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REGULARITY OF SOLUTIONS TO A CONTACT PROBLEM

RUSSELL M. BROWN, ZHONGWEI SHEN, AND PETER SHI

ABSTRACT. We consider a variational inequality for the Lamé system which models an elastic body in contact with a rigid foundation. We give conditions on the domain and the contact set which allow us to prove regularity of solutions to the variational inequality. In particular, we show that the gradient of the solution is a square integrable function on the boundary.

1. Introduction

In this note, we consider solutions to a variational inequality which provides a model for an elastic body at equilibrium, part of whose boundary is in contact with a rigid foundation. We impose boundary conditions on the contact set which model friction. Our main result is the regularity result in Theorem 2.29. We show that under certain geometric conditions on the boundary and on the interface between the free and contact set, the gradient of the solution is a square integrable function on the boundary. Thus the traction at the boundary is a function instead of a distribution in the Sobolev space $H^{-1/2}$. We hope to use this result to analyze more sophisticated models of frictional contact such as Coulomb's law of friction. This will be discussed further below.

To describe our problem more fully, let us consider $\Omega \subset \mathbf{R}^n$, $n \geq 2$, a connected bounded open set with Lipschitz boundary. We let $u: \Omega \to \mathbf{R}^n$ be a displacement field of Ω. We use ∂_j for the partial derivative of u with respect to X_j and then let

$$\epsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$$

be the strain tensor and, for λ, μ satisfying $\mu > 0$ and $\lambda + 2\mu > 0$, we let

$$\sigma_{ij}(u) = 2\mu\epsilon_{ij}(u) + \lambda\delta_{ij}\epsilon_{kk}(u)$$

be the stress tensor. (Here, and throughout this paper, we follow the convention that repeated indices are summed.) We let L be the Lamé operator given by

$$(Lu)_i = \partial_j \sigma_{ij}(u), \qquad i = 1, \dots, n.$$

We assume Ω has Lipschitz boundary. We suppose that $\partial\Omega$ has been divided into two disjoint sets, Γ_c and Γ_f, which we call the contact set and the free set. (These assumptions will be made precise and additional conditions imposed below.) We let $g \geq 0$ be a fixed function on $L^2(\Gamma_c)$ and let $K \subset H^1(\Omega)$ be the set of $\mathbf{R}^n$-valued

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functions v with $v \cdot N = 0$ on Γ_c. Here N is the unit outward normal to $\partial\Omega$. If $\tau \in L^2(\Gamma_f)$, say, we consider the problem of minimizing

$$I(v) = \frac{1}{2}\int_\Omega \sigma_{ij}(v)\epsilon_{ij}(v)\,dX - \int_{\Gamma_f} \tau_i(P)v_i(P)\,dP + \int_{\Gamma_c} g(P)|v(P)|\,dP.$$

We assume that

$$-\int_{\Gamma_f} \tau_i v_i\,dP + \int_{\Gamma_c} g|v|\,dP > 0 \tag{1.1}$$

for all $v \in \mathrm{IRD} \cap K$, $v \neq 0$. Here IRD (infinitesimal rigid displacements) denotes the set of functions $u : \Omega \to \mathbf{R}^n$ of the form $u(X) = a + BX$ where $a \in \mathbf{R}^n$ is a constant vector and $B = -B^t$ is a skew-symmetric, $n \times n$ matrix with real entries. Thanks to (1.1) we have $\lim_{\|v\|_{H^1}\to\infty} I(v) = \infty$. Thus it is elementary to show that the problem

$$\min_{v\in K} I(v) \tag{1.2}$$

has a solution. Examining the conditions for a minimum, we see that u is a solution to the minimization problem, (1.2), if and only if u satisfies the variational inequality

$$\begin{aligned}\int_\Omega \sigma_{ij}(u)\epsilon_{ij}(v-u) - \int_{\Gamma_f} \tau_i(v_i - u_i)\,dP \\ + \int_{\Gamma_c} g(|v| - |u|)\,dP \geq 0, \qquad v \in K.\end{aligned} \tag{1.3}$$

Following a standard argument (see [DL76]), one can show that two solutions to this variational inequality differ at most by an element of IRD. We let K_c be the traces on Γ_c of functions in K, $K_c = \{\psi : \ \psi = u|_{\Gamma_c},\ u \in K\}$ and assume that

$$\bar{K}_c = L^2(\Gamma_c) \cap \{u : u \cdot N = 0\}, \tag{1.4}$$

where the closure is taken in the topology of $L^2(\Gamma_c)$.

If we choose $v - u = \pm\psi,\ \psi \in C_0^\infty(\Omega)$, then the inequality (1.3) implies

$$\int_\Omega \sigma_{ij}(u)\epsilon_{ij}(\psi)\,dX = 0, \qquad \psi \in C_0^\infty(\Omega),$$

and hence v is a solution of the Lamé system. Now we choose $v - u = \pm\psi$ in (1.3) where $\psi \in K$. Then we conclude that

$$\left|\int_\Omega \sigma_{ij}(u)\epsilon_{ij}(\psi)\,dX - \int_{\Gamma_f} \tau_i\psi_i\,dP\right| \leq \int g|\psi|\,dP, \quad \psi \in K.$$

Thanks to our hypothesis (1.4), we can conclude that the expression inside the absolute values in the left-hand side of the previous inequality extends to a bounded linear functional on $L^2(\Gamma_c) \cap \{v : v \cdot N = 0\}$. Hence, there exists $h \in L^2(\Gamma_c) \cap \{v : v \cdot N = 0\}$ so that

$$\int_\Omega \sigma_{ij}(u)\epsilon_{ij}(\psi)\,dX - \int_{\Gamma_f} \tau_i\psi_i\,dP - \int_{\Gamma_c} h_i\psi_i\,dP = 0, \qquad \psi \in K. \tag{1.5}$$

In fact, it is clear that $|h| \leq g$ on $\partial\Omega$.

Thus we have that u satisfies the boundary value problem

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u \cdot N = 0 & \text{on } \Gamma_c, \\ \sigma_{\tan}(u) = h & \text{on } \Gamma_c, \\ \sigma(u)N = \tau & \text{on } \Gamma_f, \end{cases} \tag{1.6}$$

in the weak sense. Here, we have decomposed the traction at the boundary, $\sigma(v)N$, into normal and tangential components as $\sigma(v)N = \sigma_{\tan}(v) + \sigma_N(v)N$, where $\sigma_N(v) = \sigma_{ij}(v)N_iN_j$. Finally, we say u is a weak solution to (1.6) if $u \in K$ and (1.5) holds. For convenience, we summarize our discussion of existence as follows.

Theorem 1.7. *Suppose Ω is a Lipschitz domain, K_c satisfies* (1.4) *and τ and g satisfy the compatibility condition* (1.1)*. Then there is a solution to* (1.3) *which is also a weak solution to* (1.6)*. Solutions of either* (1.3) *or* (1.6) *are unique, up to an element in IRD.*

The main goal of this paper is to give conditions on Ω under which the boundary value problem (1.6) has a solution, u, with $\nabla u \in L^2(\partial\Omega)$. In contrast to traditional approaches to proving regularity, we do not differentiate the solution and show that it satisfies some equation. Instead, we consider an alternate method of proving existence which provides us with more regular solutions. Then, since solutions of (1.6) are unique (modulo elements of IRD), the regular solution must coincide with the original solution.

The main tool used in studying (1.6) is the results of Dalhberg, Kenig and Verchota [DKV88] for the Lamé system in Lipschitz domains. The argument used to pass from the mixed problem (1.6) to the ordinary traction problem are similar to ideas used by the first author in [Bro94].

The variational inequality (1.3) was discussed in the monograph of DuVaut and Lions [DL76, p. 152], where related problems are also discussed. The basic existence theory outlined above may be found in this book. They show that if u is a solution of the variational inequality (1.3), then at least formally, u satisfies the following conditions on Γ_c:

$$\begin{array}{ll} |\sigma_{\tan}(u)| \le g. & \\ \text{If } |\sigma_{\tan}(u)| < g, & \text{then } u_{\tan} = 0. \\ \text{If } |\sigma_{\tan}(u)| = g, & \text{then } u_{\tan} = -\lambda\sigma_{\tan}(u) \\ & \text{for some function } \lambda \ge 0. \end{array} \tag{1.8}$$

These conditions model frictional contact between the set Γ_c and a solid base, with g representing the magnitude of force at which slipping begins.

The monograph of DuVaut and Lions [DL76] also poses an interesting related problem, Coulomb's law of friction. In this problem, the friction bound g is not given. Instead we replace g in (1.8) by $\mathcal{F}(P)|\sigma_N(u)(P)|$, where $\mathcal{F}$ is the coefficient of friction and $\sigma_N(u)$ is the normal component of the traction. This problem has not found a satisfactory solution. DuVaut [Duv80] considered a regularized problem which gives a non-local friction law. Nečas, Jaroslav and Haslinger [NJH80] considered the problem in an infinite slab, and then Jarušek [Jar83] extended their work to more general geometries. However, these works require that the friction coefficient, $\mathcal{F}(P)$, be supported in a compact subset of the contact set. One feature of our work is that the function g in (1.8) is allowed to be nonzero near the boundary of the contact set. We also allow domains which have corners at the boundary between the contact set and the free set. On the other hand, the above authors consider the problem with $u \cdot N \geq 0$ on Γ_c instead of positing $u \cdot N = 0$. We hope that the estimates below provide a first step towards treating Coulomb's friction in more general settings.

2. A regularity result

The purpose of this section is to derive regularity for solutions to the boundary value problem (1.6). This is done under additional restrictions on the domain. Roughly speaking, we require that the contact set Γ_c lie in a hyperplane and that the contact set and the free set meet at an angle strictly less than π. We remark that the condition that Γ_c lie in a hyperplane is too strong. We expect that our method can handle smooth contact sets. What is missing is a generalization of Dalhberg, Kenig and Verchota's estimates [DKV88] for the Lamé system to elliptic systems which have smooth coefficients. To consider more general contact sets would lengthen this paper without introducing any new ideas. We remark that we do not expect that it is appropriate to study this problem with Lipschitz contact sets, since we conjecture that there are Lipschitz domains for which (1.4) fails.

Now, we precisely describe the class of domains we are considering. To do this we will use coordinate cylinders

$$Z(P,r) = \{X : \ |X_n - P_n| < (1+m)r, \ |X' - P'| < r\}$$

where $m > 0$ is constant. We assume that the coordinate systems $(X', X_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$ used to define each coordinate cylinder are related by the composition of a translation and an orthogonal transformation. We require that the domain, Ω, its boundary $\partial\Omega$ and the decomposition are as follows:

1. $\Gamma_c \cup \Gamma_f = \partial\Omega$, $\Gamma_c \cap \Gamma_f = \emptyset$.
2. For each $P \in \Gamma_f$, there is an $r > 0$ and a Lipschitz function $\phi : \ \mathbf{R}^{n-1} \to \mathbf{R}$, $\|\nabla\phi\|_\infty \leq m$, so that in some coordinate system, (X', X_n), on $\mathbf{R}^n$,
$$\Omega \cap Z(P,r) = \{(X', X_n) : \ X_n > \phi(X')\} \cap Z(P,r)$$
and
$$\begin{aligned} \partial\Omega \cap Z(P,r) &= \Gamma_f \cap Z(P,r) \\ &= \{(X', X_n) : X_n = \phi(X')\} \cap Z(P,r). \end{aligned}$$
3. Each component of Γ_c lies in a hyperplane.
4. For each $P \in \partial\Gamma_c$ (boundary taken relative to $\partial\Omega$), we have an $r > 0$ and a Lipschitz function $\phi : \ \mathbf{R}^{n-1} \to \mathbf{R}$ with $\|\nabla\phi\|_\infty \leq m$ so that in some coordinate system $(X_1, X'', X_n) \in \mathbf{R} \times \mathbf{R}^{n-2} \times \mathbf{R}$ we have
$$\begin{aligned} &\Omega \cap Z(P,r) = \{X : X_n > \phi(X'), \ X_1 > 0\} \cap Z(P,r), \\ &\Gamma_c \cap Z(P,r) = \{X : \ X_1 = 0, \ X_n > \phi(0, X'')\} \cap Z(P,r), \\ &\Gamma_f \cap Z(P,r) = \{X : \ X_1 > 0, \ X_n = \phi(X')\} \cap Z(P,r). \end{aligned}$$
See Figure 1.
5. For each P in the interior (relative to $\partial\Omega$) of Γ_c, there is an $r > 0$ and a coordinate system (X', X_n) so that
$$\Omega \cap Z(P,r) = \{X_n > 0\} \cap Z(P,r),$$
$$\partial\Omega \cap Z(P,r) = \Gamma_c \cap Z(P,r) = \{X_n = 0\} \cap Z(P,r).$$

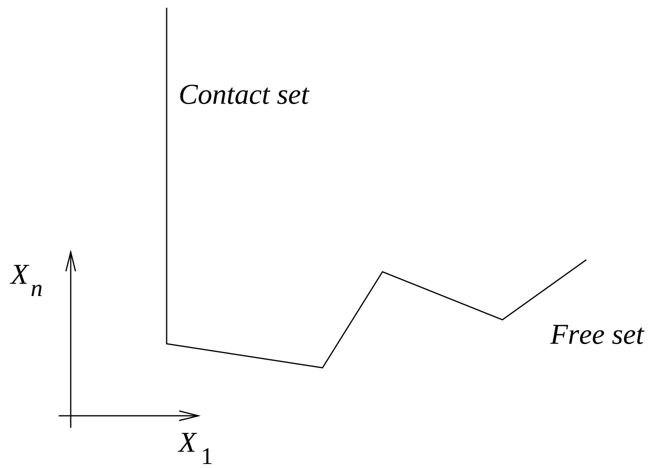

FIGURE 1. Boundary of domain near a point in $\partial\Gamma_c$

We note that domains satisfying 1) - 5) are Lipschitz domains in the sense of Dalhberg, Kenig and Verchota [DKV88]. This means that the condition 2) holds at every $P \in \partial\Omega$ (with, possibly, a different m and r). To see this we observe that near the boundary of Γ_c, we may tilt the coordinate system in condition 4) slightly to obtain that both Γ_c and Γ_f are graphs in the new coordinate system.

In the remainder of this paper, we will follow the standard practice of letting C denote constants which vary. Thus, at each occurrence, C will denote a constant which depends at most on the collection of coordinate cylinders which cover $\partial\Omega$, the Lipschitz constants of the functions which define $\partial\Omega$ in these cylinders and the constant α used to define the approach regions for the non-tangential maximal function (this is introduced below).

The restrictive condition that Γ_c lies in a hyperplane is exploited by the following lemma. Let B^+ denote the half-ball, $B(0,r) \cap \{X : X_n > 0\}$, and let $R(X', X_n) = (X', -X_n)$. The elementary proof of this lemma is omitted.

Lemma 2.1. *Let u satisfy $Lu = 0$ in B^+ with $\int_{B^+} |\nabla u|^2 \, dX < \infty$. Suppose that $u \cdot e_n = 0$ and $\sigma_{tan}(u) = 0$ on $X_n = 0$. If we extend u to $X_n < 0$ by reflection:*

$$\tilde{u} = \begin{cases} u, & X_n \geq 0, \\ R(u \circ R), & X_n < 0, \end{cases}$$

then $\tilde{u}$ satisfies $L\tilde{u} = 0$ in $B(0,r) = \{X : |X| < r\}$.

Our main estimates will be stated using the non-tangential maximal function. To define this, let Ω be a domain, $\alpha > 0$, $Q \in \partial\Omega$, and let $\Gamma_\alpha(Q) = \{X : |X - Q| < (1+\alpha)\text{dist}\,(X, \partial\Omega)\}$ denote a non-tangential approach region or "cone". Then for a function v on Ω we let $\sup_{X \in \Gamma_\alpha(Q)} |v(X)| = v^*(Q)$. We also will use v_s^* for the truncated maximal function defined by taking the supremum over the truncated cone $\Gamma_\alpha(Q) \cap B(Q, s)$.

Remark 1. The L^p-norms of non-tangential maximal functions defined using different openings are equivalent. (See, for example, the monograph of A. Torchinsky [Tor86, p. 367].)

Remark 2. Estimates for solutions in Sobolev spaces are available. If u solves $Lu = 0$, $N(u) \in L^2$, then u lies in the Sobolev space $H^{\frac{1}{2}}(u)$. As a consequence, if u is a solution and $N(\nabla u) \in L^2$, then $u \in H^{\frac{3}{2}}(\Omega)$. This estimate, for Laplace's equation, appears in Fabes's work [Fab88]. Given area-integral estimates, Fabes's argument extends easily to the Lamé system. The needed area integral estimates are due to Dahlberg, Kenig, Pipher and Verchota and will appear in [DKPV]. An extension of the argument of Dalhberg et. al. to the Stokes system was given by Shen in the appendix to Brown and Shen [BS95].

Remark 3. If v solves $Lv = 0$, L the Lamé system and $(\nabla v)^* \in L^2(\partial\Omega)$, then for a.e. $Q \in \partial\Omega$,

$$\lim_{\substack{X \to Q \\ X \in \Gamma_\alpha(Q)}} \nabla v(X) \text{ exists.}$$

This is a consequence of the work of Dalhberg, Kenig and Verchota [DKV88].

Remark 4. One may use the non-tangential maximal function and the dominated convergence theorem to justify various integrations by parts. This is done by considering a slightly smaller domain (where everything is smooth) and then taking a limit. See Verchota [Ver84] for similar arguments. This type of argument is needed in order to show that the solutions of Dahlberg, Kenig and Verchota [DKV88] are also weak solutions.

Our first step towards studying the boundary value problem (1.6) is to consider a boundary value problem with the boundary conditions $u \cdot e_n = 0$ and $\sigma_{\tan}(u) = h$ in a half space. This is probably well-known, but we include a proof for convenience.

Proposition 2.2. *Let h be a function $L^2(\mathbf{R}^{n-1})$ which takes values in $\mathbf{R}^{n-1}$. Then we may solve*

$$\begin{cases} Lv = 0, & \text{in } X_n > 0, \\ v_n = 0, & \text{on } X_n = 0, \\ \sigma_{\tan}(v) = h, & \text{on } X_n = 0. \end{cases} \tag{2.3}$$

The solution v has the estimate

$$\|(\nabla v)^*\|_{L^2(\mathbf{R}^{n-1})} \le C\|h\|_{L^2(\mathbf{R}^{n-1})}.$$

Proof. Because of the simple geometry, we may write out the solution explicitly. It is given by

$$v_j(X) = -2\int_{\mathbf{R}^{n-1}} \Gamma_{jk}(X' - Y', X_n) h_k(Y')\, dY'. \tag{2.4}$$

Here, Γ is the Kelvin matrix fundamental solution for the Lamé system which may be found in the work of Dahlberg, Kenig and Verchota [DKV88]. Using the formulas for the boundary values of derivatives of the potential v [DKV88, equation (0.6)], we can easily see that that $\sigma_{\tan}(v) = h$. One can show that $v_n = 0$ by bringing the limit inside the integral. The estimate for the non-tangential maximal function is also stated in [DKV88], though the proof in a half-space is much easier. ∎

Our next lemma gives a regularity result for the solution of the pure traction problem. This is a restatement of the results in Dalhberg, Kenig and Verchota [DKV88].

Lemma 2.5. *Suppose $Z(P,r)$ is a coordinate cylinder for $\partial\Omega$ as in condition 2 of our description of domains. Suppose $u \in H^1(\Omega \cap Z(P,r))$ and satisfies*

$$\begin{cases} Lu = 0, & \text{in } \Omega, \\ \sigma(u)N = f, & \text{in } \partial\Omega \cap Z(P,r). \end{cases}$$

If $f \in L^2(\partial\Omega \cap Z(P,r))$, then $(\nabla u)^*_{r/4} \in L^2(Z(P,r/4) \cap \partial\Omega)$ and

$$\begin{aligned} &\|(\nabla u)^*_{r/4}\|_{L^2(\partial\Omega\cap Z(P,r/4))} \\ &\quad \le C(\|f\|_{L^2(\partial\Omega\cap Z(P,r))} + r^{-\frac{1}{2}}\|\nabla u\|_{L^2(\Omega\cap Z(P,r))} + r^{-\frac{3}{2}}\|u\|_{L^2(\Omega\cap Z(P,r))}) \end{aligned}$$

Proof. By rescaling, we may assume that $r = 1$. We choose a smooth cutoff function η with $\eta = 1$ on $Z(P, 3/4)$ and $\operatorname{supp}\eta \subset Z(P,1)$. We let $F = L\eta u$ in Ω and extend F to $\mathbf{R}^n$ by setting $F = 0$ outside Ω. Since u is a solution, we have

$$\|F\|_{L^2(\mathbf{R}^n)} \le C\left(\|\nabla u\|_{L^2(\Omega\cap Z(P,1))} + \|u\|_{L^2(\Omega\cap Z(P,1))}\right).$$

Let $v = -\Gamma * F$, where Γ is the standard fundamental solution to the Lamé system (see [DKV88]). Since $F = 0$ in $Z(P, 3/4)$, we have that ∇v is locally bounded in $Z(P, 3/4)$. In particular, we have

$$\sup_{Z(P,1/4)} (\nabla v)^*_{1/4} \le C\|F\|_{L^2(\mathbf{R}^n)}. \tag{2.6}$$

If we set $w = \eta u - v$, then we have $Lw = 0$ in $Z(P,1) \cap \Omega$. Also, we have the following estimate for the traction:

$$\begin{aligned} &\|\sigma(w)N\|_{L^2(\partial(\Omega\cap Z(P,1))} \\ &\quad \le C(\|f\|_{L^2(\partial\Omega\cap Z(P,1))} + \|u\|_{L^2(\partial\Omega\cap Z(P,1))} + \|F\|_{L^2(\mathbf{R}^n)}). \end{aligned} \tag{2.7}$$

To obtain this estimate, we have used the trace theorem to bound ∇v on the boundary, which gives

$$\|\nabla v\|_{L^2(\partial(\Omega\cap Z(P,1)))} \le C(\|\nabla^2 v\|_{L^2(\Omega\cap Z(P,1))} + \|\nabla v\|_{L^2(\Omega\cap Z(P,1))}). \tag{2.8}$$

Then standard estimates for the operator $F \to \Gamma * F$ allow us to bound the right-hand side of (2.8) by $\|F\|_{L^2(\mathbf{R}^n)}$, which gives (2.7).

Next, observe that $Z(P,1) \cap \Omega$ is a Lipschitz domain. If $g \in L^2(\partial(\Omega \cap Z(P,1)))$ satisfies certain compatibility conditions, then Theorem 2.7 in [DKV88] says that there is a solution to

$$\begin{cases} Lw = 0, & \text{in } \Omega \cap Z(P,1), \\ \sigma(w)N = g & \text{in } \partial(\Omega \cap Z(P,1)), \end{cases} \tag{2.9}$$

which satisfies

$$\|(\nabla w)^*\|_{L^2} \le \|g\|_{L^2}. \tag{2.10}$$

But solutions to (2.9) are unique (modulo IRD's) in the class H^1; hence the function w we have constructed above satisfies (2.10). (Our data $g = \sigma(w)N$ must satisfy the compatibility conditions in [DKV88], because g is obtained from a solution.)

Combining the estimates (2.6) for v and (2.10) for w gives the estimate of the lemma. We remark that it does not matter that the estimate (2.10) is stated using the non-tangential maximal function for the domain $\Omega \cap Z(P,1)$. To see this, suppose that we have $Q \in Z(P,1/4)$. Then the non-tangential approach regions with vertex Q for Ω and $\Omega \cap Z(P,1)$ both have the same intersection with $B(Q,s)$ when s is sufficiently small. Thus, using interior estimates, we may dominate the truncated non-tangential maximal function for Ω by the non-tangential maximal function for the smaller domain. □

Lemma 2.11. *The solution of* (1.6) *satisfies*

$$\|(\nabla u)^*\|_{L^2(\partial\Omega)} \le C(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} + \|\tau\|_{L^2(\Gamma_f)} + \|h\|_{L^2(\Gamma_c)}). \tag{2.12}$$

Proof. We show that in a neighborhood of each $P \in \partial\Omega$, there exists an $r > 0$ so that

$$\|(\nabla u)^*_{r/4}\|_{L^2(Z(P,r/4)\cap\partial\Omega)} \le A, \tag{2.13}$$

where A is the right-hand side of (2.12). By the compactness of $\partial\Omega$, there are a finite number of points $P_1, \dots, P_N$ with

$$\bigcup_{i=1}^{N} Z(P_i, r_i/4) \supset \partial\Omega.$$

Also, interior estimates imply that

$$(\nabla u)^*(Q) \le C[(\nabla u)^*_{r/4}(Q) + r^{-n/2}\|\nabla u\|_{L^2(\Omega)}]$$

for $Q \in \partial\Omega$. The estimate (2.12) of our lemma follows from these observations and (2.13).

Thus we turn our attention to proving (2.13). There are three cases to be considered. 1) $P \in \Gamma_f$, 2) P in the interior of Γ_c and 3) $P \in \partial\Gamma_c$. In case 1), (2.13) is an immediate consequence of Lemma 2.5.

In case 2), we suppose the coordinates have been fixed so that $P = 0$ and Γ_c lies in $\{X_n = 0\}$ near P. Thus we suppose that $\Omega \cap B(0,r) = B(0,r) \cap \{X_n > 0\}$ and $\partial\Omega \cap B(0,r) = \{X_n = 0\} \cap B(0,r)$. We let $h_1 = h\chi_{\{X' :\, |X'|<r\}}$ and extend h_1 to $\mathbf{R}^{n-1}$ by setting $h_1(X') = 0$ for $|X'| > r$. We let u_1 be the solution of the boundary value problem studied in Proposition 2.2 with data h_1. According to the estimate of this proposition, we have

$$\|(\nabla u_1)^*\|_{L^2} \le C\|h_1\|_{L^2}. \tag{2.14}$$

We now let $u_2 = u - u_1$ and claim u_2 is smooth up to $B(0,r) \cap \{X_n = 0\}$. To see this, observe that since $\sigma_{\tan}(u_2) = 0$ and $u_2 \cdot e_n = 0$ when $X_n = 0$ and $|X'| < r$, we may extend u_2 by the reflection in Lemma 2.1 to obtain a solution to the Lamé system in $Z(0,r)$. Now interior estimates imply that

$$\sup_{Z(0,r/4)\cap\{X_n=0\}} (\nabla u_2)^* \le C_r \int_{Z(0,r)} |\nabla u_2|^2\, dX. \tag{2.15}$$

The estimate (2.13) follows from (2.14) and (2.15).

Now we consider case 3), $P \in \partial\Gamma_c$. Again, we assume $P = 0$ and suppose that, near 0, $\partial\Omega$ satisfies condition 4) in the description of our domains with coordinate cylinder $Z(0,r)$ and with Γ_f being given as the graph of ϕ near 0. We begin by using Proposition 2.2 (and a permutation of the coordinates) to reduce to $h = 0$ on Γ_c near 0. Thus we let

$$A_r = Z(0,r) \cap \Gamma_c$$

and let u_1 solve

$$\begin{cases} Lu_1 = 0, & \text{on } X_1 > 0, \\ u_1 \cdot e_1 = 0, & \text{on } X_1 = 0, \\ \sigma_{\tan}(u_1) = h, & \text{on } A_r, \\ \sigma_{\tan}(u_1) = 0, & \text{in } \{X_1 = 0\} \setminus A_r. \end{cases} \tag{2.16}$$

We set $u_2 = u - u_1$. We may reflect u_2 as in Lemma 2.1 to obtain $\tilde{u}_2$ in the domain

$$\tilde{D} = \{X : X_n > \tilde{\phi}(X')\} \cap Z(0,r).$$

Here, $\tilde{\phi}$ is the function obtained by extending ϕ from $X_1 \geq 0$ by requiring that ϕ be an even function in X_1. Now by Lemma 2.5, we have

$$\begin{aligned} &\|(\nabla \tilde{u}_2)^*_{r/4}\|_{L^2(Z(0,r/4)\cap\partial\tilde{D})} \\ &\leq A + C\left(\|\nabla u_1\|_{L^2(\Gamma_f \cap Z(0,r))}\right. \\ &\quad \left. + \|\nabla u_1\|_{L^2(Z(0,r)\cap\{X_1>0\})} + \|u_1\|_{L^2(Z(0,r)\cap\{X_1>0\})}\right). \end{aligned} \tag{2.17}$$

To estimate the terms on the right of (2.17), we begin with Proposition 2.2, which gives

$$\|(\nabla u_1)^*\|_{L^2(\{X_1=0\})} \leq A. \tag{2.18}$$

It is easy to estimate the L^2 norm of ∇u_1 on any bounded set in terms of the non-tangential maximal function. Thus we have

$$\|\nabla u_1\|_{L^2(Z(0,r)\cap\{X_1>0\})} \leq C\|(\nabla u_1)^*\|_{L^2(\{X_1=0\})}. \tag{2.19}$$

Using the fact that the data for u_1 is compactly supported and the formula (2.4) gives

$$\|u_1\|_{L^2(Z(0,r)\cap\{X_1>0\})} \leq A. \tag{2.20}$$

Finally, Lemma 2.25 below gives us

$$\|\nabla u_1\|_{L^2(\Gamma_f\cap Z(0,r))} \leq C\|(\nabla u_1)^*\|_{L^2(\{X_1=0\})}. \tag{2.21}$$

Combining the observations (2.17), (2.18), (2.19), (2.20) and (2.21) gives the estimate

$$\|(\nabla \tilde{u}_2)^*_{r/4}\|_{L^2(Z(0,r/4)\cap\partial\tilde{D})} \leq A. \tag{2.22}$$

To complete the proof, we need to deal with the technical complication that our non-tangential maximal function in (2.18) is taken in the domain $\{X_1 > 0\}$ and the non-tangential maximal function in (2.22) is taken in the domain $\tilde{D}$. After Lemma 2.26, we show that

$$\|(\nabla u_1)^*_{r/4}\|_{L^2(\partial\Omega\cap Z(0,r/4))} \leq C\|(\nabla u_1)^*_{\{X_1>0\}}\|_{L^2(\{X_1=0\})}. \tag{2.23}$$

Here, we temporarily introduce a subscript to indicate the domain used to form the non-tangential maximal function on the right. We also have

$$\|(\nabla u_2)^*_{r/4}\|_{L^2(Z(0,r/4)\cap\partial\Omega)} \leq C\|(\nabla \tilde{u}_2)^*_{\tilde{D}}\|_{L^2(\partial\tilde{D})} \tag{2.24}$$

by an argument similar to that used to prove (2.23). Thus, the claim (2.13) follows in case 3) from (2.18), (2.22), (2.23) and (2.24). $\square$

Lemma 2.25. *If $u^* \in L^2(\{X_1 = 0\})$ and $\Gamma \subset \{X_1 > 0\}$ is a Lipschitz surface, then*

$$\int_\Gamma |u|^2 \le C \int_{\{X_1=0\}} (u^*)^2.$$

Proof. Since surface measure on Γ is a Carleson measure, the estimate follows from the fundamental property of Carleson measures [Ste70]. The reader who is unfamiliar with Carleson measures will have no difficulty giving a direct proof of (2.21). See the argument used below to prove (2.23). $\square$

Now we compare non-tangential maximal functions taken with respect to different domains. For the next lemma, we consider two domains $\Omega \subset \tilde{\Omega}$. We let $\delta(X)$ denote the distance from X to $\partial\Omega$ and $\tilde{\delta}(X)$ the distance from X to $\partial\tilde{\Omega}$.

Lemma 2.26. *If $\Omega \subset \tilde{\Omega}$, $P \in \partial\Omega$, $\tilde{P} \in \partial\tilde{\Omega}$ and $c_0\tilde{\delta}(P) \ge |P - \tilde{P}|$, then for each $\alpha > 0$, there is a $\beta = \beta(\alpha, c_0)$ so that $\Gamma_\alpha(P) \subset \tilde{\Gamma}_\beta(\tilde{P})$.*

Proof. Let $X \in \Gamma_\alpha(P)$; then since $|P - \tilde{P}| \le c_0\tilde{\delta}(P)$, we have

$$|X - \tilde{P}| \le |X - P| + |P - \tilde{P}| \le (1 + \alpha)\delta(X) + c_0\tilde{\delta}(P). \tag{2.27}$$

Now note that

$$\tilde{\delta}(P) \le |X - P| + \tilde{\delta}(X) \le (1 + \alpha)\delta(X) + \tilde{\delta}(X) \tag{2.28}$$

since $\tilde{\delta}$ is Lipschitz (with constant 1) and $X \in \Gamma_\alpha(P)$. Since $\Omega \subset \tilde{\Omega}$, we have $\delta(X) \le \tilde{\delta}(X)$. Hence combining (2.27) and (2.28) gives

$$|X - \tilde{P}| \le [(1 + \alpha) + c_0(2 + \alpha)]\delta(X).$$

Thus $X \in \tilde{\Gamma}_\beta(\tilde{P})$ if $\beta = 2c_0 + 3\alpha$. $\square$

We now consider the estimate (2.23) for $\|(\nabla u_1)^*_{r/4}\|_{L^2(\Gamma_f \cap Z(0,r/4))}$. First, note that since this is a local estimate, we may assume that Ω is contained in $\{X_1 > 0\}$. We begin by defining $\Phi : \Gamma_f \cap Z(0, r/4) \to \{X_1 = 0\}$ by $(X_1, X'', \phi(X_1, X'')) \to (0, X'', \phi(0, X'') + X_1)$. Then $|\Phi(P) - P| \le \operatorname{dist}(P, \{X : X_1 = 0\})$. Hence, Lemma 2.26 implies that for appropriate α and β, $\Gamma_\alpha(P) \subset \tilde{\Gamma}_\beta(\Phi(P))$. Hence, $(\nabla u_1)^*_{r/4}(P) \le (\nabla u_1)^*_{\{X_1=0\}}(\Phi(P))$, and integrating this inequality gives (2.23).

We conclude with a theorem summarizing our main result:

Theorem 2.29. *Suppose that Ω is a domain satisfying the conditions 1-5 stated at the beginning of section 2. Suppose that $\tau \in L^2(\Gamma_f)$ and $g \in L^2(\Gamma_c)$ with $g \ge 0$, and that τ and g satisfy the compatibility condition* (1.1). *Then the variational problem in* (1.2) *or* (1.3) *has a solution u which satisfies*

$$\|(\nabla u)^*\|_{L^2(\partial\Omega)} \le C(\|\tau\|_{L^2(\Gamma_f)} + \|g\|_{L^2(\Gamma_f)}).$$

Proof. It is elementary to see that given our conditions on Ω, the set K satisfies the hypotheses of Theorem 1.7. Thus there exists a weak solution. According to Lemma 2.11, this solution has gradient whose non-tangential maximal function lies in L^2. If we subtract an appropriate element of IRD from u, one has the stability estimate $\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \le C(\|\tau\|_{L^2(\partial\Omega)} + \|g\|_{L^2(\partial\Omega)})$ (see [DL76]). Thus we may eliminate u and ∇u from the right-hand side of the estimate in Lemma 2.11. Also observe that we have $\|h\|_{L^2(\Gamma_c)} \le \|g\|_{L^2(\Gamma_c)}$ (see the discussion after (1.5)). This gives the estimate of the theorem. $\square$

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Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506-0027

E-mail address: `rbrown@ms.uky.edu`

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506-0027

E-mail address: `shenz@ms.uky.edu`

Department of Mathematical Sciences, Oakland University, Rochester, Michigan 48309-4401

E-mail address: `pshi@oakland.edu`