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CUSP FORMS FOR CONGRUENCE SUBGROUPS OF $Sp_n(\mathbb{Z})$ AND THETA FUNCTIONS

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ABSTRACT. In this paper we use theta functions with rational characteristic to construct cusp forms for congruence subgroups $\Gamma_g(p)$ of $Sp(g,\mathbb{Z})$.The action of the quotient group $Sp(g,\mathbb{Z}_p)$ on these forms is conjugate to the linear action of $Sp(g,\mathbb{Z}_p)$ on $(\mathbb{Z}_p)^{2g}$. We show that these forms are higher-dimensional analogues of the Fricke functions.

1. INTRODUCTION

Theta functions are a classical subject which arose during the work of Jacobi on elliptic integrals, and were later generalized to the higher-dimensional case by Riemann in his treatment of Riemann surfaces. Most of the efforts in this field have concentrated on theta functions with integral characteristics. Farkas and Kra used theta functions with rational characteristic to obtain a way of constructing cusp forms for congruence subgroups $\Gamma(p)$ of $SL_2(\mathbb{Z})$ with p an odd prime (see [FK1] and [FK2]). Further investigation in this direction in the series of papers [FKo1], [FKo2], [FKK] showed that theta functions with rational characteristics can be used to obtain mappings of modular curves $\mathbb{H}/\Gamma(p)$ to projective spaces, and the residue theorem was used to obtain equations for the image. Since we have multidimensional theta functions, it is natural to ask whether we can use these functions to obtain similar results for congruence subgroups of the Siegel modular group $Sp_n(\mathbb{Z})$, which acts on $\mathbb{H}_n$, the higher dimensional Siegel space, i.e., $\mathbb{H}_n = \{\tau \in M_{n\times n}(\mathbb{C})|^{\tau=\tau^t}_{\operatorname{Im}\tau\gg 0}\}$. In this paper we analyze carefully the action of $\Gamma_n(p)$ on the set of rational characteristics, and use our results to construct modular forms which are cusp forms for $\Gamma_n(p)$. We show that the divisors of these forms are different, and that they can be used as analogies of Fricke functions in the one-dimensional case.

2. PRELIMINARIES

First, let us briefly recall the basic properties of theta functions that we will use. For convenience we write $2n$-vectors in following form: $\varepsilon_1,\ldots,\varepsilon_n,\varepsilon'_1,\ldots,\varepsilon'_n$.

Definition 1. For $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in \mathbb{R}^{2n}$, $\tau \in \mathbb{H}_n$, and $x \in \mathbb{C}^n$, we define

$$\Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z,\tau) = \sum_{l\in\mathbb{Z}^n} \exp 2\pi i \left\{\frac{1}{2}\left(l+\frac{\varepsilon}{2}\right)^t \tau \left(l+\frac{\varepsilon}{2}\right) + \left(e+\frac{\varepsilon}{2}\right)^t\left(z+\frac{\varepsilon'}{2}\right)\right\}.$$

The series is uniformly and absolutely convergent on compact subsets of $\mathbb{C}^n\times\mathbb{H}_n$.

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We have the following main properties of theta functions which are obtained by simple manipulation with the series:

$$\Theta\begin{bmatrix}\varepsilon+2m\\ \varepsilon'+2e\end{bmatrix}(z,\tau)=\exp\pi i\{\varepsilon^t e\}\Theta\begin{bmatrix}t\\ t'\end{bmatrix}(z,\tau),$$
$$\Theta\begin{bmatrix}\varepsilon\\ -\varepsilon'\end{bmatrix}(z,\tau)=\Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(-z,\tau).$$

Let $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in\mathbb{Z}^{2n}$. Then $\Theta\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](z,\tau)$ is even or odd, depending whether $\varepsilon^t\varepsilon\equiv 0$ or $1 \bmod 2$, respectively. If $\Theta\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](z,\tau)$ is odd then clearly $\Theta\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](0,\tau)=0$ for all $\tau\in\mathbb{H}_n$.

Let us now state without proof the following fundamental result on the behavior of theta functions under a modular transformation $\gamma\in Sp_n(\mathbb{Z})$, which is known as the transformation formula: Let $\gamma=\left(\begin{smallmatrix}a&b\\ c&d\end{smallmatrix}\right)\in Sp_n(\mathbb{Z})$; then we have

Proposition 1. *For each $\gamma\in Sp_n(\mathbb{Z})$ we have*

$$\Theta\left(\gamma\circ\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\right)(0,\gamma\circ\tau)=\kappa(\gamma)\det(c\tau+d)^{1/2}\exp 2\pi i(\phi_m(\gamma))\Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau)$$

where $\kappa(\gamma)$ is an eighth root of unit whose sign depends on the choice of the square root $\det(c\tau+d)^{1/2}$, and where

$$\phi_m(\gamma)=-\frac{1}{8}(\varepsilon d^t b\varepsilon^t-2\varepsilon b^t c\varepsilon'^t+\varepsilon c^t a\varepsilon'^t-2\operatorname{diag}(a^t b)(d^t\varepsilon'-c^t\varepsilon'))$$

and

$$\gamma\circ\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}=\gamma^{-1}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}+\begin{pmatrix}\operatorname{diag}(c^t d)\\ \operatorname{diag}(a^t b)\end{pmatrix},$$

where the action $\gamma^{-1}\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]$ is a linear action; if $R\in\mathrm{Mat}_{n\times n}$ then $\operatorname{diag}R=(r_{11}\cdots r_{nn})$.

For a proof see [RF], chapter 2, or [Ig].

We define the following relation on $\mathbb{R}^{2n}$:

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\sim\begin{bmatrix}\varepsilon_1\\ \varepsilon_1'\end{bmatrix}\Leftrightarrow\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}-\begin{bmatrix}\varepsilon_1\\ \varepsilon_1'\end{bmatrix}\in 2\mathbb{Z}^{2n},$$

or equivalently (since $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]=-\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_1'\end{smallmatrix}\right]$)

$$\begin{vmatrix}\varepsilon\\ \varepsilon'\end{vmatrix}\sim\begin{vmatrix}\varepsilon\\ \varepsilon'\end{vmatrix}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\sim\begin{bmatrix}\varepsilon_1\\ \varepsilon_1'\end{bmatrix}\Leftrightarrow\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\pm\begin{bmatrix}\varepsilon_1\\ \varepsilon_1'\end{bmatrix}\in 2\mathbb{Z}^{2n}.$$

The following lemma is proved in [Ig] and [RF].

Lemma 1. *Let $\gamma\in Sp_n(\mathbb{Z})$ and $\gamma=\left(\begin{smallmatrix}a&b\\ c&d\end{smallmatrix}\right)$, and define*

$$\gamma\circ\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}=\gamma^{-1}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}+\begin{pmatrix}\operatorname{diag}(c^t d)\\ \operatorname{diag}(a^t b)\end{pmatrix}$$

for $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in\mathbb{R}^{2n}$. Then $\circ$ induces the action of $Sp_n(\mathbb{Z})$ on $\mathbb{R}^{2n}/\sim$.

It is easily shown that if $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]$ is integral then we have the induced action of $Sp_n(\mathbb{Z})$ on $\mathbb{F}_2^{2n}$. We have 2 orbits under the action of A_0 and A_1, where $A_i=\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in\{\mathbb{F}_2^{2n}|{}^t\varepsilon\varepsilon'\equiv i \bmod 2\}$, $i=0,1$. The elements of A_0 and A_1 are called respectively even and odd.

Let us assume that $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]$ has entries of the form m_i/p, $1\le i\le n$, and $m_i\in\mathbb{Z}$. Denote this set of vectors by $\mathbb{A}(p)$. Let us look on the congruence subgroup

$$\Gamma_n(p)=\{\gamma\in Sp_n(\mathbb{Z})|\gamma\equiv I \bmod p\}.$$

Then clearly we have an action of $\Gamma_n p$ on $A(p)/\sim$. In the next section we will analyze this action, and then use this analysis to construct cusp forms on $A(p)$.

3. The Action of $\Gamma_n(p)$ on $\overline{A(p)}$

Let $\overline{A(p)} = A(p)/\sim$, and denote by $[v]$ the equivalence class of $v \in A(p)$ in $\overline{A(p)}$.

Lemma 2.

$$|\overline{A(p)}| = \frac{(2p)^{2n} - 2^{2n}}{2} + 2^{2n}.$$

Proof. Define a one-to-one map $\overline{\varphi}\colon \overline{A(p)} \to \mathbb{Z}_{2p}^{2n}/\pm 1$. Let $v \in A(p)$ and assume

$$v = \begin{pmatrix} \frac{m_1}{p} \dots \frac{m_n}{p} \\ \frac{m_{n+1}}{p} \dots \frac{m_{2n}}{p} \end{pmatrix}.$$

We define $\varphi(v) = pv \bmod 2p$. φ naturally induces $\overline{\varphi}\colon \overline{A}(p) \to \mathbb{Z}_{2p}^{2n}/\pm 1$, since $v - v' \in 2\mathbb{Z}^{2n} \Rightarrow \varphi(v - v') \equiv 0 \bmod 2p$, and $v = -v' \Rightarrow \varphi(v) = -\varphi(v')$. Hence $\overline{\varphi}$ is well defined. $\overline{\varphi}$ is onto, since for each $v \in \mathbb{Z}_{2p}^{2n}$ there is $v \in A(p)$ such that $\varphi(v) = \overline{v}$. $\overline{\varphi}$ is one-to-one, since $v \simeq v' \Rightarrow \varphi(v) \neq \varphi(v')$. Hence we get

$$|\overline{A(p)}| = |\mathbb{Z}_{2p}^{2n}/\pm 1| = \frac{(2p)^{2n} - 2^{2n}}{2} + 2^{2n}. \quad \square$$

Let us observe that we have a natural action of $\mathbb{Z}_2^{2n}$ on $\overline{A(p)}$. Assume $\varepsilon_i \in \mathbb{Z}_2^{2n}$; then for each $[v] \in \overline{A(p)}$, $[v] \to [v + \varepsilon_i]$. It is easily verified that we have an action of $\mathbb{Z}_2^{2n}$ on $\mathbb{Z}_{2p}^{2n}/\pm 1$. We can also define our action of $\mathbb{Z}_2^{2n}$ on $\mathbb{Z}_{2p}^{2n}/\pm 1$. Denote by $\chi\colon \mathbb{Z}_2 \to \mathbb{Z}_{2p}$ the natural ring homomorphism which is valued by the map $1 \to p$ and has a natural extension $\chi_0\colon \mathbb{Z}_2^{2n} \to \mathbb{Z}_{2p}^{2n}$. Define the action $\overline{\omega} \circ \varepsilon = \omega + \chi_0(\varepsilon)$ for $\omega \in \mathbb{Z}_{2p}^{2n}$ and $\varepsilon \in \mathbb{Z}_2^{2n}$. It is obvious that we have an action of $\mathbb{Z}_2^{2n}$ on $\mathbb{Z}_{2p}^{2n}$, and therefore we have a group action of $\mathbb{Z}_2^{2n}$ on $\mathbb{Z}_p^{2n}/\pm 1$. From the definition of $\overline{\varphi}$ we deduce that $\overline{\varphi}$ is a $\mathbb{Z}_2^{2n}$-invariant map, since

$$\varphi(v + \varepsilon_i) = (pv + p\varepsilon_i) \bmod 2p = \varphi(v) + \chi_0(\varepsilon_i).$$

Lemma 3. *There exist $(p^{2n}+1)/2$ orbits of the group $\mathbb{Z}_2^{2n}$ on $\overline{A(p)}$ and on $\mathbb{Z}_{2p}^{2n}/\pm$. Each orbit has 2^{2n} elements.*

Proof. Since $\overline{\varphi}$ is $\mathbb{Z}_2^{2n}$ invariant, it is enough to prove the claim for $\mathbb{Z}_{2p}^{2n}/\pm 1$. Let us show first that each orbit has 2^{2n} elements; i.e, if $v \in \mathbb{Z}_{2p}^{2n}$ and $e_1, e_2 \in \mathbb{Z}_2^{2n}$, $e_1 \neq e_2$, then it is enough to show that $v + e_1 \neq \pm(v + e_2)$. Clearly $e_1 \neq e_2 \Rightarrow v + e_1 \neq v + e_2$, and $v + e_1 = -v + e_2 \to 2v = e_2 - e_1 \Rightarrow v = 0$ (e_1, e_2 are vectors with coordinates $0, p$, and p is an odd number).

Let us calculate how many orbits we have. The number of orbits equals the number of equivalence classes that we have under the action of $\mathbb{Z}_2^{2n}$ on $\mathbb{Z}_{2p}^{2n}/\pm 1$. Inside $\mathbb{Z}_{2p}^{2n}/\pm 1$ we have $\frac{(2p)^{2n} - 2^{2n}}{2} + 2^{2n}$ equivalence classes. On the other hand, each orbit has 2^{2n} elements, and therefore if X is the set of the orbits we get

$$|X| = \frac{p^{2n} - 1}{2} + 1. \quad \square$$

We denote the set of all orbits of $[v] \in \overline{A(p)}$ under the action of $\mathbb{Z}_2^{2n}$ by $\mathrm{Orb}_{\mathbb{Z}_2^{2n}}[v]$. We formulate our main observation.

Proposition 2. $\mathrm{Orb}_{\mathbb{Z}_2^{2n}}[v]$ *is invariant under the action of* $\Gamma_n(p)$ *and contains* 2 *orbits,* B_1 *and* B_2 *for the action of* $\Gamma_n(p)$ *such that* $|B_i| = 2^{n-1}(2^n+(-1)^i)$, $i = 0, 1$. *Furthermore, for each* $\alpha = [{\varepsilon \atop \varepsilon'}]$ *and* $\alpha \in A(p)$, *define* $[[\alpha]] = p^2\varepsilon\varepsilon'^t \bmod 2$; *i.e. take the standard scalar product of* ε *and* ε' *and look at the product of denominators* $\bmod 2$. *Then* $v' \in B_i \Leftrightarrow v' \in \mathrm{Orb}_{\mathbb{Z}^{2n}}[v]$ *is such that* $[[v']] = i$, $i = 0, 1$.

Proof. Let us show invariance for each $M \in \Gamma_n(p)$. We write $M = I + pL$ and $L \in M_{2g\times 2g}(\mathbb{Z})$; then

$$M \circ [\alpha] = \left[M^{-1}[{\varepsilon \atop \varepsilon'}]\right] + \begin{pmatrix}\mathrm{diag}(c^td)\\ \mathrm{diag}(a^tb)\end{pmatrix} = [\alpha] + \alpha pL' + \begin{pmatrix}\mathrm{diag}(c^td)\\ \mathrm{diag}(a^tb)\end{pmatrix}.$$

But $\alpha pL' + \left({\mathrm{diag}(c^td) \atop \mathrm{diag}(a^tb)}\right)$ is an integral vector and therefore is equivalent under $\sim$ to a vector with entries 0, 1. Therefore from the definition of $\mathrm{Orb}_{\mathbb{Z}_2^{2n}}[v]$ we get that $[\alpha] \in \mathrm{Orb}_{\mathbb{Z}_2^{2n}}$ if $[\alpha] \in \mathrm{Orb}_{\mathbb{Z}_2^{2n}}(v)$. $\square$

To prove the second assertion of the proposition, we need the following result.

Lemma 4. $Sp_n(\mathbb{Z}_2) \simeq \Gamma_n(p)/\Gamma_n(2p)$. *If* $\psi_1 \colon \Gamma_n(p) \to Sp_n(\mathbb{Z}_2)$ *denotes the corresponding homomorphism, then there exists a natural one-to-one and onto mapping* $\overline{\delta}\colon \mathrm{Orb}_{\mathbb{Z}_2^{2n}}[v] \to \mathbb{Z}_2^{2n}$. *Furthermore, for* $\alpha \in \mathrm{Orb}_{\mathbb{Z}_2^{2n}}[v]$ *we have*

$$\overline{\delta}(M \circ \alpha) = \psi_1(M) \circ \overline{\delta}(\alpha).$$

Proof. For every $M \in \Gamma_n(p)$ we define $\psi_1 \colon \Gamma_n(p) \to Sp_n(\mathbb{Z}_2)$ by $\psi_1(M) = M \bmod 2$. It is clear that $\psi_1(M) \in Sp_n(\mathbb{Z}_2)$, and $M \in \mathrm{Ker}\,\psi_1 \Leftrightarrow M = I \bmod 2$, since $M = I \bmod p \Rightarrow M = I \bmod 2p \Rightarrow M \in \Gamma_n(2p)$.

Let us show that ψ_1 is onto. It is well known that $Sp_n(\mathbb{Z}_2)$ is a symplectic group and has generators of the form $\left({E \atop 0}\ {\overline{S} \atop E}\right)$ and $\left({E \atop \overline{S}}\ {0 \atop E}\right)$, $\overline{s}_{ij} = 0, 1 \in \mathbb{Z}_2$. We can lift this system of generators to:

$$\begin{pmatrix}E_p & S\\ 0 & E_p\end{pmatrix}, \begin{pmatrix}E_p & 0\\ S & E_p\end{pmatrix} \in \Gamma_n(p),$$

where $s_{ij} = 0$ if $\overline{s}_{ij} = 0$ and $s_{ij} = p$ if $\overline{s}_{ij} = 1$. $\square$

Now for each $\alpha \in A(p)$ such that $\alpha \in \mathrm{Orb}\,\mathbb{Z}_2^{2n}[v]$ we define $\delta(\alpha) = p\alpha \bmod 2$.

It is easily verified that δ induces some $\overline{\delta}\colon \mathrm{Orb}_{\mathbb{Z}_2^{2n}}[v] \to \mathbb{Z}_2^{2n}$. This $\overline{\delta}$ is onto, since if $e_1 = \overline{\delta}(v)$ and $e_2 \in \mathbb{Z}_2^{2n}$, then $e_2 = \overline{\delta}([v + e_2 - e_1])$ by the definition of $\overline{\delta}$, and it is one-to-one since the two sets have the same number of elements. Likewise, from the definition of $\circ$ it is readily seen that

$$\overline{\delta}(M \circ \alpha) = \overline{\delta}\left(M^{-1}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} + \begin{matrix}\mathrm{diag}(c^td)\\ \mathrm{diag}(a^tbx)\end{matrix}\right) = \psi_1(M) \circ \overline{\delta}.$$

We have proved that the following diagram is commutative:

$$\begin{array}{ccc}\Gamma_n(p) \times \mathrm{Orb}_{\mathbb{Z}_2^{2n}}[v] & \longrightarrow & \mathrm{Orb}_{\mathbb{Z}_2^{2n}}[v]\\ \big\downarrow & & \big\downarrow{\scriptstyle \delta}\\ Sp_n(\mathbb{Z}_2) \times \mathbb{Z}_2^{2n} & \longrightarrow & \mathbb{Z}_2^{2n}\end{array}$$

We see from the diagram that the number of orbits is the same as the orbits of $Sp_n(\mathbb{Z}_2)$ in $\mathbb{Z}_2^{2n}$, i.e., 2. Moreover, $e_1 = [{\varepsilon \atop \varepsilon_1'}] \in \mathbb{Z}_2^{2n}$ and $e_2 = [{\varepsilon_2 \atop \varepsilon_2'}] \in \mathbb{Z}_2^{2n}$ belong to the same orbit under the action of $Sp_n(\mathbb{Z}_2) \Leftrightarrow \varepsilon_1^t\varepsilon_1' = \varepsilon_2^t\varepsilon_2' \bmod 2$ (see [RF] and [Ig]). From the definition of $\overline{\delta}$ we see that $[\alpha], [\alpha'] \in \mathrm{Orb}_{\mathbb{Z}_2^{2n}}(v)$ in the same orbit of $\Gamma_n(p) \Leftrightarrow \overline{\delta}[\alpha], \overline{\delta}[\alpha']$ belong to the same orbit of $Sp_n(\mathbb{Z}_2)$. Suppose that $\alpha = [\begin{smallmatrix}\alpha_1 \\ \alpha_1^1\end{smallmatrix}]$ and $\alpha' = [\begin{smallmatrix}\alpha_2 \\ \alpha_2^1\end{smallmatrix}]$; then $\overline{\delta}(\alpha)$ and $\overline{\delta}(\alpha')$ belong to the same orbit of $Sp_n(\mathbb{Z}_2) \Leftrightarrow p^2\alpha_1\alpha_1^{1t} \equiv p^2\alpha_2\alpha_2^{1t} \bmod 2$, and the proposition is proved. $\square$

We now formulate the second assertion. We denote

$$X_n(p) = \{[\alpha] \in \overline{A}(p) | [[\alpha]] \equiv 0 \bmod 2\},$$
$$Y_n(p) = \{[\alpha] \in \overline{A}(P) | [[\alpha]] \equiv 1 \bmod 2\}.$$

Theorem 1. (1) *$X_n(p), Y_n(p)$ are invariant under the action of $Sp_n(\mathbb{Z})$ on $\overline{A(p)}$. Moreover, there exist mappings $h_1\colon X_n(p)/\Gamma_n(p) \to \mathbb{Z}_p^{2n}/\pm 1$ and $h_2\colon Y_n(p)/\Gamma_n(p) \to \mathbb{Z}_p^{2n}/\pm 1$ which are induced by the mapping $h(\alpha) = p\alpha \bmod p$, $\alpha \in A(p)$.*

(2) *For each $M \in Sp_n(\mathbb{Z})$ and $e \in X_n(p)/\Gamma_n(p)$ there exists $h_1(M \circ [e]) = M(h_1[e])$. For $[e] \in Y_n(p)/\Gamma_n(p)$ we have $h_2(M \circ [e]) = M(h_2[e])$, where the action of $Sp_n(\mathbb{Z})/\pm 1$ on $\mathbb{Z}_p^{2n}/\pm 1$ is the natural action of $Sp_n(\mathbb{Z})$ on $\mathbb{Z}_p^{2n}$ through the map $\psi\colon Sp_n(\mathbb{Z}) \to Sp_n(\mathbb{Z}_p)$.*

(3) *Each h_i is one-to-one and onto.*

Proof. (1) If $\alpha \in A(p)$, define $h(\alpha) = p\alpha \bmod p$ and assume $\alpha \sim \alpha'$, i.e., $\alpha \pm \alpha' \in 2\mathbb{Z}^{2n}$; we obtain $h(\alpha) = \pm h(\alpha')$. Thus we obtain a mapping $\overline{h}\colon \overline{A(p)} \to \mathbb{Z}_p^{2n}/\pm 1$. Assume that, moreover, $[v'] \in \mathrm{Orb}_{\mathbb{Z}_2^{2n}}[v]$, i.e., $v - v' \in \mathbb{Z}^{2n}$. By definition we obtain $h(v) = h(v')$. Since we have shown that for each $[v] \in \overline{A}(p)$ the orbit of $[v]$ under $\Gamma_n(p)$ belongs to $\mathrm{Orb}_{\mathbb{Z}_2^{2n}}[v]$, we see that $h_1\colon X_n(p)/\Gamma_n(p) \to \mathbb{Z}_p^{2n}/\pm 1$ and $h_2\colon Y_n(p)/\Gamma_n(p) \to \mathbb{Z}_p^{2n}/\pm 1$ are well defined.

(2) Assume $M = (\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}) \in Sp_n(\mathbb{Z})$; we show that indeed $h_1(M \circ (e)) = M(h_1[e])$ for $e \in X_n(p)/\Gamma_n(p)$. We have to show the following:

$$e \in X_n(p)/\Gamma_n(p), \qquad M \circ [e] \in X_n(p)/\Gamma_n(p).$$

By definition, $v = [\begin{smallmatrix}\varepsilon \\ \varepsilon'\end{smallmatrix}] \in m_j/p$, $1 \le j \le n$, and $\varepsilon'_j = m'_j/p \Rightarrow [[v]] = \sum m_j m'_j \bmod 2$. But $M \in Sp_n(\mathbb{Z})$, so

$$[[M \circ v]] = [[M^{-1}v + \begin{smallmatrix}\mathrm{diag}(c^t d) \\ diag(a^t b)\end{smallmatrix}]].$$

We can write

$$\left[\left[M^{-1}v + \begin{matrix}\mathrm{diag}(c^t d) \\ \mathrm{diag}(a^t b)\end{matrix}\right]\right] = \left\| pM^{-1}v + \begin{matrix}\mathrm{diag}(c^t d) \\ \mathrm{diag}(a^t b)\end{matrix}\right\|,$$

where $\| \ \|$ is the standard norm of a $2n$-dimensional vector with integral entries; i.e., if $v' = [\begin{smallmatrix}\mu \\ \mu'\end{smallmatrix}]$ and $v' \in \mathbb{Z}^{2n}$, then $\|v'\| = \sum_{j=1}^n \mu_j \mu'_j \bmod 2$. Since p is odd,

$$\left\| pMv^{-1} + \begin{matrix}p\,\mathrm{diag}(c^t d) \\ p\,\mathrm{diag}(a^t b)\end{matrix}\right\| = \left\| pMv^{-1} + \begin{matrix}\mathrm{diag}(c^t d) \\ \mathrm{diag}(a^t b)\end{matrix}\right\| = \|M \circ pv\|.$$

Since pv is an integral vector and M preserves $\| \ \|$, we get $\|M \circ pv\| = \|pv\| = \|v\|$. And so we have an action of $Sp_n(\mathbb{Z})$ on $X_n(p)$ and $Y_n(p)$. Let us show that we have an action of $Sp_n(\mathbb{Z})$ on $X_n(p)/\Gamma_n(p)$; i.e., $[e]$ and $[e']$ belong to the same orbit under $\Gamma_n(p)$ and $M \in Sp_n(\mathbb{Z})$, then $M \circ [e]$ and $M \circ [e']$ belong to the same orbit too. But for $[e] = \gamma[e']$, $\gamma \in \Gamma_n(p)$ we have $M[e] = M\gamma[e'] \Rightarrow Me = \gamma' M[e']$, since $\Gamma_n(p) \triangleleft Sp_n(\mathbb{Z})$.

Let us now calculate $h_1(M \circ [e])$. Choose $v \in A(p)$ so that $[e]$ and $[v]$ belong to the same orbit under $\Gamma_n(p)$. Then

$$h_1(M \circ [v]) = h\left(M^{-1}v + \begin{matrix}\mathrm{diag}(c^t d) \\ \mathrm{diag}(a^t b)\end{matrix}\right) = h(M^{-1}v) + h\begin{pmatrix}\mathrm{diag}(c^t d) \\ \mathrm{diag}(a^t b)\end{pmatrix} = h(M^{-1}v).$$

Now $h(M^{-1}v) = \psi(M)^{-1} \circ h(v)$, where $\psi\colon Sp_n(\mathbb{Z}) \to Sp_n(\mathbb{Z}_p)$ is reduction mod p and the action of $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on $\alpha = \left[\begin{smallmatrix} m \\ m' \end{smallmatrix}\right] \in \mathbb{Z}_p^{2n}$ is the usual linear action.

(3) h_1 and h_2 are one-to-one and onto. Since for each $\alpha \in \mathbb{Z}_p^{2n}/\pm 1$ there is a $v \in A(p)$ such that $h(v) = \alpha$, it follows that $[v] \in X_n(p)$ or $[v] \in Y_n(p)$. If $v \in X_n(p)$ we can always find an element $v' \in \operatorname{Orb}_{\mathbb{Z}_p^{2n}}[v]$ such that $[v'] \in Y_n(p)$ and $h_2(v') = \alpha$. In order to show that the h_i are one-to-one it is enough to show that

$$|\mathbb{Z}_p^{2n}/\pm 1| = |X_n(p)/\Gamma_n(p)| = |Y_n(p)/\Gamma_n(p)|.$$

By Theorem 1 the number of orbits $X_n(p)/\Gamma_n(p)$ and $Y_n(p)/\Gamma_n(p)$ is precisely equal to the number $|\operatorname{Orb}_{\mathbb{Z}_2^{2n}}[v]| \Rightarrow |X_n(p)/\Gamma_n(p)| = |Y_n(p)/\Gamma_n(p)| = \frac{p^{2n-1}}{2} + 1$, and so h_1 and h_2 are one-to-one. $\square$

Before we proceed to construct automorphic forms we give a result which will show how to build the orbits effectively.

Corollary 1. *For each $\alpha \in \mathbb{Z}_p^{2n}/\pm 1$ choose $\beta \in A(p)$ such that (1) $p\beta \equiv 0 \bmod 2$ and (2) $p\beta \equiv \alpha \bmod p$ (i.e., $h(\beta) = \alpha$) and denote by $[\beta]$ the equivalence class in $\overline{A}(p)$. Then $h_2^{-1}(\alpha) = \beta + O_n$ and $h_1^{-1}(\alpha) = \beta + E_n$, where O_n denotes the odd integral characteristics and E_n denotes the even integral characteristics.*

Proof. First we note that each $\alpha \in \mathbb{Z}_p^{2n}/\pm 1$ has such a β. Moreover, h_1 and h_2 are one-to-one by the previous theorem, and $h_1^{-1}(\alpha)$ and $h_2^{-1}(\alpha)$ have $2^{n+1}(2^n \pm 1)$ respectively. It is easy to see that $\beta + E_n \subset h_1^{-1}(\alpha)$ and $\beta + O_n \subset h_2^{-1}(\alpha)$, and therefore we are done. $\square$

4. Cusp forms and some of their properties

We use the results of §3 to construct cusp forms for the group $\Gamma_n(p)$, where, as usual, p is an odd prime. For each $\alpha \in \mathbb{Z}_p^{2n}/\pm 1$ we look at $\left[\begin{smallmatrix} \varepsilon \\ \varepsilon' \end{smallmatrix}\right] \in h_1^{-1}(\alpha)$ or $\left[\begin{smallmatrix} \varepsilon \\ \varepsilon' \end{smallmatrix}\right] \in h_2^{-1}(\alpha)$ (i.e. all the functions with characteristic in $h_1^{-1}(\alpha)$ or $h_2^{-1}(\alpha)$).

Let us look at some symmetric function of $\Theta\left[\begin{smallmatrix} \varepsilon \\ \varepsilon' \end{smallmatrix}\right]^{8p}$. For each $\left[\begin{smallmatrix} \varepsilon_1 \\ \varepsilon \end{smallmatrix}\right] \in h_2^{-1}(\alpha)$ or $\left[\begin{smallmatrix} \varepsilon \\ \varepsilon' \end{smallmatrix}\right] \in h_1^{-1}(\alpha)$ we consider the polynomial

$$\prod_{\left[\begin{smallmatrix} \varepsilon \\ \varepsilon' \end{smallmatrix}\right] \in h_i^{-1}(\alpha)} \left(X - \Theta^{8p} \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (0, \tau) \right), \qquad i = 1, 2.$$

Theorem 2. *The coefficients of the polynomial are automorphic forms for the group $\Gamma_n(p)$.*

Proof. We write the transformation formula

$$\Theta \begin{bmatrix} \tilde{\varepsilon} \\ \tilde{\varepsilon}' \end{bmatrix} (0, \tau) = K(M)(c\tau + d)^{1/2} \Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (0, \gamma(\tau)),$$

where $\left[\begin{smallmatrix} \tilde{\varepsilon} \\ \tilde{\varepsilon}' \end{smallmatrix}\right] = M \circ \left[\begin{smallmatrix} \varepsilon \\ \varepsilon' \end{smallmatrix}\right] \hat{\tau} = (a\tau + b)(\tau + d)^{-1}$ and K was given at the beginning in the introduction.

Examining K, we conclude that $K(M)^{8p} = 1$ if $M \in \Gamma_n(p)$. Therefore the transformation formula will read

$$\Theta \begin{bmatrix} \tilde{\varepsilon} \\ \tilde{\varepsilon}' \end{bmatrix}^{8p} (0, \hat{\tau}) = \det(c\tau + d)^{4p} \Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}^{8p} (0, \tau).$$

Furthermore we see that $\Theta[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}]^{8p}(0,\tau)$ depends only on the image of $[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}]$ in $\overline{A(p)}$. Hence every symmetric function of $\Theta^{8p}[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}](0,\tau)$ with $[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}]\in h_i^{-1}(\alpha)$ will provide an automorphic form of $\Gamma_n(p)$ of a suitable weight. $\square$

The smallest automorphic form we get is of weight $4p$:

$$\sum_{[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}]\in h_i^{-1}(\alpha)} \Theta^{8p}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}.$$

However, we shall concentrate on the forms

$$\Theta_\alpha(\tau)=\prod_{[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}]\in h_i(\alpha)} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}^{8p}.$$

We investigate the basic properties of those forms.

Let us recall the definition of a Siegel operator and the notion of a cusp form. For any automorphic form $f\colon H_n\to\mathbb{C}$ we shall define the Siegel operator Φ which gives us a form in $\mathbb{H}_{n-1}$, i.e.,

$$\Phi(F)(\tau')=\lim_{\lambda\to\infty} f\begin{pmatrix}\tau' & 0\\ 0 & i\lambda\end{pmatrix}\tau\in\mathbb{H}_{n-1}.$$

Now using the definition of $\Theta[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}](0,\tau)$ we conclude that

$$\Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau)=\Theta\begin{bmatrix}\varepsilon_1\\ \varepsilon_1'\end{bmatrix}(0,\tau')\cdot\Theta\begin{bmatrix}\varepsilon_n\\ \varepsilon_n'\end{bmatrix}(0,\lambda);$$

here $\tau=(\begin{smallmatrix}\tau' & 0\\ 0 & i\lambda\end{smallmatrix})$ and $\varepsilon_1=\varepsilon_i$, $1\le i\le n-1$, and $\varepsilon'_{1i}=\varepsilon'_i$, $1\le i\le n-1$. Then we conclude that $\Phi(\Theta_\alpha)=0$ for each $\alpha\in\mathbb{Z}_p^{2n}/\pm1$, and therefore the following holds.

Proposition 3. *$\Theta_\alpha(\tau)$ is a cusp form for $\Gamma_n(p)$.*

Proof. Recall that $f\colon\mathbb{H}_k\to\mathbb{C}$ is a cusp form for $\Gamma_n(p)\Leftrightarrow\Phi(F(\sigma\cdot\tau))=0$ for each $\sigma\in Sp_n(\mathbb{Z}_p)$ (see [Fr1]). Since $\Phi(\Theta_\alpha(\tau))=0$ for each α, we get the result. $\square$

We next prove the following theorem:

Theorem 3. *For each $\alpha',\alpha\in\mathbb{Z}_p^{2n}/\pm1$ the divisors of $\Theta_\alpha(\tau)$ and $\Theta_{\alpha'}(\tau)$ are different in $\mathbb{H}_n/\Gamma(p)$ for $n\ge2$.*

Proof. First we recall some results of the theory of antisymmetric quadratic forms.

Lemma 5. *Let B a symplectic bilinear form nonsingular in a vector space V above F. Assume $\sigma\colon E\to E'$ is an isometry of subspaces, i.e., $\langle x,y\rangle=\langle\sigma x,\sigma y\rangle$, where $B(x,y)=\langle x,y\rangle$. Then there exists an isometry $\gamma\colon V\to V$ such that $\gamma|_E=\sigma$.*

Proof. See [A], p. 121, Theorem 3.9. $\square$

Corollary 2. *Assume $v_1,v_2\in\mathbb{Z}_p^{2n}$ and w_1,w_2 are a pair of linearly independent vectors such that $B(w_1,w_2)=B(v_1,v_2)$. Then there exists $\gamma\in Sp_n(\mathbb{Z})$ such that $w_i=\gamma v_i$.*

Proof. p is a prime number and $\mathbb{Z}_p$ is a field. Define $E=\operatorname{Span}(v_1,v_2)$ and $E'=\operatorname{Span}(w_1,w_2)$ and $\sigma v_i=w_i$; σ is an isometry, since $B\langle v_1,v_2\rangle=B\langle\sigma v_1,\sigma v_2\rangle$. There exists $\overline{\gamma}\colon\mathbb{Z}_p^{2n}\to\mathbb{Z}_p^{2n}$ such that $\overline{\gamma}\in Sp_n(\mathbb{Z}_p)$, $w_i=\overline{\gamma}v_i\Rightarrow\gamma\in Sp_n(\mathbb{Z})$ and $w_i=\gamma v_i$. $\square$

We denote by $V(f)$ the divisor of $f\colon \mathbb{H}_n \to \mathbb{C}$, and put $X_n^\alpha(p) = h_1^{-1}(\alpha)$ for $\alpha \in \mathbb{Z}_p^{2n}/\pm 1$. Then $Y_n^\alpha(p) = h_2^{-1}(\alpha), \alpha \in \mathbb{Z}_p^{2n}/\pm 1$.

We now prove the theorem. The proof will be divided into a number of steps.

Step I. We claim that

$$V\left(\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in X_n^\alpha(p)} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon\end{bmatrix}(0,\tau)\right) \neq V\left(\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in X_n^0(p)} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau)\right)$$

(the left-hand side is different from the product of even characteristics). It is enough to establish the inequality for some α_1, since if equality held for some α, then, because $Sp_n(\mathbb{Z}p)/\pm 1$ acts transitively on $\mathbb{Z}_p^{2n}/\pm 1$, we would get

$$\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in X_n^\alpha(p)} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau) = c \prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in X_n^0(p)} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau), \qquad c\in\mathbb{C}\ \forall\alpha.$$

Let us choose $\alpha_1 \in \mathbb{Z}_p^{2n}$, $\alpha_1 = \left(\begin{smallmatrix}1\cdots 1\\ 0\cdots 0\end{smallmatrix}\right)$; then the even characteristic which will correspond to α by Corollary 1 is going to be

$$\begin{bmatrix}\frac{p+1}{p} & \cdots & \frac{p+1}{p}\\ 0 & \cdots & 0\end{bmatrix}.$$

Let's look at

$$\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]=\in X_n^\alpha(p)} \Theta^{8p}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\Pi)$$

where Π is the matrix with diagonal entries τ and zeros elsewhere, and $\tau\in\mathbb{H}_1$. Every element in $X_n^\alpha(p)$ is obtained by adding all the integral characteristics; therefore all characteristics in $X_n^\alpha(p)$ should be rational, and hence $\Theta\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](0,\Pi)\neq 0$ because $\Theta\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](0,\Pi) = \prod\Theta\left[\begin{smallmatrix}\varepsilon_i\\ \varepsilon_i'\end{smallmatrix}\right](0,\tau)$, where $\left[\begin{smallmatrix}\varepsilon_i\\ \varepsilon_i'\end{smallmatrix}\right]$ is the ith column in $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]$. Since $\Theta\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](0,\tau)\neq 0$ for all $\tau\in\mathbb{H}_1$ and each $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in X_n^\alpha(p)$, the product cannot vanish. On the other hand, in the even characteristic we have an element of the form $\Theta\left[\begin{smallmatrix}1 & 0\cdots 0 & 1\\ 1 & 0\cdots 0 & 1\end{smallmatrix}\right]$ $(n\geq 2)$, since this vector has an even norm. We have $0=\Theta\left[\begin{smallmatrix}1\cdots 1\\ 0\cdots 0\end{smallmatrix}\right](0,\Pi)$ for Π diagonal, because $\Theta\left[\begin{smallmatrix}1 & 0\cdots 0 & 1\\ 1 & 0\cdots 0 & 1\end{smallmatrix}\right](0,\Pi) = \Theta\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right](0,\tau)\Theta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](0,\tau)\cdots\Theta\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right](0,\tau)$ and $\Theta\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right](0,\tau)=0$. Hence

$$V\left(\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in X_n^\alpha(p)} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon_1\end{bmatrix}\right) \neq V\left(\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in X_n^\alpha(p)} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon_1\end{bmatrix}\right) \quad \text{for } \alpha\neq 0.$$

Using the same reasoning, we show:

Step II. For $\alpha = \left[\begin{smallmatrix}1\cdots 1\\ 0\cdots 0\end{smallmatrix}\right]$ and $\alpha' = \left[\begin{smallmatrix}0\cdots 0\\ c\cdots 0\end{smallmatrix}\right]$ we have

$$V\left(\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in X_n^\alpha(p)} \theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\Pi)\right) \neq V\left(\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\in X_n^{\alpha'}(0)} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\Pi)\right).$$

Again the left-hand side does not vanish for $\Pi\in\Gamma_n(p)$ (defined as before), and the right-hand side does. On the right-hand side we will get an expression of the form

$\Theta[\begin{smallmatrix}1\\1\end{smallmatrix}](0,\tau)$, since

$$\prod_{[\begin{smallmatrix}\varepsilon\\\varepsilon'\end{smallmatrix}]\in X_n^{\alpha'}(p)} \Theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(0,\Pi) = \prod \Theta\begin{bmatrix}\varepsilon_i\\\varepsilon_i'\end{bmatrix}(0,\tau),$$

where $[\begin{smallmatrix}\varepsilon_i\\\varepsilon_i'\end{smallmatrix}]$ is the ith column in $[\begin{smallmatrix}\varepsilon\\\varepsilon'\end{smallmatrix}]$.

Step III. For $\alpha = [\begin{smallmatrix}1\cdots 1\\0\cdots 0\end{smallmatrix}]$ and $\alpha' = [\begin{smallmatrix}c\cdots 0\\0\cdots 0\end{smallmatrix}]$ we get

$$V\left(\prod_{[\begin{smallmatrix}\varepsilon\\\varepsilon'\end{smallmatrix}]\in X_n^{\alpha}(p)} \theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(0,\Pi)\right) \neq V\left(\prod_{[\begin{smallmatrix}\varepsilon\\\varepsilon'\end{smallmatrix}]\in X_n^{\alpha'}(0)} \Theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(0,\Pi)\right)$$

by reasoning exactly as in the previous steps.

Step IV. For $\alpha = [\begin{smallmatrix}1\cdots 1\\0\cdots 0\end{smallmatrix}]$ and $\alpha' = [\begin{smallmatrix}c\cdots c\\0\cdots 0\end{smallmatrix}]$ we have

$$\prod_{[\begin{smallmatrix}\varepsilon\\\varepsilon'\end{smallmatrix}]\in X_n^{\alpha}(p)} \Theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix} \neq \prod_{[\begin{smallmatrix}\varepsilon\\\varepsilon'\end{smallmatrix}]\in X_n^{\alpha}(p)} \Theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}$$

for $c \neq p-1, 1$. In order to prove that, we point out that it is enough to assume $c \equiv 0 \bmod 2$, since $\alpha \in \mathbb{Z}_p^{2n}/\pm 1$, and then $(\begin{smallmatrix}c/p & c/p\cdots c/p\\0 & 0\cdots 0\end{smallmatrix})$ will belong to $X_n^{\alpha}(p)$. We choose

$$\Pi = \begin{pmatrix}\tau_1 & & 0\\ & \ddots & \\ 0 & & \tau_n\end{pmatrix},$$

$\tau_2,\ldots,\tau_n$ constant (e.g. $\tau_j = i$), and consider

$$\prod_{[\begin{smallmatrix}\delta\\\delta'\end{smallmatrix}]\in X_n^{\alpha}(p)} \Theta\begin{bmatrix}\delta\\\delta'\end{bmatrix}(0,\pi)$$

as function of τ_1 *only*. Then the function of τ will be, up to a constant,

$$(*) \qquad \Theta\begin{bmatrix}\frac{c}{p}\\0\end{bmatrix}^{n_1} \Theta\begin{bmatrix}\frac{p-c}{p}\\0\end{bmatrix}^{n_2} \Theta\begin{bmatrix}\frac{c}{p}\\1\end{bmatrix}^{n_3} \Theta\begin{bmatrix}\frac{p-c}{p}\\1\end{bmatrix}^{n_4},$$

where n_1 is the number of times $[\begin{smallmatrix}0\\0\end{smallmatrix}]$ is the first coordinate in the integral characteristic which belongs to E_n, n_2 is the number of times the vector $[\begin{smallmatrix}1\\0\end{smallmatrix}]$ appears in the first coordinate in E_n, n_3 is the number of time $[\begin{smallmatrix}0\\1\end{smallmatrix}]$ appears in the first coordinate in E_n, and n_4 is the number of times $[\begin{smallmatrix}1\\1\end{smallmatrix}]$ appears in the first coordinate in E_n. Correspondingly, the function which will stand instead of $\prod_{[\begin{smallmatrix}\varepsilon\\\varepsilon'\end{smallmatrix}]\in X_n^{\alpha}(p)} \Theta[\begin{smallmatrix}\varepsilon\\\varepsilon'\end{smallmatrix}](0,\pi)$ will be the function of τ_1 given by

$$(**) \qquad \Theta\begin{bmatrix}\frac{p-1}{p}\\0\end{bmatrix}^{n_1} \Theta\begin{bmatrix}1/p\\0\end{bmatrix}^{n_2} \Theta\begin{bmatrix}\frac{p-1}{p}\\1\end{bmatrix}^{n_3} \Theta\begin{bmatrix}1/p\\1\end{bmatrix}^{n_4}.$$

It suffices to show that these functions, as functions of one variable, can't be equal. It is enough to calculate the order of these functions in the fundamental region $H_1/\Gamma_1(p)$, where we run toward ∞ through the line iy.

The order of infinity of $\Theta[{\varepsilon \atop \varepsilon'}](0,\tau)$ depends only on the upper characteristic, and we can write the series of $\Theta[{m/k \atop m'/k}](0,\tau)$ for $1 \le m \le k-1$ as

$$\Theta\begin{bmatrix} m/p \\ m'/p \end{bmatrix} = \left\{\zeta^{m^2/8k} \exp\left\{\frac{\pi i m m'}{2p^2} + O(1)\right\}\right\}, \quad \text{where } \zeta = e^{2\pi i\tau/p}.$$

Therefore we see that the order of the ∞ is $m^2/8p$, as in [FK1], p. 117. Then the orders in ∞ of $(*)$ and $(**)$ are given by

$$(n_3+n_1)\cdot c^2 + (n_2+n_4)(p-c)^2 \quad \text{and} \quad (n_3+n_1)(p-1)^2 + (n_2+n_4).$$

Since there is going to be an equality between the orders if the functions are equal, we obtain

$$(n_3+n_1)c^2 + (p-c)^2(n_2+n_4) - (n_3+n_1)(p-1)^2(n_2+n_4) = 0,$$

which in turn can be written as

$$(n_3+n_1)(c-(p-1))(c+p-1) - (n_2+n_4)(1-(p-c))(1+(p-c)) = 0,$$

$c \neq p-1$, and we can write

$$(n_3+n_1)(c+p-1) - (n_2+n_1)(1+p-c) = 0.$$

Opening the brackets, we get

(A)
$$(n_3+n_1)c + (n_3+n_1)p - (n_3+n_1) - (n_2+n_4) - (n_2+n_4)p + (n_2+n_4)c = 0.$$

Now we state the following fact:

Fact. $n_2 + n_4 = 2^{2n-2}$, and $n_3 + n_1 = 2^{n-1}(2^n+1) - 2^{2n-2}$.

We will prove the fact after finishing our treatment of the last expression.
So we can write (A) as

$$2^{n-1}(2^n+1)c + 2^{n-1}\cdot p - 2^{n-1}(2^n+1) = 0 \Leftarrow c = 1 - \frac{p}{2^n+1}.$$

But:
(1) $p > 2^n+1 \Rightarrow c < 0$, in contradiction to the fact that $c > 0$.
(2) $p < 2^{n+1}$, p prime$\Rightarrow c$ rational, again a contradiction.
(3) $p = 2^n+1 \Rightarrow \alpha' = [{0\cdots 0 \atop 0\cdots 0}]$. We have shown that in this case the forms are different.

Now we prove the fact. Obviously, n_3+n_1 is the number of times 0 appears in the first row and first coordinate, while n_2+n_4 is the number of times 1 appears in the first row and first coordinate. We need the following lemma:

Lemma 6. *Let n_g^0 be the number of vectors $v = [{1*\cdots* \atop *\cdots*}] \in \mathbb{Z}_2^{2g}$ with $vv^t \equiv 0 \bmod 2$, and let n_g' be the number of vectors $v = [{1*\cdots* \atop *\cdots*}] \in \mathbb{Z}_2^{2g}$ with $vv^t \equiv 1 \bmod 2$. Then $n_{g+1}^0 = 3n_g^0 + n_g'$ and $n_{g+1}' = 3n_g^1 + n_g^0$.*

Proof. Let v be a given vector of the form $[{1*\cdots* \atop 1*\cdots*}]$ of length $2(g-1)$. When we add the vectors $[{0 \atop 0}]$, $[{0 \atop 1}]$, $[{1 \atop 0}]$ from the right, v will remain even if it's even, and if it's odd we can make it even by adding $[{1 \atop 1}]$. Hence the lemma. □

Corollary 3. $n_0^g = n_1^g = 2^{2g-2}$.

Proof. By induction, and because $n_1^0 = n_0' = 1$. □

The fact follows from the formula $n_1 + n_2 + n_3 + n_4 = |E_n| = 2^{n-1}(2^n + 1)$.

We finish the last step in proving the theorem:

Step V. We pick δ, δ' in $\mathbb{Z}_p^{2n}/\pm 1$ and look on their orbits $X_n^{\delta}(p), X_n^{\delta'}(p)$. Then the following are possible:

(1) δ and δ' are independent. In this case there exists $\sigma \in Sp_n(\mathbb{Z})$ such that $\sigma\alpha = \delta$ and $\sigma\delta' = \alpha'$, for α, α' as in Steps II, III (choose them so that $B(\beta, \beta') = B(\alpha, \alpha')$). By Lemma 5 we can always do this. Then, if

$$\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in X_n^{\delta}(p)} \Theta^{8p}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau) = \prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in X_n^{\delta'}(p)} \Theta^{8p}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau),$$

we obtain from the transformation law

$$V\left(\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in X_n^{\alpha}(p)} \Theta^{8p}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau)\right) = V\left(\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in X_n^{\alpha'}(p)} \Theta^{8p}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau)\right).$$

This is a contradiction to Steps II, III.

(2) δ and δ' are linearly dependent. Then there exists $\sigma \in Sp_n(\mathbb{Z})$ such that $\sigma\delta' = \alpha'$ and $\sigma\delta = \alpha$. For α, α' as in Step IV, by the previous argument we conclude the proof of the theorem. □

Denote

$$\eta_\alpha(\tau) = \frac{\displaystyle\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in X_n^{\delta}(p)} \Theta^{8p}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau)}{\displaystyle\prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in X_n^{0}(p)} \Theta^{8p}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau)},$$

where $\eta_\alpha(\gamma\tau) = \eta_{\gamma\alpha}(\tau)$ for each γ in $Sp_n(\mathbb{Z})$, and $v(\gamma)$ is some root of unity. From the transformation formula we get $v(\gamma)^{8p^2} = 1$.

Let us consider $Q_n(p)$, the field of meromorphic functions on $H_n/\Gamma_n(p)$. It is well known that this field is a Galois extension of Q_n, the field of meromorphic functions on $H_n/Sp_n(\mathbb{Z})$. Furthermore, $Q_n(p)$ is a Galois extension of Q_n with $Sp_n(\mathbb{Z}_p)$ as Galois group.

Theorem 4. *The $\eta_\alpha^{8p^2}(\tau)$ generate $Q_n(p)$ over Q_n as fields for $\alpha \in \mathbb{Z}_p^{2n}/\pm 1$.*

Proof. It is clear that the $\eta_\alpha^{8p^2}(\tau)$ are all distinct, and for each α we have $\eta_{\gamma\circ\alpha}^{8p^2}(\tau) = \eta_\alpha^{8p^2}(\gamma\tau)$. The group $Sp_n(\mathbb{Z}_p)$ acts on the field generated by $\eta_\alpha^{8p^2}$ over Q_n; likewise, the polynomial $p(x) = \prod_{\alpha \in \mathbb{Z}_p^{2n}/\pm 1}(x - \eta_\alpha^{8p^2})$ has degree $(p^{2n} - 1)/2$, and its coefficients are in Q_n. Hence $Q_n[\eta_\alpha^{8p^2}(\tau)]$ is a Galois extension of Q_n, and

$$\psi\colon\ Sp_n(\mathbb{Z}_p)/\pm 1 \to \mathrm{Gal}(Q_n(\eta_\alpha^{8p^2}(\tau)/Q_n),$$

and ψ is a monomorphism. On the other hand,

$$Q_n[\eta_\alpha^{8p^2}(\tau)] \subset Q_n(p) \Rightarrow |Q_n[\eta_\alpha^{8p^2}]\colon Q_n| \le Sp_n(\mathbb{Z}_p) \Rightarrow Q_n[\eta_\alpha^{8p^2}(\tau)] = Q_n(p).$$

□

5. Conclusion

In this note we have tried to generalize [FK1] to higher dimensions. We constructed cusp forms and showed how we can use them to construct analogues of Fricke functions. We plan in a subsequent paper to use these automorphic forms to obtain more information about the manifolds $H_n/\Gamma_n(p)$ and their compactifications.

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