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SMALL SUBALGEBRAS OF STEENROD AND MORAVA STABILIZER ALGEBRAS

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Abstract. Let $P(j)$ (resp. $S(n)_{(j)}$) be the subalgebra of the Steenrod algebra $\mathcal{A}_p$ (resp. nth Morava stabilizer algebra) generated by reduced powers $\mathcal{P}^{p^i}$, $0 \leq i \leq j$ (resp. t_i, $1 \leq i \leq j$). In this paper we identify the dual $P(j-1)^*$ of $P(j-1)$ (resp. $S(n)_{(j)}$, for $j \leq n$) with some Frobenius kernel (resp. F_{p^n}-points) of a unipotent subgroup $G(j+1)$ of the general linear algebraic group GL_{j+1}. Using these facts, we get the additive structure of $H^*(P(1)) = \mathrm{Ext}_{P(1)}(Z/p, Z/p)$ for odd primes.

Introduction

The Adams and Adams-Novikov spectral sequences are the most powerful tools to compute stable homotopy groups of spheres. The E_2-terms of these spectral sequences are the cohomology of the Steenrod algebra or reduce essentially to the cohomology of the Morava stabilizer algebra. We study here the cohomology of some small subalgebras of these algebras for odd primes p. Let $P(j)$ be the subalgebra of the mod p Steenrod algebra $\mathcal{A}_p$, generated by the reduced powers $\mathcal{P}^{p^i}$, $0 \leq i \leq j$, and let $S(n)_{(j)}$ be the subalgebra of the nth Morava stabilizer algebra, generated by t_i, $0 < i \leq j$. We will examine a unipotent subgroup $G(j+1)$ of the general linear algebraic group GL_{j+1}, generated by matrices $\{(a_{ij} = a_{i-j,1}^{p^{j-1}})\}$ with $a_{ii} = 1$, $a_{ij} = 0$ for $i < j$. Let $k[G(j+1)]$ be its coordinate ring with coefficients the algebraic closure $k = \overline{F}_p$. Then the dual $P(j)^* \otimes k$ of $P(j) \otimes k$ is a quotient of $k[G(j+2)]$ by images of Frobenius maps (a modified Frobenius kernel), and if $j \leq n$, then $S(n)_{(j)} \otimes k$ is the dual of the group ring $k(G(j+1)(F_{p^n}))$ of F_{p^n}-points, and is the quotient of $k[G(j+1)]$ by the ideal $(x^{p^n} - x)$.

Hereafter we will always work in $P(j) \otimes k$, so we will leave out the k (and the same for $S(n)$). The cohomology $H^*(-)$ will always mean $H^*(-;k)$.

In §1 of this paper, we state the relations between this algebraic group and the Steenrod and Morava stabilizer algebras. In §2 and §3, we consider the cohomology of the Frobenius kernel and of the F_q-points respectively. The cohomology $H^*(P(1))$ is computed for $p =$ odd in §4 (when $p = 2$, it is the classical result of Adams and Liulevicius [Li]), and $H^*(P(2))$ is computed by Shimada-Iwai [S-I]). The cohomology $H^*(S(2)_{(2)})$ is computed in §5. The last section is a description of the cohomology of $S(2)$, which was already studied by Ravenel [R 1]–[R 3], Henn [H] and Gorbounov-Siegel-Symonds [G-S-S].

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We do not directly use theories of algebraic groups and their representations in these concrete cases computations. However these computations are suggested by those for the maximal unipotent group U in GL_3, which was done in joint work with Kaneda, Shimada and Tezuka [K-S-T-Y 1]. K. Shimomura and C. Peterson suggested the importance of algebraic groups for the Morava stabilizer algebra. H. Miller pointed out the similarity of cohomology of $P(1)$ and the extraspecial p-group $U(F_p)$, which was the start of this work. The author thanks those people.

1. Algebraic groups and the Steenrod algebra

Let k be the algebraic closure of F_p for an odd prime. We think of an algebraic group G as a functor $G : A \to G(A)$ from commutative k-algebras A to groups $G(A)$. For example, the additive group G_a is defined by $G_a(A) = A$. In this paper, we consider the unipotent algebraic subgroup $G(n+1)$ of the general linear group GL_{n+1}, generated by matrices (see [K-S-T-Y 2], 3.3)

$$g(a_1,\dots,a_n) = \begin{pmatrix} 1 & & & & \\ a_1 & 1 & & & \\ a_2 & a_1^p & 1 & & 0 \\ a_3 & a_2^p & a_1^{p^2} & 1 & \\ & & & & \\ a_n & a_{n-1}^p & & & a_1^{p^{n-1}} 1 \end{pmatrix}. \tag{1.1}$$

The coordinate ring $k[G]$ is a Hopf algebra defined by the homomorphisms of set-valued functors $\mathrm{Hom}(G, G_a)$. Taking $\xi_i : g(a_1,\dots,a_n) \to a_i$, its coordinate ring is $k[G(n+1)] \cong k[\xi_1,\dots,\xi_n]$ with the diagonal $\psi(\xi_i) = \sum_{j=0}^{i} \xi_j \otimes \xi_{i-j}^{p^j}, \xi_0 = 1$. For each element in the coordinate ring, we assign a weight by $\mathrm{wt}(\xi_i) = p^i - 1$, indeed, the conjugation action of the diagonal matrix $t = \mathrm{diag}(b, b^p, \dots, b^{p^{n-1}})$ is represented as $\mathrm{ad}(t)g(a_1,\dots,a_n) = g(b^{p-1}a_1,\dots,b^{p^n-1}a_n)$.

On the other hand, it is well known that the dual $P(\infty)^*$ of the subalgebra of Steenrod algebra $\mathcal{A}_p \otimes k$, generated by reduced powers, is isomorphic to $k[\xi_1,\dots,\xi_n,\dots] \cong k[G(\infty)]$ with the same diagonal map. Moreover the degree $\deg(\xi_i) = 2(p^i - 1) = 2\,\mathrm{wt}(\xi_i)$, twice the weight. Hence we may think that for a topological space X, the ordinary homology $H_*(X)$ is a $k[G(\infty) \ltimes G_m]$-comodule where G_m is the multiplicative group, and i-dimensional cohomology $H^i(X)$ is the weight i-space, i.e. $b(x) = b^i x$ for $x \in H^i(X)$ and $b \in G_m(k) = k^*$.

Let $P(n-1)$ be the sub-Hopf algebra of $P(\infty)$ generated by reduced powers $\mathcal{P}^{p^i}$, $0 \le i \le n-1$. Then its dual is

$$P(n-1)^* \cong k[\xi_1,\dots,\xi_n]/(\xi_1^{p^n}, \xi_2^{p^{n-1}}, \dots, \xi_n^p). \tag{1.2}$$

For an algebraic group G, the Frobenius kernel G_r is defined as the functor which represents the kernel of the rth Frobenius map $\sigma^r : G \to G$. Its coordinate ring $k[G_r]$ is represented as $k[G]/(x^{p^r} | x \in \text{augmentation ideal})$. Therefore $P(n-1)^*$ is a quotient algebra of $k[G(n+1)_n]$, indeed we have

Theorem 1.3. *Let* $\mathrm{GP}(n+1)$ *be the functor taking*

$$A \to g(G_a(A)_n, \dots, G_a(A)_1), \quad G_a(A)_i = (\mathrm{Ker}\,\sigma^i | A).$$

Then $k[\mathrm{GP}(n+1)] \cong P(n-1)^*$.

Remark. The subalgebra $P(n)$ of $\mathcal{A}_p$ itself is identified with the distribution $\mathrm{Dist}(G(n+2))$ (see 3.3 in [K-S-T-Y 2]).

Next consider the relation between $G(n+1)$ and the Morava stabilizer algebra. Let $K(n)_*(-)$ be Morava K-theory with coefficients $K(n)_* = Z/p[v_n, v_n^{-1}]$ with $|v_n| = 2(p^n-1)$. Recall

$$\begin{aligned} &K(n)_*K(n) \cong K(n)_*[t_1, \dots t_i, \dots]/(v_n t_i^{p^n} - v_n^{p^n} t_i) \quad \text{and} \\ &S(n) = K(n)_*K(n)/(v_n - 1) \otimes k = k[t_1, \dots]/(t_i^{p^n} - t_i). \end{aligned} \tag{1.4}$$

The diagonal map in $S(n)$ is given ([R 1], (3.1))

$$\begin{aligned} \psi(t_i) &= \sum_{0 \le j \le i} t_j \otimes t_{i-j}^{p^j} \quad \text{for } i \le n, \\ \psi(i_{n+i}) &= \sum_{0 \le j \le n+i} t_j \otimes t_{i+n-j}^{p^j} - C_{p^n}(t_i \otimes 1, t_{i-1} \otimes t_1^p, \dots, 1 \otimes t_i) \end{aligned} \tag{1.5}$$

where $C_{p^n}(x_1, \dots x_k)$ is the $\bmod p$ reduction of $p^{-1}((\sum x_i)^{p^n} - \sum(x_i^{p^n}))$. Let us write the sub-Hopf algebra

$$S(n)_{(j)} = k[t_1, \dots, t_j]/(t_i^{p^n} - t_i | 1 \le i \le j) \subset S(n). \tag{1.6}$$

One of the important theorems of Morava ([R 1], [R 2]) is the following. There exists a profinite group S_n such such that its group ring $k(S_n)$ is isomorphic to $S(n)^*$ where $S(n)^*$ is the topological dual of $S(n)$ filtered by $S(n)_{(j)}$.

On the other hand, Kaneda-Radford-Tezuka found (Theorem 2.2.2 in [T]) the relation between the group ring of the group of F_q-points and the coordinate ring; if G is a unipotent algebraic group defined over F_p, then the dual of the group ring of the finite group of its F_q-points is a quotient ring of the coordinate ring

$$k(G(F_{p^n}))^* \cong k[G]/(x^{p^n} - x | x \in \text{ augmentation ideal}). \tag{1.7}$$

Hence from (1.5), (1.6) and (1.7), we get

Theorem 1.8. *If $j \le n$, then $k(G(j+1)(F_{p^n}))^* \cong S(n)_{(j)}$. Therefore we have $H^*(S(n)_{(j)}) \cong H^*(G(j+1)(F_{p^n}))$.*

The weight for elements in $k(G(j+1)(F_{p^n}))^*$ is given by the action

$$\mathrm{ad}(t)g(a_1, \dots, a_n) = g(b^{p-1}a_1, \dots, b^{p^n-1}a_n)$$

for $t = \mathrm{dig}(b, \dots, b^{p^n-1})$. Here $b \in k^*$ such that $b^{p-1} \in F_{p^n}^*$. So the weight takes the value with modulo $(p^n-1)(p-1)$. However if $n < j$, then the weight takes only value with modulo (p^n-1) by the restrictions for the coproduct $\psi(t_{n+i})$. Hence the degree of $K(n)^*(X)$ is $2(p^n-1)$-periodic, and the action of $b \in F_{p^n}^*$ corresponds to the Adams operation.

If $j > n$, then the cohomology of $G(j+1)(F_{p^n})$ and $S(n)_{(j)}$ are quite different. Consider the case $n = 1$ and $j = \infty$. We have the spectral sequence

$$E_1 \cong \bigotimes_{i=1}^{\infty} k[b_i] \otimes \bigwedge(h_i) \Rightarrow H^*(S(1))$$

where $h_i = [t_i]$ and b_i is its Bockstein. The coproduct (1.5) shows $d_1 h_{i+1} = b_i$. So we know

$$H^*(S(1)) \cong \bigwedge(h_1), \quad \text{[R 1]}. \tag{1.9}$$

On the other hand $G(j+1)(F_p) = V(j+1)(F_p)$ where $V(j+1) \subset \mathrm{GL}_{j+1}$ is the commutative unipotent group generated by matrices $(a_{ij} = a_{i-j,1})$ with $a_{ii} = 1, a_{ij} = 0$ for $i < j$. We already know [P-Y] the decomposition $V(j+1) = \prod W_m$ (see (9) in [P-Y]) to Witt vectors, and the cohomology of Witt vectors W_m. Thus we get

$$H^*(G(\infty)(F_p)) = \operatorname*{Lim}_{j\to\infty} H^*(G(j)(F_p)) \cong \bigwedge(h_i | p \nmid i). \tag{1.10}$$

2. Cohomology of the Frobenius kernel

First, we study the cohomology of $G(3)$. There is a central extension

$$1 \to G_a \to G(3) \to G(2) \to 1 \tag{2.1}$$

where G_a (resp. $G(2)$) is generated by $g(0, a_2)$ (resp. $g(a_1, 0)$). This induces the spectral sequence

$$E_2^{*,*} = H^*(G(2)) \otimes H^*(G_a) \Rightarrow H^*(G(3)). \tag{2.2}$$

Here the E_2-term is isomorphic to

$$E_2^{*,*} = \bigotimes_{i=0}^{\infty} k[b_{1i}, b_{2i}] \otimes \wedge(h_{1i}, h_{2i})$$

where we identify $h_{ji} = [\xi_j^{p^i}]$ and $b_{ji} = \tilde{\mathcal{B}} h_{ji} = [p^{-1} \sum_{1\le k\le p-1} \binom{p}{k} \xi_j^{p^i(p-k)} \otimes \xi_j^{p^i k}]$ in the cobar complex. Here $\widetilde{\mathcal{B}} = \mathcal{B}\sigma$ is the Bockstein defined in the cobar complex (see Appendix 1 in [R 3]), which is the composition of the Frobenius map σ and the usual Bockstein $\mathcal{B}$.

Remark. In the notation [Y], [K-S-T-Y 1] $h_{ij} = x_i(j)$ and $b_{ij} = y_i(j+1)$.

Since $\psi(\xi_2) - (\xi_2 \otimes 1 + 1 \otimes \xi_2) = \xi_1 \otimes \xi_1^p$, the first differential of the spectral sequence is

$$d_2 h_{20} = h_{10} h_{11}. \tag{2.3}$$

From the naturality of $\widetilde{\mathcal{B}}$ and the fact $\widetilde{\mathcal{P}}^0 = \sigma$, we know

$$\begin{aligned} d_3 b_{20} &= \widetilde{\mathcal{B}} d_2 h_{20} = \widetilde{\mathcal{B}}(h_{10})\widetilde{\mathcal{P}}^0(h_{11}) - \widetilde{\mathcal{P}}^0(h_{10})\widetilde{\mathcal{B}}(h_{11}) \\ &= b_{10} h_{12} - b_{11} h_{11}. \end{aligned} \tag{2.4}$$

By using reduced powers $\widetilde{\mathcal{P}}^i$ in the cobar complexes (see also Appendix 1 in [R 3]) and the Cartan-Serre transgression theorem, we get

$$\begin{aligned} d_{2p^n+1} b_{20}^{p^n} &= \widetilde{\mathcal{P}}^{p^{n-1}} \cdots \widetilde{\mathcal{P}} d_3 b_{20} \\ &= b_{10}^{p^n} h_{1,n+2} - b_{11}^{p^n} h_{1,n+1}. \end{aligned} \tag{2.5}$$

The Kudo transgression theorem implies

$$\begin{aligned} d_{2p^n(p-1)+1}(b_{20}^{p^n(p-1)} \otimes d_{2p^n+1} b_{20}^{p^n}) &= \widetilde{\mathcal{B}}\widetilde{\mathcal{P}}^{p^n}(d_{2p^n+1} b_{20}^{p^n}), \\ &= b_{10}^{p^{n+1}} b_{1,n+2} - b_{1,n+1} b_{11}^{p^{n+1}}. \end{aligned} \tag{2.6}$$

Therefore by acting with the Frobenius map, we know that

$$J = (b_{1i}^{p^{m+1}} b_{1,m+i+2} - b_{1,i+1}^{p^{m+1}} b_{1,m+i+1}) \tag{2.7}$$

is zero in $H^*(G(3))$.

Next consider the cohomology of $G(3)_r$. In this case $b_{ki} = h_{ki} = 0$ if $i \geq r$. Hence the E_2-term of the spectral sequence type of (2.1) but converging to $H^*(G(3)_r)$ is

$$E_2^{*,*} \cong \bigotimes_{0 \leq i \leq r-1} k[b_{1i}, b_{2i}] \otimes \bigwedge(h_{1i}, h_{2i}). \tag{2.8}$$

From (2.5) we know that $b_{20}^{p^{r-1}}$ is a permanent cycle. By using the Frobenius maps, we also see $b_{2j}^{p^{r-j-1}}$ is a permanent cycle.

Theorem 2.9. *Let $\tilde{b}_{2j} \in H^*(G(3)_r)$ be an element which corresponds to $b_{2j}^{p^{r-j-1}}$ in the spectral sequence* (2.8)*. Then $H^*(G(3)_r)$ is finitely generated over $k[\tilde{b}_{2j} | 0 \leq j \leq r-1] \otimes k[b_{10}]$. Moreover $b_{1j}^{(p+1)^{r-j}} = 0$ for $j \geq 1$.*

Proof. From (2.7), $b_{1,k-1}^p b_{1,k-1} - b_{1,k-2}^p b_{1,k} = 0$. Take $k = r$, and we get $b_{1,r-1}^{p+1} = 0$. Since $b_{1,k-2}^{p+1} = (b_{1,k-3})^p b_{1,k-1}$, we also have $(b_{1,r-2})^{(p+1)^2} = 0$. By induction on r, we can easily see the last statement.

Let $B_1 = \bigotimes_{0 \leq i \leq r-1} k[b_{1i}]$ and $B_2 = \bigotimes_{0 \leq i \leq r-1} k[b_{2i}^{p^{r-j-1}}]$. Then we can write

$$E_2^{*,*} \cong A \otimes B_2 \text{ where } A \text{ is a finitely generated } B_1\text{-module.}$$

Since $d|B_2 = 0$, the cohomology $E_3^{*,*} = H^*(A \otimes B_2, d_2) = H(A, d_2) \otimes B_2$. It is a subquotient ring of the Noetherian ring $E_2^{*,*}$, hence it is also Noetherian. Therefore $E_s^{*,*}$ is Noetherian for finite s. Since $A^{*,t} = 0$ for $t > \sum |b_{2i}^{p^{r-j-1}}|$, we get $E_t \cong E_\infty$. This shows that $H^*(G(3)_r)$ is finite over $B_1 \otimes B_2$. q.e.d.

Next consider $G(n+1)_r$. The filtration given by maps

$$G(2) \leftarrow G(3) \leftarrow \cdots \leftarrow G(n+1) \leftarrow G_a \tag{2.10}$$

induces a spectral sequence

$$E_1^* = \bigotimes_{1 \leq i \leq n} \bigotimes_{0 \leq j \leq r-1} k[b_{ij}] \otimes \bigwedge(h_{ij}) \Rightarrow H(G(n+1)_r). \tag{2.11}$$

Before considering this spectral sequence, we study more general cases. Let U be a unipotent algebraic group and

$$1 \to G_a \to U \to U' \to 1 \tag{2.12}$$

be a central extension of U'. The induced spectral sequence for $H^*(U_r)$ has the E_2-term

$$E_2^{*,*} = H^*(U_r') \otimes \left(\bigotimes_{0 \leq j \leq r-1} k[b_j] \otimes \bigwedge(h_j) \right) \Rightarrow H^*(U_r).$$

Lemma 2.13. *There is $s > 0$ such that $b_i^{p^s}$ is a permanent cycle in the spectral sequence above. Hence $H^*(U_r)$ is Noetherian.*

Proof. By induction we assume $H^*(U_r')$ is Noetherian. The transgression theorem shows $d_{2p^i+1} b_{tp^i} = a_{i,t} \in H(U_r')$. Since $H(U_r')$ is Noetherian, there is s_t such that $a_{kt} \in (a_{0t}, \ldots, a_{st})$ for all $k > s_t$. This means $b_t^{p^k}$ is a permanent cycle. Then by arguments similar to the last parts of the proof of Theorem 2.9, we can prove the lemma. q.e.d.

Theorem 2.14. *In the spectral sequence* (2.11)*, there is an* $s > 0$ *such that* $b_{ij}^{p^s} + z$ *is a permanent cycle where* $z \in k[b_{ik}|k > j]$*. Let us write* $\tilde{b}_{ij}$ *for a corresponding element in* $H^*(G(n+1)_r)$*. Then* $H^*(G(n+1)_r)$ *is finite over* $k[\tilde{b}_{ij}|i > j$ *or* $i > [n/2]]$*.*

Proof. The first statement is immediate from Lemma 2.13. Let us write $G(n,i)$ for the kernel of the natural projection $G(n) \to G(i)$. Then we have the spectral sequences

$$E_2^{*,*} \cong H^*(G(n)_r) \otimes H^*(G_{ar}) \Rightarrow H^*(G(n+1)_r),$$
$$E(i)_2^{*,*} \cong H^*(G(n,i)_r) \otimes H^*(G_{ar}) \Rightarrow H^*(G(n+1,i)_r)$$

and induced map $i^* : E_r^{*,*} \to E(i)_r^{*,*}$. Take $n = 2i$. Then $\psi(\xi_{2i}) - (\xi_{2i} \otimes 1 - 1 \otimes \xi_{2i}) = \xi_i \otimes \xi_i^{p^i} \bmod(\xi_1, \dots, \xi_{i-1})$ implies $d_2 h_{2i,0} = h_{i,0} h_{i,i}$ in $E(i)_2$. Hence by Kudo's transgression theorem

$$\begin{aligned} &d_{2p^{i-1}(p-1)+1}(b_{2i,0}^{p^{i-1}(p-1)} \otimes d_{2p^{i-1}+1} b_{2i,0}^{p^{i-1}}) \\ &\quad = b_{i,i}^{p^i+1} - b_{i,0}^{p^i} b_{i,2i} \quad \text{in } E(i)_{2p^{i-1}(p-1)+1}. \end{aligned} \tag{1}$$

We can take as $\tilde{b}_{ii}$, the left-hand side of (1) in $H^*(G(2i)_r)$. Indeed, it is zero in $H^*(G(2i+1)_r)$ and the restriction of the image in $H^*(G(2i,i)_r)$ is the right-hand side of (1). By the Frobenius map, we can take $\tilde{b}_{ij} = 0$ for all $j \geq i$. q.e.d.

3. Cohomology of F_q-points

In this section, we assume $q = p^r$, and we study cohomology of $G(n+1)(F_q)$ comparing it to that of $G(n+1)_r$. First we study the case $G(3)$. The central extension (2.1) induces a spectral sequence such that

$$E_2^{*,*} \cong (2.8) \Rightarrow H^*(G(3)(F_q)). \tag{3.1}$$

Here the E_2-term is isomorphic to (2.8) as rings but the Frobenius map acts differently

$$\sigma h_{i,r-1} = h_{i,0}, \sigma b_{i,r-1} = b_{i,0}.$$

Hence we take the index $j \in Z/r$ for $h_{i,j}, b_{i,j}$. The differentials (2.3)–(2.6) also hold. In particular the ideal J is zero in $H^*(G(3)(F_q))$. Before studying this ideal J, we recall Quillen's theorem for cohomology of finite groups.

Given a k-algebra R, let $R(k)$ be the variety of R, that is, the set of ring maps $\mathrm{Hom}(R,k)$ endowed with the Zariski topology. Let V be a vector space over k. Given an ideal I in the symmetric algebra $S(V)$, the variety $\mathrm{Var}(I)$ is the zeros in V. Taking the coordinate homomorphism, $\mathrm{Var}(I) \cong (S(V)/I)(k)$. Quillen's stratification theorem for cohomology of finite groups is

Theorem 3.2. ([Q]) *Let* G *be a finite group and* $H(G) = H^{even}(G;k)$*. Then the variety* $H(G)(k)$ *has a stratification*

$$H(G)(k) \cong \bigvee_{A \in IE} V_A^+$$

where IE *is a set of representatives of conjugacy classes of elementary abelian* p *subgroups* A *of* G*, and* $V_A^+ = i^*(A \otimes k)^+$ *for* $i : A \subset G$ *and* $(A \otimes k)^+ = A \otimes k - \bigcup_{A' \lneqq A} A' \otimes k$*.*

We study maximal elementary abelian p-groups in $G(3)(F_q)$. The group $G(3,2)(F_q) \cong \{g(0,a)|a \in F_q\} \cong F_q$ is the center of $G(3)(F_q)$. The commutator

$$[g(a,0), g(b,0)] = g(0, a^p b - ab^p)$$

(see p. 998 [Y], where $g(a,0) = x_{12}(a)x_{23}(a^p)$). If the subgroup generated by $g(a,0)$ and $g(b,0)$ is abelian, then $a^p b - ab^p = 0$, so $a/b = (a/b)^p$, hence $a/b \in F_p^*$. Let A_a be the subgroup generated by $g(a,0)$. Then the conjugacy classes of maximal elementary abelian p-groups are written as

$$A_a \oplus G(3,2)(F_q) \cong Z/p \oplus (Z/p)^r, \quad \text{with } a \in F_q^*/F_p^*.$$

From Quillen's theorem

Theorem 3.3. *$H(G(3)(F_q))(k) \cong (\bigcup V_a) \oplus (k^r)$ with $V_a \cong A_a \otimes k \cong k$ for $a \in F_q^*/F_p^*$.*

On the other hand, we get

Theorem 3.4. *Let J be the ideal (2.7) in $S_1 = k[b_{1i}|0 \le i \le r-1]$. Then*

$$\mathrm{Var}(J) \cong \bigcup W_a \quad \textit{with } W_a \cong k, a \in F_q^*/F_p^*.$$

Proof. Let $(b_0 \cdots b_{r-1}) \in \mathrm{Var}(J) \subset (k)^r$. Then $b_0^{p^n} b_{n+1} - b_1^{p^n} b_n = 0$ so $(b_1/b_0)^{p^n} = (b_{n+1}/b_n)$. Let $b_1/b_0 = \lambda$. Then

$$(b_0, \ldots, b_{r-1}) = (b_0, \lambda b_0, \lambda^{1+p} b_0, \ldots, \lambda^{1+p+\cdots+p^{r-2}} b_0)$$

and $\lambda^{1+p+\cdots+p^{r-1}} = 1$. Since $x^{q-1} - 1 = 0$ for all $x \in F_q^*$, we can write $\lambda = x^{p-1}$ for $x \in F_q^*$. Hence the variety is

$$\mathrm{Var}(J) \cong \bigcup W_\lambda \quad \text{for } \lambda \in F_q^*/F_p^*$$

where $W_\lambda = \{(b_0, \lambda b_0, \ldots, \lambda^{1+p+\cdots+p^{r-2}} b_0 | b_0 \in k\}$. q.e.d.

Lemma 3.5. *Let π be the projection $\pi : G(3)(F_q) \to G(2)(F_q)$ and $p : S_1/J \to \pi^* H^*(G(2)(F_q))$ be the induced map. Then $p(k) : (\pi^* H(G(2)))(k) \cong \mathrm{Var}(J)$.*

Proof. Consider the commutative diagram of k-points.

$$\begin{tikzcd}
(k)^r \cong S_1(k) & \pi^*(H(G(2)(F_q))(k) \arrow[l, hook'] \arrow[r, "p(k)"'] & \mathrm{Var}(J) \cong \cup W_a \\
 & \bigcup V_a \arrow[ul, hook'] \arrow[u] & \\
 & H(G(3)(F_q))(k) \cong (\bigcup V_a \oplus (k)^r) \arrow[u] &
\end{tikzcd}$$

Since p is epic, $p(k)$ is monic. The number of subspaces V and W are the same. Hence $p(k)$ is also epic. q.e.d.

For finite groups $K' \subset K$, let $N[K' \subset K] : H^*(K') \to H^*(K)$ be the Evens norm which satisfies naturality and the multiplicative properties (for details see [E], [Y]). Let us write

$$\tilde{b}_{2j} = N[G(3,2)(F_q) \subset G(3)(F_q)](b_{2j}).$$

By the double coset formula, $\tilde{b}_{2j}|G(3,2)(F_q) = b_{2j}^{p^r}$. Then we can define

$$j : k[\tilde{b}_{2j}|0 \le j < r] \otimes S_1/(J) \to H^*(G(3)(F_q)). \tag{3.6}$$

Theorem 3.7. *The above map j is injective modulo $\sqrt{0}$, and finite.*

Proof. By Lemma 3.5, $j|1 \otimes S_1/(J)$ is injective modulo $\sqrt{0}$. By arguments similar to the last part in the proof of Theorem 2.9, we also see that the spectral sequence for $H*(G(3)(F_q))$ also has the form $E_s^{*,*} \cong A_s \otimes k[b_{2j}^{p^r}|0 \le j < r]$ for all s and for some finitely generated S_1-module A_s. Thus we have proved the theorem. q.e.d.

Next consider $G(n+1)$. Define the elements

$$\tilde{b}_{ij} = N[G(n+1,i)(F_q) \subset G(n+1)(F_q)](b_{ij})$$

so that $\tilde{b}_{ij}|G(n+1,i)(F_q) = b_{ij}^{p^{ri}}$. Then we can easily prove

Theorem 3.8. *$H(G(n+1))(F_q)$ is finite over $\bigotimes k[\tilde{b}_{ij}]$ where the index (ij) runs over the following* (i) *and* (ii)

(i) $1 \le i \le [n/2]$ *and* $0 \le 2mi \le j < (2m+1)i < r$ *for some* m,

(ii) $i > [n/2]$.

4. Cohomology of $P(1)^*$

In this section we compute $H^*(P(1)^*)$ according to the arguments in §2. We use $k[GP(3)] \cong P(1)^*$ and consider the spectral sequence type (2.8)

$$E_2^{*,*} \cong k[b_{10}, b_{11}, b_{20}] \otimes \bigwedge(h_{10}, h_{11}, h_{20}) \Rightarrow H^*(P(1)^*). \tag{4.1}$$

For ease of notations, we write $b_{1i} = b_i, h_{1i} = h_i, b_{20} = u$ and $h_{20} = z$, and moreover $S = k[b_0, b_1]$. Hence in the new notation $E_2^{*,*} = S[u] \otimes \bigwedge(h_0, h_1, z)$, and $z \in E_2^{0,1}, u \in E_2^{0,2}, S \otimes \bigwedge(h_0, h_1) = E_2^{*,0}$.

The first nonzero differential is $d_2 z = h_0 h_1$ and we get

$$E_3^{*,*} = S[u]\{1, h_0, h_1, h_0 z, h_1 z, h_0 h_1 z\}, \tag{4.2}$$

where $S[u]\{1, h_0, \dots\}$ means the free $S[u]$-module generated by $1, h_0, \dots$. The next differential is

$$d_3 u = \widetilde{\mathcal{B}}(h_0 h_1) = b_0 h_2 - b_1 h_1 = -b_1 h_1.$$

Since $d_3(u^i) = iu^{i-1} d_3(u)$, we get

$$E_4^{*,*} = k[u^p] \otimes (A/(b_1 h_1) \oplus H(A, b_1 h_1)\{u, \dots, u^{p-2}\} \oplus (\mathrm{Ker}(b_1 h_1)|A)\{u^{p-1}\})$$

where $A = E_3^{*,0} \oplus E_3^{*,1} = S\{1, h_0, h_1, h_0 z, h_1 z, h_0 h_1 z\}$, and $H(A, h_1 b_1)$ is the homology of A defined by the differential $dx = h_1 b_1 x$ for each $x \in A$. In $S \otimes \bigwedge(h_0, h_1)/(h_0 h_1)$, we see $\mathrm{Ker}(b_1 h_1) = S\{h_0, h_1\}$ and $\mathrm{Image}(b_1 h_1) = S\{h_1 b_1\}$.

$$E_4^{*,2i} \cong \begin{cases} S\{1, h_0\} \oplus S_0\{h_1\}, & i = 0, \\ (S\{h_0\} \oplus S_0\{h_1\})u^i, & 0 < i \le p-2, \\ S\{h_0, h_1\}u^{p-1}, & i = p-1, \end{cases} \tag{4.3}$$

with $S_0 = S/(b_1)$. The odd second degree parts are given for $a_i, a_{01} \in S$ by

$$d_3(a_0 h_0 + a_1 h_1 + a_{01} h_0 h_1) zu = a_0 h_0 h_1 b_1 z.$$

Hence in $S\{h_0z, h_1z, h_0h_1z\}$, we get $\mathrm{Ker}(b_1h_1) = S\{h_1z, h_0h_1z\}$ and $\mathrm{Image}(b_1h_1) = S\{h_0h_1b_1z\}$. Thus we get

$$E_4^{*,2i+1} \cong \begin{cases} (S\{h_0, h_1\} \oplus S_0\{h_0h_1\})z, & i = 0, \\ (S\{h_1\} \oplus S_0\{h_0h_1\})u^iz, & 0 < i < p-1, \\ S\{h_1, h_0h_1\}zu^{p-1}, & i = p-1. \end{cases} \tag{4.3$'$}$$

Recall that the weight introduced in §1 is always a multiple of $p-1$. So divide it by $p-1$, and let us write it $\mathrm{wt}(x)$ so that $\mathrm{wt}(h_i) = \mathrm{wt}(b_{i-1}) = p^i$ and $\mathrm{wt}(z) = p+1, \mathrm{wt}(u) = p(p+1)$. By Kudo's transgression theorem

$$d_{2p-1}(h_1b_1 \otimes u^{p-1}) = \widetilde{\mathcal{B}}\widetilde{\mathcal{P}}'(h_1b_1) = \widetilde{\mathcal{B}}(h_1)\mathcal{P}^1(b_1) = b_1b_1^p = b_1^{p+1}. \tag{4.4}$$

Therefore $E_{2p}^{*,*}$ is b_1^{p+1}-torsion. Hence there are $r \le 2p-1$ such that $d_r(x) = b_1^{p+1}h_1z$. Since $\mathrm{wt}(b_1^{p+1}h_1z) = 1 \bmod p$, we have $\mathrm{wt}(x) = 1 \bmod p$. Hence $x = ah_0u^i$ or $x = ah_1u^iz$ for $a \in S$. However x must be odd degree, since dx is even degree. Thus $x = ah_0u^i$.

Since $|a| = 2(p+1-i)$, let us write $a = \sum \lambda_j b_0^j b_1^{p+1-i-j}$. Then the weights are

$$\begin{aligned} \mathrm{wt}(b_0^jb_1^{p+1-i-j}h_0u^i) &= jp + p^2(p+1-j-i) + 1 + (p^2+p)i \\ &= (1-j)p^2 + (j+i)p + 1 \bmod p^3. \end{aligned}$$

This must be $\mathrm{wt}(b_1^{p+1}h_1z) = p^2 + 2p + 1$, hence $j = 0 \bmod p$ and $i = 2$. This is the only case. Hence we get $d_4(b_1^{p-1}h_0u^2) = b_1^{p+1}h_1z$, so we also see $d_4(h_0u^2) = b_1^2h_1z$. By similar arguments, we can prove

$$d_4(h_0u^i) = b_1^2h_1zu^{i-2}, \qquad 2 \le i \le p-1. \tag{4.5}$$

Indeed there is x such that $d_r(x) = b_1^{p+1}h_1u^j$, $r \le 2p-2j-1$, and we use $|u| = 2$ and $\mathrm{wt}(u) = p^2 + p$.

Remark 4.5. The formula (4.5) is also showed by the cohomology of Hopf algebra version of Theorem 3 in [L 2]. I. Leary proved the following; Let $0 \to Z/p \to G' \to G \to 1$ be a central extension of a finite group. We consider the induced Hochschild-Serre spectral sequence. Let us write $H^*(Z/p) \cong k[u] \otimes \bigwedge(z)$ and $d_2z = s, d_3u = s'$. If $H^*(G)$ satisfies $s'x = sx'$ for some $x' \in H^*(G)$, then $d_4(u^nx) = n(n-1)u^{n-2}zs'x'$. The formula (4.5) is the case with $x = h_0, x' = b_1, s' = b_1h_1, s = h_0h_1$.

Let $d'(x) = d_4(x)$ for $x = h_0u^i$, $2 \le i \le p-1$, and $d'(x) = 0$ for other S-module generators in (4.3) and (4.3)$'$. Let $EE_5^{*,*} = H(E_4^{*,*}, d')$. Of course $E_5^{*,*}$ is its subquotient. We get

(4.6)

$$EE_5^{*,j} \cong \begin{cases} S\{1, h_0\} \oplus S_0\{h_1\}, & j = 0, \\ (S\{h_0\} \oplus S_0\{h_1\})u, & j = 2, \\ S_0\{h_1\}u^i, & j = 2i, 2 \le i \le p-2, \\ S\{h_1\}u^{p-1}, & j = 2p-2, \end{cases}$$

$$\begin{cases} (S\{h_0\} \oplus S/(b_1^2)\{h_1\} \oplus S_0\{h_0h_1\})z, & j = 1, \\ (S/(b_1^2)\{h_1\} \oplus S_0\{h_0h_1\})zu^i, & j = 2i+1, 1 \le i < p-2, \\ (S\{h_1\} \oplus S_0\{h_0h_1\})zu^{p-2}, & j = 2p-3, \\ S\{h_1, h_0h_1\}zu^{p-1}, & j = 2p-1. \end{cases}$$

The above computations are $EE_5^{*,0} \cong E_4^{*,0}, EE_5^{*,2} \cong E_4^{*,2}$ and $EE_5^{*,4} \cong \operatorname{Ker} d'$, $E_5^{*,1} \cong E_3^{*,1}/\operatorname{Im}(d')$. The differential is

$$d' : E_4^{*,4} = (S\{h_0u\} \oplus S_0\{h_1u\}) \to E_4^{*,1} = (S\{h_0, h_1\} \oplus S_0\{h_0h_1\})z$$

by $h_0u^2 \mapsto b_1^2h_1z$; hence we know $EE_5^{*,4}$ and $EE_5^{*,1}$.

The elements $1, h_0, h_0u, h_0z$ are S-free. If $d_r(x)$ is S-free, then x is also S-free. Other S-free modules are generated by

$$B = k\{h_1u^{p-1}, h_1zu^{p-2}, h_1zu^{p-1}, h_0h_1zu^{p-1}\},$$

so the differential which hit targets in $S\{1, h_0, h_0u, h_1z\}$ are d_r with $r \le 2p-2$.

Lemma 4.7. *All non S-free nonzero elements in $EE_5^{*,*}$ are also nonzero cycles in E_{2p-1}.*

Proof. From (4.6) we know all non S-free element in $EE_5^{*,*}$ are b_1^2-torsion. Suppose that the image d_r contains a b_1^2-torsion element for $2p-2 \ge r \ge 4$. From (4.6), S-module generators are written as $u^dz^eh_0^fh_1^g$. So we can write

$$d_r(u^dz^eh_0^fh_1^g) = \sum \lambda b_0^ib_1^ju^{d''}z^{e''}h_0^{f''}h_1^{g''} \quad \text{with } j = 0 \text{ or } 1, \lambda \in Z/p.$$

Then

$$\operatorname{wt}(\text{the left of the above formula}) - \operatorname{wt}(u^{d''}) > \operatorname{wt}(u^{d-d''}) = (p^2+p)(d-d''),$$

$$\operatorname{wt}(\text{the right of the above formula}) - \operatorname{wt}(u^{d''}) = pi + p^2j + (p+1)e'' + f'' + pg''.$$

By dimensional reason, we note $d - d'' \ge i-1$ since other indexes $j, e'', \dots$ are still 0 or 1. If $d - d'' \ge 2$, then the above two weights are not equal. When $d - d'' = 1$, $i = 2$ and $j = 1$, also by dimensional reason, we know $e = 1, e'' = f'' = g'' = 0$. This case is also seen that the above two weights are not equal. Hence $\operatorname{Image}(d_r)$ does not contain b_1^2-torsion elements.

Of course, the images of b_1^2-torsion elements are also b_1^2-torsion. Hence we get the lemma. q.e.d.

If $p \ge 5$, then of course $2p-2 > 5$ and we have

Corollary 4.8. *If $p \ge 5$, then $EE_5^{*,*} \cong E_5^{*,*} \cong E_{2p-2}^{*,*}$.*

For $p = 3$, $E_5^{*,*}$ is a proper subquotient of $EE_5^{*,*}$ as stated below (4.9).

Since $E_{2p}^{*,*}$ is b_1^{p+1}-torsion, for each S-free generator a in B which is not in the $\operatorname{image}(d_r)$, $r \le 2p-1$ by dimensional reason, we know $\ker d_r|(Sa) = 0$ for some $r \le 2p-1$. The weight of free S-module generators are

$$\operatorname{wt}(b_1^ph_0) = +1, \quad \operatorname{wt}(b_1^ph_0u) = p^2+p+1, \quad \operatorname{wt}(b_1^ph_0z) = p+2 \mod p^3.$$

On the other hand the weight in B are

$$\operatorname{wt}(h_1zu^{p-2}) = -p^2+1, \quad \operatorname{wt}(h_1zu^{p-1}) = 1+p,$$
$$\operatorname{wt}(h_0h_1zu^{p-1}) = 2+p \mod p^3.$$

Therefore, we get the differential

$$\begin{aligned} d_{2p-1}(h_1u^{p-1}) &= b_1^p \text{ Kudo's transgression}, \\ d_{2p-1}(h_0h_1u^{p-1}z) &= h_0b_1^pz, \\ d_{2p-2}(h_1u^{p-2}z) &= b_1^{p-1}h_0, \\ d_{2p-2}(h_1u^{p-1}z) &= b_1^{p-1}h_0u. \end{aligned} \tag{4.9}$$

When $p = 3$, $E_5 = H(EE_5, d_{2p-2})$ here the differential d_{2p-2} is given in the above formulas. Recall $EE_5 \cong E_{2p-2}$ for $p \geq 5$. So for all $p \geq 3$, we have $E_{2p-1} \cong H(EE_5, d_{2p-2})$. Thus we have

(4.10)

$$E_{2p}^{*,j} \cong \begin{cases} S/(b_1^p)\{1\} \oplus S/(b_1^{p-1})\{h_0\} \oplus S_0\{h_1\}, & j = 0, \\ (S/(b_1^{p-1})\{h_0\} \oplus S_0\{h_1\})u, & j = 2, \\ S_0\{h_1\}u^i, & j = 2i, 2 \leq i \leq p-2, \\ 0, & j = p-1, \end{cases}$$

$$\begin{cases} (S/(b_1^{p-1})\{h_0\} \oplus S/(b_1^2)\{h_1\} \oplus S_0\{h_0h_1\})z, & j = 1, \\ (S/(b_1^2)\{h_1\} \oplus S_0\{h_0h_1\})zu^i, & j = 2i+1, 1 \leq i \leq p-3, \\ S_0\{h_0h_1\}zu^{p-2}, & j = 2p-3, \\ 0, & j = 2p-1. \end{cases}$$

Since $d_{2p+1}u^p = d_{2p+1}\widetilde{\mathcal{P}}^1 u = \widetilde{P}^1(h_1b_1) = b_1^p h_2 = 0$, we have the following theorem.

Theorem 4.11. *For all $p \geq 3$, we have isomorphisms*

$$E_\infty^{*,*} \cong E_{2p}^{*,*} \cong (4.10) \text{ and } E_\infty^{*,n} = E_\infty^{*,n+2p} \text{ for } n \geq 0.$$

Corollary 4.12. $\operatorname{gr} H^*(G(3)_2) \cong E_\infty^{*,*} \otimes k[b_{21}] \otimes \wedge(h_{21})$.

Corollary 4.13. *The cohomology $H^*(P(1))$ is multiplicatively generated by*

$$h_i, b_i, g_i' = [h_i z], k_i = [h_i u] \quad \text{for } i = 0 \text{ or } 1,$$
$$e_j = [h_1 z u^{j-2}], f_j = [h_1 u^{j-1}] \quad \text{for } 3 \leq j \leq p-1,$$
$$d_p' = [h_0 h_1 z u^{p-2}] \quad \text{and} \quad v = [u^p].$$

Unfortunately, we do not decide the multiplicative structure completely. Here we give some partial results.

Lemma 4.14. *Let $p \geq 5$. As $S = k[b_0, b_1]$-modules, $H^*(P(1)) \cong E_\infty^{*,*}$ in (4.10).*

Proof. The proof is given by routine arguments using weight and degree. We give here the proofs for cases $b_1 f_j = b_1 d_p' = 0$. The proof of other cases are given similarly. First note

$$\begin{gathered} \operatorname{wt}(h_0) = \operatorname{wt}(k_0) = 1, \quad \operatorname{wt}(h_1) = \operatorname{wt}(k_1) = 0 \quad \operatorname{mod}(p), \\ \operatorname{wt}(b_i) = 0, \quad \operatorname{wt}(g_0') = 2, \quad \operatorname{wt}(g_1') = 1 \quad \operatorname{mod}(p) \end{gathered} \tag{1}$$

for generators in $E_\infty^{*,j}$, $j \leq 2$. By (4.10), we get

$$b_1 f_j = \sum \alpha_i h_i + \beta_i k_i \quad \text{for } \alpha_i, \beta_i \in S.$$

Since $\operatorname{wt}(b_1 f_j) = 0 \operatorname{mod} p$, we know $\alpha_0 = \beta_0 = 0$. For dimensional reason, $|\alpha_1| = |\beta_1| + 2 = 2j$. The fact $\operatorname{wt}(u) > \operatorname{wt}(b_1)$ implies $\operatorname{wt}(b_1 f_j) > |\alpha_1 h_1|$ and $> |\beta_1 k_1|$. Thus $b_1 f_j = 0$.

Next consider the case $b_1 d_p'$. Its weight is $2 \operatorname{mod} p$. But there is no odd degree element of same weight in $E_\infty^{*,j}$, $j \leq 2$. Hence $b_1 d_p' = 0$. q.e.d.

Proposition 4.15. *Let $p \geq 5$. The products $x \cdot y$ for generators $x, y \in \{h_i, g'_i, k_i, e_j, f_j, d'_p\}$ are zeros except for the following cases*

$$h_1 g'_0 = -h_0 g'_1 \neq 0, \quad h_0 k_0, h_1 k_0, h_0 k_1, h_0 e_j \neq 0,$$
$$\{h_1, g'_1, k_1\} \cdot \{e_{p-2}, e_{p-1}, f_{p-2}, f_{p-1}\}, e_j e_{p+1-j}, f_j f_{p-j}, f_j e_{p+1-j}.$$

Remark. The author does not know whether the above elements are zero or nonzero. When $p = 3$ the multiplications are more complicated, for example, we know $h_0 g'_0 \neq 0, h_0 g'_1 \neq 0, g'_0 g'_1 \neq 0$.

5. Cohomology of $S(2)_{(2)}$

In this section we compute the cohomology of $S(2)_{(2)} \cong k(G(3)(F_{p^2}))^*$. Let U be the maximal unipotent subgroup of GL_3 generated by lower triangular matrices with diagonal elements 1. Denote subgroups

$$U_\alpha = \begin{pmatrix} 1 & & \\ * & 1 & \\ 0 & 0 & 1 \end{pmatrix}, \quad U_\beta = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & * & 1 \end{pmatrix}, \quad U_\gamma = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ * & 0 & 1 \end{pmatrix}$$

which are isomorphic to G_a. The cohomology of $H^*(U)$ is studied in [K-S-T-Y 1] and [Y]. We recall it before studying $H^*(G(3)(F_{p^2}))$.

The spectral sequence induced from

$$1 \to U_\gamma \to U \to U_\alpha \oplus U_\beta \to 1$$

has the E_2-term

$$E_2^{*,*} \cong H^*(U_\alpha \oplus U_\beta) \otimes H^*(U_\gamma) \Rightarrow H^*(U). \tag{5.1}$$

In [K-S-T-Y 1] $H^*(U_2)$ is computed completely. Moreover we know the spectral sequence (5.1) for low degree differentials. In the notation of [K-S-T-Y 1]

$$H^*(U_\alpha \oplus U_\beta) = \bigotimes_{i=0}^{\infty} k[y_\alpha(i+1), y_\beta(i+1)] \otimes \bigwedge(x_\alpha(i), x_\beta(i)), \quad y(i+1) = \widetilde{\mathcal{B}} x(i),$$
$$H^*(U_\gamma) = \bigotimes_{i=0}^{\infty} k[y_\gamma(i+1)] \otimes \bigwedge(x_\gamma(i)).$$

The first differential is $d_2 x_\gamma(i) = x_\alpha(i) x_\beta(i)$. Hence

$$E_3^{*,*} \cong \bigotimes_{i=0}^{\infty} P(i+1) \otimes H(i) \tag{5.2}$$

with $P(i) = k[y_\alpha(i), y_\beta(i), y_\gamma(i)]$ and

$$H(i) = k\{1, x_\alpha(i), x_\beta(i), x_\alpha(i)x_\gamma(i), x_\beta(i)x_\gamma(i), x_\alpha(i)x_\beta(i)x_\gamma(i)\}.$$

The next differential is $d_3 y_\gamma(i) = y_\alpha(i)x_\beta(i) - y_\beta(i) \cdot x_\alpha(i)$. The differential d_4 acts as $d_4(x_\xi(i) y_\gamma(i)^{j+2}) = (y_\alpha(i)x_\beta(i) - y_\beta(i)x_\alpha(i)) y_\xi(i) x_\gamma(i) y_\gamma(i)^j$ for $\xi = \alpha, \beta$. Moreover if $p > 3$, then $d_r = 0$ for $5 \leq r < 2p-2$. Thus these differentials are closed in $P(i) \otimes H(i)$. Hence if $p \geq 5$,

$$E_{2p-2}^{*,*} \cong \left(\bigotimes_{i=1} E(i)_5^{*,*} \right) \otimes H(0) \tag{5.3}$$

where $E(i)_5^{*,*}$ is a subquotient of $P(i) \otimes H(i)$ which is given (5.6)(3) on page 89 in [K-S-T-Y 1]. For $p = 3$, by arguments similar to (4.9)–(4.10), we see that E_{2p-1} is

a subquotient of the right-hand side algebra of (5.3). We know all the differentials d_{2p-2}, d_{2p-1}. They are not closed in $P(i)\otimes H(i)$, (5.8), (5.9), (5.10) in [K-S-T-Y 1].

In particular we know for all primes $p \geq 3$

Proposition 5.4 (5.11 in [K-S-T-Y 1])**.** *The $\bigotimes_{i=1}^{\infty} k[y_\xi(i)]$-module generators of $H^*(G(3))$ and in $\bigoplus_{s=0}^{2p-1} E_\infty^{*,s}$ are given by elements in $H(0)$ and for $i \geq 1$*

$$x_\xi(i), g_\xi(i) = [x_\xi(i)x_\gamma(i)], \quad k_\xi(i) = [x_\xi(i)y_\gamma(i)] \quad \textit{for } \xi = \alpha \textit{ or } \beta,$$
$$d_j(i) = [x_\alpha(i)x_\beta(i)x_\gamma(i)y_\gamma(i)^{j-2}], \qquad 3 \leq j \leq p,$$
$$c_j(i) = [(y_\alpha(i)x_\beta(i) - y_\beta(i)x_\alpha(i))x_\gamma(i)y_\gamma(i)^{j-2}], \qquad 3 \leq j \leq p-1.$$

Now consider the spectral sequence induced from (2.1)

$$E_2^{*,*} \cong k[b_0, b_1] \otimes \bigwedge(h_0, h_1) \otimes k[u_0, u_1] \otimes \bigwedge(z_0, z_1)$$
$$\Rightarrow H^*(G(3)(F_{p^2})) \cong H^*(S(2)_{(2)}).$$

The map $i^* : H^*(U) \to H^*(U(F_{p^2})) \to H^*(G(3)(F_{p^2}))$ is given by

$$i^*y_\alpha(\text{even}) = i^*y_\beta(\text{odd}) = b_1, \quad i^*y_\beta(\text{even}) = i^*y_\alpha(\text{odd}) = b_0,$$
$$i^*x_\alpha(\text{even}) = i^*x_\beta(\text{odd}) = h_0, \quad i^*x_\beta(\text{even}) = i^*x_\alpha(\text{odd}) = h_1,$$
$$i^*y_\gamma(\text{even}) = u_1, i^*y_\gamma(\text{odd}) = u_0, \quad i^*x_\gamma(\text{even}) = z_0, i^*x_\gamma(\text{odd}) = z_1.$$

The first nonzero differential is

$$d_2 z_0 = h_0 h_1, d_2 z_1 = -h_0 h_1.$$

Hence $E_3^{*,*} \cong \widetilde{E}_3^{*,*} \otimes k[u_0 + u_1] \bigotimes \bigwedge(z_0 + z_1)$ with

$$\widetilde{E}_3^{*,j} \cong \begin{cases} S\{1, h_0, h_1\}u^i, & j = 2i, \\ S\{h_0, h_1, h_0h_1\}u^i z, & j = 2i+1, \end{cases} \tag{5.5}$$

where $u = u_0, z = z_1, h_i = h_{1i}$, $S = k[b_0, b_1]$ with $b_i = b_{1i}$.

Define $\widetilde{E}_{r+1}^{*,*} = H(\widetilde{E}_r^{*,*}, d_r)$ using the facts that $d_r(\widetilde{E}_r^{*,*}) \subset \widetilde{E}_r^{*,*}$, which is shown below. Then note that $E_r^{*,*} \cong \widetilde{E}_r^{*,*} \otimes k[u_0 + u_1] \otimes \bigwedge(z_0 + z_1)$.

The next differential is $d_3 u_1 = b_0 h_0 - b_1 h_1$. Hence (see p. 88 in [K-S-T-Y 1])

$$\widetilde{E}_4^{*,j} \cong \begin{cases} S\{1\} \oplus S\{h_0, h_1\}/(b_0h_0 - b_1h_1), & j = 0, \\ (S\{h_0, h_1\}/(b_0h_0 - b_1h_1))u^i, & 0 < j = 2i < 2(p-1), \\ S\{h_0, h_1\}u^{p-1}, \end{cases}$$
$$\begin{cases} (S\{h_0, h_1\} \oplus k\{h_0h_1\})z, & j = 1, \\ (S(b_0h_0 - b_1h_1) \oplus k\{h_0h_1\})zu^i, & 1 < j = 2i+1 < 2(p-1), \\ S\{(b_0h_0 - b_1h_1), h_0h_1\}xu^{p-1}. \end{cases} \tag{5.6}$$

By naturality and (5.5) in [K-S-T-Y 1], the next differential is

$$d_4 h_j u^{i+2} = (b_0h_0 - b_1h_1)b_{j+1}zu^i.$$

Hence we get (see (5.6)(3) in [K-S-T-Y 1]) for $p > 3$

$$\widetilde{E}_5^{*,j} = \begin{cases} S\{1, h_0, h_1\}/(b_0h_0 - b_1h_1), & j = 0, \\ (S\{h_0, h_1\}/(b_0h_0 - b_1h_1))u, & j = 2, \\ 0, & 2 < j = 2i < 2(p-1), \\ S(b_0h_0 - b_1h_1)u^{p-1}, & j = 2(p-1), \end{cases}$$

$$\begin{cases} S\{h_0, h_1, h_0h_1\}/((b_0h_0 - b_1h_1)b_0, \\ \qquad\qquad (b_0h_0 - b_1h_1)b_1)z, & j = 1, \\ k\{b_0h_0 - b_1h_1, h_0h_1\}zu^i, & 1 < j = 2i+1 < 2p-3, \\ (S\{b_0h_0 - b_1h_1\} \oplus k\{h_0h_1\})zu^{p-2}, & j = 2p-3, \\ S\{b_0h_0 - b_1h_1, h_0h_1\}zu^{p-1}, & j = 2p-1. \end{cases} \tag{5.7}$$

When $p = 3$, $E_4^{*,*} = E_{2p-2}^{*,*}$ is given by (5.6) but $E_5^{*,*} = E_{2p-1}^{*,*}$ is isomorphic to the homology of the right-hand side of (5.7) by the differential d_{2p-2} defined by the following two generators.

By naturality and (5.10) in [K-S-T-Y 1], we have

$$d_{2p-2}((b_0h_0 - b_1h_1)zu^{p-2}) = b_0^p h_1 - b_1^p h_0,$$
$$d_{2p-2}(b_0h_0 - b_1h_1)zu^{p-1}) = (b_0^p h_1 - b_1^p h_0)u.$$

Kudo's transgression theorem (5.8) in [K-S-T-Y 1] gives

$$d_{2p-1}(b_0h_0 - b_1h_1)u^{p-1} = b_0^{p+1} - b_1^{p+1},$$
$$d_{2p-1}(h_0h_1zu^{p-1}) = (b_0^p h_1 - b_1^p h_0)z.$$

Therefore we get for all prime $p \geq 3$

(5.8)

$$\widetilde{E}_{2p}^{*,j} = \begin{cases} S/(b_0^{p+1} - b_1^{p+1}) \\ \qquad \oplus S\{h_0, h_1\}/(b_0h_0 - b_1h_1, b_0^p h_1 - b_1^p h_0), & j = 0, \\ S\{h_0, h_1\}/(b_0h_0 - b_1h_1, b_0^p h_1 - b_1^p h_0)u, & j = 2, \\ 0, & 2 < j = 2i \leq 2p-2; \end{cases}$$

$$\begin{cases} S\{h_0, h_1, h_0h_1\}/ \\ \qquad ((b_0h_0 - b_1h_1)b_0, (b_0h_0 - b_1h_1)b_1, \\ \qquad\quad b_0^p h_1 - b_1^p h_0)z, & j = 1, \\ k\{b_0h_0 - b_1h_1, h_0h_1\}zu^i, & 1 < j = 2i+1 < 2p-3, \\ k\{h_0h_1\}zu^{p-2}, & j = 2p-3, \\ 0, & j = 2p-1. \end{cases}$$

Since $d_{2p+1}u^p = b_0^p h_1 - b_1^p h_0 = 0$, u^p is a permanent cycle, so we get

Theorem 5.9. $E_\infty^{*,*} \cong \widetilde{E}_{2p}^{*,*} \otimes k[u_0 + u_1] \otimes \bigwedge(z_0 + z_1)$.

Corollary 5.10. *$H(S(2)_{(2)})$ is multiplicatively generated by*

$$v = [u^p], u' = [u_0 + u_1], z' = [z_0 + z_1],$$
$$h_i, b_i, g_i = [h_i z], k_i = [h_i u] \quad \textit{for } i = 0 \textit{ or } 1,$$
$$d_j = [h_0h_1zu^{j-2}] \quad \textit{for } 4 \leq j \leq p,$$
$$c_j = [(b_0h_0 - b_1h_1)zu^{j-2}] \quad \textit{for } 4 \leq j \leq p-1.$$

(*Note* $d_3 = [h_0h_1zu] = -g_0k_1$ *and* $c_3 = \beta(d_3) = k_0k_1$.)

Next consider its multiplicative structure. The weight is given modulo (p^2-1) so that

$$\mathrm{wt}(h_0) = \mathrm{wt}(b_1) = 1, \quad \mathrm{wt}(b_0) = \mathrm{wt}(h_1) = p,$$
$$\mathrm{wt}(g_0) = \mathrm{wt}(k_0) = 2+p, \quad \mathrm{wt}(g_1) = \mathrm{wt}(k_1) = 2p+1.$$

By the weight and degree consideration, we easily see that

$$h_ig_i = 0, \quad h_0g_1 + h_1g_0 = 0, \quad b_i(b_0g_0 - b_1g_1) = 0 \quad \text{for } p \geq 5. \tag{5.11}$$

Since $G(3)(F_{p^2})$ is a finite group, we can also define the usual Bockstein $\beta = \sigma^{-1}\widetilde{\beta}$ and reduced powers $\mathcal{P}^k$, which preserve weight and Frobenius maps.

(5.12)
$$\mathcal{B}h_0 = b_1, \quad \mathcal{B}g_0 = -k_0 \quad \text{and} \quad \mathcal{P}^1b_0 = b_1^p \quad (\text{[Y], Lemma 3.8, note } z = z_1, u = u_0)$$
$$(\widetilde{\mathcal{B}}h_0 = b_0, \widetilde{\mathcal{B}}g_0 = k_1 - h_1u' \text{ and } \widetilde{\mathcal{P}}^1b_0 = b_0^p \text{ with } u' = [u_0+u_1]).$$

Using these operations, we know some relations. For example

$$0 = \beta(h_0g_0) = b_1g_0 + h_0k_0 \quad \text{for } p \geq 5. \tag{5.13}$$

We can get more information from the Bockstein operation, however we give here another argument using $H^*(U_3(F_p))$.

From Proposition 5.4, the generators except for z', u' and v are in the image from $H^*(U_3)$. Consider the diagram

$$\begin{array}{ccccc} H^*(U_3) & \longrightarrow & H^*(U_3(F_{p^2})) & \xrightarrow[i^*]{} & H^*(U_3(F_p)) \\ \downarrow & & \downarrow j^* & & \\ H^*(G(3)) & \longrightarrow & H^*(G(3)(F_{p^2})). & & \end{array} \tag{5.14}$$

Ian Leary decided the multiplicative structure of $H^*(U_3(F_p))$ completely (Theorem 4.14 and 4.15 in [L 1]), while it is quite complicated. In Leary's notation, $H^*(U_3(F_p))$ is generated by elements $y, y', x, x', Y, Y', X, X', d_4, \ldots, d_p, c_4, \ldots, c_{p-1}, z$. The images of the restriction map i^* are given

$$\begin{gathered} i^*x_\alpha(i) = y, i^*x_\beta(i) = y', \quad i^*y_\alpha(i) = x, \quad i^*y_\beta(i) = x', \\ i^*g_\alpha(i) = Y, \quad i^*g_\beta(i) = -Y', \quad i^*k_\alpha(i) = -X, \quad i^*k_\beta(i) = X', \\ i^*d_j(i) = d_j, \quad i^*c_j(i) = c_j, \quad \text{for } 4 \leq j \leq p-2, \\ i^*d_{p-1}(i) = d_{p-1} - x^{p-2}y - x'^{p-2}y', \quad i^*d_p(i) = d_p - x^{p-2}X - x^{p-2}X', \\ i^*c_{p-1}(i) = c_{p-1} - x^{p-1} - x'^{p-1}. \end{gathered} \tag{5.15}$$

Here Y, Y', X are taken so that $yY' = y'Y$ and $\beta(Y) = X$ but in our notation $x_\alpha(1)g_\beta(1) = -x_\beta(1)g_\alpha(1)$ and $\beta g_\alpha(1) = -k_\beta(1)$.

In $H^*(U_3)$, Kudo's transgression and related arguments show (see (5.11) in [K-S-T-Y 1])

$$\begin{gathered} y_\alpha(2)y_\beta(1)^p - y_\beta(2)y_\alpha(1)^p = 0, \\ (y_\alpha(2)y_\beta(1)^{p-1} - y_\beta(2)y_\alpha(1)^{p-1})\{x_\xi(1), g_\xi(1), k_\xi(1) | \text{ for } \xi = \alpha \text{ or } \beta\} = 0. \end{gathered} \tag{5.16}$$

Lemma 5.17. *Let x be a homogeneous (with respect to the weight) element in $H^*(U_3)$ represented in $\bigoplus_{j=0}^{2} E_\infty^{*,j}(1) \otimes k\{1, x_\xi(2), y_\xi(2)\}$ where $E_\infty^{*,*}(1)$ is the submodule in $E_\infty^{*,*}$ defined from elements in $P(1) \otimes H(1)$ in (5.2). If $x \in \operatorname{Ker} i^*$ and $\mathrm{wt}(x) < 2p^2(\alpha+\beta)$, then $x = 0$.*

Proof. First note that $\operatorname{Ker} i^*|(S(1) = k[y_\alpha(1), y_\beta(1)])$ is the ideal $(y_\alpha(1)^p y_\beta(1) - y_\beta(1) y_\alpha(1)^p)$. Each element in this ideal is nonhomogeneous with respect to the weight. Hence suppose that $x = a y_\alpha(2) + b$ with $0 \neq a \in S(1)$, $b \in S(1)\{1, y_\beta(2)\}$. Since x is homogeneous, we have

$$x = c(a' y_\alpha(2) + b' y_\beta(1)^p) \quad \text{with } c \in S(1), a' \in k[y_\beta(1)], b' \in S(1)\{1, y_\beta(2)\}.$$

For dimensional reasons $x = c'(a'' y_\beta(1)^{p-1} y_\alpha(2) + b'' y_\beta(1)^p)$. Since x is homogeneous and in $\operatorname{Ker} i^*$, we get

$$x = c''(y_\alpha(1)^p y_\beta(2) - y_\beta(1)^p y_\alpha(2)) \quad \text{with } c'' \in S(1).$$

Thus $x = 0$ from (5.16). By similar arguments, we can also prove the lemma for the case $x \in S(1)\{x_\xi(1), g_\xi(1), k_\xi(1)\} \otimes \{1, x_\xi(2), y_\xi(2)\}$. q.e.d.

From this lemma, we can deduce all relations in $H^*(G(3)(F_p))$. For example, there is a relation $yX = xY$ for $p \geq 5$ in Theorem 2.14 [L 1]. Hence from Lemma 5.17, we get $-x_\alpha(1)k_\alpha(1) = y_\alpha(1)g_\alpha(1)$. Hence $-h_1 k_1 = b_0 g_1$, since

$$i^* x_\alpha(1) = h_1, \quad i^* y_\alpha(1) = b_0, \quad i^* g_\alpha(1) = g_1, \quad i^* k_\alpha(1) = k_1.$$

In Leary [L 1], we know

$$c_i x = \begin{cases} 0 & \text{for } i < p-1, \\ -x^p & \text{for } i = p-1. \end{cases}$$

So $i^*(c_{p-1}(1) y_\alpha(1)) = x'^{p-1} x$. Since $\mathrm{wt}(c_{p-1}(1)) = (p-1)(\alpha+\beta)$, we have

$$c_{p-1}(1) y_\alpha(1) = y_\beta(1)^{p-1} y_\alpha(2).$$

Thus we have, in $H^*(G(3)(F_{p^2}))$

$$c_i b_0 = \begin{cases} 0 & \text{for } i < p-1, \\ b_1^p & \text{for } i = p-1. \end{cases} \tag{5.18}$$

Unfortunately we cannot directly deduce $H^*(P(1))$ from $H^*(S(2)_{(2)})$ or $H^*(G(3))$. In the spectral sequence converging to $H^*(G(3))$, the differentials are given by

$$d_2 z_0 = h_0 h_1, \quad d_3 u_0 = b_0 h_2 + b_1 h_1,$$
$$d_4 h_1 u_0^2 = (b_0 h_2 - b_1 h_1) b_0 z_1.$$

Since $d_3(h_0 u_0^2) \neq 0$ in this spectral sequence, $h_0 u_0^2$ does not appear in E_4. However we already know in (4.5), $d_3(h_0 u_0^2) = 0$ and $d_4(h_0 u_0^2) = b_1^2 h_1 z_0$ in the spectral sequence converging to $H^*(P(1))$.

We know some of $H^*(G(3)_3)$ from $H^*(G(3)(F_{p^2}))$ and also a little about $H^*(P(1))$. Consider the map

$$\begin{tikzcd} H^*(G(3)_2) \arrow[r, "\sigma"'] & H^*(G(3)_3) \arrow[r] & H^*(G(3)_2) \\ & H^*(G(3)) \arrow[u] \arrow[r] & H^*(G(3)(F_{p^2})). \end{tikzcd} \tag{5.19}$$

We give an incomplete set of generators of $H^*(G(3)_3)$ from those of $\sigma H^*(G(3)_2)$ and subsets of generators of $H^*(G(3)(F_{p^2}))$ and $H^*(G(3)_2)$.

$$\begin{gathered} h_i, b_i, g_i' = [h_i z_0], \quad \text{for } i = 0 \text{ or } 1, \\ g_1 = [h_1 z_1], \quad g_2 = [h_2 z_1], \quad k_1 = [h_1 u_0], \quad k_2 = [h_2 u_0], \\ d_j = [h_1 h_2 z u^{j-2}], \quad c_j = [b_2 h_2 z_1 u_0^{j-2}], \\ h_3, b_3, g_i' \cdot k_i, e_j, f_j, d_p', v, \text{ and } z_2, u_2, z_4, u_4. \end{gathered} \tag{5.20}$$

Note that there are other generators, e.g., $g_{02} = [h_0 z_1 - h_2 z_0]$.

Finally in this section, we give a relation in $H^*(P(1))$ from that of $H^*(U_3(F_p))$. In [L 1] there is a relation $Xy' = 2xY' + x'Y$ in $H^*(U_3(F_p))$. So we get from Lemma 5.17

$$k_\alpha(1)x_\beta(1) = -2y_\alpha(1)g_\beta(1) + y_\beta(1)g_\alpha(1) \quad \text{in } H^*(U_3).$$

Hence $k_1 h_2 = -2b_0 g_2 + b_1 g_1$ in $H^*(G(3))$. Thus we have in $H^*(G(3)(F_{p^2}))$

$$k_1 h_0 = -2b_0 g_0 + b_1 g_1. \tag{5.21}$$

Note that there is an element $k_1 h_0$ from (5.20), which is represented in $E_\infty^{*,1}$. Of course the restriction image of this element to $H^*(G(3)(F_{p^2}))$ must be the right-hand side of (5.21). Recall that

$$\begin{gathered} i^* : g_{02} \to [h_0 z_1 - h_0 z_0] = 2g_0 - h_0[z_0 + z_1], \\ i^* : g_1' \to [h_1 z_0] = h_1[z_0 + z_1] - g_1. \end{gathered}$$

Moreover $b_0 h_0 - b_1 h_1 = 0$. Hence $i^*(-b_0 g_{02} - b_1 g_1)$ is the right-hand side of (5.21). Thus in $H^*(G(3))$, $h_0 k_1 = b_0 g_{02} + b_1 g_1'$. Therefore we get in $H^*(P(1))$

$$h_0 k_1 = b_1 g_1'. \tag{5.22}$$

6. Cohomology of the Morava stabilizer algebra

In this section, we study the cohomology of the Morava stabilizer algebra $S(2)$, which is more easily computed than that of $S(2)_{(j)}$. Most results in this section about cohomology of $S(2)$ are already known by Ravenel [R 1]–[R 3] and by Henn [H] in more general forms.

Let us write the quotient algebra of $S(2)_{(j)}$ as $S(2)_{(k,j)}$, letting $t_s = 0$ for $1 \le s \le k-1$. The filtration induced from injections

$$S(2)_{(2,2)} \to S(2)_{(2,3)} \to \cdots \to S(2)_{(2,j)} \to \cdots \to S(2)_{(2,\infty)}$$

gives a spectral sequence similar to (2.10), such that

$$E_2 = \bigotimes_{i \ge 2, k \in Z/2}^{\infty} k[b_{ik}] \otimes \bigwedge(h_{ik}) \Rightarrow H^*(S(2)_{(2,\infty)}). \tag{6.1}$$

From the property of the coproduct (1.5) and $t_1 = 0$, we know

$$d_2 h_{i+2,0} = b_{i,1} \bmod (h_{j,k}, b_{j-1,k} | j \le i, k \in Z/2) \quad \text{for } i \ge 2. \tag{6.2}$$

Hence we have (see Theorem 6.3.7 in [R 1])

$$E_3 \cong \bigotimes_{k \in Z/2} \bigwedge(h_{2k}, h_{3k}) \cong H^*(S(2)_{(2,\infty)}). \tag{6.3}$$

Next consider the spectral sequence

$$E_2 \cong \bigotimes_{k\in Z/2} k[b_{1k}] \otimes \bigwedge(h_{1k}, h_{2k}, h_{3k}) \Rightarrow H^*(S(2)_{(1,\infty)} = S(2)). \tag{6.4}$$

Since the algebra $B = \bigotimes_{k\in Z/2} k[b_{1k}] \otimes \bigwedge(h_{1k}, h_{2k})$ is closed under the differential from the coproduct (1.5), we get the spectral sequences

$$E_2'' \cong H(B,d) \otimes \bigotimes_{k\in Z/2} \bigwedge(h_{3k}) \Rightarrow H^*(S(2)) \tag{6.5}$$

where the differential in B is $d_2 h_{20} = h_{10}h_{11}$. Hence

(6.6)

$$A = H(B,d) \cong k[b_0, b_1]\{1, h_0, h_1, g_0, g_1, h_0g_1\} \otimes \bigwedge(z') \quad \text{with } z' = (h_{20} + h_{21}).$$

Here $g_k = [h_k z] = [h_{1k}h_{20}]$ as in Corollary 5.10. The quotient map $S(2) \to S(2)_{(2)}$ induces the map $H^*(S(2)_{(2)}) \to H^*(S(2))$. For an element x in (6.5), $d_r(x) = 0$ for all r in the spectral sequence in (6.5), because x is represented as an image from $H^*(S(2)_{(2)})$ by Corollary 5.10.

When $p \geq 5$, the relation in A are $h_i g_i = g_i g_{i+1} = h_0 g_1 + h_1 g_0 = 0$ (5.11) and the differentials in E_2'' are $d_2 h_{30} = b_1, d_2 h_{31} = b_0$. Thus we get

Theorem 6.7. (Ravenel) *For $p \geq 5$,*

$$H^*(S(2)) \cong k\{1, h_0, h_1, g_0, g_1, h_0g_1\} \otimes \bigwedge(z').$$

Corollary 6.8. *For $p \geq 5$, the inflation map $H^*(S(2)_{(2)}) \to H^*(S(2))$ is epic.*

When $p = 3$, the multiplicative structure of A is very different

$$h_i\tilde{g}_i = -b_i h_{i+1}, h_0\tilde{g}_1 - h_0\tilde{g}_1 = 0, \tilde{g}_i^2 = -b_i\tilde{g}_{i+1}, \tilde{g}_0\tilde{g}_1 = b_0 b_1. \tag{6.9}$$

Here we take new generators $\tilde{g}_0 = [h_0(z_0 - z_1)] = g_0 + h_0 z'$, $\tilde{g}_1 = [h_1(z_1 - z_0)] = -g_1 - h_1 z'$ so that $\sigma(\tilde{g}_0) = \tilde{g}_1$. This is proved by using the Massey product (page 294 in [R 2]) or by using Lemma 5.17 and the result of Leary (Theorem 2.15 in [L 1]) by the map $j^* i^{*-1}$ in (5.14), exchanging y (resp. y', x, x', Y, Y') with h_1 (resp. $h_0, b_0, b_1, g_1, -g_0$).

For $p = 3$, the differentials in E'' are given by $d_2 h_{3i} = b_{i+1} - \tilde{g}_i$ ([R 2]). This is also proved by the definition of coproduct (see [R 2]). We can prove that $b_1 - \tilde{g}_0$ is a nonzero divisor in A, e.g. for $x = a + a_0\tilde{g}_0 + a_1\tilde{g}_1 \in A$, $a, a_i \in k[b_0, b_1]$,

$$\begin{aligned} d(xh_{30}) &= (a + a_0\tilde{g}_0 + a_1\tilde{g}_1)(\tilde{g}_0 - b_1) \\ &= (a - a_0 b_1)\tilde{g}_0 + (-a_0 b_0 - a_1 b_1)\tilde{g}_1 + (a_1 b_0 - a)b_1, \end{aligned}$$

using (6.9), hence if $d(xh_{30}) = 0$, then $a = a_0 = a_1 = 0$.

Hence for the spectral sequence E'', we only need to know the homology of $A/(b_1 - \tilde{g}_0)$ with the differential $b_0 - \tilde{g}_1 = d_2(h_{31})$.

$$E_3'' \cong A/(b_1 - \tilde{g}_0, b_0 - \tilde{g}_1) \oplus (\mathrm{Ker}(b_0 - \tilde{g}_1)|A/(b_1 - \tilde{g}_0))\{h_{31}\}.$$

The algebra $A/(b_1 - \tilde{g}_0)$ is generated as a $k[b_0, b_1]$-module by $\{1, h_0, h_1, \tilde{g}_1, h_0\tilde{g}_1\}$ with the relation

$$\begin{gathered} h_0 b_1 = -b_0 h_1, \quad h_1\tilde{g}_1 = -b_1 h_0, \quad h_1 b_1 - h_0\tilde{g}_1 = 0, \\ b_1^2 = -b_0\tilde{g}_1, \quad \tilde{g}_1^2 k = -b_1^2, \quad b_1\tilde{g}_1 = b_0 b_1 \end{gathered}$$

from (6.9). Hence we have an isomorphism of $k[b_0]$-modules

$$A/(b_1-\tilde{g}_0)\cong k[b_0]\{1,h_0,h_1,b_1,\tilde{g}_1,h_0\tilde{g}_0\}\otimes\bigwedge(z'). \tag{6.10}$$

We easily see that $h_1,b_1,\tilde{g}_1\in\mathrm{Ker}(b_0-\tilde{g}_1)$ in $A/(b_1-\tilde{g}_0)$, e.g.,

$$h_1(\tilde{g}_1-b_0)=-b_1h_0-h_1b_0=0,\quad \tilde{g}_1(\tilde{g}_1-b_0)=-b_1^2+b_1^2=0.$$

For each element $0\neq x\in k[b_0]\{1,h_0\}$, it is easily proved $(\tilde{g}_1-b_0)x\neq 0$. Thus we get

Proposition 6.11. (Henn [H], [G-S-S]) *When $p=3$, as $k[b_0]$-modules*

$$H^*(S(2))\cong k[b_0]\{1,h_0,h_1,b_1,a,c_0',c_1',b_1a\}\otimes\bigwedge(z')$$

with $a=[h_1h_{31}],c_0'=[b_1h_{31}],c_1'=[g_1h_{31}]$.

The fact that a,c_i' are permanent cycles in (6.5) is also proved from Corollary 6.14 below.

Let us write as $GS_{(j)}$ an algebraic group with the group ring $k(GS_{(j)}(F_{p^2}))\cong S(2)_{(j)}$. Here we construct this group only for $j=3$. Recall that the second Witt group W_2 is defined by $W_2(A)=A\times A$ for each commutative k-algebra A with the sum

$$(a_1,a_2)+(a_1',a_2')=(a_1+a_1',a_2+a_2'-C_p(a_1,a_1'))\quad\text{for } a_i,a_i'\in A,$$

(see (1.5) for the definition of C_p). From equality (1.5), the coproduct is $\psi(t_3)=\sum_{j=0}t_j\otimes t_{3-j}^{p^j}-C_p(t_1^p\otimes 1+1\otimes t_1^p)$. Hence the coordinate ring of $GS_{(3)}$ is given by

$$\begin{aligned}k[G(4)]\otimes k[W_2]&\to k[t_1,t_2,t_3,x_1,x_2]/(t_1=x_1)\\&\supset k[t_1,t_2,t_3+x_2^p]\cong k[GS_{(3)}].\end{aligned}$$

Let $G_1(A)$ (resp. $G_2(A)$) be the subgroup of $(G(4)\times W_2)(A)$ generated by

$$\{g(a_1,a_2,a_3)\times(a_1,b_2)|a_i,b_2\in A\}\quad(\text{resp. }\{g(0,0,-b_2^p)\times(0,b_2)|b_2\in A\}). \tag{6.12}$$

Since $\mathrm{Hom}(k[G],A)\cong G(A)$, we know $GS_{(3)}\cong G_1/G_2$.

Therefore we have a short exact sequence of groups

$$0\to G_a\to GS_{(3)}\to GS_{(2)}\to 1.$$

The induced spectral sequence of F_{p^2}-points is

$$E_2^{*,*}=H^*(S(2)_{(2)})\otimes H^*(F_{p^2})\Rightarrow H^*(S(2)_{(3)}). \tag{6.13}$$

For $p=3$, since $d_2(h_{3i})=b_{i+1}-\tilde{g}_i$, for $h_{3i}\in H^*(F_{p^2})$, the following elements are permanent cycles in the above spectral sequence

$$a=[h_1h_{31}-h_0h_{30}],\quad c_1'=[b_1h_{31}+\tilde{g}_1h_{30}]\quad\text{and}\quad c_0'=[\tilde{g}_1h_{31}-\tilde{g}_0h_{30}].$$

Hence we have

Corollary 6.14. *The inflation map $H^*(S(2)_{(3)})\to H^*(S(2))$ is epic.*

The relations (6.9) are given in $S(2)_{(2)}$ and $d_2(h_{3i})=b_{i+1}-\tilde{g}_i$ for the spectral sequence (6.13). Hence $b_0^2+b_1^2=0$ in $H^*(S(2)_{(3)})$. Thus for the map $\pi:GS_{(3)}(F_9)\to G(3)(F_9)$, the variety is

$$(\pi^*k[b_1,b_0])(k)\cong\mathrm{Var}(b_1^2+b_0^2)=V_+\cup V_-\subset k\{b_0,b_1\}$$

where $V_*=\{b_0+ib_1\}$ and $V_-=\{b_0-ib\}$. Hence there are subgroups A_+,A_- which are isomorphic to $Z/3$ but not conjugate to each other in $GS_{(3)}(F_9)$.

To seek $A_\pm$, we recall the definition of b_i. Identify $G(2)(F_9) = F_9$, let x_0 (resp. x_1) in $H^1(G(2)(F_9)) = \mathrm{Hom}(F_9, k)$ be the element defined by the natural nonzero map

$$F_9 = Z/3 \oplus Z/3 = \langle 1\rangle \oplus \langle i\rangle \to \langle 1\rangle \to k \quad (\text{resp. } F_9 \to \langle i\rangle \to k).$$

Let $y_i = \beta(x_i)$. Recall that h_0 (resp. h_1) is the element induced from the identity (resp. the Frobenius map) in $\mathrm{Hom}(F_9, F_9) = H^1(F_9, F_9)$. Hence

$$h_0 = x_0 + ix_1, \quad h_1 = x_0 + i^3 x_1 = x_0 - ix_1 \quad (\text{see (3.4) in [Y]}),$$
$$b_0 = \tilde{\beta}(h_0) = y_0 - iy_1 \quad \text{and} \quad b_1 = y_0 + iy_1.$$

Since $V(b_0 + ib_1) = V(y_0 - y_1) = \mathrm{Ker}(H^*(G(2) = F_9) \to H^*(A_+))$, we know $A_+ = \langle 1 + i\rangle$ (resp. $A_- = \langle 1 - i\rangle$) in $G(2)$. Therefore in $GS_{(3)}$

$$A_+ = \langle g(1+i, 0, 0) \times (1+i, 0)\rangle, \quad A_- = \langle g(1-i, 0, 0) \times (1-i, 0)\rangle,$$

indeed, $3 \cdot g(\alpha, 0, 0) = g(0, 0, -\alpha)$ in $G(4)$ and $3 \cdot (\alpha, 0) = (0, \alpha^3)$ in W_2 for $\alpha = 1 \pm i$, hence $A_\pm$ has order 3.

For $x_\pm \in H^1(A_\pm)$ and $y_\pm = \beta(x_\pm)$, we see $x_0|A_\pm = x_\pm$, $x_1|A_\pm = \pm x_\pm$ and hence

$$h_0|A_\pm = (x_0 + ix_1)|A_\pm = (1 \pm i)x_\pm, \quad h_1|A_+ = (1 \mp i)|x_\pm,$$
$$b_0|A_+ = (1 \mp i)y_\pm, \quad b_1|A_\pm = (1 \pm i)y_\pm.$$

Note that $b_0 \pm ib_1|A_\pm = 0$.

Lemma 6.15. *If $x \in k[b_0]\{b_0, b_1\} \subset H^*(S(2)_{(3)})$ and $x|A_+ = x|A_- = 0$, then $x = 0$.*

Proof. Each element x is written $x = \lambda b_0^s + \mu b_1 b_0^{s-1}$. Then

$$x|A_+ = (\lambda + i\mu)(1+i)^s y_+^s, \quad x|A_- = (\lambda - i\mu)(1-i)^s y_-^s.$$

Therefore, we see the lemma. q.e.d.

Theorem 6.16. (Henn [H], Golbanov-Siegel-Symonds [G-S-S]) *In $H^*(S(2))$, there are relations*

$$a^2 = ah_0 = ah_1 = ac_0' = ac_1' = 0, \quad c_0h_0 = c_1h_1 = ab_1,$$
$$c_0h_1 = c_1h_0 = ab_0, \quad c_0c_1 = 0.$$

Proof. In the spectral sequence (6.5) or (6.13), we always know $a^2 = 0 \in E_\infty^{*,*}$. Hence $a^2 \in k[b_0]\{1, b_1, a, ab_1\}$. Consider the subgroup

$$Z/3 \cong B_\pm = \langle g(0, 0, 1 \pm i) \times (0, 0)\rangle \subset F_9 \subset GS_{(3)}(F_9).$$

Let $w_\pm \in H^1(B_\pm)$ be the element induced from the identity of $\mathrm{Hom}(B_\pm, B_\pm = Z/3)$. Then we see $h_{30}|B_\pm = (1+i)w_\pm$ and $h_{31}|B_\pm = (1 \mp i)w_\pm$ by arguments similar to the case $A_\pm$. Since $a = [h_1h_{31} - h_0h_{30}]$, the restriction maps of a to groups $A_+ \oplus B_+$ are

$$\begin{aligned} a|A_+ \oplus B_+ &= ((1-i)^2 - (1+i)^2)x_+w_+ \\ &= -ix_+w_+ \in k[y_+] \otimes \bigwedge(x_+, w_+) \subset H^*(A_+ \oplus B_+). \end{aligned}$$

Similarly $a|A_- \oplus B_- = ix_-w_-$. Both cases $a^2|A_+ \oplus B_+ = a^2|A_- \oplus B_- = 0$.

Let us write $a^2 = s_{00}b_0^2 + s_{10}b_1b_0 + s_{0a}b_0a + s_{1a}b_1a$. Then

$$a^2|A_\pm \oplus B_\pm = (s_{00}(1_\mp i)^2 + s_{10}(1-i)(1+i))y_\pm^2 \\ \mp i(s_{0a}(1\mp i) + s_{1a}(1\pm i))y_\pm x_\pm w_\pm.$$

Thus all s_j, s_{ja} are zero, that is $a^2 = 0$ in $H^*(S(2)_{(3)})$.

The other cases are proved by the same arguments, using the fact that $\tilde{g}_i|A_\pm = b_{i+1}|A_\pm$ since $\tilde{g}_0\tilde{g}_1 = b_0b_1, h_0b_0 = h_1b_1$ and $h_0\tilde{g}_1 = h_1\tilde{g}_0$ in $H^*(S(2)_{(2)})$ (6.9). Therefore we get relations in $H^*(S(2)_{(3)})$, so in $H^*(S(2))$. q.e.d.

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