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A GROWTH DICHOTOMY FOR O-MINIMAL EXPANSIONS OF ORDERED GROUPS

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ABSTRACT. Let $\mathfrak{R}$ be an o-minimal expansion of a divisible ordered abelian group $(R, <, +, 0, 1)$ with a distinguished positive element 1. Then the following dichotomy holds: Either there is a 0-definable binary operation $\cdot$ such that $(R, <, +, \cdot, 0, 1)$ is an ordered real closed field; or, for every definable function $f : R \to R$ there exists a 0-definable $\lambda \in \{0\} \cup \operatorname{Aut}(R, +)$ with $\lim_{x\to+\infty}[f(x) - \lambda(x)] \in R$. This has some interesting consequences regarding groups definable in o-minimal structures. In particular, for an o-minimal structure $\mathfrak{M} := (M, <, \dots)$ there are, up to definable isomorphism, at most two continuous (with respect to the product topology induced by the order) $\mathfrak{M}$-definable groups with underlying set M.

R. Poston showed in [8] that given an o-minimal expansion $\mathfrak{R}$ of $(\mathbb{R}, <, +)$, if multiplication is not definable in $\mathfrak{R}$, then for every definable function $f : \mathbb{R} \to \mathbb{R}$ there exist $r, c \in \mathbb{R}$ such that $\lim_{x\to+\infty}[f(x) - rx] = c$. In this paper, this fact is generalized appropriately for o-minimal expansions of arbitrary ordered groups.

We say that an expansion $(G, <, *, \dots)$ of an ordered group $(G, <, *)$ is **linearly bounded** (with respect to $*$) if for each definable function $f : G \to G$ there exists a definable $\lambda \in \operatorname{End}(G, *)$ such that ultimately $|f(x)| \le \lambda(x)$. (Here and throughout, **ultimately** abbreviates "for all sufficiently large positive arguments".)

We now list the main results of this paper.

Let $\mathfrak{R} := (R, <, \dots)$ be o-minimal.

Theorem A (Growth Dichotomy). *Suppose that $\mathfrak{R}$ is an expansion of an ordered group $(R, <, +)$. Then exactly one of the following holds*: (a) *$\mathfrak{R}$ is linearly bounded*; (b) *$\mathfrak{R}$ defines a binary operation $\cdot$ such that $(R, <, +, \cdot)$ is an ordered real closed field. If $\mathfrak{R}$ is linearly bounded, then for every definable $f : R \to R$ there exist $c \in R$ and a definable $\lambda \in \{0\} \cup \operatorname{Aut}(R, +)$ with $\lim_{x\to+\infty}[f(x) - \lambda(x)] = c$.*

Theorem B. *Suppose that $\mathfrak{R}$ is a linearly bounded expansion of an ordered group $(R, <, +, 0, 1)$ with $1 > 0$. Then every definable endomorphism of $(R, +)$ is 0-definable. If $\mathfrak{R}'$ (with underlying set R') is elementarily equivalent to $\mathfrak{R}$, then the ordered division ring of all $\mathfrak{R}'$-definable endomorphisms of $(R', +)$ is canonically isomorphic to the ordered division ring of all $\mathfrak{R}$-definable endomorphisms of $(R, +)$.*

The growth dichotomy imposes some surprising constraints on continuous definable groups with underlying set R. (Here and throughout, all topological notions are taken with respect to the product topologies induced by the order topology.)

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Let $+, \oplus : R^2 \to R$ be definable such that both $(R, <, +)$ and $(R, <, \oplus)$ are ordered groups. We show that if $\mathfrak{R}$ is linearly bounded with respect to both $+$ and $\oplus$, then $(R, +) \cong_{\mathrm{df}} (R, \oplus)$. (We let "$\cong_{\mathrm{df}}$" abbreviate "is definably isomorphic to".) This fact, together with Theorem A, easily yields that if there are two continuous definable groups on R which are not definably isomorphic, then $\mathfrak{R}$ expands an ordered (real closed) field; combined with Theorem A and a result of M. Otero, Y. Peterzil and A. Pillay [5], it yields the following:

Theorem C. *Up to definable isomorphism, there are at most two continuous definable groups with underlying set R.*

Assume now that $\mathfrak{R}$ expands an ordered field $(R, <, +, \cdot, 0, 1)$ and let $(R, \oplus)$ be a continuous definable group. It is an easy corollary of Theorem C that if there is no definable isomorphism from $(R, +)$ onto the positive multiplicative group $(R^{>0}, \cdot)$ then either $(R, \oplus) \cong_{\mathrm{df}} (R, +)$ or $(R, \oplus) \cong_{\mathrm{df}} (R^{>0}, \cdot)$. The situation is not so clear if $(R, +) \cong_{\mathrm{df}} (R^{>0}, \cdot)$, that is, if $\mathfrak{R}$ is **exponential**. However, we establish for the exponential case necessary and sufficient conditions in order that $(R, \oplus) \cong_{\mathrm{df}} (R, +)$ (see 4.5). These results, together with a fact due to D. Marker and Miller, are used to show that if $\mathfrak{R}$ is elementarily equivalent to a reduct of the structure $\mathbb{R}_{\mathrm{an,exp}}$ (see eġ. [3] for a definition), then $(R, \oplus) \cong_{\mathrm{df}} (R, +)$ or $(R, \oplus) \cong_{\mathrm{df}} (R^{>0}, \cdot)$.

Here is an outline of this paper. Section 1 is devoted to establishing conventions, notation and some important technical lemmas. (One of these lemmas is only stated there; its proof constitutes Section 5.) In Section 2, we prove Theorem A after developing an analog of Hardy field valuation theory. The linearly bounded case is dealt with in Section 3, where we prove Theorems B and C. Section 4 deals with the field-expansion case: The results of the preceding paragraph are established and we give a new proof of the growth dichotomy for o-minimal expansions of ordered fields.

1. Preliminaries

In this section, we establish some conventions and prove a number of technical results for later use.

Familiarity with the basic properties of o-minimal structures is assumed; in particular, we frequently invoke the Monotonicity Theorem without comment. (See [1], [2], [7].)

Given any first-order structure $\mathfrak{M}$ with underlying set M, "definable" (in $\mathfrak{M}$) means "definable with parameters" (from M) unless stated otherwise. A partial function $f : A \to M$, $A \subseteq M^n$, is said to be definable if its graph is definable. Whenever convenient, we assume without mention that any particular definable partial function is totally defined. We use "expansion" in the sense of definability; that is, a structure $\mathfrak{M}_2$ in a language L_2 is an expansion of a structure $\mathfrak{M}_1$ in a language L_1 if they have the same underlying set M and for each $n \in \mathbb{N}$, every $\mathfrak{M}_1$-definable set $A \subseteq M^n$ is definable in $\mathfrak{M}_2$. We also say in this case that $\mathfrak{M}_2$ expands $\mathfrak{M}_1$. Similarly, we use "reduct" in the sense of definability.

We will be working with several different linearly ordered sets, so we adopt the following notation: For any linearly ordered set $(X, <, 0)$ with a distinguished element denoted by 0, put $X^* := X \setminus \{0\}$, and for each $c \in X$ put

$$X^{>c} := \{\, x \in X : x > c \,\} \quad \text{and} \quad X^{\geq c} := \{\, x \in X : x \geq c \,\}.$$

Note. From now until the end of Section 4, $\mathfrak{R}$ denotes a fixed, but arbitrary, o-minimal expansion of an ordered group $(R, <, +, 0)$. We let $\oplus$ denote any definable binary operation (including possibly $+$) such that $(R, <, \oplus, 0)$ is an ordered group, with the inverse of $x \in R$ with respect to $\oplus$ denoted by $\ominus x$. Adding a constant symbol to the language of $\mathfrak{R}$ if necessary, we assume that there is a distinguished 0-definable element $1 \in (0, \infty)$.

An important fact (due to A. Pillay and C. Steinhorn, [7]) used frequently throughout this paper is that there are no proper nontrivial definable subgroups of a definable ordered group $(I, <, *)$ with $I \subseteq (-\infty, \infty)$ an interval. Easy consequences of this are that $(I, *)$ is abelian, divisible and topological, and every nontrivial definable endomorphism of $(I, *)$ is a strictly monotone, continuous automorphism of $(I, *)$. Given a continuous definable group $(R, *)$, it is also easy to show that $x * z < y * z$ and $z * x < z * y$ for all $x, y, z \in R$ with $x < y$—in other words, $(R, *)$ expands naturally to the ordered group $(R, <, *)$—*we use this fact in the sequel without further mention.* If $\mathfrak{R}$ defines a binary operation $\cdot$ such that $(R, <, +, \cdot, 0, 1)$ is an ordered ring, then $(R, <, +, \cdot, 0, 1)$ is an ordered real closed field.

We regard the set Λ $(= \Lambda(\mathfrak{R}))$ of all definable endomorphisms of $(R, +)$ as an ordered division ring, with composition as multiplication, pointwise addition in R as addition, and $\lambda \in \Lambda^{>0}$ if and only if $\lambda(1) > 0$ if and only if λ is strictly increasing. We thus regard R as an ordered (left unitary) vector space over Λ with action $(\lambda, x) \mapsto \lambda(x)$.

Since $(R, +)$ is divisible, $(\mathbb{Q}, <, +, 0, 1)$ embeds uniquely into $(\Lambda, <, +, 0, \mathrm{id}_R)$; we let λ_q denote the image of $q \in \mathbb{Q}$ under this embedding. The map $\lambda \mapsto \lambda(1) : \Lambda \to R$ is an embedding of $(\Lambda, <, +, 0, \mathrm{id}_R)$ into $(R, <, +, 0, 1)$; let D $(= D(\mathfrak{R}))$ denote the image of Λ under this map. The set D is definable if and only if $D = R$, since $(D, +)$ is a subgroup of $(R, +)$.

Example. If $\mathfrak{R} = (\mathbb{R}, <, +, 0, 1)$, then $D = \mathbb{Q}$ (hence D is not definable).

Putting $q := \lambda_q(1)$ for $q \in \mathbb{Q}$, we denote $\lambda \in \Lambda$ with $\lambda(1) = r$ by λ_r, and we also put $rx := \lambda_r(x)$ for $r \in D$ and $x \in R$. (Note that our notation "qx" for $q \in \mathbb{Q}$ is consistent with the standard use of this notation in divisible group theory.)

The ring D is isomorphic to Λ as an ordered division ring, with multiplication in D given by $(s, t) \mapsto st$ and putting $r^{-1} := \lambda_r^{-1}(1)$ for $r \in D^*$. We write $x \cdot y$ for the (not necessarily definable) function $(x, y) \mapsto xy : D \times R \to R$, and we regard R as an ordered D-vector space. Note that if $D = R$, then $(R, <, +, \cdot, 0, 1)$ is an ordered division ring; so if $D = R$ and the operation $\cdot$ is definable, then $(R, <, +, \cdot, 0, 1)$ is an ordered field.

We make a few easy observations before beginning the more difficult lemmas.

1.1. If $* : R^2 \to R$ is such that $(R, <, +, *, 0, 1)$ is an ordered field and $\mathfrak{R}$ is a reduct of an o-minimal expansion $\mathfrak{R}'$ of $(R, <, +, *, 0, 1)$, then $rx = r * x$ for each $r \in D$ and $x \in R$. (The function $x \mapsto (r * x) - rx : R \to R$ is an $\mathfrak{R}'$-definable endomorphism of $(R, +)$ taking the value 0 at $x = 1$.) Hence, if $\mathfrak{R}$ expands an ordered field whose underlying additive group is $(R, +)$, then $\mathfrak{R}$ is not linearly bounded (with respect to $+$). By a similar argument, if $*_1, *_2 : R^2 \to R$ are definable such that $(R, +, *_1, 0, 1)$ and $(R, +, *_2, 0, 1)$ are fields, then $*_1 = *_2$. Thus, if $\mathfrak{R}$ expands an ordered field with underlying additive group $(R, +)$ and multiplicative identity 1, then the multiplication of the field is 0-definable.

1.2. The function $r \mapsto r^{-1} : D^{>0} \to D^{>0}$ is an order-reversing bijection, mapping $\{ r \in D : 0 < r < s \}$ onto $\{ r \in D : r > s^{-1} \}$ for each $s \in D^{>0}$.

1.3. Let $f : R \to R$ be definable. If $y, z, a, b \in R$ are such that

$$\lim_{x\to+\infty}[f(x+y) \ominus f(x)] = a \quad \text{and} \quad \lim_{x\to+\infty}[f(x+z) \ominus f(x)] = b,$$

then

$$\lim_{x\to+\infty}[f(x+y+z) \ominus f(x)] = a \oplus b \quad \text{and} \quad \lim_{x\to+\infty}[f(x-y) \ominus f(x)] = \ominus a.$$

Thus, the sets

$$\{ y \in R : \lim_{x\to+\infty}[f(x+y) \ominus f(x)] \in R \}$$

and

$$\{ y \in R : \lim_{x\to+\infty}[f(x+y) \ominus f(x)] = 0 \}$$

are definable subgroups of $(R, +)$, and the set

$$\{ r \in R : \exists y \ \lim_{x\to+\infty}[f(x+y) \ominus f(x)] = r \}$$

is a definable subgroup of $(R, \oplus)$. Consequently, if there exist $y, r \in R$ with $y \neq 0$ and $\lim_{x\to+\infty}[f(x+y) \ominus f(x)] = r$, then

$$y \mapsto \lim_{x\to+\infty}[f(x+y) \ominus f(x)] : R \to R$$

is a definable homomorphism of $(R, +)$ into $(R, \oplus)$; if moreover $r \neq 0$, then this map is an isomorphism of $(R, +)$ onto $(R, \oplus)$. Hence, $(R, \oplus) \cong_{\text{df}} (R, +)$ if and only if there exist definable $f : R \to R$ and $y, r \in R^*$ with $\lim_{x\to+\infty}[f(x+y) \ominus f(x)] = r$.

Notation. Given $f : R \to R$ and $x, y \in R$, put $\Delta_y f(x) := f(x + y) - f(x)$. For $y = 1$, the subscript will be suppressed. Similarly, given $f : A \times R \to R$ with $A \subseteq R^m$, then $\Delta_y f(a, x) := f(a, x + y) - f(a, x)$ for all $x, y \in R$ and $a \in A$.

1.4. Let $f : R \to R$ be definable with $\lim_{x\to+\infty} \Delta f(x) = r \in R$. By 1.3, the function $Lf(y) := \lim_{x\to+\infty} \Delta_y f(x)$ is a definable endomorphism of $(R, +)$; indeed, $r \in D$ and $Lf = \lambda_r$. (The notation is intended to suggest that Lf is the "linear part" of f.) If $g : R \to R$ is also definable with $\lim_{x\to+\infty} \Delta g(x) \in R$, then $L(f+g) = Lf + Lg$; note that we thus have $L(Lf - f) = 0$. (Clearly, $L\lambda_r = \lambda_r$ for all $r \in D$.) For any definable unary function h, we have $Lh = 0$ if $\lim_{x\to+\infty} h(x) \in R$. The converse fails in general, as can be seen by considering the square root function for the positive real numbers.

1.5. The set D consists of those $r \in R$ such that there is a definable unary function f with $\lim_{x\to+\infty} \Delta f(x) = r$. Thus, if $f : R \to R$ is definable and $\lim_{x\to+\infty} |\Delta f(x)| > r$ for all $r \in D$, then $\lim_{x\to+\infty} |\Delta f(x)| = +\infty$; moreover, $\lim_{x\to+\infty} |\Delta_y f(x)| = +\infty$ for all $y \in R^*$ by 1.3. Similarly, if $\lim_{x\to+\infty} |\Delta f(x)| < r$ for all $r \in D^{>0}$, then $\lim_{x\to+\infty} \Delta_y f(x) = 0$ for all $y \in R$.

Definition. A function $f : R \to R$ is **infinitely increasing** if it is ultimately both unbounded and strictly increasing. For f definable, this is equivalent to the condition that $\lim_{x\to+\infty} f(x) = +\infty$.

1.6. Let $f : R \to R$ be definable, ultimately positive and strictly decreasing to 0 as $x \to +\infty$. Then there exist $c > 0$ and a definable decreasing homeomorphism $f^{\#} : (0, \infty) \to (0, \infty)$ such that $f^{\#} \restriction (c, \infty) = f \restriction (c, \infty)$ and $f^{\#}(f^{\#}(x)) = x$ for all $x \in (0, \infty)$. To see this, let c be the infimum of all $x > 0$ such that $f(x) < x$ and $f \restriction (x, \infty)$ is continuous and strictly decreasing. Put $d := \lim_{x \to c^+} f(x)$. The set $\{ (f(x), x) : x > c \}$ is the graph of a strictly decreasing homeomorphism h of $(0, d)$ onto (c, ∞). Now define $f^{\#}$ by

$$f^{\#}(x) := \begin{cases} h(x), & 0 < x < d, \\ -x + c + d, & d \leq x \leq c, \\ f(x), & x > c. \end{cases}$$

Given a definable function $g : R \to R$ such that $g(x) \to 0^+$ as $x \to +\infty$, the composition $f^{\#} \circ g$ is ultimately defined and $\lim_{x \to +\infty} f^{\#}(g(x)) = +\infty$. In particular, if $g : R \to R$ is definable and Δg is ultimately strictly decreasing to 0, then the (ultimately defined) function $x \mapsto (\Delta g)^{\#}(-\Delta^2 g(x))$ is infinitely increasing, where $\Delta^2 g(x)$ denotes $\Delta(\Delta g(x))$.

We come now to a key result. Roughly speaking, it states that if a definable family of unary functions has infinitely many "linear asymptotes at $+\infty$", then $\mathfrak{R}$ expands an ordered field with underlying additive group $(R, +)$.

1.7. Lemma. *Let $f : A \times R \to R$ be definable, and let $A \subseteq R^m$. Suppose that the set $\{ b \in R : \exists a \in A \, [\lim_{t \to +\infty} \Delta f(a, t) = b] \}$ is infinite. Then $D = R$, the map $x \cdot y$ is definable and $(R, <, +, \cdot, 0, 1)$ is an ordered field.*

Proof. Replacing A by the definable set $\{ a \in A : \lim_{t \to +\infty} \Delta f(a, t) \in R \}$, we may assume that $\lim_{t \to +\infty} \Delta f(a, t) \in R$ for all $a \in A$. Define $\rho : A \to R$ by $\rho(a) := \lim_{t \to +\infty} \Delta f(a, t)$. Then for all $a \in A$ and $y \in R$ we have $\rho(a) \in D$ and $\lim_{t \to +\infty} \Delta_y f(a, t) = \rho(a) y$. The functions $(a, y) \mapsto \rho(a) y : A \times R \to R$ and $\rho : A \to R$ are clearly definable.

As $\rho(A)$ is infinite, it contains an open interval $I \subseteq D$. Choose some $r \in I$, $r \neq 0$. Replacing f by $(a, t) \mapsto f(a, t) - rt$, we may assume that $0 \in I$. Since $\mathfrak{R}$ has definable Skolem functions, there is a definable function $g : I \to A$ with $\rho(g(x)) = x$ for all $x \in I$. Thus, the restriction of $x \cdot y$ to $I \times R$ is definable. Then the function $z = xy$, $(x, y) \in (0, \epsilon) \times R$, is definable for some $\epsilon > 0$ with $(0, \epsilon) \subseteq D$, say by some formula $\varphi(x, y, z)$ (allowing parameters from R). Then $\{ (x, y) \in R^2 : \varphi(x, y, 1) \}$ defines the restriction to $(0, \epsilon)$ of $x \mapsto x^{-1}$. We may assume that $\epsilon \in D$ and $x \mapsto x^{-1} \restriction (0, \epsilon)$ is continuous. Since $(0, \epsilon)$ is definably connected, the image of $(0, \epsilon)$ under $x \mapsto x^{-1}$ is definably connected, and is thus an interval $(\epsilon^{-1}, b) \subseteq R \cup \{+\infty\}$ (recall 1.2). A routine argument shows that $b = +\infty$ and thus $D = R$, since D is a subgroup of R.

Now $x \mapsto x^{-1} \restriction (\epsilon^{-1}, \infty)$ is definable, so $x \mapsto (x^{-1} - (x+1)^{-1})^{-1} - x = x^2$ is definable on $(-\infty, -\delta) \cup (\delta, \infty)$ for some $\delta > 1$. (As usual, x^2 denotes xx.) For all $x \in R$ we have $2x^2 = \lim_{t \to +\infty} [(x+t)^2 - 2t^2 + (x-t)^2]$, so in fact x^2 is definable on all of R. Then $(x, y) \mapsto xy + yx = (x+y)^2 - x^2 - y^2$ is definable on R^2. For each $r \in R$, $\{ y : ry + yr = 2ry \}$ is then a definable nontrivial subgroup of R, hence equal to R. Since r was arbitrary, the operation $\cdot$ is commutative, so $x \cdot y = \frac{1}{2}(xy + yx)$ is definable and $(R, <, +, \cdot, 0, 1)$ is an ordered field. ∎

As mentioned in the Introduction, we establish Theorem A after developing an analog of Hardy field valuation theory. A crucial part of this development is to have available suitable substitutes for the Mean Value Theorem and some of its basic consequences. We accomplish this here in a series of three lemmas. The first (1.8) is a technical necessity; the proof is best accomplished in a different context, including an essentially unavoidable change of many of the notations and conventions established so far, so we postpone it until Section 5. The second (1.9) is a mean-value/convexity result: Essentially, it says that if the 'derivative' Δf of an infinitely increasing definable function f is ultimately strictly decreasing to 0, then f is ultimately 'strictly concave down'. We then use this 'mean value theorem' to obtain (in 1.10) some important growth-rate estimates.

1.8. Lemma. *Let $f : R \to R$ be definable, continuous and infinitely increasing with $\lim_{x\to+\infty} \Delta f(x) = 0$; let $r \in D^{>0}$. Then there exists $C \in R$ such that for all $x, y > C$ with $x \neq y$, the function $t \mapsto \Delta_{rt} f(x) - r\Delta_t f(y) : R \to R$ is strictly monotone on some open interval (depending on x and y) about 0.*

1.9. Lemma. *Let $r \in D^{>1}$, and let $f : R \to R$ be definable and infinitely increasing with $\lim_{x\to+\infty} \Delta f(x) = 0$. Then there exists $C > 0$ such that if $a, b > C$ and $\Delta_r f(b) = r\Delta f(a)$, then $b < a < b + r - 1$.*

Proof. We may assume that f is continuous on R. Let C be as in 1.8. Let $a, b > C$ be such that $\Delta_r f(b) = r\Delta f(a)$. Now $\Delta_{r0} f(b) = 0 = r\Delta_0 f(a)$, so there exists $p \in (0, 1)$ such that the (continuous) function $t \mapsto \Delta_{rt} f(b) - r\Delta_t f(a)$ assumes a local extreme value at $t = p$. Thus, the function $t \mapsto \Delta_{rt} f(b + rp) - r\Delta_t f(a + p)$ assumes a local extreme value at $t = 0$. By 1.8 we have $a + p = b + rp$. The result follows. □

1.10. Lemma. *Let $r \in D^{>1}$, and let $f : R \to R$ be definable and infinitely increasing with $\lim_{x\to+\infty}[f(rx) - f(x)] = 0$. Then ultimately*

$$\Delta(f(rx)) \leq r\Delta f(rx) \leq 2\Delta f(x).$$

Proof. We may assume that f is continuous. Note that ultimately $f(rx) > f(x+1)$, so $\lim_{x\to+\infty} \Delta f(x) = 0$, and thus $\Delta_y f$ is ultimately strictly decreasing to 0 for each $y > 0$ (see 1.3).

Since $r > 0$ and $\Delta(f(rx)) = \Delta_r f(rx)$, in order to establish the leftmost inequality it suffices to show that ultimately $\Delta_r f(x) \leq r\Delta f(x)$. Suppose otherwise; then ultimately $\Delta_r f(x) > r\Delta f(x)$. Let C be as in 1.9. Let $a > C$ be such that $\Delta_r f(a) > r\Delta f(a)$. Since $\Delta_r f$ is continuous and ultimately strictly decreasing to 0, there exists $b > a$ with $\Delta_r f(b) = r\Delta f(a)$, contradicting 1.9.

Next we claim that ultimately $r\Delta f(rx) \leq 2\Delta f(x)$. Suppose not; then ultimately $r\Delta f(rx) > 2\Delta f(x)$. Let C be as in 1.9. Increasing C as necessary, we assume that f is strictly increasing on (C, ∞), the functions $\Delta_r f$ and $r\Delta f(rx)$ are strictly decreasing on (C, ∞), and $r\Delta f(rx) > 2\Delta f(x)$ for all $x > C$. Since $\Delta_r f$ and $r\Delta f(rx)$ are continuous and ultimately strictly decreasing to 0, for all sufficiently large $a > C$ there exists $b > C$ (depending on a) with $\Delta_r f(b) = r\Delta f(ra)$. By 1.9, we have $b < ra < b + r$. Since f is strictly increasing on (C, ∞), we have

$$f(ra - r) < f(b) < f(b + r) < f(ra + r),$$

so

$$f(r(a + 1)) - f(r(a - 1)) > \Delta_r f(b) = r\Delta f(ra) > 2\Delta f(a)$$

for all sufficiently large $a \in R$; that is, ultimately

$$f(r(x+1)) - f(r(x-1)) > 2\Delta f(x). \tag{$*$}$$

On the other hand, since the function $f(rx) - f(x)$ is ultimately strictly decreasing to 0, the function $f(rx) - f(x+1)$ is ultimately strictly decreasing to 0 as well; hence ultimately

$$f(r(x-1)) - f(x) > f(r(x+1)) - f(x+2)$$

and

$$\Delta_2 f(x) > f(r(x+1)) - f(r(x-1)). \tag{$**$}$$

Now $(*)$ and $(**)$ together yield that ultimately $\Delta_2 f(x) > 2\Delta f(x)$, hence ultimately $\Delta f(x+1) > \Delta f(x)$, contradicting our assumption that Δf is ultimately strictly decreasing. □

Definition. Let $I \subseteq (-\infty, \infty)$ be an interval. A binary relation P on I is a **preorder on** I if P is reflexive, transitive and for all $x, y \in I$ either $P(x, y)$ or $P(y, x)$.

Next is a variant of a result by Y. Peterzil and Starchenko.

1.11. Lemma. *Let P be a definable preorder on R. Then there exists $C \in R$ such that one of the following holds:*

1. *for all $x, y > C$, $P(x, y)$ if and only if $x \leq y$;*
2. *for all $x, y > C$, $P(x, y)$ if and only if $x \geq y$;*
3. *for all $x, y > C$, $P(x, y)$ and $P(y, x)$.*

Proof. It follows easily from 2.12 of [6] and standard "o-minimal" arguments that there exists $C \in R$ such that one of the following holds:

(i) for every $z > C$ there exists an interval I_z with $z \in I_z \subseteq (C, \infty)$ such that for all $x, y \in I_z$, $P(x, y)$ if and only if $x \leq y$;
(ii) for every $z > C$ there exists an interval I_z with $z \in I_z \subseteq (C, \infty)$ such that for all $x, y \in I_z$, $P(x, y)$ if and only if $x \geq y$;
(iii) for every $z > C$ there exists an interval I_z with $z \in I_z \subseteq I$ such that $P(x, y)$ and $P(y, x)$ for all $x, y \in I_z$.

We only do the case that (i) holds; we claim that for all $x, y > C$, $P(x, y)$ if and only if $x \leq y$. (Other cases are handled similarly.) Fix $x > C$. Since (i) holds, there exists $y > x$ such that $\neg P(y', x)$ for all $y' \in (x, y)$. Put

$$s := \sup \{\, y > x : \forall y' \in (x, y)\ \neg P(y', x) \,\}.$$

By o-minimality, s exists in $(x, +\infty]$. A routine argument (exploiting the transitivity of P) shows that $s = +\infty$. □

Remark. In the above, we used only that $\mathfrak{R}$ is an o-minimal expansion of the dense linear order $(R, <)$.

1.12. Lemma. *Let $f : R \to R$ be definable. Put $g(x, t) := f(x + t) \ominus f(x)$ for $(x, t) \in R^2$. Then there exists $C \in R$ such that one of the following holds*:

1. *$g(x, t) < g(y, t)$ for all $y > x > C$ and all $t > 0$;*
2. *$g(x, t) > g(y, t)$ for all $y > x > C$ and all $t > 0$;*
3. *$g(x, t) = g(y, t)$ for all $y, x > C$ and all $t > 0$.*

Proof. We may assume that f is continuous. Note that for every $x, y \in R$ the function $t \mapsto g(x,t) - g(y,t) : R^{>0} \to R$ is then continuous.

The relation $P \subseteq R^2$ given by

$$P(x,y) \Leftrightarrow \exists \epsilon > 0 \ \forall t \in (0,\epsilon) \ [\, g(x,t) \leq g(y,t) \,]$$

is a definable preorder on R. (That P is reflexive and transitive is clear, and for each $(x,y) \in R^2$ we have $P(x,y)$ or $P(y,x)$ by the Monotonicity Theorem.)

Fix C as in 1.11.

Suppose that 1.11.3 holds; we claim that 1.12.3 holds. Let $y > x > C$. Then $P(x,y)$ and $P(y,x)$, so there exists $\epsilon > 0$ with $g(x,t) = g(y,t)$ for all $t \in (0,\epsilon)$. Put

$$s := \sup\{\, \epsilon > 0 : \forall t \in (0,\epsilon) \ [\, g(x,t) = g(y,t) \,]\,\}.$$

It now suffices to show that $s = +\infty$. Suppose otherwise. Then there exists $\epsilon > 0$ such that for all $u \in (0,\epsilon)$ we have $g(x+s,u) = g(y+s,u)$, that is,

$$g(x, s+u) \ominus g(x,s) = g(y, s+u) \ominus g(y,s).$$

By continuity, $g(x,s) = g(y,s)$, so $g(x,t) = g(y,t)$ for all $t \in (0, s+\epsilon)$; contradiction.

Arguing similarly, we obtain the intermediate result:

$(*)$ *If* 1.11.1 *holds, then* $g(x,t) \leq g(y,t)$ *for all* $y > x > C$ *and all* $t > 0$. *If* 1.11.2 *holds, then* $g(x,t) \geq g(y,t)$ *for all* $y > x > C$ *and all* $t > 0$.

Suppose now that 1.11.1 holds; we show that 1.12.1 holds. Let $y > x > C$. Then $\neg P(y,x)$, so there exists $\epsilon > 0$ with $g(x,t) < g(y,t)$ for all $t \in (0,\epsilon)$ (again by monotonicity). Put

$$s := \sup\{\, \epsilon > 0 : \forall t \in (0,\epsilon) \ [\, g(x,t) < g(y,t) \,]\,\}.$$

It now suffices to show that $s = +\infty$. Suppose otherwise. Put $u := \frac{1}{2}s$. Then $g(x,u) < g(y,u)$ (by the definition of s) and $g(x+u,u) \leq g(y+u,u)$ by $(*)$. Hence, $g(x+u,u) \oplus g(x,u) < g(y+u,u) \oplus g(y,u)$, that is, $g(x,s) < g(y,s)$. But $g(x,s) = g(y,s)$ by continuity; contradiction.

Similarly, if 1.11.2 holds, then 1.12.2 holds. □

2. The group of definable germs at $+\infty$

Functions $f, g : R \to R$ are said to have the same **germ** (at $+\infty$) if f and g ultimately agree. We will not distinguish notationally in this section between functions and their germs, relying instead upon context. We let $\mathcal{G}$ denote the set of germs of definable unary functions; it is an abelian group under addition of germs (defined in the obvious manner). By the Monotonicity Theorem, $\mathcal{G}$ is moreover an ordered group, with $f > 0$ if and only if f is ultimately positive.

Let f be a definable unary function. If $\lim_{x\to+\infty} |f(x)| = +\infty$, then f has a definable compositional inverse f^{-1} for large enough positive x, and $f^{-1} \in \mathcal{G}$. For each $y \in R$, $\Delta_y f \in \mathcal{G}$. We regard R as lying in $\mathcal{G}$ by identifying $r \in R$ with the germ of the corresponding constant function. For $r \in D$, we let rx denote the germ of λ_r; in particular, x denotes the germ of id_R. Given $f \in \mathcal{G}$ and infinitely increasing $g : R \to R$, we let $f(g)$ denote the germ of the (ultimately defined) composition $f \circ g$; in particular, for $c \in R$ we write $f(x+c)$ for the germ of $t \mapsto f(t+c)$. (Note that for f, g as above, "$\Delta f(g)$" means "$(\Delta f)(g)$", not "$\Delta(f \circ g)$".)

Consider the relation on $\mathcal{G}^*$ given by $E(f,g)$ if and only if either: (i) there exist $r, s \in D$ with $r, s \geq 1$ such that $|g| \leq r|f|$ and $|f| \leq s|g|$; or, (ii) $\lim_{x\to+\infty} f(x) \in R^*$ and $\lim_{x\to+\infty} g(x) \in R^*$. It is easy to check that E is an equivalence relation on $\mathcal{G}^*$.

We let $\upsilon(\mathcal{G}^*)$ denote the quotient of $\mathcal{G}^*$ by E and $\upsilon : \mathcal{G}^* \to \upsilon(\mathcal{G}^*)$ denote the natural map. If $f \in \mathcal{G}$ and $\lim_{x\to+\infty} f(x) \in R^*$, then we set $\upsilon(f) := 0$. We impose a total order on $\upsilon(\mathcal{G}^*)$ by putting $\upsilon(f) < \upsilon(g)$ if $\upsilon(f) \neq \upsilon(g)$ and $|f| > r|g|$ for all $r \in D^{>0}$. Then $\upsilon(f) < 0$ if $\lim_{x\to+\infty} |f(x)| = +\infty$, and $\upsilon(f) > 0$ if $\lim_{x\to+\infty} f(x) = 0$ (by 1.5).

Remark. Without condition (ii) in the definition of E we could wind up in the undesirable situation that not all nonzero constant germs are E-equivalent. If D is cofinal in R, then (ii) is unnecessary, as is the condition "$\upsilon(f) \neq \upsilon(g)$" in the definition of the order.

It is convenient to extend υ to $\mathcal{G}$. This is done by formally adjoining to $\upsilon(\mathcal{G}^*)$ an element $\upsilon(0)$ with $\upsilon(0) > \gamma$ for all $\gamma \in \upsilon(\mathcal{G}^*)$.

We record some basic facts; verifications are left to the reader.

2.1. Proposition. *Let $f, g \in \mathcal{G}$.*

1. *If $|f| \geq |g|$, then $\upsilon(f) \leq \upsilon(g)$.*
2. *$\upsilon(f) = \upsilon(-f)$.*
3. *$\upsilon(\Delta f) = \upsilon(\Delta|f|) = \upsilon(|\Delta f|)$.*
4. *$\upsilon(rf) = \upsilon(f)$ for all $r \in D^*$.*
5. *$\upsilon(f + g) \geq \min\{\upsilon(f), \upsilon(g)\}$, with equality if $\upsilon(f) \neq \upsilon(g)$.*

We say that $f \in \mathcal{G}$ is increasing if the function f is ultimately increasing; similarly for decreasing, strictly increasing, strictly decreasing and infinitely increasing.

2.2. Lemma. *Let $f, g \in \mathcal{G}$ with g increasing and $f \geq g$. Then*

$$\lim_{x\to+\infty} \Delta f(x) \geq \lim_{x\to+\infty} \Delta g(x) \geq 0.$$

Proof. Ultimately we have $\Delta g(x) \geq 0$ (since g is increasing) and the rightmost inequality holds. We are done if $\upsilon(\Delta f) < 0$, so assume that $\upsilon(\Delta f) \geq 0$. If $f - g$ is increasing, then $\lim_{x\to+\infty} \Delta(f - g)(x) \geq 0$, so $\lim_{x\to+\infty} \Delta f(x) \geq \lim_{x\to+\infty} \Delta g(x)$. If $f - g$ is not increasing, then it is strictly decreasing to some $c \in R$, and thus $\lim_{x\to+\infty} \Delta f(x) = \lim_{x\to+\infty} \Delta g(x)$. □

2.3. Proposition. *Let $f \in \mathcal{G}$. Then*:

1. *$\upsilon(f) < \upsilon(x)$ if and only if $\upsilon(\Delta f) < 0$.*
2. *$\upsilon(f) > \upsilon(x)$ if and only if $\upsilon(\Delta f) > 0$.*
3. *$\upsilon(f) = \upsilon(x)$ if and only if $\upsilon(\Delta f) = 0$ if and only if $f = rx + u$ for some $r \in D^*$ and $u \in \mathcal{G}$ with $\upsilon(u) > \upsilon(x)$.*
4. *If $g \in \mathcal{G}$ and $\upsilon(f) = \upsilon(g) < 0$, then $f = rg + u$ for some $r \in D^*$ and $u \in \mathcal{G}$ with $\upsilon(u) > \upsilon(g)$.*

Proof. Items 1 and 2 follow immediately from 1.5 and 2.2. For 3, note that by 2.2, Lf (as in 1.4) exists; say that $Lf = rx$ and put $u := f - rx$. Then $Lu = 0$, so $\upsilon(\Delta u) > 0$ and $\upsilon(u) > \upsilon(x)$ by 2. Noting that $\upsilon(f(g^{-1})) = \upsilon(x)$, 4 follows easily from 3. □

Notation. Given $f, g : R \to R$, we write $f \approx g$ if $\lim_{x\to+\infty} [f(x) - g(x)] = 0$, in particular, if $f, g \in \mathcal{G}$ and $\upsilon(f - g) > 0$. Note that if $f \in \mathcal{G}$ with $\upsilon(f) \geq 0$, then $f \approx c$ for some $c \in R$.

2.4. Proposition. *Let $f, g \in \mathcal{G}^*$ with $0 \neq \upsilon(f) < \upsilon(g)$. Then $\upsilon(\Delta f) < \upsilon(\Delta g)$.*

Proof. First, note that it's not the case that $\upsilon(\Delta f) = \upsilon(\Delta g) = 0$ (by 2.3.3), so we show that $|\Delta f| > r|\Delta g|$ for all $r \in D^{>0}$. We may assume that $f, g > 0$. Note that for all $r \in D^{>0}$ we thus have $f - rg > 0$.
Case $\upsilon(f) < 0 \le \upsilon(g)$. Since f is infinitely increasing and $g \approx c$ for some $c \ge 0$, clearly $f \pm rg$ is infinitely increasing for all $r \in D^{>0}$. Note also that $\Delta f > 0$. Then for all $r \in D^{>0}$ we have $\Delta f \pm r\Delta g = \Delta(f \pm rg) > 0$ and $\Delta f > r|\Delta g|$.

Case $\upsilon(g) < 0$. Then $\upsilon(f) < 0$ and $\Delta f, \Delta g > 0$. For all $r \in D^{>0}$ we have $f - (r+1)g > 0$, so $f - rg > g$ and $f - rg$ is infinitely increasing, yielding $\Delta f > r\Delta g$.

Case $0 < \upsilon(f) < \upsilon(g)$. Since both f and g are strictly decreasing to 0, we have $\Delta f, \Delta g < 0$. So we must show that $-\Delta f > r(-\Delta g)$ for all $r \in D^{>0}$, that is, $\Delta(f - rg) < 0$. But this holds since $f - rg$ is strictly decreasing for all $r \in D^{>0}$. □

2.5. Proposition. *Let $f, g \in \mathcal{G}^*$ with $0 \ne \upsilon(f) = \upsilon(g)$. Then $\upsilon(\Delta f) = \upsilon(\Delta g)$.*

Proof. If $\upsilon(f) = \upsilon(g) < 0$, then $f = rg + u$ as in 2.3.4, so

$$\upsilon(\Delta f) = \min\{\upsilon(\Delta g), \upsilon(\Delta u)\} = \upsilon(\Delta g)$$

by 2.1.4 and 2.4.

Suppose now that $\upsilon(f) = \upsilon(g) > 0$. We may assume that f, g are strictly decreasing to 0 with $f > g$. Then $f - g$ is strictly decreasing to 0 and $|\Delta f| > |\Delta g|$. On the other hand, there exists $r \in D^{\ge 1}$ such that $f < rg$; then $rg - f$ is strictly decreasing to 0 and $r|\Delta g| > |\Delta f|$. □

2.6. Lemma. *Let $f \in \mathcal{G}$ and $c \in R^*$ with $\upsilon(f(x+c)) = \upsilon(f) < 0$. Then the equality $\upsilon(f(x+y)) = \upsilon(f)$ holds for all $y \in R$.*

Proof. We may assume that f is infinitely increasing; then the compositional inverse f^{-1} exists in $\mathcal{G}$ and is infinitely increasing. Put $g_y := f(f^{-1} + y)$ for $y \in R$. Then $g_y \in \mathcal{G}$, g_y is infinitely increasing, and $g_{y+z} = g_y(g_z)$ for all $y, z \in R$. The set

$$\begin{aligned}\{\, y \in R : \upsilon(f(x+y)) = \upsilon(f) \,\} &= \{\, y \in R : \upsilon(g_y) = \upsilon(x) \,\} \\ &= \{\, y \in R : \lim_{x \to +\infty} \Delta(f(f^{-1}(x) + y)) \in R \,\}\end{aligned}$$

is thus a definable subgroup of R, hence equal either to $\{0\}$ or R. □

Proof of Theorem A. Suppose that $\mathfrak{R}$ is linearly bounded, i.e. $\upsilon(x) = \min \upsilon(\mathcal{G})$. Let $f \in \mathcal{G}$. We must show that $f \in R + Dx + \{\, h \in \mathcal{G} : h \approx 0 \}$. If $\upsilon(f) \ge 0$, then $f = c + (f - c)$, where $f \approx c$. Now suppose that $\upsilon(f) < 0$, so f^{-1} exists. Then $\upsilon(f^{-1}) \ge \upsilon(x)$; hence $\upsilon(f) \le \upsilon(x)$, and $\upsilon(f) = \upsilon(x)$. By 2.3.3, we have $f = rx + u$ for some $r \in D^*$ and u with $\upsilon(u) > \upsilon(x)$. Note that $\upsilon(u) \ge 0$; if not, we would have $\upsilon(u) = \upsilon(x)$. Then $f = c + rx + h$, where $u \approx c$ and $h = u - c$.

Suppose now that $\mathfrak{R}$ is not linearly bounded. Then there exists some infinitely increasing $f \in \mathcal{G}$ with $\upsilon(f) < \upsilon(x)$. Note that $\upsilon(f(x+1)) \le \upsilon(f)$, Δf is infinitely increasing by 2.3.1, and $(\Delta f)^{-1}$ exists and is infinitely increasing.

First, consider the case that $\upsilon(f(x+1)) = \upsilon(f)$. Define $g : R \times R \to R$ by $g(a, x) := \Delta_a f((\Delta f)^{-1}(x))$. Let $a \ge 0$. By 2.6 we have $\upsilon(f) = \upsilon(f(x+a))$; hence by 2.5,

$$\upsilon(\Delta f) = \upsilon(\Delta(f(x+a))) = \upsilon(\Delta f(x+a))$$

and

$$\upsilon(\Delta f((\Delta f)^{-1}+a)) = \upsilon(x).$$

Since Δf is infinitely increasing, we have $\Delta f((\Delta f)^{-1}+a) \geq x$. Thus there exists by 2.3.3 an $r \in D^{\geq 1}$ (depending on a) with $\Delta(\Delta f((\Delta f)^{-1}+a)) \approx r$. Now

$$g(a+1,x) = g(a,x) + \Delta f((\Delta f)^{-1}(x)+a),$$

so if $\Delta g(a,x) \approx b \in R$, then $\Delta g(a+1,x) \approx b+r \geq b+1$. Noting that $\Delta g(1,x) = \Delta x = 1$, it follows that the set $\{\, b \in R : \exists a \in R\,[\,\Delta g(a,x) \approx b\,]\,\}$ is infinite. Applying 1.7, we are done with this case.

Next, suppose that $\upsilon(f(x+1)) < \upsilon(f)$. We construct an infinitely increasing $h \in \mathcal{G}$ with $\upsilon(h(x+1)) = \upsilon(h) < \upsilon(x)$; we are then done by the previous case (and 1.1).

By 2.6, $\upsilon(f(x+y)) < \upsilon(f)$ for all $y > 0$. Then for all $\epsilon > 0$ and $r \in D^{>1}$, we have $f(x+\epsilon) > rf$ and thus $f^{-1}(rx) - f^{-1} < \epsilon$. Put $g := f^{-1}$; then g is infinitely increasing, $\upsilon(g) > \upsilon(x)$, and $g(rx) \approx g$ for all $r \in D^{>1}$. By 2.3.2, Δg is strictly decreasing to 0. Put $h := (\Delta g)^{\#}(-\Delta^2 g)$ with $(\Delta g)^{\#}$ and $\Delta^2 g$ as in 1.6. Note that $\Delta g(h) = -\Delta^2 g$ and $\upsilon(h(x+1)) \leq \upsilon(h)$ (since h is infinitely increasing).

Suppose that $\upsilon(h(x+1)) < \upsilon(h)$, that is, $h(x+1) > rh$ for all $r \in D^{>0}$. By 1.10, we then have $-\Delta^2 g(x+1) < \Delta g(rh) < 2r^{-1}(-\Delta^2 g)$ for all $r \in D^{>1}$; that is, $\upsilon(\Delta^2 g(x+1)) > \upsilon(\Delta^2 g)$. But $\upsilon(g) < \upsilon(\Delta g)$ yields $\upsilon(g(x+1)) = \upsilon(g)$ by 2.1.5, and thus $\upsilon(\Delta^2 g(x+1)) = \upsilon(\Delta^2 g)$ by two applications of 2.5; a contradiction. Hence, $\upsilon(h(x+1)) = \upsilon(h)$.

Let $r \in D^{>1}$. Since $g(rx) \approx g$—that is, $\upsilon(g(rx)-g) > 0$—and $\upsilon(g), \upsilon(g(rx)) < 0$, we have $\upsilon(g) = \upsilon(g(rx))$ by 2.1.5 and $\upsilon(\Delta g) = \upsilon(\Delta(g(rx))) = \upsilon(\Delta g(rx))$ by 2.5 and 1.10. By 2.4, $\upsilon(\Delta g) < \upsilon(\Delta^2 g)$, and thus $\upsilon(\Delta g(rx)) < \upsilon(\Delta^2 g)$. Consequently, $-\Delta^2 g < \Delta g(rx)$ and $h > rx$. Since this holds for all $r \in D^{>1}$, we have $\upsilon(h) < \upsilon(x)$. □

3. The linearly bounded case

Throughout this section, we suppose that $\mathfrak{R}$ is linearly bounded.

Theorem A and 1.1 yield immediately that every structure elementarily equivalent to $\mathfrak{R}$ is linearly bounded. But more is true; the ring Λ (as in §1) is "elementarily invariant". We make this statement precise in the next result (a restatement of Theorem B). Using 1.7 and Theorem A, its proof is nearly the same as that of 4.3 and 4.4 of [4], so we omit it.

3.1. Proposition. *Every $\lambda \in \Lambda$ is 0-definable, and $\Lambda(\mathfrak{R}')$ is canonically isomorphic to Λ (as an ordered division ring) for every $\mathfrak{R}' \equiv \mathfrak{R}$.*

Here is another immediate consequence of 1.7 and Theorem A:

3.2. Proposition. *Let $f : A \times R \to R$ be definable, $A \subseteq R^m$. Then there exist $r_1, \ldots, r_l \in D$ such that for every $y \in A$, there is an $i \in \{1, \ldots, l\}$ (depending on y) with $\lim_{x\to+\infty}[f(y,x) - r_i x] \in R$.*

3.3. Proposition. *Let $\mathfrak{R}$ be linearly bounded with respect to $\oplus$. Then $(R, \oplus)$ is definably isomorphic to $(R, +)$.*

Proof. By 1.3, it suffices to show that $\lim_{x\to+\infty}[(x \oplus 1) - x] \in R^*$. Suppose otherwise. If $\lim_{x\to+\infty}[(x \oplus 1) - x] = +\infty$, then there exists by 3.2 some $r \in D^{>1}$ with $\lim_{x\to+\infty}[(x \oplus 1) - rx] \in R$. But then $\lim_{x\to+\infty}[(x \oplus 1 \oplus 1) - r^2 x] \in R$,

$\lim_{x\to+\infty}[(x\oplus 1\oplus 1\oplus 1)-r^3x]\in R$ and so on, contradicting 3.2. On the other hand, if $\lim_{x\to+\infty}[(x\oplus 1)-x]=0$, then for all $y>0$ we have $\lim_{x\to+\infty}[(x\oplus y)-x]=0$ (by 1.3) and thus ultimately $(x+1)\ominus x>y$; hence, $\lim_{x\to+\infty}[(x+1)\ominus x]=+\infty$. Since $\mathfrak{R}$ is linearly bounded with respect to $\oplus$, we get a contradiction as before. □

Remark. Let ϕ be any semialgebraic homeomorphism of $\mathbb{R}$ onto $\mathbb{R}^{>0}$, and put $x*y := \phi^{-1}(\phi(x)\phi(y))$ for $x,y\in\mathbb{R}$. Then $(\mathbb{R},<,+,*)$ is linearly bounded with respect to $*$ but not linearly bounded with respect to $+$.

Proof of Theorem C. Let $\mathfrak{M} := (M,<,+_1,+_2,+_3)$ be o-minimal such that $(M,+_i)$ is a continuous group, $i=1,2,3$. We claim that at least two of $(M,+_1)$, $(M,+_2)$ and $(M,+_3)$ are definably isomorphic. If $\mathfrak{M}$ is linearly bounded with respect to any two $+_i$, say $+_1$ and $+_2$, then $(M,+_1)\cong_{\mathrm{df}}(M,+_2)$ by 3.3. If this is not the case, then (by Theorem A) $\mathfrak{M}$ defines binary operations, say $\times_1$ and $\times_2$, such that $(M,<,+_1,\times_1)$ and $(M,<,+_2,\times_2)$ are ordered fields; then $(M,<,+_1,\times_1)$ is definably isomorphic to $(M,<,+_2,\times_2)$ by [5]. □

4. Groups definable in expansions of fields

In this section we assume that $\mathfrak{R}$ *expands an ordered field* $(R,<,+,\cdot,0,1)$.

We recall some facts, definitions and notation from [4]. A **power function** of $\mathfrak{R}$ is a definable endomorphism of $(R^{>0},\cdot)$. If a power function is not identically equal to 1 on $R^{>0}$, then it is an automorphism of $(R^{>0},\cdot)$ and is strictly monotone and differentiable with $xf'(x)=f'(1)f(x)$ for all $x>0$. The set of all power functions of $\mathfrak{R}$ is an ordered field, with addition given by pointwise multiplication in R, multiplication given by composition, and positive elements the strictly increasing power functions. A power function f is uniquely determined by its derivative at 1, and we write x^α for f, where $\alpha=f'(1)$.

We say that $\mathfrak{R}$ is **power bounded** if every definable unary function is ultimately bounded by some power function, and that $\mathfrak{R}$ is **exponential** if $(R,+)\cong_{\mathrm{df}}(R^{>0},\cdot)$.

The next result was established in [4] using Hardy field methods. We give an alternate proof using Theorem A.

4.1. Theorem (Growth Dichotomy, Field Version). $\mathfrak{R}$ *is either exponential or power bounded. If it is power bounded, then for every ultimately nonzero definable* $f:R\to R$ *there exist* $\alpha\in R$ *and* $c\in R^*$ *such that* $f(x)=cx^\alpha+o(x^\alpha)$ *as* $x\to+\infty$.

Proof. Put $\mathfrak{R}' := \big(R^{>0},<,\cdot,(f)\big)$, where f ranges over all definable functions from $R^{>0}$ into $R^{>0}$. Note that the $\mathfrak{R}'$-definable endomorphisms of $(R^{>0},\cdot)$ are precisely the power functions of $\mathfrak{R}$. By Theorem A, either $\mathfrak{R}'$ is linearly bounded or $\mathfrak{R}'$ defines a binary operation $*$ such that $(R^{>0},<,\cdot,*,1,2)$ is an ordered field. If the former, then clearly $\mathfrak{R}$ is power bounded and the asympotic condition on $\mathfrak{R}$-definable unary functions holds; so assume the latter. The operation $*$ is definable in $\mathfrak{R}$ and for each fixed $y>0$ the function $x\mapsto x*y : R^{>0}\to R^{>0}$ is a power function $x^{\alpha(y)}$ for some $\alpha(y)\in R$; so we have $\alpha(y)=(x*y)'(1)$ (where the derivative is taken with respect to x). The map $y\mapsto\alpha(y):R^{>0}\to R$ is thus $\mathfrak{R}$-definable, and one checks easily that it is an isomorphism of $(R^{>0},\cdot)$ onto $(R,+)$. Hence, $\mathfrak{R}$ is exponential. □

We now continue our investigation of definable groups.

4.2. Proposition. *If* $\mathfrak{R}$ *is power bounded, then either* $(R,\oplus)\cong_{\mathrm{df}}(R,+)$ *or* $(R,\oplus)\cong_{\mathrm{df}}(R^{>0},\cdot)$.

Proof. Let ϕ denote the homeomorphism $x \mapsto x + \sqrt{1+x^2} : R \to R^{>0}$. Put $x * y := \phi^{-1}(\phi(x)\phi(y))$ for $x, y \in R$; then $(R, *)$ is a continuous group and $(R, *) \cong_{\text{df}} (R^{>0}, \cdot)$. Since $\mathfrak{R}$ is power bounded, $(R, *) \not\cong_{\text{df}} (R, +)$ by 4.1. Now apply Theorem C. □

4.3. Corollary. *Suppose that R is the set of real algebraic numbers. Then either $(R, \oplus) \cong_{\text{df}} (R, +)$ or $(R, \oplus) \cong_{\text{df}} (R^{>0}, \cdot)$.*

Proof. It is well known that there are no isomorphisms of $(R, <, +)$ onto $(R^{>0}, <, \cdot)$, hence any isomorphism of $(R, +)$ onto $(R^{>0}, \cdot)$ must be everywhere discontinuous. Then $\mathfrak{R}$ is power bounded by 4.1 (and the Monotonicity Theorem). Apply 4.2. □

4.4. Lemma. *If there exist $c > 0$ and a definable function $f : R \to R$ with $\lim_{x\to+\infty}[f(x+c) \ominus f(x)] \neq 0$, then $(R, \oplus) \cong_{\text{df}} (R, +)$.*

Proof. Suppose that $(R, \oplus) \not\cong_{\text{df}} (R, +)$. Then $\lim_{x\to+\infty} |f(x+c) \ominus f(x)| = +\infty$ by 1.3. Now by [5] and Theorem A, $\mathfrak{R}$ is linearly bounded with respect to $\oplus$, so by 3.2 there exist $y_0 \in R$ and a definable endomorphism λ of $(R, \oplus)$ such that for all $y \geq y_0$ we have $\lim_{x\to+\infty}[f(x+y) \ominus \lambda(x)] \in R$. But then

$$\lim_{x\to+\infty}[f(x+y_0+c) \ominus f(x+y_0)] = \lim_{x\to+\infty}[f(x+c) \ominus f(x)] \in R;$$

contradiction. □

We now assume that $\mathfrak{R}$ is exponential.

There exists a unique definable solution y on R to $y' = y,\ y(0) = 1$; it behaves essentially like the real exponential function e^x, and we denote it by exp. The compositional inverse of exp (defined on $R^{>0}$) is denoted by log. For convenience, we put $\log(x) := 0$ for $x \leq 0$. (See [4] for details on the above.)

4.5. Proposition. *If there exist $\epsilon, C > 0$ and $f : R \to R$ definable and infinitely increasing such that ultimately $f(x \oplus \epsilon) \leq Cf(x)$, then $(R, \oplus) \cong_{\text{df}} (R, +)$.*

Proof. Put $g(x) := f^{-1}(\exp(x))$ for sufficiently large $x > 0$. Then ultimately $g(x + \log C) \ominus g(x) \geq \epsilon$. Now apply 4.4. □

Remark. The converse of 4.5 clearly holds.

Notation. Let e_0 denote the identity on R and put $e_{n+1}(t) := \exp(e_n(t))$ for $n \in \mathbb{N}$ and $t \in R$. Similarly, ℓ_0 denotes the identity on R and $\ell_{n+1}(t) := \log(\ell_n(t))$ for each $n \in \mathbb{N}$ and $t \in R$. We also write ℓ_{-n} instead of e_n, so given $j, k \in \mathbb{Z}$ ultimately we have $\ell_{j+k}(t) = \ell_j(\ell_k(t))$.

Definition. Let $f : R \to R$ be definable and infinitely increasing. Suppose that there exist $k, j \in \mathbb{Z}$ such that $\lim_{x\to+\infty}[\ell_k(f(x))/\ell_j(x)] = 1$. Then the integer $s = k - j$ is unique and f is said to have **level** s (we write $\text{level}(f) = s$). Also, a definable infinitely increasing function $f : R \to R$ is said to have level if there exists $s \in \mathbb{Z}$ with $\text{level}(f) = s$. (See [3] for further information on level.)

Given definable infinitely increasing unary functions f_1 and f_2 with levels s_1 and s_2 respectively, it is routine to check that $s_1 \geq s_2$ if ultimately $f_1(t) \geq f_2(t)$, and that $\text{level}(f_1 \circ f_2) = s_1 + s_2$.

4.6. Proposition. *If for every $y > 0$ the translate $x \mapsto x \oplus y$ has level, then $(R, \oplus) \cong_{\text{df}} (R, +)$.*

Proof. It suffices (by 4.5) to show that some translate $x \oplus c$ with $c > 0$ has level 0. Let x denote the identity function on R. We have $\operatorname{level}(x \oplus y) \geq \operatorname{level}(x) = 0$ for every $y > 0$. Let $c > 0$ be such that $\operatorname{level}(x \oplus c) = \min\{\operatorname{level}(x \oplus y) : y > 0\}$. Let d be such that $d \oplus d = c$. Then $2\operatorname{level}(x \oplus d) = \operatorname{level}(x \oplus c) \leq \operatorname{level}(x \oplus d)$, so $\operatorname{level}(x \oplus c) = 0$. □

4.7. Proposition. *Let $\mathfrak{M} := (M, <, +, \cdot, \dots)$ be an expansion of an ordered field, elementarily equivalent to a reduct of the structure $\mathbb{R}_{\mathrm{an,exp}}$. If $(M, *)$ is a continuous group definable in $\mathfrak{M}$, then either $(M, *) \cong_{\mathrm{df}} (M, +)$ or $(M, *) \cong_{\mathrm{df}} (M^{>0}, \cdot)$.*

Proof. By 4.1 and 4.2, we may reduce to the case that $\mathfrak{M}$ is exponential. By [3], every definable unary function has level, so $(M, *) \cong_{\mathrm{df}} (M, +)$ by 4.6. □

5. Proof of Lemma 1.8

In this section, we abandon the notation of the previous sections.

Let $\mathfrak{R}$ be an o-minimal expansion of an ordered group $(R, <, \oplus, 0, 1)$ with $1 > 0$. We denote the inverse of $x \in R$ with respect to $\oplus$ by $\ominus x$. We let $\Delta_y f(x)$ denote $f(x \oplus y) \ominus f(x)$ for a function $f : R \to R$ and $x, y \in R$, suppressing the subscript for $y = 1$. Given $t \in R$ and $\lambda \in \operatorname{Aut}(R, \oplus)$, we write $\lambda.t$ for $\lambda(t)$.

We prove the following equivalent formulation of 1.8:

5.1. *Let $f : R \to R$ be definable, continuous and infinitely increasing, and such that $\lim_{x \to +\infty} \Delta f(x) = 0$. Let $\alpha \in \operatorname{Aut}(R, \oplus)$ be definable and strictly increasing. Then there exists $C \in R$ such that for all $y > x > C$, the functions*

$$t \mapsto \alpha.\Delta_t f(y) \ominus \Delta_{\alpha.t} f(x) : R \to R, \quad t \mapsto \alpha.\Delta_t f(x) \ominus \Delta_{\alpha.t} f(y) : R \to R$$

are strictly monotone on some open interval (depending on x and y) about 0.

We have some preliminary work to do.

Since Δf is ultimately strictly decreasing to 0, there is a definable bijection from a bounded interval of R onto an unbounded interval. By 1.2 of [6] there exist definable binary operations $+$ and $\cdot$ such that $(R, <, +, \cdot, 0, 1)$ is an ordered field.

We now regard $\mathfrak{R}$ as an expansion of the ordered field $(R, <, +, \cdot, 0, 1)$, and $(R, \oplus)$ as a continuous group definable in $\mathfrak{R}$.

Since the conclusion of 5.1 is first order, we may assume throughout this section that $\mathfrak{R}$ is ω-saturated.

Recall that an element $e \in R$ is said to be **generic over** a set $A \subseteq R$ if $e \notin \operatorname{acl}(A)$. If A is a singleton $\{a\}$ and $e \in R$ is generic over A, we also say that e is generic over a. Note that since $\mathfrak{R}$ is ω-saturated, for every finite $A \subseteq R$ there exists $e \in R$ generic over A. A set $A \subseteq R$ is **independent** if, for each $a \in A$, a is generic over $A \setminus \{a\}$.

Here are some easy properties of genericity; we use them in the sequel without further mention. The proofs are left to the reader.

Let $A \subseteq R$.

1. *If A is finite and $B \subseteq R$ is A-definable, then B is finite if and only if it contains no element generic over A.*
2. *If $f : R \to R$ is A-definable and e is generic over A, then f is differentiable on an open interval containing e. If moreover $f'(e) = 0$, then f is constant on an open interval containing e.*
3. *Let $f : (a, b) \to R$ be A-definable and strictly monotone. Then $e \in (a, b)$ is generic over A if and only if $f(e)$ is generic over A.*

4. *Let $*$ be an A-definable binary operation on R^2 such that $(R,*)$ is a group. If a is generic over A and $b \in A$, then $a*b$ is generic over A.*

5.2. *Let $A \subseteq R$, let $f : R \to R$ be A-definable, and let e be generic over A. Then the function $F : R \to R$ given by $F(t) := f(t) \ominus f(e) \oplus e$ is differentiable at e.*

Proof. Since e is generic over A, there is an open interval I containing e such that f is either constant on I or strictly monotone on I. If the former, we are done, so assume that f is strictly monotone on some open interval containing e. Let d be generic over $A \cup \{e\}$; then d is generic over $A \cup \{f(e)\}$ as well. For $t \in R$ put $g_1(t) := t \oplus d \ominus f(e)$ and $g_2(t) := t \oplus e \ominus d$. Now g_1 is differentiable at $f(e)$ (since $f(e)$ is generic over $A \cup \{d \ominus f(e)\}$) and g_2 is differentiable at $d = g_1(f(e))$ (since d is generic over $A \cup \{e \ominus d\}$). Then $F = g_2 \circ g_1 \circ f$ is differentiable at e. □

Notation. Let $c \in R$ and $h, g : R \to R$ be functions. We write $h(t) \prec_c g(t)$ if $h(c) = g(c)$, and there exists $\epsilon > 0$ such that $h(t) > g(t)$ for all $t \in (c-\epsilon, c)$ and $h(t) < g(t)$ for all $t \in (c, c+\epsilon)$.

5.3. Given c, h, g as above, if h and g are continuous at c and definable, then (by the Monotonicity Theorem) we have $h(t) \prec_c g(t)$ if and only if the function $g \ominus h$ is strictly monotone increasing on some interval about c, taking the value 0 at c.

Throughout the rest of this section we let $f : R \to R$ be definable, continuous and infinitely increasing with $\lim_{x\to+\infty} \Delta f(x) = 0$; let $\alpha \in \mathrm{Aut}(R,\oplus)$ be definable and strictly increasing. We may assume that f is strictly increasing on R, and—adding constants to the language as necessary—that f and α are 0-definable.

5.4. *Let $e \in R$ be generic. Then the function $S(t) := \alpha.(t \ominus e) \oplus e$ is differentiable at e and $S'(e) > 0$.*

Proof. Noting that $S(t) = \alpha.t \ominus \alpha.e \oplus e$, we see that S is differentiable at e by 5.2. We also have $\alpha.t = S(t) \ominus e \oplus \alpha.e$ for all $t \in R$. Arguing similarly as in the proof of 5.2, we see that the function $g(t) := t \ominus e \oplus \alpha.e$ is differentiable at $e = S(e)$. Since $\alpha = g \circ S$, we have $\alpha'(e) = g'(S(e))S'(e)$. Now e is generic, and both S and α are strictly increasing, so $S'(e) > 0$. □

We now let C denote a large positive element of R which we allow to increase as convenient.

5.5. *$\Delta_t f(b) \prec_0 \Delta_t f(a)$ for all $b > a > C$.*

Proof. By 1.12, for all $b > a > C$ and all $t > 0$ we have $\Delta_t f(b) < \Delta_t f(a)$ (since Δf is ultimately strictly decreasing). On the other hand, for all $t > 0$ such that $a \ominus t > C$, we have $\Delta_t f(b \ominus t) < \Delta_t f(a \ominus t)$, that is, $\Delta_{\ominus t} f(b) > \Delta_{\ominus t} f(a)$. Hence, $\Delta_t f(b) \prec_0 \Delta_t f(a)$. □

5.6. *Let $A \subseteq R$ be finite. Then for every $e \in (C, \infty)$ generic over A there exist $c \in (C,\infty)$ generic over $A \cup \{e\}$ and an open interval I containing e such that $(\gamma + \delta) \ominus \gamma \le (c+\delta) \ominus c$ for all $\delta > 0$ and $\gamma \in I$.*

Proof. By 1.12, either:

(i) $(x+t) \ominus x \le (y+t) \ominus y$ for all $y > x > C$ and all $t > 0$; or
(ii) $(x+t) \ominus x \ge (y+t) \ominus y$ for all $y > x > C$ and all $t > 0$.

Choose c generic over $A \cup \{e\}$ with $c > e$ if (i) holds, and $C < c < e$ if (ii) holds. □

We now fix some generic $e > C$.

5.7. *For each $x > C$, the function $F_x(t) := \Delta_{t\ominus e} f(x) \oplus e$ is differentiable at $t = e$, and the function $x \mapsto F'_x(e) : (C, \infty) \to R$ is strictly decreasing.*

Proof. In the following, we will say that a property $P(t)$ holds at e^+ if $P(t)$ holds for all t in some interval (e, e') with $e' > e$.

Let a be generic over e. Noting that $F_a(t) = g(t) \ominus g(e) \oplus e$ with $g(t) := f(a \oplus t \ominus e)$ we see that F_a is differentiable at e by 5.2 (since a is generic over $a \ominus e$). This holds for arbitrary a generic over e, so the set of $x \in R$ such that F_x is not differentiable at e is finite. Hence, F_x is differentiable at $t = e$ for all $x > C$.

By 5.5, for all $b > a > C$ we have $\Delta_t f(b) \prec_0 \Delta_t f(a)$, so $F_b(t) \prec_e F_a(t)$ and $F'_b(e) \leq F'_a(e)$ (by l'Hospital's rule). Thus, $x \mapsto F'_x(e)$ is decreasing on (C, ∞), and for all $b > a > C$ we have $0 \prec_e F_a(t) \ominus F_b(t)$. Let $b > a > C$ be such that $\{e, a, b\}$ is independent. It suffices now to show that $F'_a(e) \neq F'_b(e)$.

Choose c as in 5.6 generic over $\{e, a, b\}$. Let $a_1 = a \ominus e$, $b_1 = b \ominus e$, and $c_1 = f(b) \ominus f(a) \oplus c$. Put

$$h(t) := F_a(t) \ominus F_b(t) \oplus c = f(a_1 \oplus t) \ominus f(b_1 \oplus t) \oplus c_1.$$

Then h is definable over $\{a_1, b_1, c_1\}$ and $h(t) > c$ at e^+ (since $0 \prec_e F_a(t) \ominus F_b(t)$). Now e is generic over $\{a_1, b_1, c_1\}$—if not, we would have $\{e, a, b, c\} \subseteq \operatorname{dcl}\{a_1, b_1, c_1\}$ contradicting that $\{e, a, b, c\}$ is independent—so h is differentiable at e by 5.2. Since $h(e) = c$ and e is generic and $h(t) > c$ at e^+, we must have $h'(e) > 0$. Fix $\epsilon \in (0, h'(e)/3)$. Then at e^+ we have

$$h(t) > (h'(e) - \epsilon)(t - e) + c > 2\epsilon(t - e) + c.$$

Hence,

$$(*) \qquad F_a(t) \ominus F_b(t) > (2\epsilon(t - e) + c) \ominus c \qquad \text{at } e^+.$$

Suppose that $F'_b(e) = F'_a(e) := L$. Noting that $F_a(e) = F_b(e) = e$, we have $F_a(t) \prec_e (L + \epsilon)(t - e) + e$ and $(L - \epsilon)(t - e) + e \prec_e F_b(t)$. Hence, at e^+ we have

$$\begin{aligned} F_a(t) \ominus F_b(t) &\leq ((L + \epsilon)(t - e) + e) \ominus ((L - \epsilon)(t - e) + e) \\ &= (2\epsilon(t - e) + (L - \epsilon)(t - e) + e) \ominus ((L - \epsilon)(t - e) + e) \\ &\leq (2\epsilon(t - e) + c) \ominus c. \end{aligned}$$

(The last inequality is by 5.6 and the choice of c.) This contradicts $(*)$; hence $F'_b(e) \neq F'_a(e)$. ◻

We are now ready to finish the proof of 5.1.

Let $y > x > C$, and let S be as in 5.4. Note that $S(e) = F_x(e) = F_y(e) = e$. By 5.4 and 5.7, both $S \circ F_y$ and $F_x \circ S$ are differentiable at e, with

$$(S \circ F_y)'(e) = S'(e) F'_y(e) \quad \text{and} \quad (F_x \circ S)'(e) = S'(e) F'_x(e).$$

Since $S'(e) > 0$ and $F'_y(e) < F'_x(e)$, we get $(S \circ F_y)'(e) < (F_x \circ S)'(e)$, and thus (by monotonicity and l'Hospital's rule) $S(F_y(t)) \prec_e F_x(S(t))$; unravelling this yields

$$\alpha.\Delta_{t\ominus e} f(y) \prec_e \Delta_{\alpha.(t\ominus e)} f(x).$$

Substituting $t \oplus e$ for t, we obtain $\alpha.\Delta_t f(y) \prec_0 \Delta_{\alpha.t} f(x)$, that is, the function

$$t \mapsto \alpha.\Delta_t f(y) \ominus \Delta_{\alpha.t} f(x) : R \to R$$

is strictly decreasing on some interval about 0 (recall 5.3).

By the same argument, we have $\beta.\Delta_t f(y) \prec_0 \Delta_{\beta.t} f(x)$, where β denotes the compositional inverse of α; hence $\Delta_{\alpha.t} f(y) \prec_0 \alpha.\Delta_t f(x)$ and the function

$$t \mapsto \alpha.\Delta_t f(x) \ominus \Delta_{\alpha.t} f(y) : R \to R$$

is strictly increasing on some interval about 0.

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