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# INDUCTION THEOREMS ON THE STABLE RATIONALITY OF THE CENTER OF THE RING OF GENERIC MATRICES

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Abstract. Following Procesi and Formanek, the center of the division ring of $n \times n$ generic matrices over the complex numbers $\mathbf{C}$ is stably equivalent to the fixed field under the action of $S_n$, of the function field of the group algebra of a $ZS_n$-lattice, denoted by $G_n$. We study the question of the stable rationality of the center $C_n$ over the complex numbers when $n$ is a prime, in this module theoretic setting. Let $N$ be the normalizer of an $n$-sylow subgroup of $S_n$. Let $M$ be a $ZS_n$-lattice. We show that under certain conditions on $M$, induction-restriction from $N$ to $S_n$ does not affect the stable type of the corresponding field. In particular, $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} G_n)$ are stably isomorphic and the isomorphism preserves the $S_n$-action. We further reduce the problem to the study of the localization of $G_n$ at the prime $n$; all other primes behave well. We also present new simple proofs for the stable rationality of $C_n$ over $\mathbf{C}$, in the cases $n = 5$ and $n = 7$.

## Introduction

The question of rationality of the center of the ring of generic matrices has been studied extensively over some decades, in particular for its connections to other problems. Geometrically, the center $C_n$ of 2 $n \times n$ generic matrices over an algebraically closed field is the function field of an algebraic variety under a reductive group, namely $(M_n \times M_n)/\operatorname{PGL}_n$. The classification of quotients under connected reductive groups is one of the main open problems in invariant theory. In the Brauer group setting the stable rationality of this center implies the Merkurjev-Suslin result, which says that the Brauer group of a field $k$ is generated by cyclics, provided $k$ has enough roots of unity. Le Bruyn presents an informative survey of the problem and its applications in [LL].

In 1883 Sylvester showed that for $n = 2$, $C_2$ was rational over $\mathbf{C}$, the complex numbers, [S]. In 1979 and 1980 Formanek showed that $C_n$ was rational over $\mathbf{C}$ for the case $n = 3$, [F1], and $n = 4$, [F2]. In 1984, Saltman proved that for all primes $n$, $C_n$ is retract rational over $\mathbf{C}$, [SD]. In 1991 Bessenrodt and Le Bruyn showed stable rationality for the cases $n = 5$ and 7 [BL]. Their proof, however, requires the use of a computer. In this paper we present simple proofs for the case $n = 5$ and 7, Theorems 3.3 and 3.4.

We study the problem in its module theoretic setting. If $G$ is a finite group, and $M$ is a $ZG$-lattice, then $\mathbf{C}[M]$ denotes the group algebra of the abelian group $M$, and $\mathbf{C}(M)$ its quotient field. Define the following $ZS_n$-lattices. $U_n = \sum_{i=1,\ldots,n} Zu_i$, where $gu_i = u_{g(i)}$ for all $g$ in $S_n$. $A$, usually denoted by $A_{n-1}$, is the kernel of the

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augmentation map $\varepsilon\colon U_n \to Z$. $V_n = \sum_{i,j,i\neq j} Zy_{ij}$ where $gy_{ij} = y_{g(i)g(j)}$ for all $g$ in $S_n$. $G_n$ is defined by the sequence $0 \to G_n \to V_n \to A \to 0$, where the map $V_n \to A$ is given by $y_{ij} \to u_i - u_j$. Procesi-Formanek show that $C_n \cong \mathbf{C}(G_n \oplus U_n \oplus U_n)^{S_n}$, [F1, Theorem 3]. This implies that $C_n$ is stably isomorphic to $\mathbf{C}(G_n)^{S_n}$ [L, Proposition 1.4]. Let $n$ be an odd prime. Let $H$ be an $n$-sylow subgroup of $G$, thus $H$ is cyclic of order $n$. The normalizer of $H$ in $S_n$, which we denote by $N$, is the semidirect product of $H$ by a cyclic group $C$ of order $n-1$.

The main result of this paper is:

**Theorem 2.6.** *The fields $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} G_n)$ are stably isomorphic, and the isomorphism respects the $S_n$-actions. The center $C_n$ is stably isomorphic to their fields of invariants.*

This theorem is a direct consequence of Proposition 2.4, which says that under certain conditions, induction—restriction from $N$ to $G$ does not affect the stable type of the field. This is a crucial reduction step for the problem. It implies that the stable rationality of $C_n$ depends on the structure of $G_n$ as a $ZN$-lattice. Set $n = p$, and let $Z_p$ denote the localization of $Z$ at the prime $p$.

The second result is:

**Theorem 2.7.** *Let $D$ and $R$ be invertible $ZN$-lattices. Suppose that $R$ is either $0$, or stably permutation and $Z_qC$-projective for all primes $q$ dividing $p-1$. If $(G_n \oplus R)_p \cong D_p$, then the fields $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} D)$ are stably isomorphic as $G$-fields.*

This theorem says that the stable rationality of $C_n$ in fact depends on the structure of $(G_n)_p$ as a $Z_pN$-lattice.

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## 1.

Let $G$ be a finite group, a $ZG$-lattice is finitely generated $Z$-free $ZG$-module. A permutation $ZG$-module is a $ZG$-lattice with a $Z$-basis permutated by $G$. An invertible $ZG$-module is a direct summand of a permutation module. A $ZG$-module $M$ is stably permutation if there exists permutation $ZG$-modules $P$ and $P'$, such that $M \oplus P \cong P'$.

**Definition.** A $ZG$-lattice $M$ is $G$-faithful if the map $G \to \mathrm{Aut}_Z(M)$ is injective.

**Notation.**

(1) For a $ZG$-lattice $M$ we denote by $M^* = \mathrm{Hom}_Z(M, Z)$, its dual.

(2) For any prime $q$ in $Z, Z_q$ denotes the localization of $Z$ at the ideal $(q)$, and $M_q = Z_q \otimes M$.

(3) In the remainder of this paper we will denote $S_n$ by $G$, and $n = p$ will be an odd prime, unless otherwise specified.

The following observations were made in [BL].

The map: $V_n \to A \otimes U_n$

$$y_{ij} \to (u_i - u_j) \otimes u_i$$

is a $ZG$-isomorphism.

Now tensoring the exact sequence

$$0 \to A \to U_n \to Z \to 0$$

by $A$, we get

$$0 \to A \otimes A \to U_n \otimes A \to A \to 0$$

The map $U_n \otimes A \to A$ sends $y_{ij}$ to $u_i - u_j$, so $G_n \cong A \otimes A$.

**Notation.** We let $B = A^* \otimes A$.

This lattice is used throughout the paper, in particular if let $Q$ be the field of rational numbers, then $QB \cong QG_n$.

The following proposition was proved by Bessenrodt and Le Bruyn, but not published. We present here a proof.

**Proposition 1.1.** *We have the following isomorphisms of $ZG$-lattices $B \oplus U_n \cong V_n \oplus Z$. Hence $B$ is stably permutation.*

*Proof.* The sequence $0 \to A \to U_n \to Z \to 0$ splits when localized at all primes $q$ different from $n$, dividing the order of $G$, with splitting map given by $1 \to (\sum_{i-1}^{n} u_i)/n$. So we have

$$Z_q \oplus A_q \cong (U_n)_q \text{ and taking duals } Z_q \oplus A_q^* \cong (U_n)_q.$$

Now tensoring by $A_q$ we get

$$B_q \oplus A_q \cong (V_n)_q.$$

So $B_q$ is $Z_qG$-invertible. In particular $B_q$ is $Z_qP$-invertible for all $q$-sylow subgroups $P$ of $G$. To show that $B$ is invertible it suffices by [CT, Lemma 9, section 1] to show that $B_p$ is $Z_pH$-invertible for a $p = n$-sylow subgroup of $G$. Now consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B & \longrightarrow & V_n & \longrightarrow & A^* & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & U_n & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & Z & \longrightarrow & Z & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

$(U_n)_p \cong Z_pS_n/S_{n-1}$ and $(V_n)_p \cong Z_pS_n/S_{n-2}$ are $Z_pG$-projective, since they are induced from groups of order prime to $p$, hence $X_p \cong (B \oplus U_n)_p \cong (V_n \oplus Z)_p$. So $B_p$ is $Z_pG$-invertible, and hence $Z_pH$-invertible. So $B$ is $ZG$-invertible. This implies by e.g. [B, Lemma 1.2]

$$B \oplus U_n \cong V_n \oplus Z.$$

So $B$ is stably permutation.

**Lemma 1.2.** *Let $X = Z/pZ$ be the trivial $ZG$-module with $p$ elements. Then there exists a $ZG$-exact sequence*

$$0 \to A \to A^* \to X \to 0.$$

*Proof.* $U_n^*$ is generated by $u_i^*$, where $u_i^*(u_j) = \delta_{ij}$, and $A^*$ is generated by $\mathrm{res}(u_i^*)I = 1, \ldots, n-1$, where res is restriction from $U_n$ to $A$. The map $A^* \to X$ is given by $\mathrm{res}(u_i^*) \to 1$.

*Remark.* Tensoring the sequence of Lemma 1.2 by $A$ we get

$$0 \to A \otimes A \to A^* \otimes A \to X \otimes A \to 0.$$

Identifying $A \otimes A$ with $G_n$, and setting $T = X \otimes A \cong A/pA$ we obtain

$$0 \to G_n \to B \to T \to 0.$$

Let $U = U_n/pU_n$. It is easy checked that the map $U \to X$ given by $u_i \to 1$ has kernel $T$. Thus we have a $ZG$-exact sequence

$$0 \to T \to U \to X \to 0.$$

Let $T' = A^*/pA^*$, then it is immediate that the following sequence is $ZG$-exact

$$0 \to X \to U \to T' \to 0.$$

These are important sequences which will be used in the following sections.

## 2.

Throughout the rest of this paper we adopt the following notation unless otherwise specified. We will also keep the notation used in section 1.

- $G = S_n$, $n = p$ is prime; $n$ and $p$ will be used interchangeably without mention.
- $H = p$-sylow subgroup of $G$, so $H$ is cyclic of order $p$.
- $N = N_G(H)$ will be the normalizer of $H$ in $G$, so $N = H \rtimes C$, where $C$ is cyclic of order $p-1$, and if we let $h$ and $c$ generate $H$ and $C$ respectively, then $chc^{-1} = h^a$ where $a$ is a primitive $(p-1)$st root of $1 \bmod p$.
- $F = Z/pZ$ field of $p$ elements.
- For any finite group $G$, and for any $ZG$-lattice $M$, we will denote by $\phi(M)$ the flasque class of $M$, see [CT, section 1] for definitions.
- $Z^-$ will denote the sign representation of $G$.
- $X$ will denote the trivial $ZG$-module of $p$ elements.

**Definitions.** (1) A $ZG$-lattice $M$ is quasi-permutation [EM], if there exist an exact sequence

$$0 \to M \to R \to Q \to 0$$

with $R$ and $Q$ permutation.

(2) We will say that 2 fields are stably isomorphic as $G$-fields, if the fields are stably isomorphic, and the isomorphism respects their $G$-actions.

Various versions of the following lemma have been proved in the literature, we need a slightly different version here.

**Lemma 2.1.** *Let $G$ be a finite group.*

(1) *Suppose there exists an exact sequence of $ZG$-lattices*

$$0 \to M \to N \to E \to 0$$

*with $E$ invertible and $M$, $G$-faithful, then $\mathbf{C}(E \oplus M) \cong \mathbf{C}(N)$ as $G$-fields.*

(2) *Let $G$ be a finite group, and let $M$ and $M'$ be $G$-faithful $ZG$-lattices, with $M$ quasi-permutation. Then the fields $\mathbf{C}(M \oplus M')$ and $\mathbf{C}(M')$ are stably isomorphic as $G$-fields.*

(3) *Let $M$ and $M'$ be $G$-faithful $ZG$-lattices such that $\phi(M) = \phi(M')$, then $\mathbf{C}(M)$ and $\mathbf{C}(M')$ are stably isomorphic as $G$-fields.*

(4) *If $G = S_n$ and $M$ is a $G$-faithful quasi-permutation $ZG$-lattice, then $\mathbf{C}(M)^G$ is stably rational over $\mathbf{C}$.*

*Proof.* (1) The proof is basically that of [L, Proposition 1.5]. The injection $M \to N$ induces an injection $\mathbf{C}(M) \subset \mathbf{C}(N)$. Now consider the sequence

$$0 \to \mathbf{C}(M)^* \to \mathbf{C}(M)^*N \to E \to 0.$$

Since $G$ acts faithfully on $\mathbf{C}(M)^*$, $\mathbf{C}(M)^*$ is $H^1$-trivial by Hilbert theorem 90. Since $E$ is invertible the sequence splits by [CT, Lemma 1, section 1]. Thus $\mathbf{C}(M)^*N \cong E \oplus \mathbf{C}(M)^*$. This isomorphism yields the required isomorphism of $G$-fields.

(2) Since $M$ is quasi-permutation there exist an exact sequence

$$0 \to M \to R \to Q \to 0$$

with $Q$ and $R$ permutation. By (1) $\mathbf{C}(M \oplus Q) \cong \mathbf{C}(R)$. So $\mathbf{C}(M \oplus M' \oplus Q) \cong \mathbf{C}(M' \oplus R)$. Apply [L, Proposition 1.3] to the field $l = \mathbf{C}(M')$ and the $l$-vector space $W = \sum l r_i$, where $\{r_i\}$ is a $Z$-basis for $R$ which is permuted by $G$. Then there exists elements $\{y_i\}$ in $W^G$ which form an $L$-basis for $W$. Now $W \subset l(R)$, and hence

$$l(R) = l(y_1, \ldots, y_k) = \mathbf{C}(M')(y_1, \ldots, y_k).$$

By the same argument there exist indeterminates $z_1, \ldots, z_t$, such that

$$\mathbf{C}(M \oplus M' \oplus Q) = \mathbf{C}(M \oplus M')(z_1, \ldots, z_t).$$

So $\mathbf{C}(M')(y_1, \ldots, y_k) \cong \mathbf{C}(M \oplus M')(z_1, \ldots, z_t)$.

(3) Since $\phi(M) = \phi(M')$, there exist 2 exact sequences

$$0 \to M \to E \to R \to 0 \text{ and } 0 \to M' \to E \to P \to 0$$

with $P$ and $R$ permutation by [CT, Lemma 8, section 1]. By (1) $\mathbf{C}(E) \cong \mathbf{C}(M \oplus R) \cong \mathbf{C}(M' \oplus P)$ as $G$-fields, and by (2) $\mathbf{C}(M)$ ad $\mathbf{C}(M')$ are stably isomorphic as $G$-fields.

(4) Since $M$ is quasi-permutation there is an exact sequence

$$0 \to M \to R \to Q \to 0$$

with $Q$ and $R$ permutation. By (1) $\mathbf{C}(M \oplus Q) \cong \mathbf{C}(R)$. By (2) $\mathbf{C}(M \oplus Q)$ and $\mathbf{C}(M)$ are stably isomorphic as $G$-fields. Since $\phi(U_n) = \phi(R) = 0$, $\mathbf{C}(U_n)$ and $\mathbf{C}(R)$ stably isomorphic as $G$-fields by (3). Since $\mathbf{C}(U_n)^G$ is rational over $\mathbf{C}$, the result follows.

**Proposition 2.2.** (1) *Let $G$ be a finite group and suppose there exist exact sequences*

$$0 \to E \to P \to L \to 0 \text{ and } 0 \to E' \to R \to J \oplus L \to 0$$

*with $P$ and $R$, $ZG$-permutation, $E$ and $E'$ $G$-faithful, $L$ and $J$, finite and $J = Q/mQ$ for some $ZG$-permutation module $Q$, and some positive integer $m$. Then the fields $\mathbf{C}(E')$ and $\mathbf{C}(E)$ are stably isomorphic as $G$-fields.*

(2) *Let $G = S_n$ and let $M$ be a $G$-faithful $ZG$-lattice of trivial cohomology, then $\mathbf{C}(M)^G$ is stably rational over $\mathbf{C}$. Furthermore if $M'$ is any $G$-faithful $ZG$-lattice, then the fields $\mathbf{C}(M \oplus M')$ and $\mathbf{C}(M')$ are stably isomorphic as $G$-fields.*

*Proof.* (1) We form the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E' & \longrightarrow & R & \longrightarrow & J \oplus L & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E' & \longrightarrow & M_3 & \longrightarrow & Q \oplus P & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & Q \oplus E & \longrightarrow & Q \oplus E & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $R$, $Q$ and $P$ are permutation modules, the fields $\mathbf{C}(E)$ and $\mathbf{C}(E')$ are stably isomorphic as $G$-fields, by Lemma 2.1.

(2) By [BK, Theorem 8.10] $M$ is $ZG$-projective and by [EM, Theorem 3.3] $M$ is stably permutation. The lemma follows by Lemma 2.1.

**Notation.** Let $I_{G/N}$ denote the kernel of the augmentation map $Z(G/N) \to Z$, and let $I = I_{G/N}/pI_{G/N}$.

**Proposition 2.3.** *The $Z_pG$-lattice $(I_{G/N})_p$ and the $ZG$-lattice $I$ are cohomologically trivial. Let $I^-_{G/N} = I_{G/N} \otimes Z^-$, $I^- = I \otimes Z^-$, then $(I^-_{G/N})$, $I^-$ are also cohomologically trivial.*

*Proof.* The sequence $0 \to (I_{G/N})_p \to Z_p(G/N) \to Z_p \to 0$ splits with splitting map $Z_p \to Z_p(G/N)$ given by $1 \to (\sum_{g \in G/N} g)/[G : N]$. So $Z_p(G/N) \cong Z_p \oplus (I_{G/N})_p$. By Mackey's subgroup theorem [CR1, Theorem 10.13] $\mathrm{Res}^G_H\, ZG/N \cong Z \oplus ZH^k$ for some positive integer $k$. So as a $\widehat{Z}_pH$-module $(\hat{I}_{G/N})_p$ is free by the Krull-Schmidt-Azumaya and by [CR1, Theorem 30.17] $(I_{G/N})_p$ is a free $Z_pH$-module. Now for any subgroup $K$ of $G$, $H^i(K, (I_{G/N})_p)$ injects into $H^i(H, (I_{G/N})_p) = 0$. This proves that $(I_{G/N})_p$ is cohomologically trivial. To prove that $I$ is cohomologically trivial it suffices to take the cohomology of the sequence

$$0 \to (I_{G/N})_p \to (I_{G/N})_p \to I \to 0.$$

The second statement follows since $I^-_{G/N}$, $I^-$ are isomorphic to $I_{G/N}$ and $I$ respectively as $H$-modules.

As in the preceding proof we have $FG/N \cong F \oplus I$. The following proposition is a crucial reduction step. In particular it implies Theorem 2.6 which says that $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} G_n)$ are stably isomorphic as $G$-fields, so that we can basically study $G_n$ at the $N$-level and induce up.

**Proposition 2.4.** *Let $L$ be a finite $ZG$-mdoule of exponent $p$. Suppose we have*

(1) *An $FG$-extension of $X$ or $X^-$ by $L$, or of $L$ by $X$ or $X^-$, whose middle term is $FG$-projective.*

(2) *A $ZG$-exact sequence*

$$0 \to M \to P \to L \oplus Q/mQ \to 0$$

*with $P$ and $Q$ stably permutation, where $m$ is a positive integer.*

*Then the fields $\mathbf{C}(M)$ and $\mathbf{C}(ZG \otimes_{ZN} M)$ are stably isomorphic as $G$-fields.*

*Proof.* Tensoring the sequence

$$0 \to M \to P \to L \oplus Q/mQ \to 0 \tag{s1}$$

by $ZG$ over $ZN$ we get

$$0 \to ZG \otimes_{ZN} M \to ZG \otimes_{ZN} P \to FG \otimes_{FN} L \oplus ZG \otimes_{ZN} Q/mQ \to 0. \tag{s2}$$

We form the sequence

$$0 \to Q \to Q \to Q/mQ \to 0 \tag{s3}$$

and add it to (s2) to obtain

$$\begin{aligned} 0 &\to ZG \otimes_{ZN} M \oplus Q \to ZG \otimes_{ZN} P \oplus Q \\ &\to FG \otimes_{FN} L \oplus ZG \otimes_{ZN} Q/mQ \oplus Q/mQ \to 0. \end{aligned} \tag{s4}$$

We have $FG \otimes_{FN} L \cong L \oplus I \otimes L$, and we assume that we have an $FG$-exact sequence of the form

$$0 \to L \to U \to X \to 0. \tag{1}$$

The proof extends directly to the other cases. Tensoring (1) by $I$ of $F$ preserves $FG$-exactness since $I$ is $F$-free, and we get

$$0 \to L \otimes I \to U \otimes I \to I \to 0.$$

Since $U$ is $FG$-projective, so is $U \otimes I$. By looking at the cohomology of the $ZG$-exact sequence

$$0 \to ZG \to ZG \to FG \to 0,$$

we see that $FG$ is cohomologically trivial as a $ZG$-module, therefore so is $U \otimes I$. By Proposition 2.3 so is $I$. Thus $L \otimes I$ is also $ZG$-cohomologically trivial. Now consider the exact sequence

$$0 \to M' \to M'' \to L \otimes I \to 0 \text{ with } M'' \text{ free.} \tag{s5}$$

Since $M''$ and $L \otimes I$ are cohomologically trivial so is $M'$, thus $M'$ is $ZG$-projective by [BK, Theorem 8.10], and by [EM, Theorem 3.3] it is stably permutation. Now add the sequences (s1), (s5) and

$$0 \to ZG \otimes_{ZN} Q \to ZG \otimes_{ZN} Q \to ZG \otimes_{ZN} Q/mQ \to 0$$

and form a commutative diagram with (s4),

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
0 \to & ZG \otimes_{ZN} M \oplus Q & \longrightarrow & ZG \otimes_{ZN} P \oplus Q & \longrightarrow & \begin{array}{c} FG \otimes_{FN} L \oplus \\ ZG \otimes_{ZN} Q/mQ \oplus Q/mQ \end{array} & \to 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \to & ZG \otimes_{ZN} M \oplus Q & \longrightarrow & M_3 & \longrightarrow & M'' \oplus P \oplus ZG \otimes_{ZN} Q & \to 0 \\
 & & & \uparrow & & \uparrow & \\
 & & & M' \oplus M \oplus ZG \otimes_{ZN} Q & \longrightarrow & M' \oplus M \oplus ZG \otimes_{ZN} Q & \to 0 \\
 & & & \uparrow & & \uparrow & \\
 & & & 0 & & 0 &
\end{array}$$

The result now follows by Lemma 2.1, since $M'', P, M', Q$ and $ZG \otimes_{ZN} Q$ are stably permutation for this case. A similar argument applies to the other cases.

**Corollary 2.5.** *The fields* $\mathbf{C}(A^*)$ *and* $\mathbf{C}(ZG \otimes_{ZN} A^*)$ *are stably isomorphic as $G$-fields. So are the fields* $\mathbf{C}(A^{*-})$ *and* $\mathbf{C}(ZG \otimes_{ZN} A^{*-})$.

*Proof.* To prove the first statement of the corollary, we note that there exists an exact sequence

$$0 \to A \oplus Z \to U_n \to X \to 0$$

where the map from $U_n$ to $X$ is given by $u_i \to 1$. Tensoring the sequence by $A^*$ over $Z$ we get

$$0 \to B \oplus A^* \to V \to T' \to 0,$$

where $T' = A^*/pA^*$ as in section 1.

By the remark at the end of section 1 we have an exact sequence

$$0 \to X \to U \to T' \to 0.$$

Hence by Proposition 2.4, $\mathbf{C}(A^* \oplus B)$ and $\mathbf{C}(ZG \otimes_{ZN} (A^* \oplus B))$ are stably isomorphic as $G$-fields. The result follows by Lemma 2.1 since $B$ is stably permutation. For the second statement we have the exact sequence

(1) $0 \to A^{*-} \oplus Z^- \to U_n^- \to T'^- \to 0$, where the map $U_n^- \to T'^- \to 0$ is given by $U_n \to A^* \to T' \to 0$. We also have the sequence $0 \to Z \oplus Z^- \to ZG/A_n \to J \to 0$ where $J = Z/2Z$.

Tensoring by $U_n$ we get

(2) $0 \to U_n^- \oplus U_n \to ZG \otimes_{ZA_n} U_n \to J' \to 0$ where $J' = U_n/2U_n$, and $A_n$ is the alternating group.

Adding $U_n$ to the first 2 terms of (1) and combining with (2) we get

$$0 \to A^{*-} \oplus Z^- \oplus U_n \to ZG \otimes_{ZA_n} U_n \to T'^- \oplus J' \to 0$$

since $T'^-$ and $J'$ are of relatively prime orders.

So $\mathbf{C}(A^{*-} \oplus Z^- \oplus U_n)$ and $\mathbf{C}(ZG \otimes_{ZN} (A^{*-} \oplus Z^- \oplus U_n))$ are stably isomorphic as $G$-fields, by Proposition 2.4. Now the result follows by Lemma 2.1 and by observing that $Z^-$ is quasi-permutation.

The main theorem is a direct consequence of Proposition 2.4.

**Theorem 2.6.** *The fields* $\mathbf{C}(G_n)$ *and* $\mathbf{C}(ZG \otimes_{ZN} G_n)$ *are stably isomorphic as* $G$*-fields.The center* $C_n$ *is isomorphic to their fixed fields.*

*Proof.* By the remark at the end of section 1, there is an exact sequence

$$0 \to T \to U \to X \to 0.$$

Applying Proposition 2.4 to the $ZG$-exact sequence: $0 \to G_n \to B \to T \to 0$ from section 1, we get that $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} G_n)$ are stably isomorphic as $G$-fields. By [F1, Theorem 3] and [L, Proposition 1.4] we have $C_n$ ad $\mathbf{C}(G_n)^G$ stably isomorphic.

**Theorem 2.7.** *Let* $D$ *and* $R$ *be invertible* $ZN$*-lattices. Suppose that* $R$ *is either* $0$*, or stably permutation and* $Z_qC$*-projective for all primes* $q$ *dividing* $p-1$*. If* $(G_n \oplus R)_p \cong D_p$*, then the fields* $\mathbf{C}(G_n)$ *and* $\mathbf{C}(ZG \otimes_{ZN} D)$ *are stably isomorphic as* $G$*-fields.*

*Proof.* We have $(G_n \oplus R)_p \cong D_p$. By [CR1, Lemma 31.4], this implies the existence of a $ZN$-exact sequence

$$0 \to G_n \oplus R \to D \to Y \to 0$$

where $Y$ is a finite $ZN$-module of order prime to $p$. Since $(G_n \oplus R)_p \cong D_p$, $Q(G_n \oplus R) \cong QD$ where $Q$ is the field of rational numbers. It follows directly form the definition of $G_n$, that this lattice is $ZC$-free. Therefore $(G_n \oplus R)_q$ is $Z_qC$-projective for all primes $q$ dividing $p-1$, and [B1, Theorem 1.2] implies that so is $D_q$. Since the first 2 terms of the sequence are $Z_qC$-projective for all primes $q \neq p$, $Y$ is cohomologically trivial. Therefore we have a $ZN$-exact sequence

$$0 \to Pr \to Fr \to Y \to 0$$

with $Fr$, free. Since $Y$ is cohomologically trivial, so is $Pr$, hence $Pr$ is $ZN$-projective by [BK, Theorem 8.10].

Forming a commutative diagram with the above sequences we obtain

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & G_n \oplus R & \longrightarrow & D & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G_n \oplus R & \longrightarrow & M & \longrightarrow & Fr & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & Pr & \longrightarrow & Pr & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

This implies that $G_n \oplus R \oplus Fr \cong D \oplus Pr$ since $Pr$ and $Fr$ are projective. Inducing up to $G$ we obtain

$$ZG \otimes_{ZN} (G_n \oplus R \oplus Fr) \cong ZG \otimes_{ZN} (D \oplus Pr).$$

Since $Pr$ and $Fr$ are $ZN$-projective, the induced modules are $ZG$-projective and by [EM, Theorem 3.3] they are stably permutation. The result now follows by Lemma 2.1 and Theorem 2.6.

The following lemma will be needed later.

**Lemma 2.8.** *There exist a $ZG$-lattice $W$ such that $G_n \oplus W \cong V_n \oplus Z(G/H)$. As $ZN$-lattices we have $G_n \oplus W \cong ZN^k \oplus Z(N/H)$ for some positive integer $k$.*

*Proof.* Since $H$ is cyclic and $A \cong ZH(h-1)$ as a $ZH$-module, we have the $ZH$-exact sequence

$$0 \to Z \to ZH \to A \to 0,$$

tensoring by $ZG$ over $ZH$ we get

$$0 \to Z(G/H) \to ZG \to Z(G/H) \otimes A \to 0.$$

From this sequence we form the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & I \otimes A & \longrightarrow & I \otimes A & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Z(G/H) & \longrightarrow & ZG & \longrightarrow & Z(G/H) \otimes A & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Z(G/H) & \longrightarrow & W & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $I = I_{G/H}$.

For all primes $q$ different from $p$, dividing the order of $G$, $W_q \cong (A \oplus Z(G/H))_q$, since $A_q$ is $Z_qG$-invertible by the proof of Proposition 1.1. For the prime $p$, we have

$$Z_p(G/H) \cong Z_p \oplus I_p.$$

So

$$Z_p(G/H) \otimes A_p \cong A_p \oplus I_p \otimes A_p.$$

Since $H^0(H, ZG/H \otimes A) \cong H^1(H, ZG/H) = 0$, we have

$$H^1(H, W) \cong H^0(H, I \otimes A) = 0,$$

and thus $W_p$ is $Z_pH$-invertible since $H$ is cyclic by [CT, Corollary 2 and Proposition 2, section 1]. By [B, Theorem 2.1], $W$ is invertible. We have the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G_n & \longrightarrow & V_n & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G_n & \longrightarrow & M & \longrightarrow & W & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & ZG/H & \longrightarrow & ZG/H & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $V_n, G_n, W$ and $ZG/H$ are invertible, $V_n \oplus ZG/H \cong W \oplus G_n$ by [CT, Lemma 1, section 1]. The second statement follows from the fact that $V_n \cong ZN$ and $Z(G/H) \cong Z(N/H) \oplus ZN^{k-1}$ as $ZN$-lattices.

## 3.

Recall that $N$ was defined to be the normalizer of a $p$-sylow subgroup $H$ of $G$. So $N = N_G(H) = H \rtimes C$, where $C$ is cyclic of order $p-1$, and if we let $h$ and $c$, generate $H$ and $C$ respectively than $c.h = h^a$, where $a$ is primitive $(p-1)$st root of $1 \bmod p$.

Now

$$\widehat{Z}_p N/H = \bigoplus_{k=1}^{p-1} Z_k$$

where $Z_k$ is the $\widehat{Z}_p N$-module of rank 1 with trivial $H$-action, and such that $c$ acts on 1 as $\theta^k$, where $\theta$ was chosen to be the primitive $(p-1)$-st root of 1 in $\widehat{Z}_p$ for which $\theta \equiv a \bmod p$.

We let $X_k = Z_k/pZ_k$ for $k = 1, \dots, p-1$. So $X = X_{p-1}$ is the trivial $Z_pN$-module $Z/pZ$.

**Lemma 3.1.** *Keeping the definitions of $A$ and $A^*$. The $Z_pN$-modules $A_p$ and $A_p^*$ are isomorphic as $Z_pH$-modules. Therefore $Z_pN \otimes_{Z_pH} A \cong Z_pN \otimes_{Z_pH} A^*$.*

*Proof.* As a $ZN$-module $U_n \cong ZN/C \cong ZH$ where $u_i \to h^i$ and $c.h = h^a$. Thus $A \cong ZH(h-1)$. Since $H$ is cyclic we have a $ZH$-exact sequence

$$0 \to Z \to ZH \to ZH(h-1) \to 0.$$

Equivalently

$$0 \to Z \to ZH \to A \to 0.$$

From this sequence and the sequence

$$0 \to Z \to ZH \to A^* \to 0.$$

We obtain the commutative diagram of $ZH$-modules.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Z & \longrightarrow & ZH & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ZH & \longrightarrow & M & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & A^* & \longrightarrow & A^* & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Thus $A^* \oplus ZH \cong A \oplus ZH$. By the Krull-Schmidt-Azumaya (K-S-A) $\widehat{A}_p \cong \widehat{A}_p^*$, and by [CR1, Theorem 30.17] $A_p$ and $A_p^*$. □

**Notation.** For each $m$ dividing $p-1$, we denote by $C_m$ the subgroup of $C$ generated by $c^m$, thus $C_m$ has order $(p-1)/m$, and index $m$ in $C$.

We will denote by $A_k$ the $\widehat{Z}_pN$-lattice $\widehat{A}_p^* \otimes Z_k$, for $k = 1, \ldots, p-1$.

For each $k$, we will denote by $G_k$ the $\widehat{Z}_pN$-lattice $\widehat{A}_p^* \otimes A_k$.

**Theorem 3.2.** *We have the following isomorphisms of $\widehat{Z}_pN$-lattices.*

(1) $A^* \otimes Z_1 \cong \widehat{A}_p$ *as $Z_pN$-lattices.*

(2) $(\widehat{G}_N)_p \cong \widehat{A}_p^* \otimes A_2 \cong G_2$.

(3) $\widehat{B}_p \cong G_1$.

*Proof.* By Lemma 3.1 we have $Z_pN \otimes_{Z_pH} A \cong Z_pN \otimes_{Z_pH} A^*$. Thus

$$\bigoplus_{k=1}^{p-1} A_k \cong \bigoplus_{k-1}^{p-1} \widehat{A}_P \otimes Z_k.$$

As an $Z_pH$-module $A_p^* \cong Z_p[w]$ where $w$ is a primitive $p$-th root of 1 on $Z_p$. Therefore $\widehat{A}_p^*$ is indecomposable as a $\widehat{Z}_pH$-module and a fortiori as a $Z_pN$-module. By the K-S-A, this implies that $\widehat{A}_p \cong \widehat{A}_p^* \otimes Z_j$ for some $j$. We will show that $j = 1$. Since $\mathrm{Ext}^1_{\widehat{Z}_pN}(\widehat{A}_p, \widehat{Z}_pN/H) \cong \mathrm{Ext}^1_{\widehat{Z}_pN}(\widehat{Z}_pN/H, A_p^*) \cong H^1(H, A_p^*) \cong Z/p$ we have $\mathrm{Ext}^1_{\widehat{Z}_pN}(\widehat{A}_p, Z_k) = 0$ for all $k$ except for a unique $k$ for which $\mathrm{Ext}^1_{\widehat{Z}_pN}(\widehat{A}_p, Z_k) = Z/p$. It is easily seen that $Z_k^* = \mathrm{Hom}(Z_k, Z_p) \cong Z_{p-1-k}$, for $k = 1, \ldots, p-1$.

Now taking the cohomology of the sequence

$$0 \to \widehat{A}_p \to (\widehat{U}_n)_p \to \widehat{Z}_p \to 0$$

we have $0 \to \mathrm{Hom}_N(\widehat{Z}_p, Z_k) \to \mathrm{Hom}_N((\widehat{U}_n)_p, Z_k) \to \mathrm{Hom}_N(\widehat{A}_p, Z_k) \to \mathrm{Ext}^1_N(\widehat{Z}_p, Z_k) = 0$ since $Z_k$ is invertible. Now if $k \neq p-1$ then $\mathrm{Hom}_N(\widehat{Z}_p, Z_k) \cong Z_k^N = 0$, and $\mathrm{Hom}_N((\widehat{U}_n)_p, Z_k) \cong Z_k^C \cong 0$, thus $\mathrm{Hom}_N(\widehat{A}_p, Z_k) \cong 0$. If $k = p-1$, then $\mathrm{Hom}_N(\widehat{Z}_p, Z_k) \cong \widehat{Z}_p$, $\mathrm{Hom}_N((\widehat{U}_n)_p, Z_k) \cong \widehat{Z}_p$. Since $\mathrm{Hom}_N(\widehat{A}_p, Z_k)$ is torsion free it must be 0.

Now consider the exact sequence

$$0 \to Z_k \to Z_k \to X_k \to 0.$$

From this we have $0 \to \mathrm{Hom}_N(\widehat{A}_p, Z_k) \to \mathrm{Hom}_N(\widehat{A}_p, Z_k) \to \mathrm{Hom}_N(\widehat{A}_p, X_k) \to \mathrm{Ext}^1_N(\widehat{A}_p, Z_k) \to 0$. Therefore the unique $k$ for which $\mathrm{Ext}^1_N(\widehat{A}_p, Z_k) \cong Z/p$ is the $k$ for which $\mathrm{Hom}_N(A_p, X_k) = Z/p$. Now we have a nonzero $Z_pN$-homomorphism $A_p \to X_1$ given by $h - 1 \to 1$. So $k = 1$.

Now tensoring the sequence

$$0 \to \widehat{Z}_p \to (\widehat{U}_n)_p \to A^*_p \to 0$$

by $Z_j$, and using the fact that $\widehat{A}_p \cong \widehat{A}^*_p \otimes Z_j$ we obtain

$$0 \to Z_j \to (\widehat{U}_n)_p \otimes Z_j \to \widehat{A}_p \to 0.$$

This sequence is not split, hence $j = 1$. This proves (1).

To prove (2), we tensor both sides of $\widehat{A}_p \cong \widehat{A}^*_p \otimes Z_1$ by $\widehat{A}_p$ and we get

$$(\widehat{G}_n)_p \cong \widehat{A}^*_p \otimes Z_1 \otimes \widehat{A}^*_p \otimes Z_1 \cong \widehat{A}^*_p \otimes \widehat{A}^*_p \otimes Z_2 \cong G_2.$$

To prove (3), we tensor $\widehat{A}_p \cong \widehat{A}^*_p \otimes Z_1$ by $A^*_p$ and we get

$$B_p \cong A^*_p \otimes \widehat{A}_p \cong \widehat{A}^*_p \otimes \widehat{A}^*_p \otimes Z_1 \cong G_1.$$

*Remark.* The lattice $A_2$ is an important lattice for the following reason. Consider the exact $Z_pN$-sequence

$$0 \to \widehat{Z}_p \to \widehat{Z}_pH \to A^*_p \to 0.$$

Tensoring by $A_2$ we get

$$0 \to A_2 \to \widehat{Z}_pH \otimes A_2 \to G_2 \to 0.$$

Since $ZH \cong Z(N/C)$, $\mathrm{Res}^N_C\, U_n \cong Z \oplus ZC$ by Mackey's subgroup theorem [CR1, Theorem 10.13]. As in the proof of Proposition 1.1, $(\widehat{U}_n)_q \cong A^*_q \oplus Z_q \cong A_q \oplus Z_q$ for all primes $q$ dividing $p-1$. By the K-S-A and by [CR1, Theorem 30.17] this implies that $A^*_q \cong A_q \cong Z_qC$. If we let $Q$ be the rationals this implies that $QA \cong QA^*$ as $QC$-modules. Since $Z_pC$ is a maximal order in $QC$ we have $A_p \cong A^*_p \cong Z_pC$ as $Z_pC$-modules by [CR1, Proposition 31.2]. Thus $A_k \cong \widehat{Z}_pC$ as $\widehat{Z}_pC$-modules for all $k = 1, \dots, p-1$. Hence $\widehat{Z}_pH \otimes A_2 \cong \widehat{Z}_pN$, so $\phi(A_2) = [G_2]$. By the same argument we have $\phi(A_k) = [G_k]$ for all $k = 1, \dots, p-1$.

By the above $(G_n)_q$, $(G^*_n)_q$ and $B_q$ are $Z_qC$-free for all primes $q$ dividing $p-1$. This fact will be needed later.

We will now use the machinery developed here to give new and very simple proofs for $p = 5$ and $p = 7$.

**Theorem 3.3.** *The center of the ring of* $5 \times 5$ *generic matrices is stably rational over* $C$.

*Proof.* Let $p = 5$; then

$$\widehat{Z}_pN/H = \bigoplus_{k=1}^{4} Z_k$$

and so

$$\widehat{Z}_pN \otimes_{\widehat{Z}_pH} A^*_p = A^*_p \oplus A_1 \oplus A_2 \oplus A_3.$$

Now $A_2 \cong \widehat{A}_p^* \otimes Z_2 \cong \widehat{A}_p^{*-}$, since $Z_2 \cong \widehat{Z}_p^-$. Tensoring both sides of $A_2 \cong \widehat{A}_p^* \otimes Z_2$ by $\widehat{A}_p^*$ we get by Theorem 3.2

$$(\widehat{G}_N)_p \cong (\widehat{G}_N^{*-})_p.$$

Now

$$\widehat{Z}_p N \otimes_{Z_p HC_2} A_p^* \cong \widehat{A}_p^* \oplus A_2.$$

Tensoring for $\widehat{A}_p^*$ we get

$$\widehat{Z}_p N \otimes_{Z_p HC_2} (\widehat{G}_N^*)_p \cong (\widehat{G}_N^*)_p \oplus (\widehat{G}_N^{*-})_p.$$

Therefore

$$Z_p N \otimes_{Z_p HC_2} (G_n^*)_p \cong (G_n^*)_p \oplus (G_n)_p,$$

and by [CR1, Theorem 30.17]

$$Z_p N \otimes_{Z_p HC_2} (G_n^*)_p \cong (G_n^*)_p \oplus (G_n)_p.$$

By [CT, Proposition 3, section 1], $\phi((A_n^*)) = 0$ as a $ZHC_2$-module since $C_2$ is of order 2. Since $G_n^*$ is invertible, this implies that $ZN \otimes_{ZHC_2} G_n^*$ is stably permutation as a $ZN$-module. Now adding $W_p^*$ to both sides of

$$Z_p N \otimes_{Z_p HC_2} (G_n^*)_p = (G_n^*)_p \oplus (G_n)_p,$$

we get by Lemma 2.8

$$Z_p N \otimes_{Z_p HC_2} (G_n^*)_p \oplus W_p^* \cong (G_n)_p \oplus Z_p N^k \oplus Z_p N/H.$$

Since $ZN^k \oplus ZN/H$ and $ZN \otimes_{ZHC_2} (G_n^*) \oplus W^*$ are $ZC$-free by Lemma 2.8, Theorem 2.7 implies that $\mathbf{C}(G_n \oplus ZG^k \oplus ZG/H)$ and $\mathbf{C}(ZG/N \otimes W^* \oplus ZG/HC_2 \otimes G_n^*)$ are stably equivalent as $G$-fields. By Lemma 2.1, $\mathbf{C}(G_n)$ is stably equivalent to $\mathbf{C}(ZG/N \otimes W^*)$ as $G$-fields. Now $\phi(A^*) = [G_n^*] = \Phi(W^*)$, hence $\mathbf{C}(ZG/N \otimes W^*)$ and $\mathbf{C}(ZG/N \otimes A^*)$ are stably equivalent as $G$-fields by Lemma 2.1. By Corollary 2.5 these fields are stably equivalent to $\mathbf{C}(A^*)$. Therefore by Theorem 2.6, $C_n$ is stably isomorphic to $\mathbf{C}(A^*)^G$ which is rational over $\mathbf{C}$, generated by the $(n-1)$-st elementary symmetric function [BL].

**Theorem 3.4.** *The center of the ring of* $7 \times 7$ *generic matrices is stably rational over* $\mathbf{C}$.

*Proof.* The proof is similar to that of $p = 5$. Here $p - 1 = 6$.

$$\widehat{Z}_p N/NC_3 \cong \widehat{Z}_p \oplus Z_2 \oplus Z_4$$

so

$$\widehat{Z}_p N/HC_3 \otimes \widehat{A}_p^* \oplus A_2 \oplus A_4.$$

Tensoring by $\widehat{A}_p^*$ we get by Theorem 3.2 and [CR1, Theorem 30.17]

$$Z_p N/HC_3 \otimes G_n^* \cong (G_n^* \oplus G_n \oplus B^-)_p$$

since $A_4 \cong \widehat{A}_p^-$. Now consider the exact sequence

$$0 \to B^- \to ZN/HC_2 \otimes B \to B \to 0.$$

This sequence splits at all primes different from 2, and for the prime 2, $\mathrm{Ext}^1_N(B_2, B_2^-)$ injects into $\mathrm{Ext}^1_{ZC_3}(B_2, B_2^-) = 0$ since $B_2$ is $Z_2C$-free. So $ZN/HC_2 \otimes B \cong B \oplus B^-$, and $B^-$ is stably permutation. Also since $C_3$ is of order 2, $ZN/HC_3 \otimes G_n^*$ is stably permutation by [CT, Proposition 3, section 1]. Now adding $W_p^*$ to both sides of

$$Z_pN \otimes_{Z_pHC_3} (G_n^*)_p = (G_n^*)_p \oplus (G_n)_p \oplus B_p^-$$

we get by Lemma 2.8

$$Z_pN \otimes_{Z_pHC_3} (G_n^*)_p \oplus W_p^* \cong (G_n)_p \oplus Z_pN^k \oplus Z_pN/H \oplus B_p^-.$$

Since $ZN^k \oplus ZN/H \oplus B^-$ and $ZN \otimes_{ZHC_3} (G_n^*) \oplus W^*$ are $ZC$-free, Theorem 2.7 implies that $\mathbf{C}(G_n \oplus ZG^k \oplus ZG/H \oplus ZG/H \otimes B^-)$ and $\mathbf{C}(ZG/N \otimes W^* \oplus ZG/HC_3 \otimes G_n^*)$ are stably equivalent as $G$-fields. The argument is now the same as for $p = 5$.

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