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COPIES OF c_0 AND ℓ_∞ IN TOPOLOGICAL RIESZ SPACES

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Abstract. The paper is concerned with order-topological characterizations of topological Riesz spaces, in particular spaces of measurable functions, not containing Riesz isomorphic or linearly homeomorphic copies of c_0 or ℓ_∞.

0. Introduction

The research done in this paper concentrates around generalizations of two classical theorems on Banach lattices, both of which go back to the end of the sixties. The first is due to Lozanovskii and Meyer-Nieberg, and the second to Lozanovskii (see [AB2]).

(I) *A Banach lattice X is a KB-space (or is Lebesgue and Levi) iff X contains no lattice copy of c_0 iff X contains no copy of c_0.*

(II) *A σ-Dedekind complete Banach lattice X has order continuous norm (or is Lebesgue) iff X contains no lattice copy of ℓ_∞ iff X contains no copy of ℓ_∞.*

Here and in what follows 'copy of c_0 (or ℓ_∞)' means 'isomorphic copy of c_0 (or ℓ_∞)'.

Extensions of the 'lattice copy part' of both theorems to general (locally solid) topological Riesz spaces were given in [AB1, Thm. 10.7] for (II), and [Wn] for (I). They are relatively easy and as expected: both results do extend to their natural limits. This material, with improvements, is presented in Section 2 below.

Surprisingly enough, similar extensions of the 'isomorphic copy part' of either (I) or (II) seem to be unknown. We present such extensions in this paper. Clearly, the essence of what one expects is that for a topological Riesz space X,

(i) if X is Lebesgue Levi, then X contains no copy of c_0, and
(ii) if X is σ-Dedekind complete and Lebesgue, then X contains no copy of ℓ_∞,

possibly under some completeness type assumptions on X. In view of a representation theorem from [L3], the crucial case is that of spaces of measurable functions over locally order continuous *submeasures* μ. The results we got are not that general; however, they are rather complete and fully satisfactory for spaces of measurable functions over locally finite *measures* λ. On the other hand, they are strong

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enough to imply, via the representation theorem just mentioned, results of type (i) and (ii) for general topological Riesz spaces with separating continuous dual.

Our treatment of (i) and (ii) for spaces of λ-measurable functions depends heavily on an Orlicz-Pettis type theorem for such spaces proved in this paper (Theorem 4.1), and the Orlicz Theorem asserting that *every perfectly bounded series in* $L_0(\lambda)$ *is unconditionally Cauchy.* Now, a slightly weaker version of our Orlicz-Pettis theorem (with 'Lebesgue' replacing 'σ-Lebesgue'), is known to hold also for Lebesgue spaces of measurable functions over locally order continuous submeasures (see [DL1] and [DL2]). However, the validity of the Orlicz Theorem in this case is an open question, which is the basic obstacle in establishing results of type (i) or (ii) for general topological Riesz spaces. More precisely, as we point out at the end of the paper (Section 6), the latter is equivalent to proving the following: *For each order continuous submeasure* μ*, the space* $L_0(\mu)$ *of* μ*-measurable functions contains no copy of* c_0 *(or* ℓ_∞*).* This, however, seems to be closely related to the famous Maharam control measure problem.

The plan of the rest of the paper, and some sample results, are as follows. We first review, in Section 3, the necessary facts concerning spaces of λ-measurable scalar functions in connection with topological Riesz spaces. Then, in Section 4, we prove our Orlicz-Pettis theorem, in both the 'countably additive' and 'exhaustive' forms, for spaces of λ-measurable functions having the σ-Lebesgue property. This result, besides applications given in this paper, should also be of independent interest.

The principal results of this paper are presented in Section 5. Combining our Orlicz-Pettis theorem and the Orlicz Theorem with a purely topological characterization of the Levi property, we show that *a Lebesgue Levi topological Riesz space of* λ*-measurable fuctions contains no copy of* c_0 (see Theorem 5.5). Next, with similar techniques (in fact, even more directly), we prove that *a* σ*-Lebesgue topological Riesz space of* λ*-measurable functions contains no copy of* ℓ_∞ (see Proposition 5.7).

Then, via the representation theorem from [L3], the above results are translated back into the abstract setting, yielding the following: *Let* X *be a topological Riesz space with separating continuous dual. If* X *is Lebesgue Levi, then* X *contains no copy of* c_0 (see Corollary 5.6). *If* X *is complete,* σ*-Dedekind complete, and Lebesgue, then* X *contains no copy of* ℓ_∞ (see Corollary 5.11). In particular, the Lozanovskii Theorem (II) is recaptured with a proof which we believe to be more natural than any proof known previously.

1. Some preliminary notions and facts. Throughout, we use the abbreviations TVS and TRS for Hausdorff *topological vector space* and Hausdorff *locally solid topological Riesz space*, respectively. Our terminology concerning topological (locally solid) Riesz spaces is essentially that of [AB1] and [AB2] (cf. also [F]). We point out that the adjectives 'bounded', 'convergent' and 'complete', when not accompanied by 'order' or 'Dedekind' or 'universally', are always used in their TVS-meaning.

We let

$$\mathcal{F} = \mathcal{F}(\mathbb{N}) \qquad \text{and} \qquad \mathcal{P} = \mathcal{P}(\mathbb{N})$$

stand for the family of all finite subsets of $\mathbb{N}$, and the power set of $\mathbb{N}$, respectively.

A series in a TVS is called *subseries convergent* if each of its subseries is convergent; *bounded* if the sequence of its partial sums is bounded; *perfectly bounded* if the set of all its finite sums is bounded.

As is well known, a subseries convergent series is unconditionally convergent, and the converse holds in sequentially complete spaces. Also, the range of a subseries convergent series, i.e., the set of sums of all its subseries, is compact.

We next recall a few notions and facts concerning vector-valued additive set functions. Let X be a TVS.

If $\mathcal{R}$ is a ring of sets, then a finitely additive measure $\mathbf{m} : \mathcal{R} \to X$ is said to be *exhaustive* (or strongly bounded) if $\mathbf{m}(A_n) \to 0$ for each disjoint sequence (A_n) in $\mathcal{R}$; equivalently, if for each increasing sequence (A_n) in $\mathcal{R}$, the sequence $\big(\mathbf{m}(A_n)\big)$ is Cauchy.

To every series $\sum_n x_n$ in X, we associate a finitely additive measure $\mathbf{m} : \mathcal{F} \to X$ by setting $\mathbf{m}(N) = \sum_{n\in N} x_n$ for $N \in \mathcal{F}$. Clearly, the series is perfectly bounded (resp. unconditionally Cauchy) iff the associated measure is bounded (resp. exhaustive). If the series $\sum_n x_n$ is subseries convergent, then by the associated measure we mean the countably additive measure $\mathbf{m} : \mathcal{P} \to X$ defined using the same equality as above but with $N \in \mathcal{P}$. Obviously, every countably additive measure $\mathbf{m} : \mathcal{P} \to X$ arises in this way.

Given a series $\sum_n x_n$ in X, the following conditions are equivalent: 1) the operator $(a_n) \to \sum_n a_n x_n$, defined on the (dense) subspace of c_0 consisting of finitely nonzero sequences, is continuous; 2) the set of finite sums of the form $\sum_n a_n x_n$, where $|a_n| \leqslant 1$, is bounded; 3) the set $\operatorname{conv}\{\sum_{n\in M} x_n : M \in \mathcal{F}\}$ is bounded. If these conditions hold, we say (as in [L1]) that the series is *convexly bounded*.

Similarly, given a finitely additive measure $\mathbf{m} : \mathcal{P} \to X$, the following conditions are equivalent: 1) the operator $\sum_k a_k 1_{N_k} \to \sum_k a_k \mathbf{m}(N_k)$, defined on the (dense) subspace of ℓ_∞ consisting of 'simple' sequences, is continuous; 2) the set of finite sums of the form $\sum_k a_k \mathbf{m}(N_k)$, where $|a_k| \leqslant 1$ and the N_k's are pairwise disjoint, is bounded; 3) the set $\operatorname{conv} \mathbf{m}(\mathcal{P})$ is bounded. If these conditions hold, we say that the measure $\mathbf{m}$ is *convexly bounded*.

For a continuous linear operator $J : \ell_\infty \to X$ (resp. $c_0 \to X$), the associated finitely additive measure $\mathbf{m} : \mathcal{P} \to X$ (resp. $\mathcal{F} \to X$) is defined by $\mathbf{m}(N) = J(1_N)$. Clearly, $\mathbf{m}$ is convexly bounded, and $J(a) = \int_{\mathbb{N}} a\, d\mathbf{m}$ for every $a \in \ell_\infty$ (resp. c_0).

For the result stated below, see [D1, Lemma].

1.1 Proposition. *Let X be a metrizable* TVS. *Then a finitely additive measure $\mathbf{m} : \mathcal{P} \to X$ is exhaustive iff every disjoint sequence (N_k) in $\mathcal{P}$ has a subsequence (M_k) such that $\mathbf{m}$ is countably additive on the σ-algebra in $\mathbb{N}$ generated by (M_k).*

For Banach spaces X, the result below is due to Rosenthal [R]; the c_0-case with X an F-space is due to Kalton [K]; for the general case see [D3].

1.2 Theorem. *If $J : c_0 \to X$ (resp. $J : \ell_\infty \to X$) is a continuous linear operator and $J(e_n) \not\to 0$, then there exists an infinite subset M of $\mathbb{N}$ such that $J|c_0(M)$ (resp. $J|\ell_\infty(M)$) is an isomorphism.*

Following Orlicz (see e.g. [MO]), we shall say that a TVS X has *Property* (O) if every perfectly bounded series in X is (subseries) convergent. As usual, X is said to *contain a copy* of c_0 if there exists a linear homeomorphism from the Banach space c_0 onto a subspace of X.

Evidently, if a TVS X has Property (O), then it contains no copy of c_0. For the converse to hold, some additional assumptions are needed. The result below is a slight generalization of a particular case of [L1, Thm. 1] (see also [Di]); it should be viewed as a sort of model for the similar c_0-results below.

1.3 Proposition. *Suppose a sequentially complete* TVS X *has the property that every perfectly bounded series in* X *is convexly bounded. Then* X *has Property* (O) *iff it contains no copy of* c_0.

Proof. We only have to verify the 'if' part. Suppose there is a nonconvergent (or non-Cauchy) perfectly bounded series $\sum_n x_n$ in X. By passing to suitable 'blocks' of this series, we may assume that $x_n \not\to 0$. Since the series is convexly bounded and X is sequentially complete, there exists a continuous linear operator $J : c_0 \to X$ such that $J(e_n) = x_n$ for all n. Applying Theorem 1.2, we conclude that X contains a copy of c_0. □

A sequence (or series) in a Riesz space X is said to be *disjoint* if its terms are pairwise disjoint, and a measure $\mathbf{m} : \mathcal{R} \to X$ is said to *preserve disjointness* if $\mathbf{m}(A) \wedge \mathbf{m}(B) = 0$ whenever $A \cap B = \emptyset$.

For the sake of clarity and ease of reference, we now recall a few known equivalences between some order-topological properties and their 'disjoint' counterparts. In Theorem 1.4, part (a) can be proved by generalizing the argument used in [KA, Chap. X, Sec. 4, Thm. 3], and part (b) has been obtained in [BL] (see also [Wn]). Theorem 1.5 is part of Theorem 10.1 in [AB1].

1.4 Theorem. *Let* X *be a* σ*-Dedekind complete* TRS.

(a) X *is* σ*-Lebesgue iff* X *is* disjointly σ-Lebesgue*, that is, every order convergent disjoint positive series in* X *is convergent.*
(b) X *is* σ*-Levi iff* X *is* disjointly σ-Levi*, that is, every bounded disjoint positive series in* X *is order convergent.*

By definition, a TRS X is *pre-Lebesgue* if every positive increasing and order bounded sequence in X is Cauchy.

1.5 Theorem. *A* TRS X *is pre-Lebesgue iff it is* disjointly pre-Lebesgue*, that is, every order bounded disjoint positive sequence in* X *converges to zero or, equivalently, every order bounded disjoint positive series in* X *is Cauchy.*

Remark. The definition and the above characterizations of the pre-Lebesgue property can be expressed as follows: *A* TRS X *is pre-Lebesgue iff every order bounded finitely additive measure* $\mathbf{m} : \mathcal{F} \to X$ *is exhaustive iff every order bounded and disjointness preserving finitely additive measure* $\mathbf{m} : \mathcal{F} \to X$ *is exhaustive.*

The proof of the following result has been inspired by the proofs of Lemma 12.5 and Theorem 13.2 in [AB1].

1.6 Proposition. *Let* $X = (X, \tau)$ *be a* σ*-Fatou* TRS. *If* (x_n) *is a Cauchy sequence in* X *such that* $\inf_{n\in N} |x_n| = 0$ *for every infinite set* $N \subset \mathbb{N}$*, then* $x_n \to 0$ (τ).

Proof. We may assume that all $x_n \geqslant 0$.

Take any σ-Fatou τ-neighborhood V_0 of zero in X, and next pick a sequence (V_n) of σ-Fatou τ-neighborhoods of zero in X such that $V_{n+1} + V_{n+1} \subset V_n$ for $n = 0, 1, \ldots$. Select a subsequence (y_n) of (x_n) so that $y_{n+1} - y_n \in V_{n+1}$ for all n. We claim that $y_n \in V_n$ for every n. In fact, if $z_r = \inf_{n\leqslant p\leqslant r} y_p$ for $r > n$, then

$$\begin{aligned} 0 \leqslant y_n - z_r &\leqslant \sup_{n\leqslant p\leqslant r} |y_n - y_p| \\ &\leqslant \sum_{p=n}^{r-1} |y_{p+1} - y_p| \in V_{n+1} + \cdots + V_r \subset V_n \end{aligned}$$

so that $y_n - z_r \in V_n$. Moreover, by the assumption on (x_n), we have $\inf_{p\geqslant n} y_p = 0$, whence $y_n - z_r \uparrow y_n$. Since V_n is σ-order closed, we conclude that $y_n \in V_n$.

Now, since the sequence (x_n) is τ-Cauchy, there is n_0 such that $x_n - x_m \in V_1$ for $m, n \geqslant n_0$. It follows that $x_n = (x_n - y_n) + y_n \in V_1 + V_n \subset V_0$ for all $n \geqslant n_0$. Thus $x_n \to 0$ (τ). □

We will need the following consequence of Proposition 1.6 at one crucial point in the proof of Theorem 4.1 below.

1.7 Corollary. *Let $X = (X, \tau)$ be a σ-Fatou* TRS*, and let ρ be an arbitrary Hausdorff locally solid topology on X. If (x_n) is a τ-Cauchy sequence in X and $x_n \to x$ (ρ) for some $x \in X$, then also $x_n \to x$ (τ).*

Proof. Apply the preceding proposition to the sequence $\big(|x_n - x|\big)$. □

2. Positive and lattice copies of c_0 and ℓ_∞

We found it convenient to formulate our results using 'positive' and 'disjoint' versions of Property (O) of Orlicz. We shall say that a TRS X has the *positive Property* (O) if every bounded positive series in X is convergent; the *disjoint Property* (O) if every bounded disjoint series in X is convergent. Clearly, the latter condition is satisfied if it is so for the bounded *positive* disjoint series in X.

Note that if a positive or disjoint series is bounded, then it is perfectly bounded. In consequence, 'convergent' can be replaced in the above conditions by 'subseries convergent'.

2.1 Proposition. *The following are equivalent for a* TRS *X.*

(a) *X is σ-Lebesgue σ-Levi.*
(b) *Every bounded increasing positive sequence in X is convergent.*
(c) *X has the positive Property (O).*
(d) *X is σ-Dedekind complete and has the disjoint Property (O).*

Proof. The equivalence of (a) and (b) is easily verified, and (c) is simply a reformulation of (b). Finally, the equivalence of (a) and (d) follows from Theorem 1.4. □

Recall that a TRS X is said to have the *Monotone Completeness Property*, MCP (resp. σ-MCP), if every positive increasing Cauchy net (resp. sequence) is convergent in X. It is easily seen and worth noting that every positive increasing Cauchy net is bounded.

2.2 Proposition. *The following are equivalent for a* TRS *X.*

(a) *X is Lebesgue Levi.*
(b) *Every bounded increasing positive net in X is convergent.*
(c) *X has the MCP and is σ-Lebesgue σ-Levi.*

Proof. The equivalence of (a) and (b) is easy to verify, and (b) implies (c) is trivial.

(c) $\Longrightarrow$ (b): Suppose (b) fails. Then, taking the MCP into account, there exists a bounded increasing positive net in X which is not Cauchy. By a standard argument (cf. [AB1], Proof of Thm. 10.1), it follows that there is also a bounded increasing positive *sequence* in X which is not Cauchy. Thus X is not σ-Lebesgue σ-Levi. □

We shall say that a TRS X *contains a positive copy* of c_0 if there is a positive linear homeomorphism from the Banach lattice c_0 onto a subspace of X, and that

X *contains a lattice copy* of c_0 if there is a homeomorphic Riesz isomorphism from c_0 onto a sublattice of X. Similar terminology will be used with ℓ_∞ replacing c_0.

Trivially, if X has the positive Property (O), then it contains no positive copy of c_0; if X has the disjoint Property (O), then it contains no lattice copy of c_0. It is a remarkable fact that these implications are reversible. The essential part of the two theorems below is the equivalence of conditions (b) and (d). Results of this type have been known in Banach lattices for a long time, see e.g. [AB2, Sec. 14], and for general TRS's a little weaker results were obtained by Wnuk [Wn]. Note that in contrast to [Wn], in our treatment the Lebesgue-Levi case follows directly from its σ-counterpart.

2.3 Lemma. *Let (x_n) be a sequence in a* TRS *X such that the series $\sum_n |x_n|$ is bounded. Then the series $\sum_n x_n$ is convexly bounded. Moreover, if X has the σ-MCP, then the series $\sum_n a_n x_n$ converges in X for every $a = (a_n) \in c_0$, and the linear operator $J : c_0 \to X$ defined by $J(a) = \sum_n a_n x_n$ is continuous.*

Proof. (cf. [Wn], Proof of Lemma 1). All the assertions can be easily deduced from the inequalities

$$\begin{aligned} \left|\sum_{n\in M} a_n x_n\right| &\leqslant \sum_{n\in M}(a_n x_n)^+ + \sum_{n\in M}(a_n x_n)^- \\ &= \sum_{n\in M}|a_n x_n| \leqslant \sup_{n\in M}|a_n| \sum_{n\in M}|x_n|, \end{aligned}$$

where $a = (a_n) \in c_0$ and $M \in \mathcal{F}$. □

2.4 Theorem. *The following are equivalent for a* TRS *X.*

(a) *X has the σ-MCP and contains no positive copy of c_0.*
(b) *X has the σ-MCP and contains no lattice copy of c_0.*
(c) *X has the σ-MCP and the disjoint Property (O).*
(d) *X has the positive Property (O).*

Proof. The implications (d) $\Longrightarrow$ (a) and (a) $\Longrightarrow$ (b) are obvious.

(b) $\Longrightarrow$ (c) (cf. [Wn], Proof of Thm. 1): If (c) fails, then there is a bounded disjoint positive series $\sum_n x_n$ in X which is not Cauchy. By passing to suitable 'blocks' of this series, we may assume that $x_n \notin U$ for all n and some solid neighborhood U of zero in X. By Lemma 2.3, we may define a continuous linear operator $J : c_0 \to X$ by $J(a) = \sum_n a_n x_n$ for $a = (a_n)$, and it is clear that J preserves absolute value. Moreover, as is easily seen, $J(a) \notin U$ if $\|a\|_\infty = 1$. Thus J is a homeomorphic Riesz embedding of c_0 into X, contradicting (b).

(c) $\Longrightarrow$ (d): In view of Theorem 1.5, the disjoint Property (O) implies that X is pre-Lebesgue. Next, the latter along with the σ-MCP implies that X is σ-Dedekind complete. Now, appealing to Proposition 2.1, we conclude that X has the positive Property (O). □

Remark. A direct proof that (a) implies (d) in Theorem 2.4 is worth noting: Assuming (d) fails, we argue as in the proof that (b) implies (c) above, and arrive at a positive operator $J : c_0 \to X$ with $Je_n = x_n \not\to 0$. By Theorem 1.2, there exists an infinite subset M of $\mathbb{N}$ such that the restriction of J to the sublattice $c_0(M) \cong c_0$ is a positive isomorphic embedding, contradicting (a).

By combining Proposition 2.2 and Theorem 2.4, we have the following.

2.5 Theorem. *The following are equivalent for a* TRS X.

(a) *X has the MCP and contains no positive copy of c_0.*
(b) *X has the MCP and contains no lattice copy of c_0.*
(c) *X has the MCP and the disjoint Property (O).*
(d) *X is Lebesgue Levi.*

The next two results concern the non-existence of positive or lattice copies of ℓ_∞, and are a slight refinement of Theorem 10.7 in [AB1]; see also the historical comments therein, compare also [L1, Cor. A to Thm. 1].

Recall that a Riesz space X is said to be *disjointly σ-Dedekind complete* if every order bounded disjoint sequence has a supremum in X.

2.6 Proposition. *The following are equivalent for a disjointly σ-Dedekind complete* TRS X.

(a) *X is pre-Lebesgue.*
(b) *Every finitely additive measure $\mathbf{m} : \mathcal{P} \to X_+$ is exhaustive.*
(c) *Every disjointness preserving finitely additive measure $\mathbf{m} : \mathcal{P} \to X_+$ is exhaustive.*

Proof. For (a) $\Longrightarrow$ (b), see the Remark after Theorem 1.5; (b) $\Longrightarrow$ (c) is trivial.

(c) $\Longrightarrow$ (a): If (a) fails then, in view of Theorem 1.5, there exists an order bounded disjoint sequence (x_n) in X_+ such that $x_n \not\to 0$. Since X is disjointly σ-Dedekind complete, we may define a finitely additive measure $\mathbf{m} : \mathcal{P} \to X_+$ by

$$(*) \qquad \mathbf{m}(N) = \text{(o)-}\sum_{n\in N} x_n = \sup_{n\in N} x_n.$$

Then $\mathbf{m}$ is as required in (d), but is not exhaustive. □

2.7 Theorem. *The following are equivalent for a disjointly σ-Dedekind complete* TRS X.

(a) *X is pre-Lebesgue.*
(b) *X contains no positive copy of ℓ_∞.*
(c) *X contains no lattice copy of ℓ_∞.*

Proof. (a) $\Longrightarrow$ (b) by Proposition 2.6, and (b) $\Longrightarrow$ (c) is trivial.

(c) $\Longrightarrow$ (a): The argument below is the same as in the proof of [AB1, Thm. 10.7], with some simplifications. Denying (a) and using Theorem 1.5, we find an order bounded disjoint sequence (x_n) in X_+ and a solid neighborhood V of zero in X such that $x_n \notin 2V$ for every n. Since X is disjointly σ-Dedekind complete, we may define a linear operator $J : \ell_\infty \to X$ by

$$J(a) = \text{(o)-}\sum_n a_n^+ x_n - \text{(o)-}\sum_n a_n^- x_n \qquad \text{for } a = (a_n) \in \ell_\infty.$$

If $\|a\|_\infty \leqslant 1$, then $|J(a)| \leqslant \sup_n x_n$; hence J is continuous. If $\|a\|_\infty = 1$, then $J(a) \notin V$; hence also J^{-1} is continuous. Finally, J is clearly a Riesz isomorphism onto its range. (Note that $J(a) = \int_{\mathbb{N}} a\, d\mathbf{m}$, where $\mathbf{m}$ is the measure defined by formula $(*)$.) □

Remark. Let $\mathbf{m} : \mathcal{P} \to X_+$ be the measure associated with a lattice embedding $J : \ell_\infty \to X$. Although such $\mathbf{m}$'s preserve disjointness, they need not be given by formula $(*)$ with $x_n = \mathbf{m}(\{n\})$. To see this, let $X = \ell_\infty(\mathbb{N} \cup \{0\})$, and let $\varphi : \mathcal{P} \to \{0, 1\}$ be a finitely additive measure vanishing on the singletons and with

$\varphi(\mathbb{N}) = 1$. Then the measure $\mathbf{m} : \mathcal{P} \to X_+$ defined by $\mathbf{m}(N) = \varphi(N)e_0 + 1_N$ is disjointness preserving and so represents a lattice embedding of ℓ_∞ into X. Since $x_n = e_n$ for $n \in \mathbb{N}$, we have $\mathbf{m}(\mathbb{N}) \neq \sup_{n\in\mathbb{N}} x_n = 1_{\mathbb{N}}$.

Recall that a Riesz space X is *relatively uniformly complete* if, for every $u \in X_+$, the ideal A_u generated by u is a Banach space under the norm whose closed unit ball is equal to the order interval $[-u, u]$. (This definition is equivalent to that given in [AB1, p. 4].)

The last result in this section is a complement to Theorem 2.7.

2.8 Proposition. *The following conditions are equivalent for a relatively uniformly complete* TRS X.

(a) *X contains no positive copy of ℓ_∞.*
(b) *Every finitely additive measure $\mathbf{m} : \mathcal{P} \to X_+$ is exhaustive.*

Proof. Only (a) implies (b) has to be shown. Suppose a finitely additive measure $\mathbf{m} : \mathcal{P} \to X_+$ is not exhaustive. Then without loss of generality it can be assumed that $\mathbf{m}(\{n\}) \not\to 0$. Denote $u = \mathbf{m}(\mathbb{N})$. By assumption, the ideal A_u is a Banach space under the norm defined as the Minkowski functional of the interval $[-u, u]$. Now, treating $\mathbf{m}$ as a measure with values in A_u, the integral $J(a) = \int_{\mathbb{N}} a \, d\mathbf{m} \in A_u$ exists for every $a \in \ell_\infty$. Since the image by J of the unit ball in ℓ_∞ is contained in $[-u, u]$, a bounded subset of X, the operator J is continuous as a mapping from ℓ_∞ into X. Finally, since $J(e_n) = \mathbf{m}(\{n\}) \not\to 0$, by Theorem 1.2 there exists an infinite subset M of $\mathbb{N}$ such that J is an isomorphic (and positive) embedding of $\ell_\infty(M) \cong \ell_\infty$ into X. Thus X contains a positive copy of ℓ_∞, contradicting (a). $\square$

Remark. By a result of Geiler and Veksler [VG, Thm. 3], a Riesz space is σ-Dedekind complete iff it is Archimedean, relatively uniformly complete and disjointly σ-Dedekind complete.

3. Spaces of scalar measurable functions

Throughout the rest of the paper,

$$(S, \Sigma, \lambda) \quad \text{is a locally finite measure space.}$$

Recall that *locally finite* (or semi-finite) means that every $A \in \Sigma$ with $\lambda(A) > 0$ contains a $B \in \Sigma$ such that $0 < \lambda(B) < \infty$. We denote

$$\Sigma_{\mathrm{f}}(\lambda) = \text{all sets in } \Sigma \text{ of finite } \lambda \text{ measure,}$$

and

$$\Sigma_{\mathrm{s}}(\lambda) = \text{all sets in } \Sigma \text{ of } \sigma\text{-finite } \lambda \text{ measure.}$$

Clearly, $\Sigma_{\mathrm{f}}(\lambda)$ is an ideal, and $\Sigma_{\mathrm{s}}(\lambda)$ is a σ-ideal in Σ; in particular, both are directed upward by inclusion.

We denote by $L_0(\lambda) = L_0(S, \Sigma, \lambda)$ the Riesz space (or vector lattice) of all (λ-equivalence classes of) measurable scalar functions on S, often referred to as λ-measurable functions. We equip $L_0(\lambda)$ with the locally solid vector topology τ_λ

of convergence in measure λ on all sets $A \in \Sigma_{\mathrm{f}}(\lambda)$. A base of solid neighborhoods at zero for τ_λ consists of the sets

$$U(A,\varepsilon) = \big\{f \in L_0(\lambda) \,:\, \lambda\{s \in A \,:\, |f(s)| \geqslant \varepsilon\} < \varepsilon\big\}, \quad \text{where } A \in \Sigma_{\mathrm{f}}(\lambda),\ \varepsilon > 0.$$

Of course, $f_n \to f\ (\tau_\lambda)$ iff $\lim_n \lambda(\{s \in A \,:\, |f(s) - f_n(s)| \geqslant \varepsilon\}) = 0$, $\forall\, A \in \Sigma_{\mathrm{f}}(\lambda)$.

3.1 Proposition. *$L_0(\lambda)$ is a Hausdorff σ-universally complete* TRS *having the positive Property (O). It is metrizable iff λ is σ-finite.*

By a TRS *of λ-measurable functions* we shall mean a solid subspace X of $L_0(\lambda)$ equipped with a Hausdorff locally solid topology. For such X and each $A \in \Sigma$, the band projection P_A in X is defined by $P_A(f) = f1_A$, and X_A stands for the range of P_A; thus

$$X_A = P_A(X) = \{f \in X \,:\, f = f1_A\}.$$

Although, in general, we do not assume the inclusion $X \subset L_0(\lambda)$ to be continuous, it is often so in practice; the case of importance for us is indicated in Proposition 3.4 below.

3.2 Proposition. *If the measure λ is σ-finite, then $L_0(\lambda)$ has the countable sup property. In consequence, if a* TRS *X of λ-measurable functions is σ-Lebesgue, or σ-Levi, or σ-Fatou, then it is Lebesgue, or Levi, or Fatou, respectively.*

In a few instances, we will need measures having additional properties. We declare that a (locally finite) measure λ is *of type (C)* (resp. (SC)) if the space $L_0(\lambda)$ is complete (resp. sequentially complete). For the following characterization of measures of type (C), also called *Maharam measures*, see [F].

3.3 Proposition. *A measure λ is of type (C) iff its measure algebra $\Sigma/\mathcal{N}(\lambda)$ is Dedekind complete iff $L_0(\lambda)$ is a universally complete Lebesgue Levi* TRS.

3.4 Proposition. *If X is a σ-Fatou* TRS *of λ-measurable functions, then X is continuously included in $L_0(\lambda)$.*

Proof. This can be obtained by applying [AB1, Thm. 24.3] or [L3, Cor. 12] to the bands X_A, where $A \in \Sigma_{\mathrm{f}}(\lambda)$. □

The next result is a special case of the Representation Theorem 2.7 in [L3].

3.5 Theorem. *If a* TRS *X is Lebesgue (and Dedekind complete), and has separating continuous dual, then there exists a measure space (S,Σ,λ) of type (C) such that X is continuously included as an order dense (and solid) sublattice in $L_0(\lambda)$.*

We will also need a (rather well known) topological consequence of the σ-Levi property stated in Proposition 3.6 (a) below.

We shall say that a TRS X of λ-measurable functions is *boundedly closed* (resp. *boundedly sequentially closed*) *in* $L_0(\lambda)$ if, for every bounded subset of X, its closure (resp. sequential closure) in $L_0(\lambda)$ is a subset of X. Note that a band X_A of X, where $A \in \Sigma_{\mathrm{s}}(\lambda)$, is boundedly closed iff it is boundedly sequentially closed.

3.6 Proposition. *Let $X = (X,\tau)$ be a* TRS *of λ-measurable functions.*

(a) *If X is σ-Levi, then every band X_A, where $A \in \Sigma_{\mathrm{s}}(\lambda)$, is boundedly closed in $L_0(\lambda)$.*

(b) *If $X \subset L_0(\lambda)$ continuously and is boundedly sequentially closed in $L_0(\lambda)$, then X is σ-Levi.*
(c) *If X is Levi, then X is boundedly closed in $L_0(\lambda)$.*
(d) *If λ is of type (C), $X \subset L_0(\lambda)$ continuously, and X is boundedly closed in $L_0(\lambda)$, then X is Levi.*

Proof. (a): Let $A \in \Sigma_s(\lambda)$. We have to show that if (f_n) is a bounded sequence in X_A converging in $L_0(\lambda)$ to some f, then $f \in X$. By passing to a subsequence, we may assume that $f_n \to f$ λ-a.e.

We may also assume that the support of X_A is equal to A. By combining this with the Egoroff theorem, we find a Σ-partition $\{A_k : k \in \mathbb{N}\}$ of A such that, for every k: 1) $1_{A_k} \in X$, and 2) $\lim_n f_n 1_{A_k} = f 1_{A_k} =: g_k$ uniformly. From 2) it follows that $|f_n 1_{A_k} - g_k| \leqslant 1_{A_k}$ for large k whence, in view of 1) and since X is solid, $g_k \in X_{A_k}$. Thus (g_k) is a disjoint sequence in X_A. Moreover, 1) implies that, on each of the bands X_{A_k}, the topology τ is weaker than the topology of uniform convergence. In consequence, we infer from 2) that $\tau\text{-}\lim_n f_n 1_{A_k} = g_k$ for every k. From this it is easily seen that the sequence $h_n = \sum_{k=1}^n g_k$ is bounded in X and, clearly, $h_n \to f$ in $L_0(\lambda)$. Now, the sequence $(|h_n|)$ is bounded in X and $|h_n| \uparrow |f|$. Since X_A is σ-Levi, we conclude that $|f|$, whence also f, is in X_A.

(b): Let (f_n) be a bounded increasing positive sequence in X. Then it is bounded in $L_0(\lambda)$ and, as $L_0(\lambda)$ is σ-Lebesgue σ-Levi, converges in $L_0(\lambda)$ to its supremum f therein. By the assumption on X, we have $f \in X$.

(c): Let (f_i) be a bounded net in X converging in $L_0(\lambda)$ to f. Then, by part (a), $f 1_A \in X$ for every $A \in \Sigma_s(\lambda)$. The net $(|f| 1_A)$ in X is increasing and bounded. (The latter is 'automatic' because $\Sigma_s(\lambda)$ is a σ-ideal in Σ.) By the Levi property, it has a supremum g in X. Clearly, $|f| 1_A = g 1_A$ for every $A \in \Sigma_s(\lambda)$. In consequence, $|f| = g \in X$, whence also $f \in X$.

(d): Let (f_i) be a bounded increasing positive net in X. Then it is also bounded in $L_0(\lambda)$. Since λ is of type (C), $L_0(\lambda)$ is Levi, hence the net (f_i) has a supremum f in $L_0(\lambda)$. Then $f_i \to f$ (τ_λ), and as X is boundedly closed in $L_0(\lambda)$, we get that $f \in X$. Clearly, $f = \sup_i f_i$ in X. □

4. An Orlicz-Pettis Theorem

We establish here the following.

4.1 Theorem. *Let $X = (X, \tau)$ be a σ-Lebesgue* TRS *of λ-measurable functions, and $\mathbf{m} : \mathcal{P} \to X$ a finitely additive measure.*

(a) *If $\mathbf{m}$ is τ_λ-countably additive, then it is τ-countably additive.*
(b) *If $\mathbf{m}$ is τ_λ-exhaustive, then it is τ-exhaustive.*

The proof of this result will require some preparations.

Given a finitely additive measure $\mathbf{m} : \mathcal{R} \to X \subset L_0(\lambda)$, we define $\Sigma_{ca}(\mathbf{m})$ and $\Sigma_{exh}(\mathbf{m})$ to be the classes of sets $A \in \Sigma$ such that the measure

$$\mathbf{m}_A = P_A \circ \mathbf{m} : \mathcal{R} \to X$$

is countably additive or exhaustive, respectively.

4.2 Lemma. *Let X be a* TRS *of λ-measurable functions, $\mathcal{R}$ a σ-ring of sets, and $\mathbf{m} : \mathcal{R} \to X$ a finitely additive measure. If X is σ-Lebesgue, then $\Sigma_{ca}(\mathbf{m})$ and $\Sigma_{exh}(\mathbf{m})$ are σ-ideals in Σ.*

Proof. Clearly, both $\Sigma_{\mathrm{ca}}(\mathbf{m})$ and $\Sigma_{\mathrm{exh}}(\mathbf{m})$ are ideals in Σ. Let (A_n) be an increasing sequence in $\Sigma_{\mathrm{ca}}(\mathbf{m})$ (resp. $\Sigma_{\mathrm{exh}}(\mathbf{m})$) with union A. Then $\mathbf{m}_{A_n}(N) \to \mathbf{m}_A(N)$ in X for every $N \in \mathcal{R}$. By the Nikodym theorem (resp. the Brooks-Jewett theorem, see [D1]), $\mathbf{m}_A$ is countably additive (resp. exhaustive). Thus $A \in \Sigma_{\mathrm{ca}}(\mathbf{m})$ (resp. $A \in \Sigma_{\mathrm{exh}}(\mathbf{m})$). □

In the three lemmas that follow, we tacitly assume that (S, Σ, λ) and X are as in Theorem 4.1. We recall that a *submeasure* on Σ is a set function $\mu : \Sigma \to \overline{\mathbb{R}}_+$ that is nondecreasing, subadditive, and vanishing at the empty set. The submeasure μ is said to be *order continuous* (o.c.) if $\mu(A_n) \to 0$ whenever $A_n \in \Sigma$ and $A_n \downarrow \emptyset$. For an o.c. submeasure μ, the topology τ_μ of convergence in submeasure μ on S is defined the same way, and has similar properties, as if μ were a finite measure.

The first lemma below is obvious when λ is a finite measure (or an o.c. submeasure), by the 'ε–δ-continuity' of η with respect to λ in this case.

4.3 Lemma. *Let η be an o.c. submeasure on Σ such that $\eta(A) = 0$ whenever $\lambda(A) = 0$. If a finitely additive measure $\mathbf{m} : \mathcal{P} \to L_0(\lambda)$ is exhaustive, then it is also exhaustive in the topology of convergence in submeasure η.*

Proof. Let (N_k) be a disjoint sequence in $\mathcal{P}$. By Corollary 5.2 below, $\mathbf{m}(N_k)(s) \to 0$ for λ-a.e. $s \in S$. Since η vanishes on all λ-null sets, we also have $\mathbf{m}(N_k)(s) \to 0$ for η-a.e. $s \in S$. As η is o.c., $\mathbf{m}(N_k) \to 0$ in submeasure η. □

For $f \in L_0(\lambda)$ and $A \in \Sigma$, we denote $\operatorname{supp} f = \{s \in S : f(s) \neq 0\}$. As before,

$$X_A = P_A(X) = \{g \in X : \operatorname{supp} g \subset A\}.$$

We consider X_A with the topology induced from X.

4.4 Lemma. *Let $h \in X$ and let p be a continuous monotone F-seminorm on X. Denote $H = \operatorname{supp} h$ and define $\eta : \Sigma \to \mathbb{R}_+$ by $\eta(A) = p(h1_A)$. Then η is an o.c. submeasure on Σ. Moreover, for every $f \in X_H$, we have $f = 0$ η-a.e. iff $p(f) = 0$.*

Proof. We only have to verify the second assertion. Let $f \in X_H$ and $F = \operatorname{supp} f$.

If $f = 0$ η-a.e., then $\eta(F) = 0$, or $p(h1_F) = 0$. Also, there exists in X_H a sequence $0 \leqslant f_n \uparrow |f|$ such that each f_n is of the form $\sum_{j=1}^k c_j h 1_{A_j}$, with $c_j \in \mathbb{R}_+$ and disjoint $A_j \subset F$. Clearly, $p(f_n) = 0$ for each n, and since p is σ-Lebesgue, we conclude that $p(f) = 0$.

Conversely, assume $p(f) = 0$. There exist a sequence of sets $F_n \uparrow F$ and a sequence of numbers $a_n > 0$ such that $a_n h 1_{F_n} \leqslant |f|$ for each n. Then $\eta(F_n) = 0$ for all n and, in consequence, $\eta(F) = 0$. □

4.5 Lemma. *Let $\mathbf{m} : \mathcal{P} \to X$ be a τ_λ-exhaustive finitely additive measure. Then* $\operatorname{supp} h \in \Sigma_{\mathrm{exh}}(\mathbf{m})$ *for every $h \in X$.*

Proof. Take any $h \in X$ and denote $H = \operatorname{supp} h$. Let p be a continuous monotone F-seminorm on X, and define η as in Lemma 4.4. According to that lemma, the null space N of $p|X_H$ coincides with those f in X_H that are zero η-a.e. Now, consider the quotient space $\widehat{X}_H = X_H/N$, and equip it with two quotient topologies: τ_η, the topology of convergence in submeasure η; and τ_p, the topology defined by the quotient F-norm $\hat{p}$ corresponding to the F-seminorm p. Both τ_η and τ_p are locally solid and metrizable, and τ_p is σ-Lebesgue. Let Q denote the quotient map from X_H onto $\widehat{X}_H$. By Lemma 4.3, the measure $Q \circ \mathbf{m}_H : \mathcal{P} \to (\widehat{X}_H, \tau_\eta)$ is exhaustive.

Let $H_n = \{s \in S : n^{-1} \leqslant |h(s)| \leqslant n\}$ for $n \in \mathbb{N}$. Clearly, on each of the subspaces $Q(X_{H_n})$ of $\widehat{X}_H$ the topology τ_p is weaker than the topology of (η-a.e.) uniform convergence. By [D2, Thm. 2.2], the countably additive version of the Orlicz-Pettis theorem 'from τ_η to τ_p' holds for each of the subspaces $Q(X_{H_n})$. That is, if a series is subseries convergent in $Q(X_{H_n})$ for τ_η, then it does so for τ_p. Since $H_n \uparrow H$, by applying (an analogue of) Lemma 4.2 we infer that it is so for the whole space $\widehat{X}_H$. Combining this with Proposition 1.1 shows that $Q \circ \mathbf{m}_H$ is exhaustive when $\widehat{X}_H$ is considered with the topology τ_p. Since p was arbitrary, $\mathbf{m}_H : \mathcal{P} \to X_H \subset X$ is exhaustive for the original topology of X. $\square$

Proof of Theorem 4.1. Let $\mathbf{m} : \mathcal{P} \to X$ be a finitely additive measure. First assume $\mathbf{m}$ is τ_λ-exhaustive. Fix a disjoint sequence (N_k) in $\mathcal{P}$. By Lemmas 4.5 and 4.2, $F = \bigcup_k \operatorname{supp} \mathbf{m}(N_k) \in \Sigma_{\mathrm{exh}}(\mathbf{m})$. That is, $\mathbf{m}_F : \mathcal{P} \to X$ is exhaustive. Therefore, $\mathbf{m}(N_k) = \mathbf{m}_F(N_k) \to 0$ as $k \to \infty$.

Now, assume $\mathbf{m}$ is τ_λ-countably additive. Then, by the preceding part of the proof, $\mathbf{m}$ is τ-exhaustive. Thus, if (N_k) is a disjoint sequence in $\mathcal{P}$ with union N, then the series $\sum_k \mathbf{m}(N_k)$ is τ-Cauchy, and τ_λ-convergent to $\mathbf{m}(N)$. Applying Corollary 1.7 we see that the series is also τ-convergent to $\mathbf{m}(N)$. That is, $\mathbf{m}$ is τ-countably additive. $\square$

4.6 Corollary. *Let $X = (X, \tau)$ be a σ-Lebesgue* TRS *of λ-measurable functions, and let ρ be another Hausdorff σ-Lebesgue topology on X. Let $\mathbf{m} : \mathcal{P} \to X$ be a finitely additive measure.*

(a) $\mathbf{m}$ *is ρ-exhaustive iff it is τ-exhaustive.*
(b) $\mathbf{m}$ *is ρ-countably additive iff it is τ-countably additive.*

In particular, subseries convergence coincides for all Hausdorff σ-Lebesgue topologies on X.

Proof. Observe that, by Proposition 3.4, both (X, τ) and (X, ρ) are continuously included in $L_0(\lambda)$, and apply Theorem 4.1. $\square$

Remark. Theorem 4.1, with 'σ-Lebesgue' strengthened to 'Lebesgue', which makes the proof much easier, remains valid even for spaces of Bochner measurable functions over a locally order continuous submeasure space (see [DL2]). Moreover, for the case of scalar functions, this weaker form of Theorem 4.1 is a particular case of the Orlicz-Pettis theorem for general topological Riesz spaces established in [DL1].

5. Isomorphic copies of c_0 and ℓ_∞

Let us start by calling the reader's attention to the fact that, by Proposition 3.4, every TRS of λ-measurable functions having the σ-Lebesgue property is continuously included in $L_0(\lambda)$. This will be used below without explicit mentioning.

Besides the Orlicz-Pettis theorem proved in Section 4, our main tool in the present section is the following result of Orlicz (see [O1, Hilfsatz], [O2, Thm. 8] and [MO, Thms. 1 and 3]). Originally, it was formulated for finite measures λ, but its extension to general (locally finite) measures λ is straightforward.

5.1 Theorem. *If a series $\sum_n f_n$ in $L_0(\lambda)$ is perfectly bounded, then*

$$\sum_{n=1}^{\infty} |f_n(s)|^2 < \infty \qquad \lambda\text{-a.e. on } S.$$

In consequence, a series in $L_0(\lambda)$ is unconditionally Cauchy if (and only if) it is perfectly bounded.

5.2 Corollary. *Let $\mathcal{R}$ be a ring of sets. Then every bounded finitely additive measure $\mathbf{m} : \mathcal{R} \to L_0(\lambda)$ is exhaustive. In fact, $\mathbf{m}(A_n) \to 0$ λ-a.e. for every disjoint sequence (A_n) in $\mathcal{R}$.*

Proof. Since $\mathbf{m}$ is bounded, the series $\sum_n \mathbf{m}(A_n)$ in $L_0(\lambda)$ is perfectly bounded. By Theorem 5.1, $\sum_n |\mathbf{m}(A_n)(s)|^2 < \infty$ λ-a.e., whence $\mathbf{m}(A_n) \to 0$ λ-a.e. □

We extend Theorem 5.1 and Corollary 5.2 to some other spaces of measurable functions. Our approach is essentially that of [D2]. We first prove the following.

5.3 Lemma. *Let X be a sequentially complete Lebesgue* TRS *of λ-measurable functions. If a series $\sum_n f_n$ in X is such that for every $A \in \Sigma_s(\lambda)$ the series $\sum_n f_n 1_A$ is subseries convergent in X, then so is the series $\sum_n f_n$.*

Proof. It is enough to verify the following: If a sequence (g_n) in X is such that, for every $A \in \Sigma_s(\lambda)$, the sequence $(1_A g_n)$ is Cauchy, then so is the sequence (g_n).

In fact, since $\Sigma_s(\lambda)$ is a σ-ideal, it is easy to see that the sequences $(1_A g_n)$ are equi-Cauchy for $A \in \Sigma_s(\lambda)$. That is, given a neighborhood V of zero in X, there is n_0 such that $1_A g_n - 1_A g_m \in V$ for all $A \in \Sigma_s(\lambda)$ and $n, m \geqslant n_0$. On the other hand, by the Lebesgue property, $\lim_A g 1_A = g$ for every $g \in X$. Therefore, assuming as we may that V is closed, we get $g_n - g_m \in V$ for all $n, m \geqslant n_0$. □

5.4 Theorem. *If a* TRS *X of λ-measurable functions is sequentially complete, Lebesgue, and σ-Levi, then X has Property (O).*

Proof. Let $\sum_n f_n$ be a perfectly bounded series in X. Applying Corollary 5.2, we see that for every $A \in \Sigma_s(\lambda)$ the series $\sum_n f_n 1_A$ is subseries convergent in $L_0(\lambda)$. Since, by Proposition 3.6 (a), X_A is boundedly closed in $L_0(\lambda)$, the series $\sum_n f_n 1_A$ is in fact subseries convergent in X for the topology inherited from $L_0(\lambda)$. It is subseries convergent in the original topology of X by Theorem 4.1. To complete the proof, apply Lemma 5.3. □

Remark. By a similar (though somewhat simpler) argument it can be seen that *if λ is of type (SC) and X is a σ-Lebesgue* TRS *of λ-measurable functions which is boundedly sequentially closed in $L_0(\lambda)$, then X has Property (O).*

Part (d) $\Longrightarrow$ (e) of the next result has quite a number of predecessors. Here are some of them: [MO, Thm. 3], [S], [Wo], [C, Thm. 5], [D2, Thm. 3.5], [L2, Thm. 2.12 B].

5.5 Theorem. *Let X be a* TRS *of λ-measurable functions. If X has the MCP, then the following conditions are equivalent.*

(a) *X contains no copy of c_0.*
(b) *X contains no lattice copy of c_0.*
(c) *X has the disjoint Property (O).*
(d) *X is σ-Lebesgue σ-Levi.*
(e) *X has Property (O).*

Proof. The implications (e) $\Longrightarrow$ (a) $\Longrightarrow$ (b) are trivial; (b), (c) and (d) are equivalent by Theorem 2.4; and (d) $\Longrightarrow$ (e) is a consequence of Theorem 5.4 because (d) together with the MCP means X is Lebesgue Levi (see Proposition 2.2), whence also complete. □

If a TRS X is σ-Lebesgue and has the MCP, then it is Lebesgue and Dedekind complete. Therefore, if X has also separating continuous dual, then via Theorem 3.5 it can be realized as a TRS of measurable functions over a locally finite measure space. In view of this, the following is an immediate consequence of Theorem 5.5.

5.6 Corollary. *Let X be a* TRS *with separating continuous dual. If X has the MCP, then conditions* (a) *through* (e) *as in Theorem 5.5 are mutually equivalent.*

We now come to results of Lozanovskii type (see [AB2, Thm. 14.9]).

5.7 Proposition. *If a* TRS *X of λ-measurable functions is σ-Lebesgue, then every bounded finitely additive measure $\mathbf{m} : \mathcal{P} \to X$ is exhaustive. In particular, X contains no copy of ℓ_∞.*

Proof. By Corollary 5.2, $\mathbf{m}$ is exhaustive in the topology induced from $L_0(\lambda)$. Hence, by Theorem 4.1, $\mathbf{m}$ is exhaustive for the original topology of X. □

5.8 Theorem. *Let X be a* TRS *of λ-measurable functions having the σ-MCP. Then the following conditions are equivalent.*

(a) *X contains no copy of ℓ_∞.*
(b) *X contains no lattice copy of ℓ_∞.*
(c) *X is σ-Lebesgue.*
(d) *Every bounded finitely additive measure $\mathbf{m} : \mathcal{P} \to X$ is exhaustive.*

Proof. The implications (d) $\Longrightarrow$ (a) $\Longrightarrow$ (b) are obvious. If (b) is assumed then, by Theorem 2.7, X is pre-Lebesgue. Since X has also the σ-MCP, we conclude that X is σ-Lebesgue, i.e. (c) holds. Finally, (c) $\Longrightarrow$ (d) by Proposition 5.7. □

5.9 Corollary. *If a locally convex* TRS *X is σ-Dedekind complete and σ-Lebesgue, then every bounded finitely additive measure $\mathbf{m} : \mathcal{P} \to X$ is exhaustive. In particular, X contains no copy of ℓ_∞.*

Proof. The topological completion $\widetilde{X}$ of X is locally convex, Dedekind complete, and Lebesgue. By Theorem 3.5, we can represent $\widetilde{X}$ as a space of measurable functions over a locally finite measure space. To finish, apply Proposition 5.7. □

The two corollaries below generalize the Lozanovskii Theorem and are derived from Corollary 5.9 and Theorem 5.8 in much the same way as Corollary 5.6 was from Theorem 5.5.

5.10 Corollary. *Let X be a locally convex* TRS *which is σ-Dedekind complete and has the σ-MCP. Then conditions* (a) *through* (d) *as in Theorem 5.8 are mutually equivalent.*

5.11 Corollary. *Let X be a σ-Dedekind complete* TRS *with separating continuous dual and having the MCP. Then conditions* (a) *through* (d) *as in Theorem 5.8 are mutually equivalent.*

Remark. Corollary 5.10 can also be obtained by a reduction to the original setting of the Lozanovskii Theorem. Of course, only (c) $\Longrightarrow$ (d) is to be verified. (Note that the σ-MCP will not be needed here.)

Let P be a family of continuous Riesz seminorms defining the topology of X. For every $p \in P$, let X_p be the completion of the quotient of (X, p) by the null space of p. Then, by [LA, Thm. 4.1], each X_p is still σ-Dedekind complete and σ-Lebesgue, i.e. a Banach lattice with order continuous norm. Now, consider X

as a Riesz subspace of the product $\prod_{p\in P} X_p$, and let $\mathbf{m} : \mathcal{P} \to X$ be a bounded finitely additive measure. By the Lozanovskii Theorem, none of the spaces X_p contains a copy of ℓ_∞. Therefore, for every projection $\pi_p : X \to X_p$ the measure $\pi_p \circ \mathbf{m} : \mathcal{P} \to X_p$ is exhaustive. It follows that also $\mathbf{m} : \mathcal{P} \to X$ is exhaustive.

6. Comments. As was already mentioned in the Introduction, our dependence on the the Orlicz Theorem (5.1) restricts the results about isomorphic copies of c_0 and ℓ_∞ given in Section 5 to TRS's of measurable functions over a *measure* space. In contrast to this, similar results about lattice copies of c_0 and ℓ_∞ in Section 2 are valid in general TRS's which, by the Representation Theorem 2.7 in [L3], can quite often be viewed as spaces of measurable functions over a locally order continuous *submeasure* space. (See [DL2] for more information on such submeasure spaces.) Now, whether or not the results in Section 5 can be extended to general TRS's depends on the answers to the following questions.

Problem. *Is it true that, for every order continuous submeasure μ,*

1. *the space $L_0(\mu)$ contains no copy of c_0 (resp. ℓ_∞)?*
2. *the space $L_0(\mu)$ has Property (O) (resp. every bounded finitely additive measure $\mathbf{m} : \mathcal{P} \to L_0(\mu)$ is exhaustive)?*

Indeed, by essentially the same type of arguments as those used in Section 5 (in the case of underlying measure spaces), one can see that:

(1′) If the spaces $L_0(\mu)$ contain no copy of c_0 (resp. ℓ_∞), neither does any Lebesgue Levi (resp. Lebesgue σ-Dedekind complete) TRS's X.

(2′) If the spaces $L_0(\mu)$ have Property (O), so does every Lebesgue Levi TRS X. Likewise for the property that every bounded finitely additive measure $\mathbf{m}$ from $\mathcal{P}$ to $L_0(\mu)$ or X is exhaustive.

Let us give a sketch of proof for (1′). Suppose J is an isomorphic embedding of c_0 or ℓ_∞ into X. By applying the Representation Theorem mentioned above, we can realize X as a TRS of μ-measurable functions for a locally order continuous submeasure μ such that $L_0(\mu)$ is complete and $X \subset L_0(\mu)$ continuously. Then the vector measure associated with J is exhaustive into $L_0(\mu)$, or else we could produce a copy of c_0 (or ℓ_∞) in $L_0(\mu)$ using Theorem 1.2. It follows that the series $\sum_n J(e_n)$ is subseries convergent in $L_0(\mu)$. By Proposition 3.6 (suitably extended) and the Orlicz-Pettis Theorem in [DL1], it is also subseries convergent in X which is clearly impossible. This ends the proof in the c_0-case. In the ℓ_∞-case the argument is even simpler due to the fact that the associated vector measure is directly defined on $\mathcal{P}$.

Remark. There is one more question that we would like to ask here: *Is it true that, for every order continuous submeasure μ, the space $L_0(\mu)$ has the bounded multiplier property?* Note that if a space $L_0(\mu)$ has this property, then the statements (1) and (2) above are equivalent for $L_0(\mu)$.

Clearly, if the famous Maharam problem admits a positive solution, i.e., every order continuous submeasure is equivalent to a finite measure, then also all of the above questions have affirmative answers.

References

[AB1] C. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, Academic Press, 1978. MR **58:**12271

[AB2] ———, *Positive operators*, Academic Press, 1985. MR **87h:**47086

[BL] G. Buskes and I. Labuda, *On Levi-like properties and some of their applications in Riesz space theory*, Canad. Math. Bull. **31** (1988), 477–486. MR **90a:**46009

[C] A. Costé, *Convergence des séries dans les espaces F-normés de fonctions mesurables*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. **19** (1971), 131–134. MR **44:**5425

[Di] S. Díaz, *On Schwartz's C-spaces and Orlicz's O-spaces*, Colloq. Math. **64** (1993), 245–251.

[D1] L. Drewnowski, *Equivalence of Brooks-Jewett, Vitali-Hahn-Saks and Nikodym theorems*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. **20** (1972), 725–731. MR **47:**431

[D2] ———, *On subseries convergence in some function spaces*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. **22** (1974), 797–803. MR **51:**1366

[D3] ———, *Un théorème sur les opèrateurs de $l_\infty(\Gamma)$*, C. R. Acad. Sci. Paris Ser. A **281** (1975), 967–969. MR **52:**6486

[DL1] L. Drewnowski and I. Labuda, *The Orlicz-Pettis theorem for topological Riesz spaces*, Proc. Amer. Math. Soc. (to appear).

[DL2] ———, *Spaces of Bochner measurable functions*, Preprint (1996).

[F] D. H. Fremlin, *Topological Riesz Spaces and Measure Theory*, Cambridge Univ. Press, Cambridge, 1974. MR **56:**12824

[K] N. J. Kalton, *Exhaustive operators and vector measures*, Proc. Edinburgh Math. Soc. **19** (1974), 291–300. MR **52:**8936

[KA] L. Kantorovich and G. Akilov, *Functional Analysis* (2nd, ed.), Pergamon Press, Oxford, 1982. MR **83h:**46002

[L1] I. Labuda, *Exhaustive measures in arbitrary topological vector spaces*, Studia Math. **58** (1976), 239–248. MR **53:**8789

[L2] ———, *Spaces of measurable functions*, Comment. Math. (Tomus spec. in honorem Ladislai Orlicz II) (1979), 217–249. MR **81j:**46037

[L3] ———, *Submeasures and locally solid topologies on Riesz spaces*, Math. Z. **195** (1987), 179–196. MR **88j:**46009

[LA] E. Langford and C. D. Aliprantis, *Regularity properties of quotient Riesz seminorms*, Indagationes Math. **78** (1975), 199–212. MR **51:**6356

[MO] W. Matuszewska and W. Orlicz, *A note on modular spaces. IX*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. **16** (1968), 801–808. MR **39:**3278

[O1] W. Orlicz, *Über die Divergenz von allgemeinen Orthogonalreihen*, Studia Math. **4** (1933), 27–32.

[O2] ———, *On a class of asymptotically divergent sequences of functions*, Studia Math. **12** (1951), 286–307. MR **13:**936d

[R] H. P. Rosenthal, *On relatively disjoint families of measures, with some applications to Banach space theory*, Studia Math. **37** (1970), 13–36. MR **42:**5015

[S] L. Schwartz, *Un théorème de convergence dans les L^p, $0 \leqslant p < \infty$*, C. R. Acad. Sci. Paris **268 A** (1969), 704–706. MR **39:**1958

[VG] A. I. Veksler and V. A. Geiler, *Order and disjoint completeness of linear partially ordered spaces*, Siberian Math. J. (1972), 30–35. MR **45:**5713

[Wn] W. Wnuk, *Locally solid Riesz spaces not containing c_0*, Bull. Polish Acad. Sci., Mathematics **36** (1988), 51–56. MR **90g:**46015

[Wo] W. A. Woyczyński, *Sur la convergence des séries dans les espaces de type (L)*, C. R. Acad. Sci. Paris **268 A** (1969), 1254–1257. MR **39:**6049

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