A WEAK-TYPE INEQUALITY FOR DIFFERENTIALLY SUBORDINATE HARMONIC FUNCTIONS

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Abstract. Assuming an extra condition, we decrease the constant in the sharp inequality of Burkholder \( \mu(|v| \geq 1) \leq 2\|u\|_1 \) for two harmonic functions \( u \) and \( v \). That is, we prove the sharp weak-type inequality \( \mu(|v| \geq 1) \leq K\|u\|_1 \) under the assumptions that \( |v(\xi)| \leq |u(\xi)| \), \( |\nabla v| \leq |\nabla u| \) and the extra assumption that \( \nabla u \cdot \nabla v = 0 \). Here \( \mu \) is the harmonic measure with respect to \( \xi \) and the constant \( K \) is the one found by Davis to be the best constant in Kolmogorov’s weak-type inequality for conjugate functions.

Let \( D \) be a domain in \( \mathbb{R}^n \) where \( n \) is a positive integer. Let \( D_0 \) be a bounded subdomain of \( D \) with \( \partial D_0 \subseteq D \) and \( \xi \in D_0 \). Let \( \mu \) be the harmonic measure on \( \partial D_0 \) with respect to \( \xi \). Let \( K \) be the constant given by
\[
K = \frac{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots}.
\]

Theorem. If \( u \) and \( v \) are harmonic functions on \( D \) such that

(i) \[ |v(\xi)| \leq |u(\xi)|, \]
(ii) \[ |\nabla v| \leq |\nabla u| \] on \( D \),
(iii) \[ \nabla u \cdot \nabla v = 0 \] on \( D \),

then
\[
\mu(|v| \geq 1) \leq K \int_{\partial D_0} |u| \, d\mu.
\]

Remarks. 1. The constant \( K \) was discovered by Davis [6]. He showed that \( K \) is the best constant in Kolmogorov’s weak-type inequality for conjugate functions [9] or, equivalently, for the special case of the inequality above in which \( D \) is the open unit disk of \( \mathbb{R}^2 \), \( D_0 \) is an open disk with center 0 and radius \( r < 1 \), \( \xi = 0 \), \( v(0) = 0 \), and \( u \) and \( v \) are harmonic in \( D \) and satisfy the Cauchy-Riemann equations. Also, see Baernstein [2] for related sharp inequalities.

2. Dropping the classical conjugacy condition and working in \( \mathbb{R}^n \), Burkholder [4] proved the sharp inequality
\[
\mu(|u| + |v| \geq 1) \leq 2 \int_{\partial D_0} |u| \, d\mu
\]
for harmonic functions \( u \) and \( v \) that satisfy the assumptions (i) and (ii) of the theorem. In fact, he proved his inequality for Hilbert-space valued \( u \) and \( v \).
Example 13.1 of [5], he showed that 2 is the best constant even for the inequality  
\[ \mu(|v| \geq 1) \leq 2\|u\|_1 \]  
where \[ \|u\|_p = \sup_{D_0} \int_{\partial D_0} |u|^p \, d\mu. \]

3. Using (i) and (ii), Burkholder [4] also proved that  
\[ \|v\|_p \leq (p^* - 1)\|u\|_p \]
for \(1 < p < \infty\) where \(p^* = \max\{p, p/(p-1)\}\). It is not yet known whether the constant \(p^* - 1\) is best possible in this setting. However, using (i), (ii), and the extra assumption (iii), Bañuelos and Wang [3] proved the inequality  
\[ \|v\|_p \leq \cot(\pi/2p^*)\|u\|_p \]
for \(1 < p < \infty\). This is a sharp inequality since it is already sharp in the classical M. Riesz case [11] in which \(v(0) = 0\) and \(v\) is the harmonic function conjugate to \(u\) on the open unit disk of the plane (see Pichorides [10] and the independent work of Brian Cole that is described in Gamelin [7]).

Outline of the proof of the theorem. Consider the function \(V\) on \(\mathbb{R}^2\) given by  
\[ V(x, y) = \begin{cases}  
-K|x| & \text{if } |y| < 1, \\
1 - K|x| & \text{if } |y| \geq 1. 
\end{cases} \]

We observe that  
\[ \mu(|v| \geq 1) - K \int_{\partial D_0} |u| \, d\mu = \int_{\partial D_0} V(u, v) \, d\mu. \]

The following lemma will be proved later:

**Main Lemma.** There is a continuous function \(U\) on \(\mathbb{R}^2\) such that

(a) \(V \leq U\) on \(\mathbb{R}^2\),  
(b) \(U(u, v)\) is superharmonic on \(D\),  
(c) \(U(x, y) \leq 0\) if \(|x| \geq |y|\).

Then from (a) and (b) we get  
\[ \int_{\partial D_0} V(u, v) \, d\mu \leq \int_{\partial D_0} U(u, v) \, d\mu \leq U(u(\xi), v(\xi)) \]

because \(\mu\) is the harmonic measure on \(\partial D_0\) with respect to \(\xi\). Finally, (c) and the assumption (i) imply that \(U(u(\xi), v(\xi)) \leq 0\), which proves the theorem.

Before we prove the lemma we define another function \(W\) on \(\mathbb{R}^2\) and establish some properties of \(W\). We will use basic facts about harmonic functions, which can be found in [1] and [8].

Put \(H = \{ (\alpha, \beta) : \beta > 0 \}, \ S = \{ (x, y) : |y| < 1 \}\) and \(S^+ = \{ (x, y) \in S : x > 0 \}\). Also, put \((x, y) = x + iy = z, J(x + iy) = y, (\alpha, \beta) = \alpha + i\beta = \zeta,\) and define a function \(W\) on \(H\) by

\[ W(\alpha, \beta) = W(\zeta) = \frac{2}{\pi^2} \int_{-\infty}^{\infty} \frac{\beta|\log |t||}{(\alpha - t)^2 + \beta^2} \, dt. \]

Observe that \(W\) is the harmonic function on \(H\) that vanishes as \(\beta \to \infty\), and satisfies

\[ \lim_{(\alpha, \beta) \to (t, 0)} W(\alpha, \beta) = \frac{2}{\pi} |\log |t|| \quad \text{if } t \neq 0. \]
Using \( \pi^2/8 = \sum_{k=0}^{\infty} (2k + 1)^{-2} \), we have that
\[
W(0, 1) = \frac{4}{\pi^2} \int_0^\infty \frac{|\log t|}{t^2 + 1} \, dt
\]
\[
= \frac{4}{\pi^2} \int_{-\infty}^\infty \frac{|s|e^s}{e^{2s} + 1} \, ds
\]
\[
= \frac{8}{\pi^2} \int_0^\infty se^{-s} \sum_{k=0}^{\infty} (-e^{-2s})^k \, ds
\]
\[
= \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^2} = \frac{1}{K}.
\]

Consider the conformal map \( \varphi \) on \( S \) given by
\[
\varphi(z) = ie^{\pi z/2} = \exp \frac{\pi}{2}(z + i).
\]
Observe that \( \varphi(i) = -1, \varphi(-i) = 1, \varphi(-\infty + iy) = 0, \varphi(\infty + iy) = \infty \) and \( \varphi(0) = i \).
Hence \( \varphi \) maps the strip \( S \) onto the upper half plane \( H \). Define \( W: \mathbb{R}^2 \to \mathbb{R} \) by
\[
W(x, y) = \begin{cases} 
|x| & \text{if } |y| \geq 1, \\
W(\varphi(x, y)) & \text{if } |y| < 1,
\end{cases}
\]
and notice that the restriction of \( W \) to \( S \) is harmonic since this restriction is the real part of an analytic function. For \( x_0 \in \mathbb{R} \) we have \( \varphi(x_0, \pm 1) = \pm e^{\pi x_0/2} \neq 0 \), thus
\[
\lim_{(x, y) \to (x_0, \pm 1)} W(x, y) = \frac{2}{\pi} [\log |\varphi(x_0, \pm 1)|] = |x_0| = W(x_0, \pm 1).
\]
Hence \( W \) is continuous on \( \mathbb{R}^2 \) as is the function \( U \) defined by
\[
U(x, y) = 1 - K W(x, y) \quad \text{for } (x, y) \in \mathbb{R}^2.
\]

\textbf{Lemma 1.} If \( (x, y) \in S \), then \( W(x, y) = W(-x, y) = W(x, -y) \) and
\[
W_x(0, y) = W_y(x, 0) = W_{xy}(x, 0) = W_{xy}(0, y) = 0.
\]
\textit{Proof.} In (1) we use the change of variable \( t = -s \) to get \( W(-\alpha, \beta) = W(\alpha, \beta) \).
Also, in the reformulation of \( \mathcal{W} \)
\[
\mathcal{W}(\zeta) = \frac{4}{\pi^2} \int_0^\infty \frac{\zeta |\log t|}{t^2 - \zeta^2} \, dt
\]
we use the change of variable \( t = 1/s \) to get \( \mathcal{W}(1/\bar{\zeta}) = \mathcal{W}(\zeta) \). With \( \varphi(x, y) = \zeta = \alpha + i\beta \) we get \( \varphi(-x, y) = 1/\bar{\zeta} \) and \( \varphi(x, -y) = -\alpha + i\beta \). The symmetry of \( W \) and the rest of the lemma follow.

\textbf{Lemma 2.} \( \lim_{x \to -\infty} \frac{W(x, y) - x}{(x, y) \in S} = 0 \).

\textit{Proof.} \( \varphi(x, y) = \zeta \) we have \( x = \frac{2}{\pi} \log |\zeta| \), hence \( x \to \infty \) if and only if \( |\zeta| \to \infty \) and the lemma is equivalent to
\[
\lim_{|\zeta| \to \infty} \left[ \mathcal{W}(\zeta) - \frac{2}{\pi} \log |\zeta| \right] = 0.
\]
(2)
On $H$, the harmonic function $\zeta \mapsto \frac{2}{\pi} \log |\zeta|$ can be represented by its Poisson integral. Therefore, by (1),

$$W(\zeta) - \frac{2}{\pi} \log |\zeta| = \frac{2}{\pi^2} \int_{-\infty}^{\infty} \left[ \frac{\beta |t|}{(\alpha - t)^2 + \beta^2} - \frac{\beta |t|}{(\alpha - t)^2 + \beta^2} \right] dt$$

$$= \frac{4}{\pi^2} \int_{-1}^{1} \frac{\beta |t|}{(\alpha - t)^2 + \beta^2} dt \to 0 \text{ as } |\zeta| \to \infty$$

which proves (2), hence the lemma.

**Lemma 3.** $\lim_{x \to \infty} W_{xx}(x, y) = \lim_{x \to \infty} W_{xy}(x, y) = 0$.

**Proof.** Consider the continuous function $G$ on $\overline{S^+}$ given by $G(x, y) = W(x, y) - x$.

Observe that $G$ is harmonic on $S^+$ and $G(x, \pm 1) = 0$.

We consider a conformal map $\psi$ given by $\psi(z) = \frac{2}{\pi} \log(1 + \sqrt{2})$. One checks that $\psi(\frac{2}{\pi} \log(1 + \sqrt{2})) = i$, $\psi(0) = \infty$, $\psi(-i) = -1$, $\psi(\infty) = 0$ and $\psi(i) = 1$. Thus $S^+$ is mapped onto $H$ under $\psi$.

We define a harmonic function $F(\alpha, \beta)$ on $H$ by $G = F \circ \psi$. For $|t| < 1$ we have

$$\lim_{(\alpha, \beta) \to (t, 0)} F(\alpha, \beta) = 0.$$  

Indeed, we have from Lemma 2 that if $t = 0$, then

$$0 = \lim_{(x, y) \to (x, y), (x, y) \in S^+} G(x, y) = \lim_{(\alpha, \beta) \to (0, 0)} F(x, y).$$

Also, for $t \neq 0$, since $\psi^{-1}(t, 0) = (c, \pm 1)$ for some $c$ and $G(c, \pm 1) = 0$, the limit (3) follows from the continuity of $G$.

Applying the Schwarz reflection principle, we see that the functions $F_\alpha(\alpha, \beta)$, $F_\beta(\alpha, \beta)$, $F_{\alpha\alpha}(\alpha, \beta)$, $F_{\alpha\beta}(\alpha, \beta)$ and $F_{\beta\beta}(\alpha, \beta)$ tend to certain limits as $(\alpha, \beta)$ tends to $(0, 0)$.

Now from the basic identities

$|\cos iz|^2 = \sinh^2 x + \cosh^2 y$ and $|\sin iz|^2 = \sin^2 x + \sin^2 y$

we observe that

$$\lim_{x \to \infty} \cos \frac{\pi}{2} iz = \lim_{x \to \infty} \sin \frac{\pi}{2} iz = \infty$$

and

$$\lim_{x \to \infty} \tan \frac{\pi}{2} iz = 1.$$  

Differentiating $\psi(z) = \frac{2}{\pi} \log(\frac{2}{\pi} iz) = -1$, we get

$$\psi'(z) \cos \frac{\pi}{2} iz + \frac{\pi}{2} \psi(z) \sin \frac{\pi}{2} iz = 0,$$

$$\psi''(z) \sin \frac{\pi}{2} iz - \frac{\pi}{2} \psi'(z) \cos \frac{\pi}{2} iz + \frac{\pi^2}{4} \psi(z) \sin \frac{\pi}{2} iz = 0.$$  

Hence, if $z = (x, y) \in S^+$ and $|z| \to \infty$, then $\lim \psi(z) = \lim \psi'(z) = \lim \psi''(z) = 0$.

Writing $\psi(x + iy) = \alpha(x, y) + i\beta(x, y)$, we see that as $(x, y) \in S^+$ and $x \to \infty$ all the functions $\alpha$, $\beta$, $\alpha_x$, $\beta_x$, $\alpha_{xx}$, $\beta_{xx}$ tend to 0 because $\psi' = \alpha + i\beta$ and $\psi'' = \alpha_{xx} + i\beta_{xx}$.

From the Cauchy-Riemann equations we have $\alpha_y = -\beta_x$ and $\beta_y = \alpha_x$, so $\alpha_{xy} = -\beta_{xx}$ and $\beta_{xy} = \alpha_{xx}$. Thus, using the chain rule and omitting the argument $(x, y)$,
we have

\[
G_x = \alpha_x F_\alpha (\alpha, \beta) + \beta_x F_\beta (\alpha, \beta),
\]

\[
G_{xx} = (\alpha_x)^2 F_\alpha (\alpha, \beta) + (\beta_x)^2 F_\beta (\alpha, \beta) + 2\alpha_x \beta_x F_{\alpha\beta} (\alpha, \beta)
\]

\[
+ \alpha_x F_\alpha (\alpha, \beta) + \beta_x F_\beta (\alpha, \beta),
\]

\[
G_{xy} = -\alpha_x \beta_x F_{\alpha\beta} (\alpha, \beta) + \alpha_x \beta_x F_{\beta\beta} (\alpha, \beta) + [(\alpha_x)^2 - (\beta_x)^2] F_{\alpha\beta} (\alpha, \beta)
\]

\[
- \beta_x F_\alpha (\alpha, \beta) + \alpha_x F_\beta (\alpha, \beta).
\]

It follows that

\[
\lim_{(x,y) \to S^+} G_{xx} (x,y) = \lim_{(x,y) \to S^+} G_{xy} (x,y) = 0.
\]

This proves the lemma because \(G_{xx} = W_{xx}\) and \(G_{xy} = W_{xy}\).

**Lemma 4.** Consider the function \(A\) on \(H\) given by

\[
A(x, y) = \frac{1}{\pi} \int_{-1}^{1} \frac{y|t|}{(x-t)^2 + y^2} dt.
\]

Then we have

\[
\liminf_{(x,y) \to (0,0)} A_{xx} (x,y) \geq 0 \quad \text{and} \quad \limsup_{(x,y) \to (0,0)} A_{xy} (x,y) \leq 0.
\]

**Proof.** Differentiating under the integral sign and then integrating by parts, we get

\[
\pi A_x (x, y) = \int_{0}^{1} t \frac{\partial}{\partial t} \left[ \frac{y}{(x+t)^2 + y^2} - \frac{y}{(x-t)^2 + y^2} \right] dt
\]

\[
= \frac{y}{(x+1)^2 + y^2} - \frac{y}{(x-1)^2 + y^2} - \int_{0}^{1} \left[ \frac{y}{(x+t)^2 + y^2} - \frac{y}{(x-t)^2 + y^2} \right] dt.
\]

Differentiating under the integral again, we get

\[
\pi A_{xx} (x, y) = -\frac{2(x+1)y}{[(x+1)^2 + y^2]^2} + \frac{2(x-1)y}{[(x-1)^2 + y^2]^2}
\]

\[
- \frac{y}{(x+1)^2 + y^2} - \frac{y}{(x-1)^2 + y^2} + \frac{2y}{x^2 + y^2}
\]

and

\[
\pi A_{xy} (x, y) = \frac{(x+1)^2 - y^2}{[(x+1)^2 + y^2]^2} - \frac{(x-1)^2 - y^2}{[(x-1)^2 + y^2]^2}
\]

\[
+ \frac{x-1}{(x+1)^2 + y^2} + \frac{x+1}{(x-1)^2 + y^2} - \frac{2x}{x^2 + y^2}.
\]

Since \(y > 0\) we have

\[
\liminf_{(x,y) \to (0,0)} A_{xx} (x,y) = \frac{1}{\pi} \liminf_{(x,y) \to (0,0)} \frac{2y}{x^2 + y^2} \geq 0.
\]

Also,

\[
\limsup_{(x,y) \to (0,0)} A_{xy} (x,y) = \frac{1}{\pi} \limsup_{x \to 0} \left( \frac{-2x}{x^2 + y^2} \right) \leq 0.
\]
Lemma 5.

\[ \liminf_{(x,y) \to (0,-1)} W_{xx}(x,y) \geq 0 \quad \text{and} \quad \limsup_{(x,y) \to (0,-1)} W_{xy}(x,y) \leq 0. \]

Proof. Let \( A \) be the function given in Lemma 4. Define \( B(x,y) \) on \( S \) by

\[ B(x,y) = W(x,y) - A(x,y + 1). \]

Observe that \( B \) is harmonic on \( S \) and if \( |x_0| < 1 \), then

\[ \lim_{(x,y) \to (x_0,1)} B(x,y) = |x_0| - \lim_{(x,y) \to (x_0,0)} A(x,y) = 0. \]

Applying the Schwarz reflection principle we get a harmonic extension \( B^* \) of \( B \) over \( S \cup \{(x,-1) : |x| < 1\} \cup \{(x,y) : x \in \mathbb{R}, -3 < y < -1\} \).

Note that \( B^*(x,-1) = 0 \) for \( |x| < 1 \). Thus \( B_{xx}^*(0,-1) = 0 \). Both \( W \) and \( A \) are symmetric with respect to \( y \)-axis, hence so is \( B^* \). Thus \( B_{xy}^*(0,-1) = 0 \). From Lemma 4 and the limit

\[ \lim_{(x,y) \to (0,-1)} B_{xx}(x,y) = B_{xx}^*(0,-1) = 0 \]

we get

\[ \liminf_{(x,y) \to (0,-1)} W_{xx}(x,y) = \lim_{(x,y) \to (0,-1)} B_{xx}(x,y) + \liminf_{(x,y) \to (0,0)} A_{xx}(x,y) \geq 0. \]

The inequality about \( \limsup |W_{xy}| \) is obtained similarly.

Lemma 6. If \( x_0 > 0 \), then

\[ \lim_{(x,y) \to (x_0,-1)} W_{xx}(x,y) = 0 \quad \text{and} \quad \lim_{(x,y) \to (x_0,-1)} W_{xy}(x,y) \leq 0. \]

Proof. Let \( x_0 > 0 \). Define a harmonic function \( C \) on \( S \) by \( C(x,y) = W(x,y) - x \). Observe that for \( x_0 \geq 0 \) we have

\[ \lim_{(x,y) \to (x_0,-1)} C(x,y) = 0. \]

We apply the Schwarz reflection principle to get a harmonic extension \( C^* \) of \( C \) over \( S \cup \{(x,-1) : x > 0\} \cup \{(x,y) : x \in \mathbb{R}, -3 < y < -1\} \).

If \( x > 0 \), then \( C^*(x,-1) = 0 \), hence \( C_{xx}^*(x,-1) = 0 \) and

\[ \lim_{(x,y) \to (x_0,-1)} W_{xx}(x,y) = \lim_{(x,y) \to (x_0,-1)} C_{xx}(x,y) = C_{xx}(x_0,-1) = 0 \]

which proves the first part of the lemma.

For the second part of the lemma we apply the Maximum Principle to \( C_{xy}^* \) over \( \Omega = \{(x,y) : x > 0 \text{ and } -2 < y < 0\} \) to get \( C_{xy}^*(x_0,-1) \leq 0 \) which yields

\[ \lim_{(x,y) \to (x_0,-1)} W_{xy}(x,y) = \lim_{(x,y) \to (x_0,-1)} C_{xy}(x,y) = C_{xy}(x_0,-1) \leq 0. \]

Now we will check the boundary conditions of the Maximum Principle. For \( -3 < y < -1 \) we have \( C^*(x,y) = -C(x,-y-2) \), hence

\[ C_{xy}^*(x,y) = C_{xy}(x,-y-2) = W_{xy}(x,-y-2). \]
From Lemma 1 we get $C_{xy}^*(0, y) = 0$ if $0 < |y + 1| < 1$. Also, from Lemma 1 $W_{xy}(x, 0) = 0$. Thus if $x_1 > 0$ and $|y_0 + 1| = 1$, then

$$\limsup_{(x, y) \to (x_1, y_0)} C_{xy}^*(x, y) = C_{xy}^*(x_1, y_0) = W_{xy}(x_1, 0) = 0.$$ 

And using Lemma 3, we have

$$\limsup_{x \to -\infty} C_{xy}^*(x, y) = \lim_{x \to -\infty} W_{xy}(x, y) = 0$$

because $C_{xy}^*(x, y) = W_{xy}(x, -y - 2)$ for $-2 < y < -1$ and $C_{xy}^*$ is continuous on $\Omega$. Finally, the second part of Lemma 5 implies that

$$\limsup_{(x, y) \to (0, -1)} C_{xy}^*(x, y) = \limsup_{x \to 0} W_{xy}(x, y) \leq 0.$$ 

This checks all the boundary conditions and finishes the proof of the lemma.

**Lemma 7.** $W_{xx} \geq 0$ on $S$.

**Proof.** We apply the Maximum Principle to the harmonic function $-W_{xx}$ over $S$. Observe, from Lemma 1, that $W_{xx}(-x, y) = W_{xx}(x, -y) = W_{xx}(x, y)$.

It remains to check the boundary conditions. The first part of Lemma 3 implies

$$\limsup_{|x| \to \infty} [-W_{xx}(x, y)] = - \lim_{(x, y) \in S} W_{xx}(x, y) = 0.$$ 

For $x_0 \neq 0$, the first part of Lemma 6 gives

$$\limsup_{(x, y) \to (x_0, \pm 1)} [-W_{xx}(x, y)] = - \lim_{(x, y) \to (x_0, -1)} W_{xx}(x, y) = 0$$

Finally, the first part of Lemma 5 gives

$$\limsup_{(x, y) \to (0, \pm 1)} [-W_{xx}(x, y)] = - \lim_{(x, y) \to (0, -1)} W_{xx}(x, y) \leq 0.$$ 

This proves the lemma.

**Lemma 8.** $W_{xy} \leq 0$ on $\Omega = \{(x, y) : x > 0 \text{ and } -1 < y < 0\}$.

**Proof.** Note that $W_{xy}$ is harmonic on $\Omega$ and it is continuous on $S$. For $x_0 > 0$, Lemma 1 implies

$$\limsup_{(x, y) \to (x_0, 0)} W_{xy}(x, y) = W_{xy}(x_0, 0) = 0$$

and the second part of Lemma 6 implies

$$\limsup_{(x, y) \to (x_0, -1)} W_{xy}(x, y) = \lim_{(x, y) \to (x_0, -1)} W_{xy}(x, y) \leq 0.$$ 

Let $-1 < y_0 \leq 0$. Lemma 1 gives

$$\limsup_{(x, y) \to (0, y_0)} W_{xy}(x, y) = W_{xy}(0, y_0) = 0.$$
Also, from the second part of Lemma 5,

\[
\limsup_{(x,y) \to (0,0)} W_{xy}(x,y) = \limsup_{(x,y) \to (0,1)} W_{xy}(x,y) \leq 0,
\]

and from the second part of Lemma 3

\[
\limsup_{x \to \infty} W_{xy}(x,y) = \lim_{x \to \infty} W_{xy}(x,y) = 0.
\]

Therefore we can apply the Maximum Principle and the lemma follows.

**Proof of Main Lemma.** We have defined the continuous function \( U \) on \( \mathbb{R}^2 \). It remains to show the properties (a), (b) and (c) of the function \( U \).

**Proof of (a).** By the definitions we have \( U(x,y) = V(x,y) \) if \( |y| \geq 1 \). Also, \( W(0,0) = \mathcal{W}(\varphi(0,0)) = W(0,1) = \frac{1}{K} \). Thus, if \( |y| < 1 \), then

\[
U(x,y) = 1 - KW(x,y) = -K[W(x,y) - W(0,0)].
\]

Hence the property (a) follows if \(-K|x| \leq -K[W(x,y) - W(0,0)] \) on \( S \). By the symmetry of \( W \) it suffices to show

\[
E(x,y) \leq 0 \quad \text{if} \quad (x,y) \in \mathbb{S}^\tau
\]

where \( E(x,y) = W(x,y) - W(0,0) - |x| \). Using Lemma 2 we have

\[
\limsup_{x \to \infty} E(x,y) = -W(0,0) < 0.
\]

Also \( E(x, \pm 1) = -W(0,0) < 0 \) for \( x \geq 0 \). Since \( W \) is harmonic on \( S \) we have \( W_{xx} + W_{yy} = 0 \) thus \( W_{yy} = -W_{xx} \leq 0 \) on \( S \) by Lemma 7. Hence, for \( |y| < 1 \), \( E_{yy}(0, y) = W_{yy}(0, y) \leq 0 \) and \( E(0, y) \) is a concave function on \( y \). But \( E_y(0,0) = W_y(0,0) = 0 \) from Lemma 1. Thus \( E(0, y) \leq E(0, 0) = 0 \) for \( |y| < 1 \). Because \( E \) is continuous on \( \mathbb{S}^\tau \) the Maximum Principle proves the inequality (5), hence the property (a).

**Proof of (b).** By (4) the property (b) becomes

\[
W(u, v) \text{ is subharmonic on } D.
\]

Arguing similarly as in the proof of (a) one gets

\[
W(x,y) \geq |x| \quad \text{if} \quad |y| < 1.
\]

Now we put \( w = W(u, v) \) on \( D \). When \( |v| > 1 \) clearly \( w = |u| \) is subharmonic because \( u \) is harmonic. When \( |v| < 1 \), writing \( W_x \) for \( W_x(u, v) \) etc., we have

\[
\Delta w = W_{xx}|\nabla u|^2 + W_{yy}|\nabla v|^2 + 2W_{xy}\nabla u \cdot \nabla v + W_x \Delta u + W_y \Delta v
\]

\[
= W_{xx}(|\nabla u|^2 - |\nabla v|^2) \geq 0,
\]

hence \( w \) is subharmonic. In the above we used the assumptions (ii) and (iii), Lemma 7 and the harmonicity of \( u \) and \( v \). When \( |v| = 1 \) at \( \eta \in D \) we have, for all small \( r > 0 \), that

\[
\text{Avg}(w; \eta, r) \geq \text{Avg}(|u|; \eta, r) \geq |u(\eta)| = w(\eta),
\]
thus \( w \) is subharmonic at \( \eta \). In the above we used the inequality (7). Also
\[
\text{Avg}(w; \eta, r) \text{ is the average of } w \text{ over the ball } \{ \lambda \in D : |\lambda - \eta| < r \} \text{ with respect to the Lebesgue measure in } \mathbb{R}^n.
\]
This proves (6), hence (b).

**Proof of (c).** By (4) the property (c) of \( U \) follows from
\[
W(x, y) \geq W(0, 0) \quad \text{if} \quad |x| \geq |y|.
\]
Let \( I_0 = [0, \infty) \) and for \(-1 \leq a < 0\), put \( I_a = [0, -\frac{1}{a}] \). Define \( \Phi_a \) by \( \Phi_a(t) = W(t, at) \) for \( t \in I_a \). Then for \( t \) in the interior of \( I_a \)
\[
\Phi_a'(t) = W_x(t, at) + aW_y(t, at)
\]
and
\[
\Phi_a''(t) = W_{xx}(t, at) + a^2 W_{yy}(t, at) + 2a W_{xy}(t, at)
\]
\[
= (1 - a^2) W_{xx}(t, at) + 2a W_{xy}(t, at)
\]
\[
\geq 0
\]
because \( W \) is harmonic, \( W_{xx}(t, at) \geq 0 \) by Lemma 7 and because \( W_{xy}(t, at) \leq 0 \) by Lemma 8. Observe that \( \Phi_a'(0) = W_x(0, 0) + aW_y(0, 0) = 0 \) by Lemma 1. Hence \( \Phi_a(t) \geq \Phi_a(0) \) for \( t \in I_a \). Thus \( W(t, at) \geq W(0, 0) \) if \(-1 \leq a \leq 0 \) and \( t \in I_a \). But
\[
\{(x, y) : x \geq -y \text{ and } -1 < y \leq 0\} = \{(t, at) : -1 \leq a \leq 0 \text{ and } t \in I_a\}.
\]
Using the symmetry of \( W \) we have
\[
W(x, y) \geq W(0, 0) \quad \text{if} \quad |x| \geq |y| \quad \text{and} \quad |y| < 1.
\]
Also, if \( |x| \geq |y| \) and \( |y| \geq 1 \), then
\[
W(x, y) = |x| \geq 1 > \frac{1}{K} = W(0, 0).
\]
This proves (8), hence (c).

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