^*-algebras are Baire spaces [5, (3.4.13)], a straightforward argument yields the existence of some dense subset D of S consisting of points that are also open in S.

Since G is a countable amenable group,

$$\bigcup_{\pi\in\widehat{G}} \operatorname{supp}(\pi \otimes \overline{\pi})$$

is a dense subset of S by [19, Theorem]. Let $\mathcal{C}(S)$ denote the set of all closed subsets of S, endowed with Fell's topology [10, p. 427]. By [18, Proposition 2], the mapping

$$\pi \to \operatorname{supp}(\pi \otimes \overline{\pi}), \widehat{G} \to \mathcal{C}(S)$$

is continuous. It follows that

$$V = \{\pi \in \widehat{G} : \operatorname{supp}(\pi \otimes \overline{\pi}) \cap D \neq \varnothing\}$$

is non-empty and open in $\widehat{G}$.

Now, as points in D are open in S, $\mathrm{supp}(\pi \otimes \overline{\pi})$ contains a finite dimensional subrepresentation for each $\pi \in V$. However, $\pi \otimes \overline{\pi}$ then also contains the trivial representation 1_G. This can be seen as follows. Suppose that τ is finite dimensional and that $\tau \leq \pi \otimes \overline{\pi}$. Then

$$1_G \leq \tau \otimes \overline{\tau} \leq \pi \otimes \overline{\pi \otimes \tau},$$

and $\pi \otimes \tau$ is a (finite) direct sum of irreducible representations $\rho_1, \ldots, \rho_n$. Thus $1_G \leq \pi \otimes \overline{\rho_i}$ for some i, which is impossible unless $\rho_i \sim \pi$ (see [17, Proposition 2.4]). As is well-known, $1_G \leq \pi \otimes \overline{\pi}$ forces π to be finite dimensional. Hence every $\pi \in V$ is finite dimensional.

Finally, for a discrete FC-group G, the existence of a non-empty open subset in $\widehat{G}$ consisting of finite dimensional representations implies that G has a finite commutator subgroup. In fact, in this case the left regular representation of G has a subrepresentation of type I, and then the commutator subgroup has to be finite by [16, Satz 1] (see also [32, Theorem 3]). □

Lemma 4.6. *Let G be a locally compact group with open centre. Then, for each $a \in G$,*

$$\mathrm{ind}_{C(a)}^G 1_{C(a)} \prec \gamma_G,$$

where $C(a)$ denotes the centralizer of a in G.

Proof. Since $\mathrm{ind}_{C(a)}^G 1_{C(a)}$ is the cyclic representation of G defined by the positive definite function $\chi_{C(a)}$, the characteristic function of $C(a)$, and since $C(a)$ is open, it suffices to show that given any compact subset K of $G \setminus C(a)$, there is a positive definite function φ associated to γ_G such that $\varphi(x) = 1$ for all $x \in C(a)$ and $\varphi(x) = 0$ for all $x \in K$. Now, since

$$C = \{a^{-1}x^{-1}ax : x \in K\}$$

is compact and $e \notin C$, we find an open neighborhood V of e, contained in the centre of G, such that $CV \cap V = \varnothing$. Let

$$\varphi(x) = |V|^{-1} \langle \gamma_G(x)\chi_{aV}, \chi_{aV} \rangle$$

for $x \in G$. Then it is easily verified that $\varphi(x) = 1$ for all $x \in C(a)$ and $\varphi(x) = 0$ for all $x \in K$. □

Lemma 4.7. *Suppose that G is second countable and contains an open normal subgroup N such that G/N is abelian and every irreducible representation of N is finite dimensional. If $\mathrm{supp}\, \gamma_G$ is countable, then every irreducible representation of G is finite dimensional.*

Proof. Given an irreducible representation π of G, there exist a subgroup H of G and a finite dimensional irreducible representation τ of H such that $N \subseteq H$ and

$$\pi \sim \mathrm{ind}_H^G \tau.$$

In fact, this has been shown in [7, Theorem 3.2.3] as an application of representation theory of crossed product C^*-algebras. We prefer to outline a direct argument for this within the framework of Mackey's unitary representation theory of group extensions [26].

Choose $\gamma \in \widehat{N}$ such that $\pi|N \sim G(\gamma)$, the G-orbit of γ in $\widehat{N}$ under the action of G. Let S denote the stability subgroup of γ. By [26, Theorems 8.2 and 8.3], there exist a multiplier ω on S/N, an irreducible ω-representation ρ of S in $\mathcal{H}_\gamma$ and an irreducible ω-representation σ of S/N such that

$$\pi = \operatorname{ind}_S^G(\rho \otimes \sigma).$$

Now, an irreducible ω-representation of an abelian group is weakly equivalent to the ω-representation induced from some one-dimensional ω-representation of a certain subgroup. Hence there exist a subgroup H of S containing N and a one-dimensional ω-representation λ of H/N such that

$$\rho \otimes \sigma \sim \rho \otimes \operatorname{ind}_H^S \lambda = \operatorname{ind}_H^S(\rho|H \otimes \lambda).$$

Let $\pi = \rho|H \otimes \lambda$, a finite dimensional ordinary representation. Then

$$\pi \sim \operatorname{ind}_S^G(\operatorname{ind}_H^S(\rho|H \otimes \lambda)) = \operatorname{ind}_H^G \tau,$$

as required.

It follows that

$$\begin{aligned}\pi \otimes \overline{\pi} = \operatorname{ind}_H^G(\tau \otimes \overline{\pi}|H) &\succ \operatorname{ind}_H^G(\tau \otimes \overline{\tau})\\ &\succ \operatorname{ind}_H^G 1_H \sim \widehat{G/H}.\end{aligned}$$

On the other hand, $\pi \otimes \overline{\pi}$ is weakly contained in γ_G since G is amenable. Now, as $\operatorname{supp} \gamma_G$ is countable and H is open and G/H is abelian, H must have finite index in G. Thus, $\operatorname{ind}_H^G \tau$ is finite dimensional and hence so is π. □

Theorem 4.8. *Suppose that G is a Lie group with countably many connected components. If $A_{\gamma_G}(G)$ is w^*-closed in $B(G)$, then $G/Z(G)$ is compact.*

Proof. For any normal subgroup N of G, let $q_N : G \to G/N$ denote the quotient homomorphism. Since $A_{\gamma_G}(G)$ is w^*-closed in $B(G)$ and G is second countable, and since the connected component G_0 of e is open, $\operatorname{supp} \gamma_{G_0}$ is countable by Corollary 4.3. Theorem 3.4 of [21] yields that $G_0 = V \times C$, the direct product of a vector group V and a compact group C. Clearly, C is normal in G.

We claim that $V = G_0/C$ is contained in the centre of G/C. To that end, fix $a \in G$ and consider the open subgroup H of G generated by a and G_0. Then $\operatorname{supp} \gamma_H$ is countable (Corollary 4.3), and since

$$\gamma_{G/C} \circ q_C \prec \gamma_H$$

due to the compactness of C [19, Remark 1], it follows that $\gamma_{H/C}$ has a countable support. Since G_0/C is a vector group and $(H/C)/(G_0/C) = H/G_0$ is abelian, an application of Lemma 3.2 in [21] shows that G_0/C is in the centre of H/C. Since a is arbitrary, G_0/C is central in G/C.

Next we are going to prove that G/G_0 has a finite commutator subgroup. Passing to $F = G/C$, we know that γ_F has countable support, and since V is central in F, by [21, Lemma 1.1]

$$\gamma_{F/V} \circ q_V \prec \gamma_F,$$

so that $\operatorname{supp} \gamma_{F/V}$ is countable. Let $D = F/V$ and denote by D_f the finite conjugacy class subgroup of D. Then, by [20, Theorem 1.8]

$$\lambda_{D/D_f} \circ q_{D_f} \prec \gamma_D,$$

where λ_{D/D_f} is the regular representation of D/D_f. Thus the reduced dual of D/D_f is countable, and hence D/D_f is finite (see [2, 34]). At this stage we know in particular that G is amenable, so that $\gamma_{G/N} \circ q_N \prec \gamma_G$ for each closed normal subgroup N of G [21, Lemma 1.1]. Also, γ_{D_f} has countable support and therefore the commutator subgroup D'_f of D_f is finite by Lemma 4.5. Let N be the inverse image of D'_f in G and let $E = G/N$. Then γ_E has countable support, and E possesses an abelian normal subgroup A of finite index, namely $A = D_f/D'_f$. We apply Lemma 4.6 to E and obtain that

$$\mathrm{ind}_{C(x)}^{E} 1_{C(x)} \prec \gamma_E$$

for every $x \in E$. It follows that

$$\mathrm{ind}_{C(x)\cap A}^{A} 1_{C(x)\cap A} \prec (\mathrm{ind}_{C(x)}^{E} 1_{C(x)})|A \prec \gamma_E|A,$$

and $\gamma_E|A$ has countable support since γ_E does and E/A is finite. However, A being abelian, countability of

$$(A/C(x) \cap A)\hat{} = \mathrm{supp}(\mathrm{ind}_{C(x)\cap A}^{A} 1_{C(x)\cap A})$$

implies that $C(x) \cap A$ is of finite index in A. Hence $C(x)$ has finite index in E for every $x \in E$. Therefore G/N is an FC-group, and Lemma 4.5 implies that E has a finite commutator subgroup. Recalling that $N/G_0 = D'_f$ is finite, we conclude that G/G_0 has a finite commutator subgroup.

Since G/G_0 has a finite commutator subgroup, there exists a normal subgroup N of G such that $G_0 \subseteq N$, N/G_0 is finite and G/N is abelian. Now, $G_0 = V \times C$, and this implies that all the irreducible representations of N are finite dimensional. Since $\mathrm{supp}\,\gamma_G$ is countable, Lemma 4.7 shows that G has only finite dimensional irreducible representations.

On the other hand, G has a relatively compact commutator subgroup. To see this, notice first that since V is contained in the centre of N/C, N/C has a finite commutator subgroup. Denoting its inverse in G by K, N/K is isomorphic to V. By arguments that have previously been used, V is central in G/K, and applying Lemma 4.6 again, this time to G/K, we conclude that G/K is a group with finite conjugacy classes.

Thus G is a group with relatively compact conjugacy classes all of whose irreducible representations are finite dimensional. Since, in addition, G is a Lie group, combining Theorem 2 of [27] and Lemma 5.4 of [24] shows that $G/Z(G)$ is compact. □

5. w^*-continuity of the restriction map

Let G be a locally compact group, and let H be a closed subgroup of G. In this section we study the question of when the restriction map

$$\Phi : B(G) \to B(H), \varphi \to \varphi|H$$

is continuous for the w^*-topologies. Clearly, if H is open then $C^*(H)$ is a subalgebra of $C^*(G)$ and hence Φ is w^*-continuous. We are going to establish the following converse.

Theorem 5.1. *Let H be a closed subgroup of the locally compact group G. If the restriction map $B(G) \to B(H)$ is continuous for the w^*- topologies, then H is open in G.*

Let $M(G)$ denote the algebra of all bounded measures on G. The canonical embedding of $L^1(G)$ into $M(G)$ extends to an isometric *-homomorphism of $C^*(G)$

into $C^*(M(G))$, the enveloping C^*-algebra of $M(G)$. Therefore we may (and shall) identify $C^*(G)$ with a closed two-sided ideal of $C^*(M(G))$.

Let H be a closed subgroup of G. For $f \in L^1(H)$, let $\mu_f \in M(G)$ denote the measure on G defined by f. The mapping $f \to \mu_f$ from $L^1(H)$ into $M(G)$ extends to a *-homomorphism $\Psi : C^*(H) \to C^*(M(G))$. By general principles, the restriction map $\Phi : B(G) \to B(H)$ is w^*-continuous if and only if Φ is the transpose of some continuous linear mapping $\Theta : C^*(H) \to C^*(G)$. It is easy to verify that, in this case, Θ has to agree with Ψ on $L^1(H)$. Hence we have the following lemma.

Lemma 5.2. *The restriction map $\Phi : B(G) \to B(H)$ is w^*-continuous if and only if the range of $\Psi : C^*(H) \to C^*(M(G))$ is contained in $C^*(G)$.*

Remark 5.3. It is interesting to notice that the homomorphism $\Psi : C^*(H) \to C^*(M(G))$ need not always be injective (see [4]).

Now, let ρ denote the right regular representation of G on $L^2(G)$ and $C^*_r(M(G))$ the C^*-subalgebra of $\mathcal{L}(L^2(G))$ generated by the set of all operators $\rho(\mu)$, $\mu \in M(G)$. The mapping $f \to \mu_f$ from $L^1(H)$ into $M(G)$ extends to a *-homomorphism

$$\Psi_r : C^*(H) \to C^*_r(M(G)).$$

It is clear that Ψ_r is the composition of Ψ and the canonical homomorphism from $C^*(M(G))$ onto $C^*_r(M(G))$.

In view of Lemma 5.2, we thus observe that Theorem 5.1 will be a consequence of the following stronger result.

Theorem 5.4. *Let H be a closed subgroup of the locally compact group G. If the range of the homomorphism $\Psi_r : C^*(H) \to C^*_r(M(G))$ is contained in $C^*_r(G)$, then H is open in G.*

The proof of Theorem 5.4 depends on two elementary lemmas, the first of which appears also in [25]. For the sake of completeness, however, we give a very short and different proof.

Lemma 5.5. *Let G be a locally compact group and let $T \in C^*_r(G)$. Suppose that K is a compact subset of G and regard $L^2(K)$ as a closed subspace of $L^2(G)$ in the usual manner. Then the restriction*

$$T|L^2(K) : L^2(K) \to L^2(G)$$

of T to $L^2(K)$ is a compact operator.

Proof. Of course, it suffices to prove the statement for operators T of the form $T = \rho(f)$ where $f \in C_c(G)$. Let $K' = \operatorname{supp} f \cdot K$. Then $Tg \in L^2(K')$ for all $g \in L^2(K)$. Choose $f_j \in C_c(K')$ and $g_j \in C_c(K)$ such that, as $n \to \infty$,

$$\int_G \int_G \left| \sum_{j=1}^n f_j(x) g_j(y) - f(xy) \right|^2 dx dy \to 0.$$

It is straightforward to verify that this implies that $T : L^2(K) \to L^2(K')$ is a norm limit of finite rank operators. □

Lemma 5.6. *Let (X, μ) be a probability measure space without atoms. Then the canonical embedding $L^2(X) \to L^1(X)$ fails to be compact.*

Proof. It suffices to show that there is an orthonormal sequence $(f_n)_n$ in $L^2(X)$ such that $\|f_n\|_\infty \leq 1$. Indeed, we then have

$$2 = \|f_n - f_m\|_2^2 = \int_X |f_n(x) - f_m(x)|^2 d\mu(x) \leq \|f_n - f_m\|_1$$

for all $n, m \in \mathbb{N}$, $n \neq m$, and hence no subsequence of $(f_n)_n$ can converge in $L^1(X)$.

Since μ has no atoms, there exists, for every $0 \leq r \leq 1$, a measurable subset A of X with $\mu(A) = r$ (see [13, Section 4.1, Exercise 1]). Therefore, we can choose inductively, for every $n \in \mathbb{N}$, disjoint measurable subsets

$$A_1^{(n)}, A_2^{(n)}, \ldots, A_{2^{n-1}}^{(n)}$$

of X with the following properties:

(1) $\mu(A_i^{(n)}) = \frac{1}{2^{n-1}}$ for $i = 1, \ldots, 2^{n-1}$.
(2) $A_{2i-1}^{(n+1)}, A_{2i}^{(n+1)} \subseteq A_i^{(n)}$ for $i = 1, \ldots, 2^{n-1}$.

For each $n \in \mathbb{N}$, define a function $f_n \in L^2(X)$ (a kind of Rademacher function) by setting

$$f_n(x) = (-1)^i \text{ for } x \in A_i^{(n)} \text{ and } i = 1, \ldots, 2^{n-1}.$$

By construction and (1) and (2), we obviously have

$$\|f_n\|_\infty = \|f_n\|_2 = 1 \text{ and } \int_X f_n(x) f_m(x) d\mu(x) = 0, \qquad n \neq m.$$

This completes the proof. □

Proof of Theorem 5.4. Let H be a closed subgroup of G and suppose $\Psi_r(C^*(H))$ is contained in $C_r^*(G)$. Fix a Bruhat function β on G for H (see [31, Chapter 8]), that is, a non-negative continuous function β on G with the following properties:

(i) For every compact subset K of G, β agrees on KH with the restriction of some function from $C_c(G)$.
(ii) $\int_H \beta(xh) dh = 1$ for all $x \in G$.

Let Δ_G and Δ_H denote the modular functions of G and H, respectively, and define a function q on G by

$$q(x) = \int_H \beta(xh) \Delta_G(h) \Delta_H(h)^{-1} dh.$$

Since $q(xh) = q(x)\Delta_G(h)^{-1}\Delta_H(h)$ for all $x \in G$ and $h \in H$, there exists a quasi-invariant measure $d_q\dot{x}$ on $\dot{G} = G/H$ such that

$$\int_G \left(\frac{f(xh)}{q(xh)} dh \right) d_q\dot{x} = \int_G f(x) dx$$

for all $f \in L^1(G)$ [31, Chapter 8].

Now let $\dot{K} \subseteq \dot{G}$ be a compact neighbourhood of $\dot{e}$ in $\dot{G}$. We are going to prove that $\dot{K}$ is finite. Clearly, this will imply that H is open. Let $\pi : G \to \dot{G}$ denote the quotient map and choose a compact neighbourhood K of e in G so that $\pi(K) = \dot{K}$. Fix a continuous function f with compact support on H such that

$$\int_H f(h) \Delta_G(h)^{-1} dh \neq 0.$$

Let $K' = KH \cap \operatorname{supp}\beta$, which is a compact subset of G. By Lemma 5.5, the restriction of $\rho(\mu_f)$, the right convolution operator defined by μ_f, to $L^2(K')$ is a compact operator. Define a linear mapping

$$T : L^2(\dot{K}, d_q\dot{x}) \to L^2(K')$$

by $T\dot{g}(x) = \dot{g}(\dot{x})\beta(x)^{1/2}$ for $\dot{g} \in L^2(\dot{K}, d_q\dot{x})$ and $x \in K$. Since

$$\int_H \frac{\beta(xh)}{q(xh)}dh = \frac{1}{q(x)}\int_H \beta(xh)\Delta_G(h)\Delta_H(h)^{-1}dh = 1$$

for all $x \in G$, we have

$$\begin{aligned}\|T\dot{g}\|_2^2 &= \int_{\dot{G}}\left(\int_H \frac{|\dot{g}(\dot{x})|^2\beta(xh)}{q(xh)}dh\right)d_q\dot{x}\\ &= \int_{\dot{G}} |\dot{g}(\dot{x})|^2 d_q\dot{x} = \|\dot{g}\|_2^2.\end{aligned}$$

Thus T is a bounded linear operator. Set

$$q_1(x) = \int_H \beta(xh)^{1/2}\Delta_G(h)\Delta_H(h)^{-1}dh$$

for $x \in G$. Then, for all $x \in G$ and $h \in H$,

$$q_1(xh) = q_1(x)\Delta_G(h)^{-1}\Delta_H(h).$$

Hence there exists a quasi-invariant measure $d_{q_1}\dot{x}$ on $\dot{G}$ such that

$$\int_{\dot{G}}\left(\int_H \frac{q(xh)}{q_1(xh)}dh\right)d_{q_1}\dot{x} = \int_G g(x)dx$$

for all $g \in L^1(G)$. Let

$$S : L^1(G) \to L^1(\dot{G}, d_{q_1}\dot{x})$$

be the linear operator defined by

$$Sg(\dot{x}) = \int_H \frac{g(xh)}{q_1(xh)}dh$$

for $g \in C_c(G)$. S is bounded since

$$\|Sg\|_{L^1(\dot{G}, d_{q_1}\dot{x})} \le \int_{\dot{G}}\left(\int_H \frac{|g(xh)|}{q_1(xh)}dh\right)d_{q_1}\dot{x} = \|g\|_1.$$

Observe next that $\rho(\mu_f)$ maps $L^2(K')$ into $L^2(K'')$ for some compact subset K'' of G. Hence $\rho(\mu_f)T$ map $L^2(\dot{K}, d_q\dot{x})$ into $L^1(G)$, and we may consider the bounded linear operator

$$S\rho(\mu_f)T : L^2(\dot{K}, d_q\dot{x}) \to L^1(\dot{G}, d_{q_1}\dot{x}).$$

For $\dot{g} \in C(\dot{K})$, we have

$$\begin{aligned}S\rho(\mu_{f|H})T(\dot{g})(\dot{x}) &= \dot{g}(\dot{x})\int_H \frac{1}{q_1(xh)}\left(\int_H \beta(xhk)^{1/2}f(k)dk\right)dh\\ &= \dot{g}(\dot{x})\int_H f(k)\left(\int_H \frac{\beta(xhk)^{1/2}}{q_1(xh)}dh\right)dk.\end{aligned}$$

However, for $x \in G$ and $k \in K$,

$$\int_H \frac{\beta(xhk)^{1/2}}{q_1(xh)} dh = \Delta_H(k)^{-1} \int_H \frac{\beta(xh)^{1/2}}{q_1(xhk^{-1})} dh$$
$$= \Delta_G(k)^{-1} \int_H \frac{\beta(xh)^{1/2}}{q(xh)} dh = \Delta_G(k)^{-1}.$$

Thus, setting $\alpha = \int_H f(k)\Delta_G(k)^{-1} dk \neq 0$, we obtain that

$$S\rho(\mu_{f|H})T(\dot{g}) = \alpha \dot{g}$$

for all $\dot{g} \in L^2(\dot{K}, d_q\dot{x})$. Now, by hypothesis and Lemma 5.5, the restriction of $\rho(\mu_{f|H})$ to $L^2(K')$ is compact. Hence

$$\alpha I : L^2(\dot{K}, d_q\dot{x}) \to L^1(\dot{K}, d_{q_1}\dot{x})$$

is a compact operator.

Finally, notice that since

$$d_{q_1}\dot{x} = \frac{q_1(x)}{q(x)} d_q\dot{x}$$

and $\frac{q_1}{q}$ is a strictly positive continuous function on the compact set $\dot{K}$, the corresponding L^1-spaces are equal and the L^1-norms are equivalent. Hence the embedding

$$L^2(\dot{K}, d_q\dot{x}) \to L^1(\dot{K}, d_q\dot{x})$$

is compact. By Lemma 5.6, $\dot{K}$ has to have atoms. However, this implies that $\dot{K}$ is finite. □

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