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CONNECTED FINITE LOOP SPACES WITH MAXIMAL TORI

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Abstract. Finite loop spaces are a generalization of compact Lie groups. However, they do not enjoy all of the nice properties of compact Lie groups. For example, having a maximal torus is a quite distinguished property. Actually, an old conjecture, due to Wilkerson, says that every connected finite loop space with a maximal torus is equivalent to a compact connected Lie group. We give some more evidence for this conjecture by showing that the associated action of the Weyl group on the maximal torus always represents the Weyl group as a crystallographic group. We also develop the notion of normalizers of maximal tori for connected finite loop spaces, and prove for a large class of connected finite loop spaces that a connected finite loop space with maximal torus is equivalent to a compact connected Lie group if it has the right normalizer of the maximal torus. Actually, in the cases under consideration the information about the Weyl group is sufficient to give the answer. All this is done by first studying the analogous local problems.

1. Introduction

A *loop space* $L := (L, BL, e)$ is a triple consisting of two spaces L and BL, which is pointed, and an equivalence $e : \Omega BL \to L$ between the loop space of BL and L. The space BL is called the *classifying space* of L. A loop space L is called *finite* if L is $\mathbb{Z}$–finite, i.e. the integral cohomology $H^*(L;\mathbb{Z})$ is a finitely generated graded $\mathbb{Z}$–module. A finite loop space L is called *connected* if L is a connected space. For a compact Lie group G the triple (G, BG, e) is a finite loop space, where $e : \Omega BG \to G$ is the obvious equivalence.

Following an old idea of Rector [19], several notions of (Lie) group theory can be given in homotopy theoretic terms by means of classifying spaces (e.g. see [19], [18], [5], [6], [13]). For example, a *homomorphism* $f : L \to M$ of finite loop spaces is a pointed map $Bf : BL \to BM$. The homotopy fiber of Bf is denoted by M/L. A homomorphism $f : L \to M$ is an *isomorphism* if Bf is a homotopy equivalence. In particular, we are interested in maximal tori and the associated Weyl groups of connected finite loop spaces. We will use the following definition:

1.1 Definition. Let L be a connected finite loop space. A *maximal torus* of L is a homomorphism $f : T \to L$ from a torus T into L which satisfies the following two conditions:

1. The homogeneous space L/T is $\mathbb{Z}$–finite.

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2. The rank $rk(L)$ of L equals the rank $rk(T)$ of T. The rank is defined to be the transcendence degree of the polynomial ring $H^*(BL;\mathbb{Q})$ over $\mathbb{Q}$.

The *Weyl monoid* $W_L := W_L(T)$ of a maximal torus $f : T \to L$ is the monoid

$$W_L(T) := \{\ [w : BT \to BT]\ |Bf \circ w \simeq Bf\}$$

of all homotopy classes of self maps of BT stabilizing the structure map $Bf : BT \to BL$.

The classical maximal torus $T_G \to G$ of a compact connected Lie group G passes to a fibration $G/T_G \to BT \to BG$, and an element $w \in W_G$ of the Weyl group W_G of G give rise to a homotopy commutative diagram

$$\begin{tikzcd} BT_G \arrow[rr, "w"] \arrow[dr] & & BT_G \arrow[dl] \\ & BG & \end{tikzcd}$$

Hence, the above definitions are the extract of the classical situation in terms of classifying spaces. We also show that every two maximal tori of a finite loop space are conjugate and that the definition of the Weyl group does not really depend on the chosen maximal torus (Proposition 3.5).

Usually, when talking about finite loop spaces, the $\mathbb{Z}$–finiteness condition is replaced by the condition of being homotopy equivalent to a finite CW–complex, which is a slightly stronger condition. For our purpose $\mathbb{Z}$–finiteness is sufficient.

Having a maximal torus is a quite distinguished property for finite loop spaces. It is well known that there exist finite loop spaces which don't have a maximal torus. Examples may be found in [19] or [18]. Actually, an old conjecture, which we could trace back to [20], says that a connected finite loop space a with maximal torus is isomorphic to a compact connected Lie group (as loop space). We want to give some more evidence for this conjecture. An integral representation $W \to Gl(n,\mathbb{Z})$ of a finite group W is called crystallographic, if the associated rational representation $W \to Gl(n,\mathbb{Q})$ represents W as a reflection group. We define $LT_L := H_2(BT_L,\mathbb{Z})$ and $L^*T_L := H^2(BT_L,\mathbb{Z})$.

1.2 Theorem. *Let L be a finite loop space with maximal torus $T \to L$. Then the following hold:*

1. *The Weyl monoid W_L is a group.*
2. *The induced action of W_L on L^*T_L represents W_L as a crystallographic group.*
3. *We have $H^*(BL;\mathbb{Q}) \cong H^*(BT;\mathbb{Q})^{W_L}$.*

The representation $W_L \to Gl(L^*T_L)$ is called the associated representation.

Examples of crystallographic representations $W \to Gl(n,\mathbb{Z})$ are given by actions of Weyl groups of compact connected Lie groups on the 2–dimensional homology or cohomology of the classifying space of the maximal torus. We say that a connected finite loop space L with maximal torus $T_L \to L$ has the *same Weyl group type* as a compact connected Lie group G if $rk(L) = rk(G) =: n$ and if the two associated representations $W_L, W_G \to Gl(n,\mathbb{Z})$ are conjugate. The following question is a weaker and slightly different form of the above mentioned conjecture. Let L be a connected finite loop space with the same Weyl group type as the compact connected Lie group G. Does this imply that L and G are isomorphic as loop spaces? This conjecture can't be true. The compact connected Lie groups $SO(2n+1)$

and $Sp(n)$ are not isomorphic as loop spaces, because the existence of a homotopy equivalence $BSO(2n+1) \simeq BSp(n)$ would imply that $SO(2n+1)$ and $Sp(n)$ were isomorphic as Lie groups [16]. But on the other hand they have the same Weyl group type.

The right invariant to distinguish the isomorphism types of connected finite loop spaces with maximal tori should be given by the normalizer of the maximal torus. Some evidence for this conjecture comes from [16], where it is proved that two compact connected Lie groups G and H are isomorphic (as Lie groups) if and only if the classifying spaces are homotopy equivalent if and only if the classifying spaces of the two normalizers of the maximal tori are equivalent if and only if the normalizers itself are isomorphic (as Lie groups) for the loop spaces associated with compact connected Lie groups.

1.3 Definition. Let L be a connected finite loop space with maximal torus $T_L \to L$, The *normalizer* of T_L is a monomorphism $N(T_L) \to L$ of finite loop spaces such that $N(T_L)$ fits up to homotopy into a fibration

$$BT_L \to BN(T_L) \to BW_L$$

and such that

$$\begin{tikzcd} BT_L \arrow[r] \arrow[d] & BL \\ BN(T_L) \arrow[ur] & \end{tikzcd}$$

commutes up to homotopy.

1.4 Proposition. *Let L be a connected finite loop space with maximal torus $T_L \to L$. Then there exists a normalizer $N(T_L)$ of the maximal torus, and $N(T_L)$ is isomorphic to a uniquely determined compact Lie group as loop space.*

In order to proceed with the maximal torus conjecture we first want to fix some notation about compact connected Lie groups. For every compact connected Lie group G there exists a finite covering $K \to G_s \times T \to G$ of compact Lie groups, where G_s is simply connected, where T is a torus, and where $K \subset G_s \times T$ is a finite central subgroup.

1.5 Definition. Let G be a compact connected Lie group.

1. G satisfies condition (T_p) if $H^*(G;\mathbb{Z})$ or, equivalently, $H^*(BG;\mathbb{Z})$ is p–torsion free.
2. G satisfies condition (I_p) if $H^*(BG;\mathbb{F}_p) \cong H^*(BT_G;\mathbb{F}_p)^{W_G}$.
3. G satisfies condition (C_p) if G satisfies (T_p) and G_s the condition (I_p).

In [15] a compact connected Lie group G is called p–convenient if it satisfies (C_p). This condition was used there to prove homotopy uniqueness results for classifying spaces of compact connected Lie groups. We will apply similar methods here to classify the isomorphism type of compact connected Lie groups as loop spaces using the normalizer. Each of the above three conditions is weaker than the following one. But for an odd prime all three conditions are equivalent [15]. We also notice that $U(2)$ satisfies (I_2) but not (C_2), because $SU(2)$ does not satisfy (T_2). A complete list of simple simply connected compact Lie groups satisfying (C_p) may be found in [15, Chapter 1].

1.6 Theorem. *Let G be a compact connected Lie group satisfying conditions (C_2) and (T_p) for all primes. Let L be a connected finite loop space with maximal torus $T_L \to L$. Then the following statements are equivalent:*

1. *L has the same Weyl group type as G.*
2. *$N(T_G)$ and $N(T_L)$ are isomorphic as Lie groups.*
3. *$BN(T_G)$ and $BN(T_L)$ are homotopy equivalent.*
4. *BG and BL are homotopy equivalent.*

1.7 *Remark.* There is an interesting observation related to Proposition 1.4. For a given extension $N(T)$ of an operation of a Weyl group W on a maximal torus T it is not known if there exists a connected finite loop space with these data. For example, there are two extensions of the permutation representation of the symmetric group $\Sigma_n \cong W_{U(n)}$ on $(S^1)^n = T_{U(n)}$, namely the semidirect product and a nontrivial extension. But Theorem 1.6 says that only the semidirect product can occur as the normalizer of a maximal torus of a connected finite loop space.

To study finite loop spaces, Dwyer and Wilkerson introduced the notion of p–compact groups in their influential paper [5]. A loop space X is a *p–compact group* if the classifying space BX is p–complete and if X is $\mathbb{F}_p$–finite, i.e. $H^*(X;\mathbb{F}_p)$ is finite [5]. The completion of a connected finite loop space always gives a p–compact group. For nonconnected finite loop spaces one has to assume that the group of the components is a finite p–group.

The p–compact groups behave very much like compact Lie groups. For example, a p–compact group always has a maximal torus in the sense of [5]. Dwyer and Wilkerson also constructed a normalizer $N(T_X)$ of the maximal torus $T_X \to X$ which satisfies the analogous properties as in Lemma 1.3 of [5] (see also Section 2). The classifying space $BN(T_X)$ fits into a fibration $BT_X \to BN(T_X) \to BW_X$. In particular, $BN(T_X)$ is a fiberwise complete space with respect to this fibration. For details see Section 2.

For a fibration $F \to E \to B$ we denote by E_p° the fiberwise p–adic completion [4].

1.8 Proposition. *Let L be a connected finite loop space.*

1. *For any prime p the triple $(L_p^\wedge, BL_p^\wedge, e_p^\wedge)$ is a p–compact group.*
2. *If $T \to L$ is a maximal torus of L, then completion gives rise to a maximal torus $T_p^\wedge \to L_p^\wedge$ and induces a natural isomorphism $W_L \cong W_{L_p^\wedge}$ and a homotopy equivalence $BN(T_L)_p^\circ \simeq BN({T_L}_p^\wedge)$.*

The Weyl group W_X of a p–compact group X acts on $L^*{T_X}_p^\wedge := H^*(BT_X;\mathbb{Z}_p^\wedge)$ and gives rise to a representation $W_X \to Gl(L^*{T_X}_p^\wedge)$. By the above lemma the completion of a connected finite loop space with maximal torus gives a connected p–compact group with maximal torus. We say that a connected p–compact group X has the *p–adic Weyl group type* of a compact connected Lie group G if $rk(X) = rk(G) =: n$ and if the associated representations $W_X, W_G \to Gl(n,\mathbb{Z}_p^\wedge)$ are conjugate.

The next statement is the completed version of Theorem 1.6.

1.9 Theorem. *Let G be a compact connected Lie group satisfying condition (T_p) if p is odd or (C_2) if $p = 2$. Let X be a connected p–compact group. Then the following conditions are equivalent:*

1. *X has the same p–adic Weyl group type as G.*

2. *$BN(T_G)^\circ_p$ and $BN(T_X)$ are homotopy equivalent.*
3. *$BG^\wedge_p$ and BX are homotopy equivalent, i.e. $G^\wedge_p$ and X are isomorphic p–compact groups.*

Using the p–adic result, we are able to prove Theorem 1.6.

Proof of Theorem 1.6. Let us assume that (1) is satisfied, i.e. L and G have the same Weyl group type. The Weyl group type of a connected finite loop space determines the p–adic Weyl group type of the associated connected p–compact group. By Theorem 1.9 this implies that $BL^\wedge_p \simeq BG^\wedge_p$ for every prime p. Moreover, we also have a rational equivalence, because both spaces are rationally a product of Eilenberg–Mac Lane spaces, and because $H^*(BL;\mathbb{Q}) \cong H^*(BT_G;\mathbb{Q})^{W_G} \cong H^*(BG;\mathbb{Q})$ (Theorem 1.2). That is to say that BL and BG have the same adic genus. By assumption L has a maximal torus, which implies that there exists a compact connected Lie group H such that $BL \simeq BH$ [14]. Because the two associated representations of the Weyl groups are isomorphic, we also have $K(BG) \cong K(BT_G)^{W_G} \cong K(BT_H)^{W_H} \cong K(BH)$ as λ–rings. The compact connected Lie group G satisfies condition (C_2). Because every quotient of a Spin–group has 2–torsion in the integral cohomology, the finite cover $G_s \times T$ of G cannot contain a factor isomorphic to $Spin(n)$ for some n. In this situation we can apply [15, 1.6] to conclude that $BX \simeq BG$. This is condition (4).

If $BL \simeq BG$, then $BN(T_G)^\circ_p \simeq BN(T_L)^\circ_p$ for any prime (Theorem 1.9). This is really an equivalence of the associated fibrations, and implies therefore that $BN(T_G) \simeq BN(T_L)$ [16, proof of 3.6], which is condition (3). By [16, Theorem A], a homotopy equivalence $BN(T_G) \simeq BN(T_L)$ establishes an isomorphism $N(T_G) \cong N(T_L)$ as compact Lie groups, which is condition (2).

If $N(T_G) \cong N(T_L)$ as compact Lie groups, then it is obvious that G and L have the same Weyl group type. This completes a circle of implications. □

The paper is organized as follows: Section 2 is devoted to p–compact groups. From [5] we will recall how notions of (Lie) group theory translate to p–compact groups. We also recall necessary results from [6] and [13]. In Section 3 we discuss the (integral) Weyl group of finite loop spaces and prove Theorem 1.2, Proposition 1.4 and Proposition 1.8. Section 4 provides some more results about p–compact groups necessary for the proof of Theorem 1.9. In particular, we study homotopy classes of liftings of maps into a connected p–compact group to the maximal torus. The last three sections are devoted to the proof of Theorem 1.9, which is split into several cases. First we discuss uniqueness results for products of unitary groups (Section 5), then for simply connected compact Lie groups (Section 6), and finally in the general case (Section 7). Section 7 also contains a proof of Theorem 1.9.

Completion and localization are always meant in the sense of Bousfield and Kan [4].

2. p–COMPACT GROUPS

In this section we recall the basic notions and the fundamental results about p–compact groups from [5]. Most of the notions are motivated by classical Lie group theory and by passing to classifying spaces. To keep things short, and because the analogy to compact Lie groups is discussed in [5], [6] and [13], we omit motivations.

2.1 Homomorphisms, monomorphisms, isomorphisms, subgroups and exact sequences. A *homomorphism* $f : Y \to X$ of p–compact groups is a pointed map $BY \to BX$. The map f is called an *isomorphism* if $Bf : BY \to BX$ is an equivalence. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of p–compact groups is *short exact* if the associated sequence $BX \xrightarrow{Bf} BY \xrightarrow{Bg} BZ$ is a fibration up to homotopy. A *monomorphism* of p–compact groups is a map $BX \to BY$ whose homotopy fiber, denoted by Y/X, is $\mathbb{F}_p$–finite. A subgroup $Y \to X$ of a p–compact group X is a monomorphism of p–compact groups.

2.2 p–compact toral groups. A *p–compact toral group* P is a p–compact group P fitting into a short exact sequence $T \to P \to \pi$, where T is a p–adic torus and where π is a finite p–group (as a p–compact group), i.e. $B\pi \simeq K(\pi, 1)$ and π is an honest finite group. For every p-compact toral group P, there exists a locally finite group P_∞, which actually is the union of finite groups, and a map $BP_\infty \to BP$ which becomes an equivalence after completion. $P_\infty \to P$ is called the p–discrete approximation of P [5, 6.4].

2.3 Conjugation and subconjugation. Two homomorphisms $f, g : Y \to X$ of p–compact groups are called conjugate if the induced maps $Bf, Bg : BY \to BX$ are freely homotopic.

For a homomorphism $f : Y \to X$ of p–compact groups and for a p–compact toral subgroup $i : P \to X$ we say that P is subconjugate to Y if there exists a homomorphism $j : P \to Y$ such that fj and i are conjugate.

2.4 Centralizers. For a homomorphism $f : Y \to X$ between p–compact groups, the *centralizer* $C_X(f(Y))$ is defined to be the loop space given by the triple

$$C_X(f(Y)) := (\Omega map(BY, BX)_{Bf}, map(BY, BX)_{Bf}, id) \ .$$

The evaluation at the basepoint $ev : map(BY, BX)_{Bf} \to BX$ establishes a homomorphism $C_X(f(Y)) \to X$ of loop spaces. If Y is a p–compact toral group, then the centralizer $C_X(f(Y))$ is again a p–compact group and the evaluation $C_X(f(Y)) \to X$ is a monomorphism [5, 5.1, 5.2 and 6.1].

2.5 Maximal tori. The maximal torus of a p–compact group X is a monomorphism $T_X \to X$ of a p-compact torus into X such that the centralizer $C_X(T_X)$ is a p-compact toral group whose component of the unit is given by T_X [5]. Because T_X centralizes itself, there is a homomorphism $T_X \to C_X(T_X)$

2.6 Theorem [5, 8.11, 8.13 and 9.1]. *Let X be a p–compact group.*

1. *X has a maximal torus $T_X \to X$.*
2. *Any subtorus $T \to X$ of X is subconjugated to the maximal torus $T_X \to X$.*
3. *Any two maximal tori of X are conjugate.*
4. *If X is connected, then $T_X \to C_X(T_X)$ is an isomorphism.*
5. *If X is connected, then every finite cyclic subgroup $\mathbb{Z}/p^n \to X$ of X is subconjugate to T_X.*

2.7 Proposition. *Let X be a p–compact group, and let $T \to X$ be a monomorphism of a p-compact torus into X. Then the following conditions are equivalent:*

1. *$rk(T) = rk(X)$.*
2. *The homomorphism $T \to X$ is a maximal torus.*

Proof. Let $T \to X$ be a subtorus satisfying condition (1). By Theorem 2.6, there exists a maximal torus $T_X \to X$, and $T \to X$ lifts up to conjugation to a monomorphism $T \to T_X$. This is an isomorphism because of the rank condition, and $T \to X$ is a maximal torus.

If $T_X \to X$ is a maximal torus, then $rk(X) = rk(T)$ (Theorem 2.9 (2)), and hence $T \to X$ satisfies condition (1). □

2.8 Weyl spaces and Weyl groups. Let $T_X \to X$ be a maximal torus of a p–compact group. We think of $BT_X \to BX$ as being a fibration. The Weyl space $\mathcal{W}_T(X)$ is defined to be the mapping space of all fiber maps over the identity on BX. Then each component of $\mathcal{W}_T(X)$ is contractible and the Weyl group $W_T(X) := \pi_0(\mathcal{W}_T(X))$ is a finite group under composition [5, 9.5].

2.9 Theorem [5, 9.5 and 9.7]. *Let $T_X \to X$ be the maximal torus of a connected p–compact group X.*

1. *The action of W_X on BT_X induces representations*
$$W_X \to Aut(H^2(BT_X;\mathbb{Z}_p^\wedge)\otimes\mathbb{Q}) \cong Gl(n,\mathbb{Q}_p^\wedge)$$
and
$$W_X \to Aut(H_2(BT_X;\mathbb{Z}_p^\wedge)\otimes\mathbb{Q}) \cong Gl(n,\mathbb{Q}_p^\wedge)$$
which are monomorphisms whose images are generated by pseudo reflections.
2. *The map $H^*_{\mathbb{Q}_p^\wedge}(BX) \to H^*_{\mathbb{Q}_p^\wedge}(BT_X)^{W_X}$ is an isomorphism.*

Here, we define $H^*_{\mathbb{Q}_p^\wedge}(-) := H^*(-;\mathbb{Z}_p^\wedge)\otimes\mathbb{Q}$.

2.10 Proposition. *Let X be a connected p–compact group. Then $H^2(BX;\mathbb{Z}_p^\wedge) \to H^2(BT_X;\mathbb{Z}_p^\wedge)^{W_X}$ is an isomorphism.*

Proof. The statement is true for simply connected X, because $H^2(BT_X;\mathbb{Z}_p^\wedge)^{W_X} \subset H^2_{\mathbb{Q}_p^\wedge}(BT_X)^{W_X} \cong H^2_{\mathbb{Q}_p^\wedge}(BX) = 0$ in this case.

For the general case, we recall [13] that there exists a diagram of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, equal] & T_{X_s}\times T \arrow[r, "q"] \arrow[d] & T_X \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K \arrow[r] & X_s\times T \arrow[r, "q"] & X \arrow[r] & 1
\end{tikzcd}$$

of p–compact groups, where X_s is simply connected , where T is a p-compact torus and where $K \to X_s \times T$ is a central finite subgroup. The projection induces an isomorphism $W_{X_s} \cong W_X =: W$ between the Weyl groups, which we identify via this map. Then, the upper horizontal line is W equivariant (up to homotopy).

Applying the Serre spectral sequence to both fibrations and passing to invariants establishes a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^2(BT_X;\mathbb{Z}_p^\wedge)^W \arrow[r] & H^2(BT_{X_s}\times BT;\mathbb{Z}_p^\wedge)^W \arrow[r] & H^2(BK;\mathbb{Z}_p^\wedge) \\
0 \arrow[r] & H^2(BX;\mathbb{Z}_p^\wedge) \arrow[r] \arrow[u] & H^2(BX_s\times BT;\mathbb{Z}_p^\wedge) \arrow[r] \arrow[u] & H^2(BK;\mathbb{Z}_p^\wedge) \arrow[u, equal]
\end{tikzcd}$$

Both rows are exact. The middle terms in both rows are isomorphic to $H^2(BT;\mathbb{Z}_p^\wedge)$. For the top row, this follows from Theorem 2.9. Hence, the right and middle vertical

arrows are isomorphisms, and so is the left one by the five lemma. This finishes the proof. □

2.11 Proposition. *Let p be an odd prime and X a p–compact group. If the associated representation $W_X \to Gl(H^2(BT_X;\mathbb{Z}_p^\wedge)$ is a pseudo reflection group, then X is connected.*

Proof. Because W_X is a p–adic pseudo reflection group, it is generated by elements of order coprime to p. By [13, 3.8] there exists an epimorphism $W_X \to \pi_0(X)$. Because $\pi_0(X)$ is a finite p–group, this homomorphism is also the trivial map. Thus, X is connected. □

2.12 *Remark.* Let X be a p–compact group. The proof of Proposition 2.11 shows that X is connected if any homomorphism $W_X \to \pi$, from the Weyl group into a finite p–group, always has a kernel which is not a pseudo reflection group. For example, this is true if $p = 2$ and if X has the 2–adic Weyl group type of G, where G is a quotient of a product of $SU(n)$'s, $n \geq 3$, and of a torus. In particular this is true if G satisfies condition (C_2).

2.13 Normalizers and p–normalizers of maximal tori and p–toral Sylow subgroups. Let $i : T_X \to X$ be a maximal torus of a p–compact group X. Again we think of $BT_X \to BX$ as being a fibration. The Weyl space $\mathcal{W}_X$ acts on BT_X via fiber maps. This establishes a monoid homomorphism $\mathcal{W}_X \to HE(BT_X)$, where $HE(BT_X)$ denotes the monoid of all self-equivalences of BT_X. Passing to classifying spaces establishes a map $B\mathcal{W}_X \to BHE(BT_X)$, which can be thought of as being a classifying map of a fibration $BT_X \to BN(T_X) \to B\mathcal{W}_X$. The total space gives the classifying space of the normalizer $N(T_X)$ of T_X. This is always a finite extension of the p-compact torus T_X.

Let $\mathcal{W}_p$ be the union of those components of $\mathcal{W}_X$ corresponding to a p–Sylow subgroup W_p of W_X. The restriction of the above construction to $\mathcal{W}_p$ gives the classifying space of the p–normalizer $N_p(T_X)$.

Since the action of $\mathcal{W}_X$ respects the map $BT_X \to BX$, the monomorphism $T_X \to X$ extends to a loop map $N(T_X) \to X$.

2.14 Proposition. *For a p–compact group X, the map*

$$H^*(BX;\mathbb{F}_p) \to H^*(BN_p(T_X);\mathbb{F}_p)$$

is a monomorphism.

Proof. This follows from [5, proof of 2.3 and 9.13]. □

2.15 Centers. A subgroup $Z \to X$ of a p–compact group X is called *central* [5] if the homomorphism $C_X(Z) \to X$ is an isomorphism. The *center* $Z(X)$ of X is the maximal central subgroup of X [6, 1.2], [13, 4.3, 4.4]. To give an explicit definition we use a result of Dwyer and Wilkerson [6, 1.3]. For every p–compact group X, the centralizer $C_X(X)$ is a p–compact group and $Z(X) := C_X(X) \to X$ is the center of X.

For every p–compact group X there exists a short exact sequence $Z(X) \to X \to X/Z(X) =: PX$ of p–compact groups, and, if X is connected, the quotient PX has a trivial center [13, 4.7]. We call a p–compact group X *centerfree* if $Z(X)$ is the trivial group.

For connected p–compact groups, there is another description of the center in terms of the fixed point set of the Weyl group action on the maximal torus. Let

$T_X \to X$ be the maximal torus of a connected p–compact group X. The Weyl group W_X acts on the p-discrete approximation $\check{T} \subset T_X$ of T_X. Here, we can consider $\check{T}$ as an honest locally finite group with an honest W_X–action. Then we define $T_X^{W_X}$ to be the p–compact group given by the equivalence $BT_X^{W_X} \simeq B(\check{T}^{W_X})_p^\wedge$ between the classifying spaces.

2.16 Proposition. *Let X be a connected p–compact group. If one of the following three conditions is satisfied, then $T_X^{W_X} \to X$ is the center of X.*

1. *p is odd.*
2. *$T_X^{W_X}$ is connected; i.e. $T_X^{W_X}$ is a p-compact torus.*
3. *$H^*(BX;\mathbb{F}_p) \cong H^*(BT_X;\mathbb{F}_p)^{W_X}$.*

Proof. If p is odd, this is proved in [6, 7.7]. If $T_X^{W_X}$ is a p-compact torus, then $C := C_X(T_X^{W_X})$ is connected [13, 3.11] with Weyl group $W_C = W_X$. Therefore, $C \to X$ induces an isomorphism in rational cohomology between the classifying spaces (Theorem 2.9) and between the spaces itself, and is even an isomorphism between connected p–compact groups [13, 3.7].

If $H^*(BX;\mathbb{F}_p) \cong H^*(BT_X;\mathbb{F}_p)^{W_X}$, then

$$H^*(BC_X(T_X^{W_X});\mathbb{F}_p) \cong H^*(BT_X;\mathbb{F}_p)^{W'}$$

[15, 10.1]. Here, the subgroup $W' \subset W_X$ consists of all elements acting trivially on $T_X^{W_X}$. That is to say that $W' = W_X$. Therefore, $BC_X(T_X^{W_X}) \to BX$ is a homotopy equivalence and $C_X(T_X^{W_X}) \to X$ an isomorphism. □

3. The Weyl group of a connected finite loop space

In this section L always denotes a connected finite loop space with maximal torus $f : T_L \to L$. We are going to discuss the Weyl group W_L of L. In particular we will prove Theorem 1.2.

Passing to completion at a prime p, we get a p–compact group $L_p^\wedge$ with maximal torus ${T_L}_p^\wedge \to L_p^\wedge$ and Weyl group $W_{L_p^\wedge}$. These pieces satisfy the identity $H^*_{\mathbb{Q}_p^\wedge}(BL_p^\wedge) \cong H^*_{\mathbb{Q}_p^\wedge}(B{T_L}_p^\wedge)^{W_{L_p^\wedge}}$ by Theorem 2.9. That is to say that the monomorphism $H^*_{\mathbb{Q}_p^\wedge}(BL_p^\wedge) \to H^*_{\mathbb{Q}_p^\wedge}(B{T_L}_p^\wedge)$ is a Galois extension of integral domains which are integrally closed with Galois group $W_{L_p^\wedge}$. An integral domain is called integrally closed if it is integrally closed in its field of fractions.

To get global information we have to study the rational situation. First we have to clarify and to fix some notions and notations about integral ring extensions. We denote by $FF(A)$ the field of fractions of an integral domain A. An integral extension $A \to B$ of integral domains is called normal, separable or Galois if the associated extension $FF(A) \to FF(B)$ of the field of fractions is normal, separable or Galois.

3.1 Lemma. *Let $A \to B$ be a Galois extension of integral domains A and B, and let W be the Galois group of $FF(A) \to FF(B)$. If A and B are integrally closed, then $A = B^W$.*

Proof. Because $A \to B$ is an integral extension and because B is integrally closed, B is also the integral closure of A in $FF(B)$. Therefore, by naturality, W acts on B.

If $b \in B^W$, then b is integral over A and $b \in FF(A)$. This implies that $b \in A$, and hence $A = B^W$. □

3.2 Proposition. *Let L be a connected finite loop space with maximal torus $f : T_L \to L$. The map $H^*(BL;\mathbb{Q}) \to H^*(BT_L)$ is a monomorphism and an integral Galois extension of integrally closed domains.*

Proof. Let $A := H^*(BL;\mathbb{Q})$ and $B := H^*(BT_L;\mathbb{Q})$. Then, A and B are polynomial rings on $n = rk(L) = rk(T_L)$ generators. In particular, they are integrally closed domains and noetherian. Because the homogeneous space L/T_L is $\mathbb{Z}$–finite, a Serre spectral sequence argument shows that B is a finitely generated A–module. Because A and B have the same transcendence degree over $\mathbb{Q}$, the map $A \to B$ is an inclusion. This shows that $A \to B$ is an integral ring extension [10, XI, §1].

The characteristic of $\mathbb{Q}$ is zero, and therefore all extensions are separable. So it only remains to show that $A \to B$ is normal. Tensoring with $\mathbb{Z}_p^\wedge$ yields $A \otimes \mathbb{Z}_p^\wedge \cong H^*_{\mathbb{Q}_p^\wedge}(BL_p^\wedge)$ and $B \otimes \mathbb{Z}_p^\wedge \cong H^*_{\mathbb{Q}_p^\wedge}(BT_{Lp}{}^\wedge)$. Therefore, by Theorem 2.9, $A \otimes \mathbb{Z}_p^\wedge \cong (B \otimes \mathbb{Z}_p^\wedge)^{W_{L_p^\wedge}} \to B \otimes \mathbb{Z}_p^\wedge$ is an integral Galois extension of integrally closed domains. Now we can proceed as in [18, 3.5] to show that $A \to B$ is normal and hence a Galois extension. □

Proof of Proposition 1.8. Part (1) is obvious, as $\mathbb{Z}$–finiteness implies $\mathbb{F}_p$–finiteness and for a simply connected space Y we have $\Omega(Y_p^\wedge) \simeq (\Omega Y)_p^\wedge$ [4, VI, 6.5]. For (2), the same argument shows that $T_{Lp}{}^\wedge \to L_p^\wedge$ is a maximal torus. The isomorphism between the Weyl groups is shown in the proof of Theorem 1.2 below. □

Proof of Theorem 1.2. Let $W_\mathbb{Q}$ be the Galois group of the extension $H^*(BL;\mathbb{Q}) \to H^*(BT_L;\mathbb{Q})$. The Weyl group $W_p^\wedge$ of the p–compact group $L_p^\wedge$ is the Galois group of $H^*(BL;\mathbb{Q}) \otimes \mathbb{Z}_p^\wedge \to H^*(BT_L;\mathbb{Q}) \otimes \mathbb{Z}_p^\wedge$ (Theorem 2.9). So, the functor $\otimes \mathbb{Z}_p^\wedge$ induces a group homomorphism $W_\mathbb{Q} \to W_p^\wedge$. Because the completion $T_p^\wedge \to L_p^\wedge$ of the maximal torus of L is a maximal torus of $L_p^\wedge$ (Proposition 1.8), both groups $W_\mathbb{Q}$ and $W_p^\wedge$ are faithfully represented by the action on $H^*(BT;\mathbb{Q}) \otimes \mathbb{Z}_p^\wedge \cong H^*(BT_p^\wedge;\mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$. Because

$$(H^*(BT;\mathbb{Q}) \otimes \mathbb{Z}_p^\wedge)^{W_\mathbb{Q}} \cong H^*(BL;\mathbb{Q}) \otimes \mathbb{Z}_p^\wedge \cong (H^*(BT_p^\wedge;\mathbb{Z}_p^\wedge) \otimes \mathbb{Q})^{W_p^\wedge},$$

the homomorphism $W_\mathbb{Q} \to W_p^\wedge$ is an isomorphism.

In the diagram

$$\begin{tikzcd}
1 \arrow[r] & W \arrow[r] \arrow[d] & {[BT, BT]} \arrow[r] \arrow[d] & {[BT, BL]} \arrow[d] \\
1 \arrow[r] & W_p^\wedge \arrow[r] & {[BT_p^\wedge, BT_p^\wedge]} \arrow[r] & {[BT_p^\wedge, BL_p^\wedge]}
\end{tikzcd}$$

both rows are exact. For the top row this follows from the definition, and for the bottom row from Theorem 2.9. Hence, the third term in the rows contains the quotient of the first two terms as a subset. Here, W denotes the Weyl monoid of L. Therefore, p–completion induces for every prime p a monoid homomorphism $W \to W_p^\wedge$ which is injective. This map is also surjective. For let $w_p \in W_p^\wedge$ for some prime p. We choose elements $w_0 \in W_\mathbb{Q}$ and $w_q \in W_q^\wedge$ corresponding under the isomorphisms $W_q^\wedge \cong W_\mathbb{Q} \cong W_p^\wedge$. We realize w_0 as a self map of $BT_\mathbb{Q}$ over $BL_\mathbb{Q}$, and let w_q denote the associated self map of $BT_q^\wedge$ over $BL_q^\wedge$. Since $H^2(BT_p^\wedge;\mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \cong H^2(BT_\mathbb{Q};\mathbb{Q}) \otimes \mathbb{Z}_p^\wedge$ for all primes, an arithmetic square argument shows that there exists a self map $w : BT \to BT$ over BL such that $w_q^\wedge \simeq w_q \in [BT_q^\wedge, BT_q^\wedge]$. In particular, we have $W \cong W_p^\wedge$, and W is a group, which is part (1).

Because W is a subgroup of $[BT, BT]$, the integral cohomology group $H^2(BT;\mathbb{Z})$ gives an invariant sublattice of $H^2(BT;\mathbb{Q})$. Moreover, we have $H^*(BT;\mathbb{Q})^W \cong H^*(BT;\mathbb{Q})^{W_\mathbb{Q}} \cong H^*(BL;\mathbb{Q})$. Because $H^*(BL;\mathbb{Q})$ is a polynomial ring, the action of W on $H^2(BT;\mathbb{Q})$ represents W as a reflection group. This proves the last two parts of the statement. □

The proof of Theorem 1.2 shows that the integral Weyl group W_L and the p–adic Weyl groups $W_{L^\wedge_p}$ of a connected finite loop space L with maximal torus $T_L \to L$ are isomorphic for every prime p. We identify all these Weyl groups via this isomorphism. Using this fact one also can construct an integral normalizer of the maximal torus $T_L \to L$, which is the claim of Proposition 1.4.

Fibrations of the form $BT_L \to Y \to BW_L$ are classified by cohomology classes in $H^3(BW_L; LT_L)$, where W_L acts on $LT_L = H_2(BT_L;\mathbb{Z})$ via the action on the maximal torus. Fibrations over BW with fiber $BT_{L^\wedge_p}$ are classified by cohomology classes in $H^3(BW_L; LT_{L^\wedge_p})$. In both cases this follows from obstruction theory. Because W_L is a finite group we have an isomorphism $H^3(BW_L; LT_L) \cong \prod_p H^3(BW_L; LT_{L^\wedge_p})$. This isomorphism is given by the product over all p–adic fiberwise completions. Hence there exists a fibration $BT_L \to BN \to BW$, such that fiberwise completion induces an equivalence $BN^\circ_p \simeq BN(T_{L^\wedge_p})$. Moreover, by [16, 3.2], the space BN is the classifying space of a uniquely determined compact Lie group N fitting into a short exact sequence $1 \to T_L \to N \to W \to 1$. In particular, the group N gives a finite loop space, which we call the normalizer of the maximal torus $T_L \to L$ of L.

Next we want to construct a map $BN \to BL$. By the construction of N, for every prime p, there exists a map $BN \to BL^\wedge_p$ which is an extension of the map $BT_L \to BL^\wedge_p$ and which induces an isomorphism $H^*_{\mathbb{Q}^\wedge_p}(BN) \cong H^*_{\mathbb{Q}^\wedge_p}(BL)$. The isomorphism between the cohomology groups follows from the Serre spectral sequence for calculating the cohomology of BN and Theorem 2.9. The same argument using Theorem 1.2 instead of Theorem 2.9 shows that there exists an isomorphism $H^*(BN;\mathbb{Q}) \cong H^*(BL;\mathbb{Q})$. Because $BL_\mathbb{Q}$ is a product of Eilenberg-Mac Lane spaces, this last isomorphism can be realized by a map $BN \to BL_\mathbb{Q}$. All coherence conditions for using an arithmetic square are controlled by cohomology and are therefore satisfied by construction. This establishes a map $BN \to BL$ which, again by construction, is an extension of $BT_L \to BL$. We collect the properties of the group N, now called $N(T_L)$, in the following statement, which also includes Proposition 1.4.

3.4 Proposition. *Let L be a connected finite loop space with maximal torus $T_L \to L$ and Weyl group W_L. Then the following hold:*

1. *The triple $(N(T_L), BN(T_L), e)$ is a finite loop space. In particular, $N(T_L)$ is isomorphic to a uniquely determined compact Lie group given as a finite extension of T_L by W_L.*
2. *There exists a homomorphism $N(T_L) \to L$ of finite loop spaces which is an extension of $T_L \to L$.*

Finally we show, as promised in the introduction, that any two maximal tori of a finite loop space are conjugate.

3.5 Proposition. *Let $T \xrightarrow{g} L$ be a homomorphism of a torus into a connected finite loop space L with maximal torus $T_L \xrightarrow{f} L$. Then T is subconjugate to $T_L \xrightarrow{f} L$.*

Proof. Let $A := H^*(BL;\mathbb{Q})$, $B := H^*(BT_L;\mathbb{Q})$ and $C := H^*(BT;\mathbb{Q})$, and let $A_p^\wedge$, $B_p^\wedge$ and $C_p^\wedge$ denote the associated cohomology groups given by the theory $H^*_{\mathbb{Q}_p^\wedge}(\ )$. After completion at a prime p, we already know that T is subconjugate to ${T_L}_p^\wedge \to L_p^\wedge$ via a homomorphism $T_p^\wedge \xrightarrow{h_p^\wedge} {T_L}_p^\wedge$. Therefore, the diagram

$$(*) \qquad \begin{tikzcd} B \arrow[r, "Bh^*"] & C_p^\wedge \\ A \arrow[u, "Bf^*"] \arrow[r] & A_p^\wedge \arrow[u, "Bg^*"'] \end{tikzcd}$$

commutes. Let $p(t) \in A[t]$ be a monic polynomial which splits into linear factors over B. By the diagram $(*)$, the polynomial $Bg^*(p(t)) \in C[t]$ also splits into linear factors after completion at every prime. Now we can argue as in the proof of [18, 3.5] to show that $Bg^*(p(t))$ already splits into linear factors over C. Because $A \to B$ is an integral Galois extension, there exists an extension $\phi : B \to C$ of $Bg^* : A \to C$. Because all involved spaces are rationally products of Eilenberg–Mac Lane spaces, the map ϕ has a geometric realization $Bh_0 : BT_0 \to (BT_L)_0$ which is a lift of $BT_0 \xrightarrow{Bg} BL_0$.

Over the adeles, the maps Bh_0 and $Bh_p^\wedge$ differ only by an Weyl group element [2]. Hence we can assume that they are equal over the adeles. In this case, the coherence conditions for glueing all the maps together by an arithmetic square are satisfied. We get a map $Bh : BT \to BT_L$, such that the diagram

$$\begin{tikzcd} BT \arrow[rr, "Bh"] \arrow[dr, "Bg"'] & & BT_L \arrow[dl, "Bf"] \\ & BL & \end{tikzcd}$$

commutes up to homotopy. □

The following corollary is obvious.

3.6 Corollary. *Let $T_1, T_2 \to L$ be two maximal tori of a connected finite loop space. Then T_1 and T_2 are conjugate.*

In particular, this corollary says that the definition of the Weyl group does not depend in an essential way on the chosen maximal torus.

4. The map $[\ \ , BT_X] \to [\ \ , BX]$

In this section, X is a p–compact group with a fixed maximal torus $f : T_X \to X$. For any p-compact toral group P, the Weyl group W_X acts on the set $[BP, BT_X]$ of homotopy classes of maps and establishes a map

$$F : [BP, BT_X]/W_X \to [BP, BX].$$

4.1 Proposition. *Let P be a p-compact toral group. The map*

$$F : [BP, BT_X]/W_X \to [BP, BX]$$

is an injection.

Remark. In [12, 2.3] a similar but slightly weaker result is proved with different methods.

For the proof of this result we will use the following construction. Let A be a p–compact abelian group and $\alpha : A \to T_X$ a homomorphism. Both classifying spaces BA and BT_X are loop spaces and carry a multiplication μ. Because α is a homomorphism, this establish a commutative diagram

$$\begin{tikzcd} BA \times BA \arrow[r, "B\alpha \times id"] \arrow[d, "\mu"] & BT_X \times BA \arrow[r, "id \times B\alpha"] \arrow[d, "\mu(id \times B\alpha)"] & BT_X \times BT_X \arrow[d, "\mu"] \\ BA \arrow[r, "B\alpha"] & BT_X \arrow[r, equal] & BT_X \end{tikzcd}$$

Taking adjoints in the vertical line yields a diagram

$$(*) \qquad \begin{tikzcd} BA \arrow[r, "B\alpha"] \arrow[d] & BT_X \arrow[r, "Bf"] \arrow[d] & BX \\ map(BA, BA)_{id} \arrow[r] & map(BA, BT_X)_{B\alpha} \arrow[r] & map(BA, BX)_{Bf \circ B\alpha} \arrow[u, "e"'] \end{tikzcd}$$

The right vertical arrow is given by the evaluation. Now let $w \in W_X$ be an element of the Weyl group. The diagram

$$\begin{tikzcd} BT_X \times BT_X \arrow[r, "\mu"] \arrow[d, "w \times w"] & BT_X \arrow[d, "w"] \\ BT_X \times BT_X \arrow[r, "\mu"] & BT_X \end{tikzcd}$$

commutes up to homotopy and, taking adjoints again and combining it with $(*)$, establishes another homotopy commutative diagram, namely

$$\begin{tikzcd} BA \arrow[r, "B\alpha"] \arrow[d, equal] & BT_X \arrow[r] \arrow[d, "w"] & map(BA, BT_X)_{B\alpha} \arrow[r] \arrow[d, "map(id,w)"] & map(BA, BX)_{Bf \circ B\alpha} \arrow[d, equal] \\ BA \arrow[r, "w \circ B\alpha"] & BT_X \arrow[r] & map(BA, BT_X)_{w \circ B\alpha} \arrow[r] & map(BA, BX)_{Bf \circ w \circ B\alpha} \end{tikzcd}$$

The component of $map(BA, BT_X)$ in the bottom row is determined by the homotopy commutative square

$$\begin{tikzcd} BT_X \arrow[r] \arrow[d, "w"] & map(BA, BX)_{Bf \circ B\alpha} \arrow[d] \\ BT_X \arrow[r] & map(BA, BX)_{Bf \circ w \circ B\alpha}. \end{tikzcd}$$

Now we are prepared for the proof of Proposition 4.1.

Proof of Proposition 4.1. Let $\alpha, \beta : P \to T_X$ be two homomorphism such that $f \circ \alpha$ and $f \circ \beta$ are conjugate. First of all this implies that α and β have the same kernel which we can divide out. This follows from [17, §2]. There only the case of maps with source given by the classifying space of a honest p–toral group is discussed, but the same arguments are applicable in our situation. Hence, we can assume that α and β are monomorphisms and that P is abelian as a subgroup of T_X [13, 3.1

and 3.5]. We get the following diagram:

$$\begin{array}{ccccccc} BT_X & \longrightarrow & map(BP,BX)_{B\alpha} & \longrightarrow & map(BP,BX)_{f\circ B\alpha} & \xrightarrow{e} & BX \\ \downarrow{\scriptstyle w} & & & & \| & & \| \\ BT_X & \longrightarrow & map(BP,BT_X)_{B\beta} & \longrightarrow & map(BP,BX)_{f\circ B\beta} & \xrightarrow{e} & BX. \end{array}$$

The top and the bottom row give two maximal tori of $map(BP,BX)_{f\circ B\alpha}$. Hence the left vertical arrow exists and is given by a self equivalence. Because the outer square commutes, this self equivalence is given by an element $w \in W_X$. The above considerations imply that $w \circ B\alpha \simeq B\beta$, which finishes the proof. □

Next we will discuss under which conditions the map is also a surjection. We have the following generalization of well known results of Borel.

4.2 Theorem. *Let X be a connected p–compact group with maximal torus $f : T_X \to X$. Then the following hold:*

1. *The map $H^*(BX;\mathbb{Z}_p^\wedge) \xrightarrow{Bf^*} H^*(BT_X;\mathbb{Z}_p^\wedge)^{W_X}$ is an isomorphism if and only if $H^*(BX;\mathbb{Z}_p^\wedge)$ is p–torsion free.*
2. *If $H^*(BX;\mathbb{Z}_p^\wedge)$ is p–torsion free, every elementary abelian subgroup $V \to X$ is subconjugate to T_X.*
3. *Let A be an abelian p–compact group. If $H^*(BX;\mathbb{F}_p) \cong H^*(BT_X;\mathbb{F}_p)^{W_X}$, then the map $F : [BA, BT_X]/W_X \to [BA, BX]$ is a bijection.*

Proof. First let us assume that $H^*(X;\mathbb{Z}_p^\wedge)$ is not torsion free and that $H^*(BX,\mathbb{Z}_p^\wedge)$ is torsion free. Let $x \in H^*(X;\mathbb{Z}_p^\wedge)$ be a torsion class of minimal degree. Then, in the Serre spectral sequence of the fibration $X \to * \to BX$ this class can't be killed by any differential, which leads to a contradiction. Hence, if $H^*(BX;\mathbb{Z}_p^\wedge)$ is p–torsion free then $H^*(X;\mathbb{Z}_p^\wedge)$ is also p–torsion free. Moreover, $H^*(X;\mathbb{F}_p) \cong E(\overline{y}_1,...,\overline{y}_r)$ is an exterior algebra, $H^*(BX;\mathbb{F}_p) \cong \mathbb{F}_p[\overline{x}_1,...,\overline{x}_r]$ and $H^*(BX;\mathbb{Z}_p^\wedge) \cong \mathbb{Z}_p^\wedge[x_1,...,x_r]$ are polynomial algebras, r is equal to the rank $rk_{\mathbb{Q}}(X)$ of X and $H^*(BX;\mathbb{Z}_p^\wedge) \xrightarrow{Bf^*} H^*(BT_X;\mathbb{Z}_p^\wedge)^{W_X}$ is a monomorphism. All these facts follow from results of Hopf and Borel (e.g. see [9]).

Let $y \in H^*(BX;\mathbb{Z}_p^\wedge)$. If $Bf^*(y) \equiv 0 \bmod p$ then $y \equiv 0 \bmod p$. Otherwise we would get an algebraic relation among the generators of the image of $H^*(BX;\mathbb{F}_p) \to H^*(BT_X;\mathbb{F}_p)$, which contradicts the fact that $H^*(BT_X;\mathbb{F}_p)$ is a finitely generated module over $H^*(BX;\mathbb{F}_p)$. On the other hand, for every $x \in H^*(BT_X;\mathbb{Z}_p^\wedge)^{W_X}$, there exist $n \in \mathbb{N}$ and $z \in H^*(BX;\mathbb{Z}_p^\wedge)$ such that $Bf^*(z) = p^n x$. This follows from Theorem 2.9. If $n > 0$, then z is divisible by p by the above considerations, and $Bf^*(p^{-1}z) = p^{-1}x$. This proves (1).

By (1) the map $H^*(BX;\mathbb{F}_p) \to H^*(BT_X;\mathbb{F}_p)$ is an injection. For every elementary abelian p–group V, the algebra $H^*(BV;\mathbb{F}_p)$ is an injective object in the category of unstable algebras over the Steenrod algebra. Thus, every map $H^*(BX;\mathbb{F}_p) \to H^*(BV;\mathbb{F}_p)$ lifts to a map $H^*(BT_X;\mathbb{F}_p) \to H^*(BV;\mathbb{F}_p)$. Because in this situation the mod–p cohomology classifies the maps up to homotopy and because every algebraic map has a realization [11], this proves part (2).

The third part is proved in [15, 10.1]. □

5. p–compact groups with the Weyl group type of unitary groups

If X is a connected p–compact group with the same p–adic Weyl group type as a compact connected Lie group G, then both have the same rank, and we can identify the two maximal tori as well as the Weyl groups. That is to say, there exists a homomorphism $T_G \to X$ which is a maximal torus with Weyl group W_G. Here, we have to complete T_G in order to get a p-compact torus. As already said in Section 4, the Weyl group W_G acts on the set $[BY, BT_G]$ of homotopy classes of maps for any p–compact group Y.

5.1 Theorem. *Let G be a product of unitary groups, and let X be a connected p–compact group with the same p–adic Weyl group type as G. Then for any abelian p–compact group A the following hold:*

1. $[BA, BT_{G_p}{}^{\wedge}] \to [BA, BX]$ *is surjective.*
2. $[BA, BT_{G_p}{}^{\wedge}]/W_G \to [BA, BX]$ *is a bijection.*
3. *For any homomorphism $g : A \to T_G$, the centralizer $C_X(g)$ is connected and has the p–adic Weyl group type of $C_G(g)$, which is a product of unitary groups.*

Proof. By Proposition 4.1, the first statement implies the second one.

To prove (1) and (3) we first assume that A is a finite abelian p–compact group, i.e A is really a finite abelian p–group. Hence, A is isomorphic to a product $A \cong A_1 \times \cdots \times A_r$ such that $A_i \cong \mathbb{Z}/p^{k_i}$. Moreover, every map $BA \to BT_{G_p}{}^{\wedge}$ lifts to a map $BA \to BT_G$, and any map $BA \to BG_p^{\wedge}$ lifts to BG. By [7] in both cases the lifts are induced by a homomorphism $A \to T_G$ or $A \to G$ of groups. For the proof of (1) and (3) we will use an induction over the number of factors of A.

Let $A \cong \mathbb{Z}/p^k$. Then the map $g : A \to X$ factors through the maximal torus, because every finite cyclic group is subconjugated to the maximal torus (Theorem 2.6). This proves (1) in the case of a cyclic group.

Because A is a subgroup of T_G, the centralizer $C_X(A)$ contains T_G, which also plays the role of the maximal torus of $C_X(A)$. Let W_C denote the Weyl group of $C_X(A)$, which is isomorphic to the isotropy group $Iso(g) := \{w \in W_G | w \circ Bg \simeq Bg\}$. By the assumptions this is also the Weyl group of $C_G(A)$, which is again a product of unitary groups. Therefore, $T_G^{W_C}$ is a p-compact torus and contains A. Moreover, $T_G^{W_C} \to C_X(A)$ is the center (Proposition 2.16 (2)), while $C_X(A) \cong C_{C_X(A)}(T_G^{W_C}) \cong C_X(T_G^{W_C})$ is a connected p–compact group [13, 3.11] and has the p–adic Weyl group type of $C_G(A)$, which is a product of unitary groups. This is condition (3).

Now let A be an abelian finite p–group. Then A splits into a product $A_0 \times A_1$ such that $A_1 \cong \mathbb{Z}/p^k$ and such that A_0 has less factors than A. Let $g : A \to X$ be a homomorphism. By the induction hypothesis the restriction $g|_{A_0} : A_0 \to X$ is subconjugate to the maximal torus, and $C_X(g(A_0))$ is connected and again of the p–adic Weyl group type of a product of unitary groups. Passing to adjoints yields a homomorphism $\overline{g} : A_1 \to C_X(g(A_0))$. We can again apply the induction hypothesis to show that A_1 is subconjugate to T_G in $C_X(g(A_0))$ which proves the first statement, and also shows that $C_X(g(A)) = C_{C_X(g(A_0))}(\overline{g}(A_1))$ is connected and of the p–adic Weyl group type of a product of unitary groups, which is part (3).

Finally let A be an abelian p-compact toral group. Let $A(k) \to A$ be the subgroup of elements of order p^k. Then $A(\infty) := \bigcup_k A(k) \to A$ is a p–discrete

approximation. The map $map(BA, BX) \to \varprojlim_k map(BA(k), BX)$ is an equivalence. Moreover, for every fixed homomorphism $f : A \to X$, the sequence $map(BA(k), BX)_{Bf|_{BA(k)}}$ stabilizes [5, 6.1, 6.7]. This proves the statement in the general case. □

For odd primes there is a more general result of this form.

5.2 Theorem. *Let p be an odd prime, G a compact connected Lie group satisfying condition (T_p), and X a connected p–compact group with the same p–adic Weyl group type as G. Then the following hold:*

1. *$[BA, BT_G{}_p^\wedge] \to [BA, BX]$ is an epimorphism.*
2. *$[BA, BT_G{}_p^\wedge]/W_G \to [BA, BX]$ is an isomorphism.*
3. *For any homomorphism $g : A \to T_G$, the centralizer $C_X(g)$ is connected and has the p–adic Weyl group type of $C_G(g)$. Moreover, $C_G(g)$ satisfies condition (T_p).*

Proof. We argue as in the above proof. Only one argument has to be replaced. We use the same notation as above. In this case $T_G^{W_C}$ may not be a torus. But by [15, 10.1] we have

$$H^*(BC_G(g(A)); \mathbb{F}_p) \cong H^*(BT_G; \mathbb{F}_p)^{Iso(g)},$$

where $Iso(g) := \{w \in W_G | w \circ Bg \simeq Bg\}$ is the isotropy group of g. In particular, $C_G(g(A))$ is connected with Weyl group $W_C \cong Iso(g)$, the mod–p cohomology is concentrated in even degrees, and $C_X(g(A))$ is a p–compact group with the same Weyl group type as $C_G(g(A))$. Because W_C is a reflection group and because p is odd, $C_X(g(A))$ is connected (Lemma 2.11). Using this argument, the induction of the proof of Theorem 5.1 works and gives a proof of the statement. □

One part of Theorem 1.9 for products of unitary groups is contained in the following statement:

Theorem 5.3. *Let X be a connected p–compact group with maximal torus, and let G be a product of unitary groups. If $BN(T_X)$ and $BN(T_G)_p^\circ$ are homotopy equivalent, then the following hold:*

1. *BX is p–torsion free, i.e. the p–adic cohomology is p–torsion free.*
2. *$H^*(BX; \mathbb{Z}_p^\wedge) \cong H^*(BT_X; \mathbb{Z}_p^\wedge)^{W_X}$.*
3. *BX and $BG_p^\wedge$ are homotopy equivalent.*

Proof. Because the two normalizers $BN(T_X)_p^\circ$ and $BN(T_G)_p^\circ$ are equivalent, the space X has the same p–adic Weyl group type as G. The mod–p cohomology of $BN(T_G)$ is detected by elementary abelian p–subgroups [8]. The map $H^*(BX; \mathbb{F}_p) \to H^*(BN(T_X); \mathbb{F}_p)$ is a monomorphism (Proposition 2.14). Thus, the cohomology $H^*(BX; \mathbb{F}_p)$ is also detected by elementary abelian p–subgroups. By Theorem 5.1, we know that every elementary abelian p–subgroup is subconjugate to the maximal torus. This implies that $H^*(BX; \mathbb{F}_p)$ is concentrated in even degrees and that $H^*(BX; \mathbb{Z}_p^\wedge)$ is p–torsion free. By Theorem 4.2, it follows that

$$H^*(BX; \mathbb{Z}_p^\wedge) \cong H^*(BT_G; \mathbb{Z}_p^\wedge)^{W_G}.$$

Because $H^*(BT_G; \mathbb{Z}_p^\wedge)^{W_G} \otimes \mathbb{F}_p \cong H^*(BT_G; \mathbb{F}_p)^{W_G}$ we get isomorphisms

$$H^*(BX, \mathbb{F}_p) \cong H^*(BT_G; \mathbb{F}_p)^{W_G} \cong H^*(BG; \mathbb{F}_p).$$

Because G is a product of unitary groups we can apply [15, 1.3] to conclude that BX and $BG_p^\wedge$ are homotopy equivalent. That is what we need for all three statements. □

For later purposes we need for odd primes a slightly more general result.

5.4 Theorem. *Let p be an odd prime and G a product of unitary groups and $SU(p)$'s. Let X be a connected p–compact group with the same p–adic Weyl group type as G. If $BN(T_X)$ and $BN(T_G)_p^\circ$ are homotopy equivalent, then the following hold:*

1. *BX is p–torsion free, i.e. the p–adic cohomology is p–torsion free.*
2. *$H^*(BX;\mathbb{Z}_p^\wedge) \cong H^*(BT_X;\mathbb{Z}_p^\wedge)^{W_X}$.*
3. *BX and $BG_p^\wedge$ are homotopy equivalent.*

Proof. The proof is the same as above. We only have to ensure that the mod–p cohomology of $BN(T_G)$ is detected by elementary abelian subgroups. But this is also true for $SU(p)$ [15, 12.6]. □

5.5 Proposition. *Let G be a product of unitary groups, or, if p is odd, let G be a compact connected Lie group satisfying condition (T_p). If X is a connected p–compact group having the same p–adic Weyl group type as G, then $BN(T_X)$ and $B(NT_G)_p^\circ$ are homotopy equivalent.*

Proof. For the proof we have to discuss extensions, i.e. fibrations, of the form

$$(*) \qquad {BT_G}_p^\wedge \to BN_p^\circ \to BW_G.$$

Fibrations of this form are classified by cohomology classes in $H^3(BW_G;\pi_2({BT_G}_p^\wedge))$ [16, 3.1]. If p is odd and if G satisfies the condition (T_p), the obstruction group vanishes [15, 5.3], and hence there is only one fibration up to fiber homotopy equivalence. Hence, we have $BN(T_X) \simeq B(NT_G)_p^\circ$.

Now let G be a product of unitary groups. Difficulties arise only for $p = 2$, because in this case there is more than one extension in general. By Proposition 2.10, the map $H^2(BN(T_X);\mathbb{F}_p) \to H^2(BT_X;\mathbb{F}_p)^{W_G}$ is an epimorphism. We can apply [15, 11.5], which implies that $BN(T_X)$ is equivalent to the fiberwise completion of the classifying space of the semidirect product $T_G \rtimes W_G \cong N(T_G)$. This proves the statement for products of unitary groups. □

Finally we are prepared to prove Theorem 1.9 for products of unitary groups:

Proof of Theorem 1.9 *for products of unitary groups.* Condition (1) implies (2) by Proposition 5.5, (2) implies (3) by Theorem 5.2, and (3) implies (1) because X and G are isomorphic p–compact groups. □

6. Connected p–compact group with the Weyl group type of pseudo simply connected or simply connected compact Lie groups.

A pseudo simply connected compact Lie group is a product of unitary groups and simple simply connected compact Lie groups not isomorphic to any $SU(n)$. That is why we replace the special unitary factors in a simply connected compact Lie group by the associated unitary groups.

6.1 Proposition. *Let G be a pseudo simply connected compact Lie group satisfying the condition (C_p). Then there exist an elementary abelian subgroup $V \to T_G$, a compact connected Lie group H which is a product of unitary groups and $SU(p)$'s, and an exact sequence*

$$1 \to K \to H \to C_G(V) \to 1,$$

where K is a finite central subgroup of H of order coprime to p. The centralizer $C_G(V)$ is of maximal rank and the index of $|W_G : W_{C_G(V)}|$ is also coprime to p. The two groups H and $C_G(V)$ have the same Weyl group type, $BN(T_H)^\circ_p \simeq BN(T_{C_G(V)})^\circ_p$, and the mod–$p$ cohomology of $BN(T_G)$ is detected by elementary abelian p–groups.

Proof. This statement is Lemma 5.2 of [15]. Only the last three assertions need a remark. They follow from what we have already said and from the fact that the mod–p cohomolgy of $BN(T_H)$ is detected by elementary abelian subgroups [8], [15, 12.6]. □

6.2 Theorem. *Let G be a pseudo simply connected compact Lie group satisfying condition (C_p). Let X be a connected p–compact group with the same Weyl group type as G. Then the following hold:*

1. $H^*(BX; \mathbb{Z}_p^\wedge) \cong H^*(BT_G; \mathbb{Z}_p^\wedge)^{W_G}$.
2. *$BN(T_G)^\circ_p$ and $BN(T_X)$ are homotopy equivalent.*
3. *BX and $BG_p^\wedge$ are homotopy equivalent.*

Proof. For $p = 2$, a pseudo simply connected compact Lie group is a product of unitary groups. This case we already discussed in the last section. Therefore we assume that p is odd. Part (2) follows from Proposition 5.5. The mod–p cohomology of $BN(T_X) \simeq BN(T_G)^\circ_p$ is detected by elementary abelian subgroups. This follows from Proposition 6.1. Now, using Theorem 5.2 instead of Theorem 5.1, we can argue analogously as in Theorem 5.3 or Theorem 5.4 to show that $H^*(BX; \mathbb{Z}_p^\wedge) \cong H^*(BT_G; \mathbb{Z}_p^\wedge)^{W_G}$, and, using again [15, 1.3], that $BX \simeq BG_p^\wedge$. This is part (3) and also implies part (1). □

To conclude this section we discuss the case of simply connected compact Lie groups.

6.3 Theorem. *Let G be a simply connected compact Lie group satisfying condition (C_p). Let X be a connected p–compact group with the same Weyl group type as G. Then the following hold:*

1. $H^*(BX; \mathbb{Z}_p^\wedge) \cong H^*(BT_G; \mathbb{Z}_p^\wedge)^{W_G}$.
2. *$BN(T_G)^\circ_p$ and $BN(T_X)$ are homotopy equivalent.*
3. *BX and $BG_p^\wedge$ are homotopy equivalent.*

Proof. We split G into a product $G_1 \times G_2$, where G_1 is the product of all factors isomorphic to any $SU(n)$ and G_2 collects the other factors. By the assumptions, $SU(2)$ does not occur for $p = 2$. By Proposition 2.16, we have $Z(X) \cong (T_G^{W_G})^\wedge_p = Z(G_1)^\wedge_p \times Z(G_2)^\wedge_p$.

For each n there exists an exact sequence $1 \to Z(SU(n)) \to SU(n) \times S^1 \to U(n) \to 1$. Let T be a torus with one S^1 for each factor in G_1, and let $Z(G_1) \to T$ be the inclusion established by the above exact sequence. This gives rise to

commutative diagrams of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & Z(G_1) \arrow[r] \arrow[d, equal] & T_{G_1} \times T \times T_{G_2} \arrow[r] \arrow[d] & T_{\overline{G}_1} \times T_{G_2} := T_{\overline{G}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z(G_1) \arrow[r] & G_1 \times T \times G_2 \arrow[r] & \overline{G}_1 \times G_2 := \overline{G} \arrow[r] & 1
\end{tikzcd}$$

and

$$1 \to G \to \overline{G} \to \overline{T} \to 1,$$

where $Z(G_1) \to G_2$ is the trivial homomorphism, and where $\overline{T} := T/Z(G_1)$. The quotient $\overline{G}$ is a pseudo simply connected compact Lie group which also satisfies condition (C_p). Moreover, the epimorphism $G \times T \to \overline{G}$ and the inclusion induce an isomorphism $W_G \cong W_{\overline{G}}$, and $T_G \times T \to T_{\overline{G}}$ and $T_G \to T_{\overline{G}}$ are equivariant.

Working with X instead of G, one gets similar exact sequences, namely

$$\begin{tikzcd}
1 \arrow[r] & Z(G_1)_p^\wedge \arrow[r] \arrow[d, equal] & T_{G_1} \times T \times T_{G_2} \arrow[r] \arrow[d] & T_{\overline{G}_1} \times T_{G_2} := T_{\overline{G}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z(G_1)_p^\wedge \arrow[r] & X \times T_p^\wedge \arrow[r] & \overline{X} \arrow[r] & 1
\end{tikzcd}$$

and

$$1 \to X \to \overline{X} \to \overline{T} \to 1.$$

This shows that $\overline{X}$ has the same Weyl group type as $\overline{G}$ and hence that $B\overline{X} \simeq B\overline{G}_p^\wedge$. The maps $B\overline{X} \to B\overline{T}$ and $B\overline{G} \to B\overline{T}$ are totally determined by $H^2(\ ;\mathbb{Z}_p^\wedge)$. And by construction, this establishes a commutative diagram

$$\begin{tikzcd}
BG_p^\wedge \arrow[r] \arrow[d] & B\overline{G}_p^\wedge \arrow[r] \arrow[d] & B\overline{T}_p^\wedge \arrow[d] \\
BX \arrow[r] & B\overline{X} \arrow[r] & B\overline{T}_p^\wedge
\end{tikzcd}$$

where the right and left vertical arrows are homotopy equivalences. And so is the left one, which is part (3). The first and the second statement obviously follow from this fact. □

7. Connected p–compact group with the Weyl group type of a general compact connected Lie group

In this section we want to prove a version of Theorem 6.3 for general compact connected Lie groups. For this we need some preliminary remarks.

Let G be a compact connected Lie group. Then, the inclusion $T_G \subset G$ of a maximal torus of G induces an isomorphism $\pi_1(T_G)_{W_G} \cong \pi_1(G)$. Here, $\pi_1(T_G)_{W_G}$ denotes the covariants, i.e. the quotient of $\pi_1(T_G)$ by the sublattice generated by all elements of the form $a - w(a)$ for $a \in \pi_1(T_G)$ and $w \in W_G$. This follows from [1, Proposition 5.47] and the detailed description of the kernel of $\pi_1(T_G) \to \pi_1(G)$ also given in [1, Section 5]. Since $\pi_1({T_G}_p^\wedge) \cong \pi_1(T_G) \otimes \mathbb{Z}_p^\wedge \cong H_2(BT;\mathbb{Z}_p^\wedge) \cong H_2(B{T_G}_p^\wedge;\mathbb{Z})$ are isomorphisms of W_G–modules, since $\pi_1(G_p^\wedge) \cong \pi_1(G) \otimes \mathbb{Z}_p^\wedge \cong$

$H_2(BG;\mathbb{Z}_p^\wedge) \cong H_2(BG_p^\wedge;\mathbb{Z}_p^\wedge)$, and since passing to covariants obviously commutes with tensoring with $\mathbb{Z}_p^\wedge$, we get an isomorphism $H_2(BT_{Gp}{}^\wedge;\mathbb{Z})_{W_G} \cong H_2(BG_p^\wedge;\mathbb{Z})$.

Lemma 7.1. *Let G be a compact connected Lie group satisfying condition (C_p). Let X be a connected p–compact group with the same Weyl group type as G. Then the following hold:*

1. *$H_2(BX;\mathbb{Z})$ is torsion free.*
2. *For all r, the maps $BT_{Gp}{}^\wedge \to BG_p^\wedge$ and $BT_{Gp}{}^\wedge \to BX$ induce isomorphisms $H^2(BX;\mathbb{Z}/p^r) \cong H^2(BT_{Gp}{}^\wedge;\mathbb{Z}/p^r)^{W_G} \cong H^2(BG_p^\wedge;\mathbb{Z}/p^r)$.*

Proof. Let $L := H_2(BT_{Gp}{}^\wedge;\mathbb{Z})$ and let $W := W_G$. Since G satisfies condition (C_p), we have $H^*(BG_p^\wedge;\mathbb{Z}_p^\wedge) \cong H^*(BT_{Gp}{}^\wedge;\mathbb{Z}_p^\wedge)^W$. In particular, $H^*(BG_p^\wedge;\mathbb{Z}_p^\wedge)$ is torsion free, and so is $L_W \cong H_2(BG_p^\wedge;\mathbb{Z}_p^\wedge)$.

The maximal torus $BT_{Gp}{}^\wedge \to BX$ of the connected p–compact group X induces epimorphisms $L_W \to \pi_1(X)$ and $L_W \to H_2(BX;\mathbb{Z})$. The kernel of this map is finite, since rationalizing gives an isomorphism $L_W \otimes \mathbb{Q} \cong H_2(BX;\mathbb{Z}) \otimes \mathbb{Q}$. This follows by dualizing the isomorphism of Proposition 2.9 (2) and from the fact that dualizing transforms invariants into covariants. Because L_W is torsion free, the kernel is trivial and $L_W \to H_2(BX;\mathbb{Z})$ an isomorphism.

Since G satisfies condition (C_p), we have $H^*(BG_p^\wedge;\mathbb{Z}/p^r) \cong H^*(BT_{Gp}^\wedge;\mathbb{Z}/p^r)^W$ for all r [15, 4.2]. An application of the universal coefficient theorem now proves the second part. □

As mentioned in the introduction, for every compact connected Lie group G, there exists a finite covering $1 \to K \to \widetilde{G} \xrightarrow{q} G \to 1$, where $\widetilde{G} \cong G_s \times T$ is a product of a simply connected Lie group G_s and a torus T. The group $K \subset \widetilde{G}$ is a central subgroup. We call this covering *universal finite*. The group $\widetilde{G}$ is uniquely determined, but not the map q. One can compose q with a finite self-covering of T.

Every universal finite covering establishes two associated commutative diagrams of exact sequences of compact Lie groups:

$$\begin{array}{ccccc}
K_s & \longrightarrow & G_s & \longrightarrow & G_s/K_s := \overline{G}_s \\
\downarrow & & \downarrow & & \downarrow \\
K & \longrightarrow & G_s \times T & \longrightarrow & G \\
\downarrow & & \downarrow & & \downarrow \\
\overline{K} = K/K_s & \longrightarrow & T & \longrightarrow & T/K_s := \overline{T}
\end{array}$$

and

$$\begin{array}{ccccc}
K_s & = & K_s & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
K & \longrightarrow & G_s \times T & \longrightarrow & G \\
\downarrow & & \downarrow & & \| \\
\overline{K} & \longrightarrow & \overline{G}_s \times T & \longrightarrow & G\ ,
\end{array}$$

where $K_s := K \cap (G_s \times \{0\})$. Because $H^1(\overline{G}_s;\mathbb{Z}) = 0$, the sequence $\overline{G}_s \longrightarrow G \longrightarrow \overline{T}$ induces isomorphisms $H^1(\overline{T};\mathbb{Z}) \cong H^1(G;\mathbb{Z})$ and $H^2(B\overline{T};\mathbb{Z}) \cong H^2(BG;\mathbb{Z})$. On the

other hand, these isomorphisms determine the maps $G \longrightarrow \overline{T}$ and $BG \longrightarrow B\overline{T}$, and therefore also the fibration $B\overline{G}_s \longrightarrow BG \longrightarrow B\overline{T}$.

If G satisfies condition (T_p), then, by [15, 4.1], $BG_s \times BT$ and $B\overline{G}_s \times T$ as well as BG_s and $B\overline{G}_s$ are p–torsion free. The order $|K_s|$ of K_s is coprime to p, because $\pi_2(B\overline{G}_s) \cong H_2(B\overline{G}; \mathbb{Z}) \cong K_s$.

7.2 Theorem. *Let G be a compact connected Lie group satisfying condition (C_p). Let X be a connected p–compact group with the same Weyl group type as G. Then the following hold:*

1. $H^*(BX; \mathbb{Z}_p^\wedge) \cong H^*(BT_G; \mathbb{Z}_p^\wedge)^{W_G}$.
2. *$BN(T_G)_p^\circ$ and $BN(T_X)$ are homotopy equivalent.*
3. *BX and $BG_p^\wedge$ are homotopy equivalent.*

Proof. The finite covering

$$K \longrightarrow G_s \times T \longrightarrow G,$$

G_s simply connected and T a torus, establishes a fibration

$$BK_p \longrightarrow BG_{s_p}^\wedge \times BT_p^\wedge \longrightarrow BG_p^\wedge,$$

where K_p is the p–Sylow subgroup of K. This fibration is classified by a map $BG_p^\wedge \longrightarrow BBK_p$. By Lemma 7.1, there is a corresponding map $BX \longrightarrow BBK_p$ classifying the fibration

$$BK_p \to BY \to BX.$$

The total space BY is simply connected and p–complete. Hence, $Y := \Omega BY$ is a connected p–compact group [13, 3.3], and $Y \to X$ is a finite covering of connected p–compact groups. The fibration fits into

$$\begin{tikzcd}
BK_p \arrow[r] \arrow[d, equal] & BG_{s_p}^\wedge \times BT_p^\wedge \arrow[r] & BG_p^\wedge \\
BK_p \arrow[r] \arrow[d, equal] & BT_{G_{s_p}^\wedge} \times BT_p^\wedge \arrow[r] \arrow[u] \arrow[d] & BT_{G_p}^\wedge \arrow[u] \arrow[d] \\
BK_p \arrow[r] & BY \arrow[r] & BX
\end{tikzcd}$$

The Weyl group $W_G = W_{G_s}$ acts as Weyl group on $BG_{sp}^\wedge \times BT_p^\wedge \to BY$. Thus, Y has the same p–adic Weyl group type as $G_{sp}^\wedge \times T_p^\wedge$. By Theorem 6.3, the spaces $BG_s \times BT$ and BY are homotopy equivalent. Moreover, this equivalence fits into a diagram

$$(*) \qquad \begin{tikzcd}
BK_p \arrow[r] \arrow[d, equal] & BG_{s_p}^\wedge \times BT_p^\wedge \arrow[r] \arrow[d, "\simeq"] & BG_p^\wedge \\
BK_p \arrow[r] & BY \arrow[r] & BX
\end{tikzcd}$$

We have to fill in the arrow in the right column. This is done by the following trick. The top row is a principal fibration [3, 7.2]. The composition $BK_p \to BY \to BX$ is null homotopic and, because X has finite mod–p cohomology, the map $BK_p \to map(BK_p, BX)_{const}$ is an equivalence [17]. By a lemma of Zabrodsky [21] (for details see also [3, 9.8 and the proof of 9.7]), we can fill in the arrow in the right

column so as to make the diagram commutative up to homotopy. This arrow is an equivalence, which is the third statement. The first two statements follow obviously from the third. □

Proof of Theorem 1.9. Part (1) implies (3) by Theorem 7.2. The implications from (3) to (2) and from (2) to (1) are obvious. □

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