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COMPLICATED DYNAMICS OF PARABOLIC EQUATIONS WITH SIMPLE GRADIENT DEPENDENCE

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Abstract. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Given positive integers n, k and $q_l \leq l$, $l = 1, \ldots, k$, consider the semilinear parabolic equation

$$\text{(E)}\quad \begin{aligned} &u_t = u_{xx} + u_{yy} + a(x,y)u + \sum_{l=1}^{k} a_l(x,y) u^{l-q_l} (u_y)^{q_l}, && t > 0, (x,y) \in \Omega, \\ &u = 0, && t > 0, (x,y) \in \partial\Omega. \end{aligned}$$

where $a(x,y)$ and $a_l(x,y)$ are smooth functions. By refining and extending previous results of Poláčik we show that arbitrary k-jets of vector fields in $\mathbb{R}^n$ can be realized in equations of the form (E). In particular, taking $q_l \equiv 1$ we see that very complicated (chaotic) behavior is possible for reaction-diffusion-convection equations with linear dependence on ∇u.

Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. We consider the semilinear parabolic equation

$$(1)\qquad \begin{aligned} &u_t = u_{xx} + u_{yy} + a(x,y)u + f(x,y,u,u_y), && t > 0,\ (x,y) \in \Omega, \\ &u = 0, && t > 0,\ (x,y) \in \partial\Omega, \end{aligned}$$

where $a\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f\colon \mathbb{R}^4 \to \mathbb{R}$, $(x,y,s,w) \mapsto f(x,y,s,w)$, is a smooth function (more precisely, f is continuous together with all its partial derivatives with respect to (s,w)).

Set $X = L^p(\Omega)$, $p > 2$, and let $A\colon W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to X$ be the linear operator defined as

$$-Au = u_{xx} + u_{yy} + a(x,y)u.$$

Being a sectorial operator, A generates the family X^α, $\alpha \geq 0$, of fractional power spaces. Choosing α with $\frac{1}{2} + \frac{1}{p} < \alpha < 1$, we have that $X^\alpha \subset C^1(\bar{\Omega})$ with continuous inclusion.

Define the Nemitski operator $\hat{f}\colon X^\alpha \to X$ by

$$\hat{f}(u)(x,y) = f\left(x, y, u(x,y), u_y(x,y)\right).$$

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Then equation (1) can be rewritten in the form

$$u_t + Au = \hat{f}(u). \tag{2}$$

It is well known that (2) defines a dynamical system (more precisely, a local semiflow) on X^α.

In his recent paper [11], Poláčik showed that the dynamics of equation (2) can be very complicated. He proved in fact that a dense set of dynamics of ordinary differential equations on $\mathbb{R}^n$ can be 'simulated' by such parabolic equations.

In order to describe Poláčik's results, let us introduce some terminology. Let $X_1 := \ker A$, and suppose $X_1 \neq \{0\}$. Since A has compact resolvent, X_1 is finite dimensional, and since A as an operator in $L^2(\Omega)$ is selfadjoint, we can find an $L^2(\Omega)$-orthonormal basis of eigenfunctions $\phi_1, \ldots, \phi_n$ for X_1 so that the spectral projection P for the spectral set $\{0\}$ is the orthogonal projection given by

$$Pu = \sum_{i=1}^{n} \phi_i \int_\Omega u\phi_i \, dx.$$

The space X is then decomposed as the sum of two A-invariant subspaces, namely $X = X_1 \oplus X_2$, where $X_2 = \ker P$.

Let $J_0^k(X_1)$ denote the set of all k-jets on X_1 mapping 0 to itself. Equivalently, $h \in J_0^k(X_1)$ if and only if h is a polynomial on X_1 of order $\leq k$ with $h(0) = 0$. Using the basis $\phi_1, \ldots, \phi_n$ of X_1, we see that h can also be regarded as a polynomial on $\mathbb{R}^n$ of order $\leq k$.

Now Poláčik proved (see [11, Theorem 2.2] that, given any positive integers n and k, there is a convex smooth bounded domain $\Omega \subset \mathbb{R}^2$ and a smooth function $a = a(x, y)$ such that the corresponding operator A has n-dimensional kernel X_1. Moreover, there is a neighborhood $\mathcal{B}$ of zero in $J_0^k(X_1)$ such that every jet $h \in \mathcal{B}$ can be realized on a local center manifold of equation (2) for an appropriate choice of the nonlinearity. More precisely, if $h \in \mathcal{B}$ then there is a nonlinearity $f = f(x, y, w)$ of the form

$$f(x, y, w) = \sum_{l=1}^{k} a_l(x, y) w^l, \quad \text{with } a_l \in H^2(\Omega), \tag{3}$$

there is an open neighborhood U of zero in X_1 and a C^k-map $\sigma\colon U \to X_2 \cap X^\alpha$ with $\sigma(0) = 0$ with the properties that the manifold

$$W = \{\, u_1 + \sigma(u_1) \mid u_1 \in U \,\}$$

is locally invariant with respect to (2) and the k-th order Taylor polynomial at zero of the map

$$u_1 \mapsto P\hat{f}(u_1 + \sigma(u_1))\colon U \to X_1$$

is equal to h. The proof of [11, Theorem 2.2] actually implies that a dense set of vector fields of ordinary differential equations on a bounded set of $\mathbb{R}^n$ can be realized (up to flow equivalence) on local center manifolds of equations of the form (2) with f of the form (3). See [11, Corollary 2.5] for a precise statement.

The form (3) of the nonlinearity f means that equation (1) (or, equivalently, equation (2)) depends on high powers of the gradient of the solution u. On the other hand, when modelling scientific phenomena by equation (1), one usually tries to make the convection terms (i.e. terms depending on ∇u) as simple as possible, e.g. linear in ∇u. Therefore the question arises if such systems can also exhibit complex dynamics. The purpose of this paper is to give a positive answer to this question. More precisely, by refining some crucial arguments in [11] we shall prove that the jet realization result [11, Theorem 2.2] holds for nonlinearities of the form

$$f(x,y,s,w)=\sum_{l=1}^{k}a_l(x,y)s^{p_l}w^{q_l},\quad \text{with } a_l\in H^2(\Omega),$$

where for every l with $1\le l\le k$, q_l is an arbitrarily prescribed integer with $1\le q_l\le l$ and $p_l:=l-q_l$. In particular, choosing $q_l\equiv 1$ we obtain a jet realization result for nonlinearities with linear gradient dependence. This gives a positive answer to a question posed by Poláčik in [11]. We also prove that the above mentioned vector field realization result [11, Corollary 2.5] continues to hold for nonlinearities with prescribed (e.g. linear) gradient dependence.

The main results

The aim of this section is the proof of the following result:

Theorem 1. *For n and $k\in\mathbb{N}$, and arbitrary integers $q_1,\dots,q_k$ such that $1\le q_l\le l$ for $l=1,\dots,k$, let $\mathcal{E}=\mathcal{E}(q_1,\dots,q_k)$ be the set of all functions $f\colon\mathbb{R}^4\to\mathbb{R}$ of the form*

$$f(x,y,s,w)=\sum_{l=1}^{k}a_l(x,y)s^{l-q_l}w^{q_l},\quad (x,y,s,w)\in\mathbb{R}^4,$$

where $a_l\in H^2(\Omega)$ for $l=1,\dots,k$.

There exist a convex bounded domain $\Omega\subset\mathbb{R}^2$ with smooth boundary and a C^∞ function $a(x,y)$ such that:

1. *The operator A has n-dimensional kernel;*
2. *There is an open neighborhood $\mathcal{B}$ of 0 in $J_0^k(X_1)$ such that every jet $h\in\mathcal{B}$ can be realized in (1) by a nonlinearity $f\in\mathcal{E}$; that is, every $h\in\mathcal{B}$ can be realized in an equation of the form*

$$\begin{aligned}&u_t=u_{xx}+u_{yy}+a(x,y)u+\sum_{l=1}^{k}a_l(x,y)u^{l-q_l}(u_y)^{q_l}, && t>0,(x,y)\in\Omega,\\ &u=0, && t>0,(x,y)\in\partial\Omega.\end{aligned}$$

Remark. Choosing $q_l=l$ for all $l=1,\dots,k$, we obtain Poláčik's result [11, Theorem 2.2]. On the other hand, choosing $q_l=1$ for all $l=1,\dots,k$, we obtain a jet realization result for nonlinearities which are polynomials in u and which are linear functions of u_y.

Theorem 1 can be strengthened as follows:

Theorem 2. *For n and $k\in\mathbb{N}$, and arbitrary integers $q_1,\dots,q_k$ such that $1\le q_k\le l$ for $l=1,\dots,k$, there exist a convex bounded domain $\Omega\subset\mathbb{R}^2$ with smooth boundary and a C^∞ function $a(x,y)$ such that:*

1. *The operator A has n-dimensional kernel;*
2. *For every $m\ge 0$ there is an open neighborhood $\mathcal{B}$ of 0 in $J_0^k(X_1)$ such that every C^m-family of jets in $\mathcal{B}$ can be realized in (1) by a family of nonlinearities in $\mathcal{E}(q_1,\dots,q_k)$.*

Choosing $q_l = l$ for all $l = 1, \ldots, k$, we obtain [11, Theorem 2.4]. Similarly as in [11, Corollary 2.5], Theorem 2 implies that a dense set of vector fields in $\mathbb{R}^n$ can be realized, up to flow equivalence, in equation (1) by nonlinearities which have prescribed polynomial dependence on u and u_y. More precisely, in the case of linear gradient dependence, we have the following result:

Theorem 3. *Let $n \in \mathbb{N}$ be arbitrary, B be an open ball in $\mathbb{R}^n$ containing 0, and ${}_0C^1(\bar{B})$ be the Banach space of all C^1-maps $g\colon \bar{B} \to \mathbb{R}^n$ with $g(0) = 0$, endowed with the C^1-norm. There is a dense set $\mathcal{D}$ in ${}_0C^1(\bar{B})$ with the property that for every $g \in \mathcal{D}$ there exist a smooth bounded domain $\Omega \subset \mathbb{R}^2$, a smooth function $a = a(x,y)$ and a nonlinearity f of the form*

$$f(x,y,s,w) = \sum_{l=1}^{k} a_l(x,y)s^{l-1}w, \quad (x,y,s,w) \in \mathbb{R}^4,$$

for some $k \in \mathbb{N}$, such that the flow of the equation

$$\dot{v} = g(v), \quad v \in B,$$

is C^1-equivalent to the flow of equation (1) (with the given Ω, a and f) restricted to some locally invariant (center) manifold.

Proof. This theorem follows from Theorem 2 in exactly the same way as [11, Corollary 2.5] follows from [11, Theorem 2.4]. ☐

Remark. In the statements of Theorems 1 - 3 the coefficients $a_l(x,y)$ can be assumed C^∞-smooth. This follows from the remark following condition (IC) below.

Readers interested in other realization results are referred to the References. In particular, the recent paper [2] proves realization of jets and dense vector fields in the class of (nonpolynomial) spatially homogeneous nonlinearities, i.e. functions of the form $f = f(u, \nabla u)$. Such functions do depend on ∇u in a complicated, largely arbitrary way, but, on the other hand, they are independent of the space variables. Note that neither the methods in [2] nor the methods used in the present paper seem to yield any general jet realization results in the class of functions which are spatially homogeneous and have prescribed polynomial dependence on u and ∇u.

Let us now proceed to the proof of Theorems 1 and 2.

Identifying functions $f = f(x,y,s,w) = \sum_{l=1}^{k} a_l(x,y)s^{l-q_l}w^{q_l} \in \mathcal{E}(q_1, \ldots, q_k)$ with $(a_l)_{1 \le l \le k}$ (thus identifying $\mathcal{E}(q_1, \ldots, q_k)$ with the Hilbert space $(H^2(\Omega))^k$) and using the arguments contained in sections 2 and 3 of [11], we see that, given a domain $\Omega \subset \mathbb{R}^2$ and a function a, statements (2) of Theorem 1 and Theorem 2 hold provided the following surjectivity condition is satisfied:

(SC). *For every polynomial function $H\colon X_1 \to X_1$ of degree $\le k$, $H(0) = 0$, there is an $f \in \mathcal{E}(q_1, \ldots, q_k)$ such that*

$$\sum_{j=1}^{n} \phi_j \int_\Omega \phi_j(x,y) f(x,y,u_1(x,y), u_{1y}(x,y))\,dxdy = H(u_1) \tag{4}$$

for all $u_1 \in X_1$.

We shall now find an equivalent, but more convenient form of this condition.

Let us introduce coordinates $(r_1, ..., r_n)$ in X_1, with respect to the basis $\phi_1, \ldots,$ ϕ_n, so that $u_1 \in X_1$ can be written in the form $u_1 = r_1\phi_1 + \cdots + r_n\phi_n$. Then, if $H \in J_0^k(X_1)$, there are uniquely determined real coefficients $\rho_{j\beta}$, $j = 1, ..., n$, $\beta \in \mathbb{N}^n$, $|\beta| \le k$, such that

$$H\left(\sum_{i=1}^n r_i\phi_i\right) = \sum_{j=1}^n \left(\sum_{l=1}^k \sum_{|\beta|=l} \rho_{j\beta} r^\beta\right)\phi_j.$$

Thus (4) reads

$$\int_\Omega \phi_j(x,y) f\left(x, y, \sum_{i=1}^n r_i\phi_i(x,y), \sum_{i=1}^n r_i\phi_{iy}(x,y)\right) dxdy = \sum_{l=1}^k \sum_{|\beta|=l} \rho_{j\beta} r^\beta,$$

for $j = 1, ..., n$. The nonlinearity f has the form

$$f(x, y, u, u_y) = \sum_{l=1}^k a_l(x,y) u^{l-q_l} u_y^{q_l},$$

so (4) becomes

$$\int_\Omega \phi_j \sum_{l=1}^k a_l \left(\sum_{j=1}^n r_j\phi_j\right)^{l-q_l} \left(\sum_{i=1}^n r_i\phi_{iy}\right)^{q_l} dxdy = \sum_{l=1}^k \sum_{|\beta|=l} \rho_{j\beta} r^\beta, \quad j = 1, ..., n. \tag{5}$$

Consequently (SC) is satisfied, provided we can find functions $a_1, \ldots, a_k \in H^2(\Omega)$ such that (5) holds for all $(r_1, ..., r_n) \in \mathbb{R}^n$. The left hand side of (5) can be manipulated in the following way:

$$\begin{aligned}
&\int_\Omega \phi_j \sum_{l=1}^k a_l \left(\sum_{j=1}^n r_j\phi_j\right)^{l-q_l} \left(\sum_{i=1}^n r_i\phi_{iy}\right)^{q_l} dxdy \\
&= \int_\Omega \phi_j \sum_{l=1}^k a_l \left(\sum_{|\alpha|=l-q_l} \frac{(l-q_l)!}{\alpha!} r^\alpha \phi^\alpha\right)\left(\sum_{|\gamma|=q_l} \frac{q_l!}{\gamma!} r^\gamma \phi_y^\gamma\right) dxdy \\
&= \sum_{l=1}^k \sum_{|\alpha|=l-q_l} \sum_{|\gamma|=q_l} \frac{(l-q_l)!}{\alpha!}\frac{q_l!}{\gamma!} r^{\alpha+\gamma} \int_\Omega a_l\phi_j\phi^\alpha\phi_y^\gamma \, dxdy \\
&= \sum_{l=1}^k \sum_{|\beta|=l} \left(\sum_{\substack{\alpha+\gamma=\beta \\ |\alpha|=l-q_l, |\gamma|=q_l}} \frac{(l-q_l)!q_l!}{\alpha!\gamma!} \int_\Omega a_l\phi_j\phi^\alpha\phi_y^\gamma \, dxdy\right) r^\beta.
\end{aligned}$$

Equating coefficients, we therefore see that (SC) is satisfied if and only if for every $l = 1, \ldots, k$, for all $\rho_{j\beta} \in \mathbb{R}$, $j = 1, \ldots, n$, $\beta \in \mathbb{N}_0^n$, $|\beta| = l$, there is $a_l(x,y) \in H^2(\Omega)$ such that, for all $j = 1, \ldots, n$, and for all $\beta \in \mathbb{N}_0^n$ with $|\beta| = l$,

$$\int_\Omega a_l \left(\sum_{\substack{\alpha+\gamma=\beta \\ |\alpha|=l-q_l, |\gamma|=q_l}} \frac{(l-q_l)!q_l!}{\alpha!\gamma!} \phi_j\phi^\alpha\phi_y^\gamma\right) dxdy = \rho_{j\beta}. \tag{6}$$

Using the density of $H^2(\Omega)$ in $L^2(\Omega)$, it is easy to see that this latter condition is satisfied if and only if for for every $l = 1, \dots, k$ the functions

$$\left\{ \sum_{\substack{\alpha+\gamma=\beta \\ |\alpha|=l-q_l, |\gamma|=q_l}} \frac{(l-q_l)!q_l!}{\alpha!\gamma!} \phi_j \phi^\alpha \phi_y^\gamma \right\}_{\substack{j=1,\dots,n \\ |\beta|=l}}$$

are linearly independent.

Now we introduce the following notations: given $\gamma, \beta \in \mathbb{N}_0^n$, we say that $\gamma \leq \beta$ iff $\gamma_i \leq \beta_i$, $i = 1, \dots, n$. Moreover, set

$$\epsilon_j := (\underbrace{0, ..., 0, 1}_{j}, 0, ..., 0) \in \mathbb{N}_0^n.$$

With these notations, the independence condition reads:

(IC). *For every $l = 1, \dots, k$ and for every q, $1 \leq q \leq l$, the functions*

$$\left\{ \sum_{\substack{\gamma \leq \beta \\ |\gamma|=q}} \frac{1}{\gamma!(\beta-\gamma)!} \phi^{\beta-\gamma+\epsilon_j} \phi_y^\gamma \right\}_{\substack{j=1,\dots,n \\ |\beta|=l}}$$

are linearly independent.

Remark. If (IC) is satisfied then the functions a_l in (6) can actually be chosen to belong to any dense subspace of $L^2(\Omega)$, e.g. they can be chosen as smooth as Ω is. On the other hand, these functions cannot be chosen to be constant, in general. The reason for this is that the subspace of functions in $\mathcal{E}(q_1, \dots, q_k)$ with spatially constant coefficients has dimension k, while, for $\dim X_1 > 1$, the space of jets $J_0^k(X_1)$ has dimension $> k$, so the surjectivity condition (SC) is not satisfied in this case.

It follows that in order to prove Theorems 1 and 2 we only have to find a smooth convex bounded domain Ω and a smooth function $a(x, y)$ in such a way that condition (IC) is satisfied.

As in [11] we begin by considering the eigenvalue problem

$$\begin{aligned} u_{xx} + u_{yy} + a(x)u + \mu u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{7}$$

on the square $\Omega =]0, \pi[\times]0, \pi[$. Let $\lambda_j(a)$, $j \in \mathbb{N}$, be the increasing sequence of the eigenvalues of the problem

$$\begin{aligned} u_{xx} + a(x)u + \lambda u &= 0 \quad \text{in }]0, \pi[, \\ u &= 0 \quad \text{in } \{0, \pi\}, \end{aligned}$$

and $\psi_j(a)$, $j \in \mathbb{N}$, be the corresponding $L^2(0, \pi)$-normalized eigenfunctions. Choose positive integers $m_1 > m_2 \cdots > m_n$ which are $(k+1)$-conditionally rationally independent. As in [11], let $a = a(x)$ be a C^∞ function on $[0, \pi]$ such that

$$a(x) = 0 \quad \text{on } [0, \delta], \text{ for some } \delta > 0;$$

$$\lambda_j(a) = -m_j^2 \quad \text{for } j = 1, \dots, n;$$

$$\lambda_{n+1}(a) > 0.$$

It follows that the eigenvalue problem (7) has $\mu = 0$ as an eigenvalue. The corresponding eigenspace is n-dimensional and is spanned by the eigenfunctions

$$\phi_j(x,y) = \sin(m_j y)\psi_j(a)(x), \quad j = 1, \dots, s, \ (x,y) \in \Omega. \tag{8}$$

We have the following

Proposition 1. *The eigenfunctions in* (8) *satisfy the independence condition* (IC).

Proof. Fix $l = 1, \dots, k$ and suppose

$$\sum_{\substack{j=1,\dots,n \\ |\beta|=l}} C_{j\beta} \sum_{\substack{\gamma\le\beta \\ |\gamma|=q}} \frac{1}{\gamma!(\beta-\gamma)!} \phi^{\beta-\gamma+\epsilon_j}\phi_y^{\gamma} = 0.$$

Since $a(x) = 0$ for $x \in [0,\delta]$, we see that

$$\phi_j(x,y) = \eta_j(x)\xi_j(y), \quad x \in [0,\delta], y \in [0,\pi],$$

where

$$\eta_j(x) = d_j \sinh(m_j x), \quad d_j \neq 0,$$

$$\xi_j(y) = \sin(m_j y)$$

for $j = 1, \dots, n$. Set

$$\eta(x) = (\eta_1(x), \dots, \eta_n(x)),$$

$$\xi(y) = (\xi_1(y), \dots, \xi_n(y)).$$

Thus, by analyticity,

$$\begin{aligned} &\sum_{\substack{c\in\mathbb{N}^n \\ |c|=l+1}} \sum_{\substack{j=1,\dots,n \\ |\beta|=l \\ \beta+\epsilon_j=c}} C_{j\beta} \sum_{\substack{\gamma\le\beta \\ |\gamma|=q}} \frac{1}{\gamma!(\beta-\gamma)!} \phi^{\beta-\gamma+\epsilon_j}(x,y)\phi_y^{\gamma}(x,y) \\ &= \sum_{\substack{c\in\mathbb{N}^n \\ |c|=l+1}} \eta^c(x) \sum_{\substack{j=1,\dots,n \\ |\beta|=l \\ \beta+\epsilon_j=c}} C_{j\beta} \sum_{\substack{\gamma\le\beta \\ |\gamma|=q}} \frac{1}{\gamma!(\beta-\gamma)!} \xi^{c-\gamma}(y)\xi_y^{\gamma}(y) = 0 \quad \text{for } (x,y) \in \mathbb{R}^2. \end{aligned}$$

By the arguments in [11, pp. 41-42] we obtain that the functions $(\eta^c)_{|c|=l+1}$ are linearly independent, so

$$\begin{aligned} &\sum_{\substack{j=1,\dots,n \\ |\beta|=l \\ \beta+\epsilon_j=c}} C_{j\beta} \sum_{\substack{\gamma\le\beta \\ |\gamma|=q}} \frac{1}{\gamma!(\beta-\gamma)!} \xi^{c-\gamma}(y)\xi_y^{\gamma}(y) \\ &= \sum_{|\gamma|=q} \left(\sum_{\substack{j=1,\dots,n \\ |\beta|=l,\beta\ge\gamma \\ \beta+\epsilon_j=c}} C_{j\beta} \frac{1}{\gamma!(\beta-\gamma)!} \right) \xi^{c-\gamma}(y)\xi_y^{\gamma}(y) = 0 \quad \text{for } |c| = l+1 \text{ and } y \in \mathbb{R}. \end{aligned}$$

By Lemma 1 in the Appendix the functions $(\xi^{c-\gamma}\xi_y^{\gamma})_{|\gamma|=q}$ are linearly independent, so

$$\sum_{\substack{j=1,\dots,n \\ |\beta|=l,\beta\ge\gamma \\ \beta+\epsilon_j=c}} C_{j\beta} \frac{1}{(\beta-\gamma)!} = 0 \quad \text{for } |\gamma| = q \text{ and } |c| = l+1.$$

Now it is easy to see that

$$\sum_{\substack{j=1,\dots,n\\|\beta|=l,\beta\geq\gamma\\ \beta+\epsilon_j=c}} C_{j\beta}\frac{1}{(\beta-\gamma)!}=\sum_{\substack{j=1,\dots,n\\ \gamma+\epsilon_j\leq c}} C_{j,c-\epsilon_j}\frac{1}{(c-\epsilon_j-\gamma)!}$$

$$=\sum_{\substack{j=1,\dots,n\\ \gamma+\epsilon_j\leq c}} C_{j,c-\epsilon_j}\frac{(c_j-\gamma_j)}{(c-\gamma)!},$$

so

$$\sum_{\substack{j=1,\dots,n\\ \gamma+\epsilon_j\leq c}} C_{j,c-\epsilon_j}(c_j-\gamma_j)=0 \quad \text{for } |\gamma|=q \text{ and } |c|=l+1. \tag{9}$$

Now fix $c\in\mathbb{N}_0^n$, $|c|=l+1$. If $\gamma\not\leq c$, then the sum in (9) is over an empty set of elements; if $\gamma\leq c$, but $\gamma+\epsilon_j\not\leq c$, then $\gamma_j+1\not\leq c_j$ and so $\gamma_j=c_j$. Thus (9) is equivalent to the statement that for every $c\in\mathbb{N}_0^n$ with $|c|=\sum_{i=1}^n c_i=l+1$

$$\sum_{\substack{j=1,\dots,n\\ \epsilon_j\leq c}} C_{j,c-\epsilon_j}(c_j-\gamma_j)=0 \quad \text{whenever } \gamma\leq c \text{ and } |\gamma|=q. \tag{10}$$

Let $c\in\mathbb{N}_0^n$ with $|c|=l+1$ be arbitrary. We will show that (10) implies that $C_{j,c-\epsilon_j}=0$ for all j such that $c\geq\epsilon_j$. This will conclude the proof of the proposition. Permuting components, we may assume that, for some s, $1\leq s\leq n$, $c_j\geq 1$ if $1\leq j\leq s$ and $c_j=0$ if $s+1\leq j\leq n$. Then whenever $\gamma\leq c$, we also have $\gamma_j=0$ for $s+1\leq j\leq n$. Therefore we only have to prove the following assertion:

(A). *If $s\in\mathbb{N}$, $q\in\mathbb{N}$, $c\in\mathbb{N}^s$ and $a\in\mathbb{R}^s$ are such that $q<|c|$ and*

$$\sum_{j=1}^s a_j(c_j-\gamma_j)=0 \qquad \textit{whenever } \gamma\in\mathbb{N}_0^s,\ \gamma\leq c \textit{ and } |\gamma|=q,$$

then $a_j=0$ for all $j=1,\dots,s$.

In order to prove (A) we apply Lemma 2 in the Appendix with $p=|c|-q$. Let x^i, $i=1,\dots,s$, be as in that lemma, and set $\gamma^i:=c-x^i$, $i=1,\dots,s$. It follows that $|\gamma^i|=p$ and $\gamma^i\leq c$ for all i and that the matrix $\left(c_j-\gamma^i_j\right)_{1\leq i,j\leq s}=\left(x^i_j\right)_{1\leq i,j\leq s}$ is regular. Assertion (A) follows immediately, and so the proposition is proved. □

So far we have proved that there is a smooth function $a=a(x)$ (i.e. independent of y) such that the surjectivity condition (SC) is satisfied. In particular, taking $q_l=l$ for all $l=1,\dots,k$, we obtain the results in [11, section IV A] as a special case. However, we also claim that there are a smooth convex bounded domain Ω and a smooth function $a(x,y)$ such that condition (SR) is satisfied. In order to prove this claim, we proceed exactly as in [11, section IV B] but defining the functions $\chi_{j\beta}$ as follows:

$$\chi_{j\beta}=\sum_{\substack{\gamma\leq\beta\\|\gamma|=q}}\frac{1}{\gamma!(\beta-\gamma)!}\phi^{\beta-\gamma+\epsilon_j}\phi_y^{\gamma} \quad \text{for } j=1,\dots,n,\ |\beta|=l$$

with $\phi_j=\phi_j(a,s,x,y)$, where a and s are as in [11, section IV B]. No other changes are necessary. This proves the claim and completes the proof of Theorems 1 and 2.

Appendix

We shall now prove the auxiliary results used in the proof of Proposition 1.

Recall that $m_1, \ldots, m_n \in \mathbb{R}$ are said to be *M-conditionally rationally independent* iff whenever $\gamma_j \in \mathbb{Z}$, $j = 1, \ldots, n$, are such that $|\gamma_j| \le M$, for $j = 1, \ldots, n$ and

$$\sum_{j=1}^{n} \gamma_j m_j = 0,$$

then $\gamma_j = 0$, for $j = 1, \ldots, n$.

Lemma 1. *Suppose that $d \in \mathbb{N}$ and $m_1, \ldots, m_n \in \mathbb{R}$ are d-conditionally rationally independent. For every multi-index $\gamma \in \mathbb{N}_0^n$, define the function f_γ by*

$$f(x) = \prod_{k=1}^{n} \cot(m_k x)^{\gamma_k}.$$

Then the functions $\{f_\gamma\}_{|\gamma| \le d}$ are linearly independent on every open interval $I \subset \mathbb{R}$ on which these functions are defined.

Proof. We have to prove that whenever

$$\sum_{|\gamma| \le d} a_\gamma \mathrm{i}^{-|\gamma|} f_\gamma = 0 \quad \text{on } I, \tag{11}$$

then

$$a_\gamma = 0 \quad \text{for all } |\gamma| \le d. \tag{12}$$

Since

$$\cot y = \mathrm{i}\frac{1 + e^{-2\mathrm{i}y}}{1 - e^{-2\mathrm{i}y}},$$

we obtain that (11) is equivalent to

$$\sum_{|\gamma| \le d} a_\gamma \prod_{k=1}^{n} \frac{(1 + e^{-2\mathrm{i}m_k x})^{\gamma_k}}{(1 - e^{-2\mathrm{i}m_k x})^{\gamma_k}} = 0 \quad \text{for } x \in I. \tag{13}$$

By analyticity, (13) is equivalent to

$$\sum_{|\gamma| \le d} a_\gamma \prod_{k=1}^{n} (1 + e^{m_k x})^{\gamma_k} \prod_{k=1}^{n} (1 - e^{m_k x})^{d - \gamma_k} = 0 \quad \text{for } x \in \mathbb{C}. \tag{14}$$

Using the notation

$$\binom{\gamma}{\mathbf{k}} = \prod_{j=1}^{n} \binom{\gamma_j}{k_j}$$

for $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ and setting

$$\mathbf{d} = (\underbrace{d, \ldots, d}_{n \text{ times}}),$$

we see that the right hand side of (14) is equal to

$$\sum_{|\gamma| \le d} a_\gamma \sum_{\substack{\mathbf{k},\mathbf{l} \in \mathbb{N}_0^n \\ \mathbf{k} \le \gamma,\, \mathbf{l} \le \mathbf{d} - \gamma}} (-1)^{|\mathbf{l}|} \binom{\gamma}{\mathbf{k}} \binom{\mathbf{d} - \gamma}{\mathbf{l}} e^{\sum_j m_j (k_j + l_j) x}.$$

Thus (14) is equivalent to

$$\sum_{\mathbf{r}\in\mathbb{N}_0^n}\left(\sum_{|\gamma|\le d} a_\gamma \sum_{\substack{\mathbf{k},\mathbf{l}\in\mathbb{N}_0^n\\ \mathbf{k}\le\gamma,\,\mathbf{l}\le\mathbf{d}-\gamma\\ \mathbf{k}+\mathbf{l}=\mathbf{r}}} (-1)^{|\mathbf{l}|}\binom{\gamma}{\mathbf{k}}\binom{\mathbf{d}-\gamma}{\mathbf{l}}\right) e^{(\sum_j m_j r_j)x} = 0 \quad \text{on } \mathbb{C}. \tag{15}$$

Since $m_1, \ldots, m_n$ are d-conditionally rationally independent, it follows that

$$\sum_j m_j r_j \neq \sum_j m_j s_j \quad \text{whenever } \mathbf{r}\neq\mathbf{s},\ |r_j|\le d,\ |s_j|\le d \text{ for } j=1,\ldots,n.$$

The linear independence of the functions $(e^{cx})_{c\in\mathbb{R}}$ now implies that (11) is equivalent to

(16)

$$\sum_{|\gamma|\le d} a_\gamma \sum_{\substack{\mathbf{k},\mathbf{l}\in\mathbb{N}_0^n\\ \mathbf{k}\le\gamma,\,\mathbf{l}\le\mathbf{d}-\gamma\\ \mathbf{k}+\mathbf{l}=\mathbf{r}}} (-1)^{|\mathbf{l}|}\binom{\gamma}{\mathbf{k}}\binom{\mathbf{d}-\gamma}{\mathbf{l}} = 0 \quad \text{whenever } |r_j|\le d \text{ for } j=1,\ldots,n.$$

On the other hand, using similar arguments, we also obtain the following equivalences:

$$a_\gamma = 0 \quad \text{for all } |\gamma|\le d$$

if and only if

$$\sum_{|\gamma|\le d} a_\gamma y^\gamma = 0 \quad \text{for } y\in\mathbb{R}^n$$

if and only if

$$\sum_{|\gamma|\le d} a_\gamma \prod_{k=1}^n \frac{(1+x_k)^{\gamma_k}}{(1-x_k)^{\gamma_k}} = 0 \quad \text{whenever } x\in\mathbb{R}^n \text{ with } x_j\neq 1, \text{ for } j=1,\ldots\ n$$

if and only if

$$\sum_{|\gamma|\le d} a_\gamma \prod_{k=1}^n (1+x_k)^{\gamma_k} \prod_{k=1}^n (1-x_k)^{d-\gamma_k} = 0 \quad \text{for } x\in\mathbb{R}^n$$

if and only if

$$\sum_{\mathbf{r}\in\mathbb{N}_0^n}\left(\sum_{|\gamma|\le d} a_\gamma \sum_{\substack{\mathbf{k},\mathbf{l}\in\mathbb{N}_0^n\\ \mathbf{k}\le\gamma,\,\mathbf{l}\le\mathbf{d}-\gamma\\ \mathbf{k}+\mathbf{l}=\mathbf{r}}} (-1)^{|\mathbf{l}|}\binom{\gamma}{\mathbf{k}}\binom{\mathbf{d}-\gamma}{\mathbf{l}}\right) x^{\mathbf{r}} = 0 \quad \text{for } x\in\mathbb{R}^n$$

if and only if

$$\sum_{|\gamma|\le d} a_\gamma \sum_{\substack{\mathbf{k},\mathbf{l}\in\mathbb{N}_0^n\\ \mathbf{k}\le\gamma,\,\mathbf{l}\le\mathbf{d}-\gamma\\ \mathbf{k}+\mathbf{l}=\mathbf{r}}} (-1)^{|\mathbf{l}|}\binom{\gamma}{\mathbf{k}}\binom{\mathbf{d}-\gamma}{\mathbf{l}} = 0 \quad \text{whenever } |r_j|\le d \text{ for } j=1,\ldots,n.$$

We thus obtain that (11) is equivalent to (16) and (16) is equivalent to (12). This concludes the proof of the lemma. □

Remark. The lemma may not hold if $m_1, \dots, m_n \in \mathbb{R}$ are not d-conditionally rationally independent. In fact, using the formula

$$2 \cot 2x \cot x - (\cot x)^2 + 1 = 0$$

and choosing $n = 2$, $m_1 = 2$, $m_2 = 1$ and $d = 2$, we see that $\{f_\gamma\}_{|\gamma| \le d}$ are linearly dependent on every open interval $I \subset \mathbb{R}$ on which these functions are defined.

Lemma 2. *If $s \in \mathbb{N}$, $p \in \mathbb{N}$ and $c \in \mathbb{N}^s$ are such that $p < |c|$, then there exist vectors $x^1, \dots, x^s \in \mathbb{N}_0^s$ such that:*

1. *$x^1, \dots, x^s$ are linearly independent;*
2. *$x^i \le c$, $i = 1, \dots, s$;*
3. *$|x^i| = p$, $i = 1, \dots, s$.*

Proof. Define

$$C := \{\, x \in \mathbb{R}^s \mid |x| := \sum_{j=1}^{s} x_j = p, \text{ and } 0 \le x \le c \,\}.$$

C is convex, closed and bounded, so C is the convex hull of the set C^* of the extremal points of C.

We claim that for every $x \in C^*$ there is an index $i = i(x)$ such that $x_j \in \{0, c_j\}$ for all j with $j \ne i$. In fact, suppose there are $x \in C^*$ and indices i and j with $i \ne j$, $0 < x_i < c_i$ and $0 < x_j < c_j$. Define y, $z \in \mathbb{R}^s$ in the following way:

$$\begin{aligned} y_k = z_k &= x_k, \quad k \ne i, j, \\ y_i = x_i + \epsilon&, \quad y_j = x_j - \epsilon, \\ z_i = x_i - \epsilon&, \quad z_j = x_j + \epsilon. \end{aligned}$$

We have $|y| = |z| = p = |x|$, and, if ϵ is sufficiently small, y, $z \in C$; but $x = (1/2)(y + z)$, contradicting the fact that $x \in C^*$. This proves the claim. Since $c \in \mathbb{N}_0^s$, the claim implies for every $x \in C^*$ that $x_j \in \mathbb{N}_0$ for all $j \ne i$, where $i = i(x)$. Since $|x| = p \in \mathbb{N}$, it also follows that $x_i \in \mathbb{N}_0$. So we conclude that, if $x \in C^*$, then $x \in \mathbb{N}_0^s$. If we show that $\dim \operatorname{span} C^* = s$, then the lemma will be proved. Consider the hyperplane

$$H := \{\, x \in \mathbb{R}^s \mid |x| = \sum_{j=1}^{s} x_i = 0 \,\}.$$

Clearly $\dim H = s - 1$. Let $\lambda := p/|c|$ (notice that $0 < \lambda < 1$), and set $a := \lambda c$; then $|a| = p$, and, since $0 < a_i < c_i$ for all i, it follows that $a \le c$, so that $a \in C$. Now write C^* as $C^* = \{x_\alpha\}_{\alpha \in A}$. Then there are $\{\lambda_\alpha\}_{\alpha \in A}$, $\lambda_\alpha \ge 0$ for all α, $\lambda_\alpha = 0$ for almost all α, $\sum_\alpha \lambda_\alpha = 1$, such that $a = \sum_\alpha \lambda_\alpha x_\alpha$. Let $y^1, ..., y^{s-1} \in H$ be linearly independent. Choose a sufficiently small $\delta > 0$, so that $0 < \delta y_i^j + a_i < c_i$, for all $j = 1, \dots, s-1$, $i = 1, \dots, s$. Then $\delta y^j + a \in C$ for $j = 1, \dots, s-1$. For every $j = 1, \dots, s-1$ there are $\{\lambda_\alpha^j\}_{\alpha \in A}$ such that $\lambda_\alpha^j \ge 0$ for all α, $\lambda_\alpha^j = 0$ for almost all α, $\sum_\alpha \lambda_\alpha^j = 1$ and $y^j = \sum_\alpha \lambda_\alpha^j x_\alpha$. It follows that

$$y^j = \sum_\alpha \frac{(\lambda_\alpha^j - \lambda_\alpha)}{\delta} x_\alpha,$$

so that $y^j \in \operatorname{span} C^*$ for $j = 1, \ldots, s-1$. If $a \in \operatorname{span}\{ y_1, ..., y_{s-1} \}$, then $a \in H$, and then $|a| = 0$, but this is impossible since $|a| = p > 0$. Thus we obtain

$$\operatorname{span} C^* = \operatorname{span}\{ a, y_1, ..., y_{s-1} \} = \mathbb{R}^s.$$

This concludes the proof. □

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