

PROJECTIVITY, TRANSITIVITY AND AF-TELESCOPES

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ABSTRACT. Continuing our study of projective C^* -algebras, we establish a projective transitivity theorem generalizing the classical Glimm-Kadison result. This leads to a short proof of Glimm's theorem that every C^* -algebra not of type I contains a C^* -subalgebra which has the Fermion algebra as a quotient. Moreover, we are able to identify this subalgebra as a generalized mapping telescope over the Fermion algebra.

We next prove what we call the multiplier realization theorem. This is a technical result, relating projective subalgebras of a multiplier algebra $M(A)$ to subalgebras of $M(E)$, whenever A is a C^* -subalgebra of the corona algebra $C(E) = M(E)/E$. We developed this to obtain a closure theorem for projective C^* -algebras, but it has other consequences, one of which is that if A is an extension of an MF (matricial field) algebra (in the sense of Blackadar and Kirchberg) by a projective C^* -algebra, then A is MF.

The last part of the paper contains a proof of the projectivity of the mapping telescope over any AF (inductive limit of finite-dimensional) C^* -algebra. Translated to generators, this says that in some cases it is possible to lift an infinite sequence of elements, satisfying infinitely many relations, from a quotient of any C^* -algebra.

1. INTRODUCTION

Recall from [17] that a C^* -algebra P is *projective*, if for every pair of C^* -algebras B, C such that $\pi : B \rightarrow C$ is a surjective morphism (throughout the paper morphism means *-homomorphism), and for each morphism $\varphi : P \rightarrow C$, there is a morphism $\psi : P \rightarrow B$ such that $\pi \circ \psi = \varphi$. In diagrammatic notation:

$$\begin{array}{ccc} & & B \\ & \nearrow \psi & \downarrow \pi \\ P & \xrightarrow{\varphi} & C \end{array}$$

This definition and the basic properties of projective C^* -algebras are due to Effros and Kaminker [9]. There is also a definition of a projective morphism due to Blackadar [3]. It was proved in [18, 2.2] (and we shall use this fact repeatedly) that it suffices to show that morphisms lift from corona algebras to multiplier algebras.

Received by the editors November 7, 1994.

1991 *Mathematics Subject Classification*. Primary 46L05.

Key words and phrases. Projectivity, transitivity, multipliers, telescopes, Bratteli diagram, Glimm's theorem, MF algebra.

This research was made possible through a NATO Collaboration Grant (# 920177). Both authors also acknowledge the support of their respective science foundations: NFS (# DMS-9215024) and SNF; and the second author recalls with gratitude the hospitality offered (twice!) by the Department of Mathematics at the University of New Mexico.

Thus, in the separable case, P is projective if for every σ -unital C^* -algebra A with multiplier algebra $M(A)$ and corona algebra $C(A) = M(A)/A$, we can solve the lifting problem

$$\begin{array}{ccc}
 & M(A) & \\
 \psi \nearrow & & \downarrow \pi \\
 P & \xrightarrow{\varphi} & C(A)
 \end{array}$$

Evidently projective C^* -algebras must be rather special: If \mathbf{CP} denotes the mapping cone over P , i.e.

$$\mathbf{CP} = C_0([0, 1]) \otimes P = C_0([0, 1], P),$$

then the map $f \rightarrow f(1)$ is a surjection of \mathbf{CP} onto P . Therefore, if P is projective, it has an embedding as a C^* -subalgebra of \mathbf{CP} . Now \mathbf{CP} is contractible — in any conceivable sense — and thus so is P . In particular, $K_0(P) = K_1(P) = 0$. Moreover, since \mathbf{CP} contains no non-zero projections, P is never unital.

Another peculiar property of a projective C^* -algebra P is that it must have a separating family of finite-dimensional representations (see [14]), i.e. P must be *residually finite-dimensional*. (See [10] for equivalent formulations of this condition.) To show this, at least in the separable case, write the separable Hilbert space \mathfrak{H} as the inductive limit of n -dimensional subspaces \mathfrak{H}_n , $n \in \mathbb{N}$, and in the direct product C^* -algebra of matrix algebras $\prod \mathbb{B}(\mathfrak{H}_n)$ take the C^* -subalgebra A of all strong* convergent sequences (relative to the embeddings $\mathfrak{H}_n \rightarrow \mathfrak{H}_{n+1}$) and the closed ideal I of A consisting of sequences converging to zero. Then $A/I = \mathbb{B}(\mathfrak{H})$. Evidently P has a faithful representation $\varphi : P \rightarrow \mathbb{B}(\mathfrak{H})$, thus, being projective, also an embedding as a C^* -subalgebra of A , which is clearly residually finite-dimensional.

It is the purpose of this paper to show that — despite the above-mentioned restrictions — there is a rich and fascinating supply of projective C^* -algebras. Indeed, every mapping telescope of an AF-algebra is projective by Theorem 7.2. The strategic importance of projective algebras lies in the fact that they provide an algebraic setting of lifting problems, which otherwise have a tendency to degenerate into single operator theory.

A major detour through the structure of corona algebras is unavoidable on the way to our goal of proving various C^* -algebras to be projective. This is not a bad thing, as the focus moves from representation theory, with $\mathfrak{K}(\mathfrak{H})$, $\mathfrak{B}(\mathfrak{H})$, $\mathfrak{Q}(\mathfrak{H})$ playing the starring roles, to almost multiplicative maps and asymptotic morphisms, where corona algebras such as $\prod M_n / \bigoplus M_n$ and $C_b([1, \infty[, B) / C_0([1, \infty[, B)$ (assuming B unital) are prominent. We will discuss connections with asymptotic morphisms, and so with the Connes-Higson E -theory, in a future paper, but do include here a result that gives some evidence for a conjecture of Blackadar and Kirchberg [3, §7] regarding MF algebras.

2. THREE ITEMS TO RECALL

Universal C^* -algebras. Given a set G of generators, and a set R of relations between elements in G , there is a universal algebra $\langle G \mid R \rangle$ generated by G satisfying R . If the relations R contain or imply norm restrictions on the generators, there is a *universal C^* -algebra* $C^*\langle G \mid R \rangle$, which surjects onto any other C^* -algebra A that

is generated (as a C^* -algebra) by elements $\{a_g \in A \mid g \in G\}$ satisfying the relations R . Often enough, universal C^* -algebras are hopelessly complicated. Consider e.g.

$$C^*\langle x \mid \|x\| \leq 1 \rangle,$$

which, although it is projective, will surject onto any singly generated C^* -algebra. Sometimes they are quite harmless though. Thus

$$C^*\langle x \mid \|x\| \leq 1, x^2 = 0 \rangle \cong \mathbf{CM}_2,$$

cf. [16, 1.4]. Note that we have defined universality in the category of non-unital (i.e. not necessarily unital) C^* -algebras. Note also that to define a morphism π from a universal C^* -algebra $C^*\langle G \mid R \rangle$ into a C^* -algebra A , it suffices to define $\pi(g)$ in A for each g in G , and then check that the $\pi(g)$'s satisfy the relations R .

Split Extensions. As usual we say that a C^* -algebra A is an *extension* of P by Q , if there is a short exact sequence

$$0 \rightarrow P \xrightarrow{\iota} A \xrightarrow{\pi} Q \rightarrow 0.$$

As shown by Busby ([5], see also [25, Ch. 3]), extensions are classified by the set of morphisms from Q into the corona algebra $M(P)/P$. In this paper we shall mostly deal with extensions where the quotient Q is projective, in which case, of course, the extension *splits*, i.e. there is an injective morphism $\lambda : Q \rightarrow A$, such that $\pi \circ \lambda = \text{id}$. For easy reference we note the following well-known result:

2.1. Proposition. *There is a bijective correspondence between split extensions (with specified splitting) A of P by Q , and morphisms $\theta : Q \rightarrow M(P)$, given in one direction by*

$$A = \{(q, m) \in Q \oplus M(P) \mid \theta(q) - m \in P\},$$

where $\iota(p) = (0, p)$, $\pi(q, m) = q$ and $\lambda(q) = (q, \theta(q))$; and in the other by setting

$$\theta(q)p = \lambda(q)\iota(p) \quad (q \in Q, p \in P).$$

Proof. Straightforward computations. \square

2.2. Proposition. *If A is a split extension of P by Q given by the morphism $\theta : Q \rightarrow M(P)$, and B is another C^* -algebra, there is a bijective correspondence between morphisms $\pi : A \rightarrow B$ and pairs (φ, ψ) of morphisms $\varphi : P \rightarrow B$ and $\psi : Q \rightarrow B$ satisfying*

$$\varphi(p\theta(q)) = \varphi(p)\psi(q) \quad (p \in P, q \in Q).$$

Proof. Since A is the universal C^* -algebra with generators $P \cup Q$ and relations $p \cdot q = p\theta(q)$, $p \in P$, $q \in Q$, this follows from the remark above. \square

Telescope Algebras. Let $A = \overline{\bigcup A_n}$ be an inductive limit of a sequence of C^* -algebras (A_n) , where the embeddings $A_1 \hookrightarrow A_2 \hookrightarrow \dots$ simply are regarded as inclusion maps. If all the algebras A_n are unital and the embeddings are unit-preserving, we talk about a *unital* inductive limit. Following Brown (see [24, 5.2]), we define the *mapping telescope* on (A_n) as the C^* -algebra

$$\mathbf{T}(A) = \{f \in C_0([0, \infty], A) \mid t \leq n \Rightarrow f(t) \in A_n\}.$$

If we let $\mathbf{T}(A_1, A_2, \dots, A_n)$ equal

$$\left\{ f \in C_0([0, \infty], A_n) \mid \begin{array}{l} t \leq k \Rightarrow f(t) \in A_k \text{ and} \\ t \geq n \Rightarrow f(t) = f(n) \end{array} \right\},$$

then we have embeddings

$$\mathbf{T}(A_1) \subset \mathbf{T}(A_1, A_2) \subset \cdots \subset \mathbf{T}(A),$$

and it is easily verified that the infinite telescope $\mathbf{T}(A)$ is the inductive limit of the finite telescopes $\mathbf{T}(A_1, \dots, A_n)$. The relevance of telescopes for $*$ -algebras should be obvious — even to the layman.

2.3. Remark. Clearly the telescope C^* -algebra $\mathbf{T}(A)$ depends not only on A , but also on the sequence (A_n) , so that the notation is highly ambiguous. In the interest of brevity we shall nevertheless retain the compact symbol $\mathbf{T}(A)$ instead of the more correct $\mathbf{T}(A_1, A_2, \dots)$. For the finite telescopes, however, the longer designation $\mathbf{T}(A_1, \dots, A_n)$ will be necessary.

If for each n we identify $]0, 1]$ with $]n - 1, n]$, there is a natural embedding of the cone \mathbf{CA}_n as a closed ideal of the telescope algebra $\mathbf{T}(A_1, \dots, A_n)$, where

$$\mathbf{CA}_n = \{f \in \mathbf{T}(A_1, \dots, A_n) \mid t \leq n - 1 \Rightarrow f(t) = 0\}.$$

Assuming that we have a unital inductive limit, this leads to the following.

2.4. Proposition. *Each finite telescope $\mathbf{T}(A_1, \dots, A_n)$ is the split extension of \mathbf{CA}_n by $\mathbf{T}(A_1, \dots, A_{n-1})$ determined by the morphism θ of $\mathbf{T}(A_1, \dots, A_{n-1})$ into $M(\mathbf{CA}_n)$ ($= C_b(]0, 1], A_n$) given by $\theta f(t) = f(n - 1)$ for $t \in]0, 1]$.*

Proof. Recall from Proposition 2.1 that the split extension B determined by θ is

$$B = \{(f, m) \in \mathbf{T}(A_1, \dots, A_{n-1}) \oplus C_b(]0, 1], A_n) \mid m - \theta(f) \in \mathbf{CA}_n\}.$$

Since θf is a constant map with value in A_{n-1} , this means that m must be continuous on $[0, 1]$ with $m(0) = f(n - 1)$. Define $\gamma : B \rightarrow \mathbf{T}(A_1, \dots, A_n)$ by

$$\gamma(f, m)(t) = \begin{cases} f(t), & 0 < t \leq n - 1, \\ m(t + 1 - n), & n - 1 \leq t \leq n, \\ m(t), & n \leq t. \end{cases}$$

Elementary, albeit somewhat lengthy computations show that γ is a $*$ -isomorphism of B onto $\mathbf{T}(A_1, \dots, A_n)$. Its inverse is given by restriction: $\gamma^{-1}(f) = (rf, m)$, where

$$rf(t) = \begin{cases} f(t), & 0 < t \leq n - 1, \\ f(n - 1), & n - 1 \leq t, \end{cases}$$

and

$$m(t) = f(n - 1 + t) - f(n - 1).$$

□

Combining Propositions 2.2 and 2.4 we have

2.5. Corollary. *Given a unital inductive limit $A = \overline{\bigcup A_n}$ and a sequence of morphisms $\varphi_n : \mathbf{CA}_n \rightarrow B$ into some C^* -algebra B , satisfying*

$$\varphi_m(f)\varphi_n(g) = \varphi_n(f(1)g) \quad (m < n),$$

there is a morphism $\varphi : \mathbf{T}(A) \rightarrow B$ such that $\varphi|_{\mathbf{CA}_n} = \varphi_n$ for all n , where each cone \mathbf{CA}_n is regarded as an ideal in $\mathbf{T}(A_1, \dots, A_n)$.

3. PRELIMINARY LIFTING RESULTS

We have shown previously, [16, 4.2], that the mapping cone over a finite-dimensional C^* -algebra is projective. We shall need a slightly stronger result in which some elements are constrained to be less than some given orthogonal elements “upstairs”. Contained in this is a shorter proof of the projectivity of \mathbf{CM}_n .

If not otherwise specified, A will denote a C^* -algebra, I a closed ideal and $\pi : A \rightarrow A/I$ the quotient map.

3.1. Lemma. *In any C^* -algebra, if $a^*a \leq b^*b$ and $cc^* \leq dd^*$, then for each x ,*

$$\|axc\| \leq \|bxd\|.$$

Proof. We have

$$\begin{aligned} \|axc\|^2 &= \|c^*x^*a^*axc\| \leq \|c^*x^*b^*bxc\| \\ &= \|bxc^*x^*b^*\| \leq \|bxdd^*x^*b^*\| = \|bxd\|^2. \end{aligned}$$

□

3.2. Lemma. *Suppose that a , b and c are elements in a C^* -algebra, such that $a^*a \leq b$ and $cc^* \leq b$. Then the limit*

$$x = \lim a \left(\frac{1}{n}1 + b \right)^{-1/2} c$$

exists (in norm). Moreover:

- (i) $x^*x \leq c^*c$;
- (ii) $xx^* \leq aa^*$;
- (iii) if $cc^* = b$ then $xx^* = aa^*$;
- (iv) if $c = b^{1/2}$ then $x = a$.

Proof. See [21, 1.1.4 and 1.1.5], or use the previous lemma plus functional calculus. □

The following two-sided version of Combes’ order lifting theorem (cf. [21, 1.5.10]) was proved by Davidson [8, 2.4]. Here is a short proof.

3.3. Theorem. *Suppose that a and b are positive elements in A and $y \in A/I$ such that $y^*y \leq \pi(a)$ and $yy^* \leq \pi(b)$. Then there is a lift x in A of y with $x^*x \leq a$ and $xx^* \leq b$.*

Proof. Let z be any lift of y and put $c = (z^*z - a)_+ + a$, so that $z^*z \leq c$, $a \leq c$, and $\pi(c) = \pi(a)$. By Lemma 3.2 (i) and (iv) we have $x_0 = \lim z(\frac{1}{n}1 + c)^{-1/2}a^{1/2}$ in A with $x_0^*x_0 \leq a$ and $\pi(x_0) = y$. Now put $d = (x_0x_0^* - b)_+ + b$, so that $x_0x_0^* \leq d$, $b \leq d$, and $\pi(d) = \pi(b)$. By Lemma 3.2 (i), (ii) and (iv) we have $x = \lim b^{1/2}(\frac{1}{n}1 + d)^{-1/2}x_0$ in A with $xx^* \leq b$, $x^*x \leq x_0^*x_0 \leq a$, and $\pi(x) = y$, as desired. □

3.4. Proposition. *Suppose that k_1, \dots, k_n are mutually orthogonal, positive elements in A/I of norm at most one. Then there are elements h_1, h_2, \dots, h_n in A with the same properties, such that $\pi(h_j) = k_j$ for all j .*

Proof. Put $b = \sum_{j=2}^n 2^{-j}k_j$, and let x be a self-adjoint element in A with $\pi(x) = k_1 - b$. Define $f(t) = (t \vee 0) \wedge 1$ and $g(t) = -(t \wedge 0)$, and put $h_1 = f(x)$. Then $\pi(h_1) = f(k_1 - b) = k_1$, and if A_1 denotes the two-sided annihilator of h_1 , then

k_2, \dots, k_n belong to $\pi(A_1)$, since $g(x) \in A_1$ and $\pi(g(x)) = b$. The argument now proceeds by induction. As can be seen from [19, 6.5], the argument can be used to lift a whole sequence of orthogonal elements. \square

3.5. Theorem. *Suppose that $\varphi : \mathbf{CM}_n \rightarrow A/I$ is a morphism of the mapping cone over some \mathbb{M}_n , and suppose we have chosen mutually orthogonal elements h_1, \dots, h_n in A with $0 \leq h_j \leq 1$ and $\pi(h_j) = \varphi(\text{id} \otimes e_{jj})$ for $1 \leq j \leq n$. Then there is a morphism $\psi : \mathbf{CM}_n \rightarrow A$ with $\pi \circ \psi = \varphi$, such that $\psi(\text{id} \otimes e_{jj}) \leq h_j$ for all j .*

Proof. Recall from [16, 2.7] that \mathbf{CM}_n is the universal C^* -algebra generated by the contractions $a_j (= \text{id} \otimes e_{j1})$, $2 \leq j \leq n$, subject to the relations

$$(*) \quad \begin{aligned} \|a_j\| &\leq 1, && \text{for all } j, \\ a_j^* a_k &= 0, && \text{if } j \neq k, \\ a_j^* a_j &= a_k^* a_k, && \text{for all } j, k, \\ a_j^2 &= 0 && \text{for all } j. \end{aligned}$$

Applying Theorem 3.3 we find a lift y_n in A of the element $\varphi(a_n)$ such that

$$y_n^* y_n \leq h_1^2, \quad y_n y_n^* \leq h_n^2.$$

Applying it again we find a lift y_{n-1} of $\varphi(a_{n-1})$ satisfying

$$y_{n-1}^* y_{n-1} \leq y_n^* y_n, \quad y_{n-1} y_{n-1}^* \leq h_{n-1}^2.$$

Continuing by induction, we end up with elements y_2, y_3, \dots, y_n in A such that

$$\begin{aligned} \pi(y_j) &= \varphi(a_j), \\ y_j y_j^* &\leq h_j^2 && (2 \leq j \leq n), \\ y_2^* y_2 &\leq \dots \leq y_n^* y_n \leq h_1^2. \end{aligned}$$

Except for the penultimate condition, these elements satisfy the relations (*). We correct them by setting

$$x_j = \lim y_j \left(\frac{1}{n} 1 + y_j^* y_j \right)^{-1/2} (y_2^* y_2)^{1/2},$$

for $2 \leq j \leq n$, which exist in A by Lemma 3.2 and satisfy the relations (*). By universality (cf. Section 1) there is therefore a morphism $\psi : \mathbf{CM}_n \rightarrow A$ given by $\psi(a_j) = x_j$, $2 \leq j \leq n$. Since $\pi(x_j) = \varphi(a_j)$, and the a_j 's are generators, it follows that $\pi \circ \psi = \varphi$, and clearly

$$\psi(\text{id} \otimes e_{jj}) = \psi((a_j a_j^*)^{1/2}) = (x_j x_j^*)^{1/2} \leq (y_j y_j^*)^{1/2} \leq h_j,$$

since the square root is operator monotone. \square

3.6. Corollary. *Let F be a finite-dimensional C^* -algebra and let p_1, \dots, p_n be a set of mutually orthogonal, one-dimensional projections in F , summing to the identity. Assume that $\varphi : \mathbf{CF} \rightarrow A/I$ is a morphism, and h_1, \dots, h_n is a set of mutually orthogonal elements in A with $0 \leq h_j \leq 1$ and $\pi(h_j) = \varphi(\text{id} \otimes p_j)$ for all j . Then there is a morphism $\psi : \mathbf{CF} \rightarrow A$ such that $\pi \circ \psi = \varphi$ and $\psi(\text{id} \otimes p_j) \leq h_j$ for all j .*

3.7. Theorem. *Let F be a finite-dimensional C^* -algebra and let q_1, \dots, q_m be an orthogonal family of projections (of any dimensions) summing to the identity in F . Given any morphism $\varphi : \mathbf{CF} \rightarrow A/I$ and mutually orthogonal hereditary C^* -subalgebras A_1, \dots, A_m of A such that $\varphi(\text{id} \otimes q_j) \in \pi(A_j)$ for $1 \leq j \leq m$, there is a morphism $\psi : \mathbf{CF} \rightarrow A$ such that $\pi \circ \psi = \varphi$ and $\psi(\text{id} \otimes q_j) \in A_j$ for all j .*

Proof. Let p_1, \dots, p_n be an orthogonal family of one-dimensional projections in F , summing to the identity and subordinate to the q_j 's. By Lemma 3.4 there are mutually orthogonal elements h_1, \dots, h_n in A with $0 \leq h_i \leq 1$ and $\pi(h_i) = \varphi(\text{id} \otimes p_i)$ for all i , and such that $h_i \in A_j$ whenever $p_i \leq q_j$. Now apply Corollary 3.6 and the fact that the A_j 's are hereditary. \square

3.8. Corollary. *The mapping cone \mathbf{CF} over any finite-dimensional C^* -algebra F is projective.*

A key consequence (see [17, 3.3]) of the fact that \mathbf{CM}_n is projective is that the class of (σ -unital) projective C^* -algebras is closed under tensoring with matrix algebras. Similarly, the fact that the cone over $\mathbb{C} \oplus \mathbb{C}$ is projective has as a consequence that the (σ -unital) projectives are stable under direct sums. The more elementary fact that if A and B are C^* -algebras and $\tilde{A} = \tilde{B}$, then both or none of A and B are projective, is verified directly.

These closure properties were used in [18] to show that $C_0(X) \otimes \mathbb{M}_n$ is projective whenever X is a finite tree. We can now handle more general subhomogeneous C^* -algebras over finite trees. While this will follow from more general results later, we give an example to show how the more precise lifting results involving \mathbf{CM}_n are useful.

3.9. Example. Let B denote the universal C^* -algebra generated by h_1, \dots, h_n and a_2, \dots, a_n , subject to the relations

$$\begin{aligned} 0 \leq h_j \leq 1, & \quad \text{for all } j, \\ \|a_j\| \leq 1, & \quad \text{for all } j, \\ h_j h_k = 0, & \quad \text{if } j \neq k, \\ a_j^* a_j = a_k^* a_k, & \quad \text{for all } j, k, \\ a_j h_1 = h_j a_j = a_j, & \quad \text{for all } j. \end{aligned}$$

It is easily seen that there is a surjection

$$\varphi : B \rightarrow \{f \in C_0([0, 2], \mathbb{M}_n) \mid f(t) \text{ is diagonal if } t \leq 1\}$$

such that

$$\varphi(h_j) = r \otimes e_{jj}, \quad \varphi(a_j) = s \otimes e_{j1},$$

where $r(t) = t \wedge 1$ and $s(t) = (t - 1) \vee 0$. In fact φ is an isomorphism.

To prove injectivity (see [18, 4.3]) assume – for ease of notation – that h_1, \dots, h_n and a_2, \dots, a_n are operators on a Hilbert space \mathfrak{H} (still satisfying the relations, of course) and generate an irreducible C^* -algebra. The element $\sum h_j$ is central, and thus $\sum h_j = \alpha 1$ for some scalar α with $0 \leq \alpha \leq 1$. But since $\sum h_j$ acts as a unit against all the a_j 's we must have $\alpha \neq 0$. There are three cases to consider.

If $\alpha < 1$, the relation $a_j h_j = a_j$ implies that $a_j = 0$ for all j . The only time the orthogonal elements h_j can act irreducibly is when $\dim \mathfrak{H} = 1$, so $h_j = \alpha$ for some j and $h_k = 0$ for $k \neq j$. The representation is therefore the pull-back of φ of a subrepresentation of evaluation at α .

If $\alpha = 1$ the element below is central, so for some scalar β in $[0, 1]$ we have

$$a_2^* a_2 + \sum_{j=2}^n a_j a_j^* = \beta 1.$$

When $\beta = 0$ we have $a_j = 0$ for $2 \leq j \leq n$, and we proceed as in the first case to show that the representation is the pull-back of φ of a subrepresentation of evaluation at 1.

In the third case $\alpha = 1$ and $0 < \beta \leq 1$. Then we have two sets of mutually orthogonal projections, h_1, \dots, h_n , and $\beta^{-1}a_2^*a_2, \beta^{-1}a_2a_2^*, \dots, \beta^{-1}a_n a_n^*, \beta^{-1}a_n a_n^*$, both summing to 1. Since $a_2^*a_2 \leq h_1$ and $a_j a_j^* \leq h_j$ for $2 \leq j \leq n$, this forces $a_2^*a_2 = \beta h_1$ and $a_j a_j^* = \beta h_j$ for $2 \leq j \leq n$. Therefore $C^*(a_2, \dots, a_n)' = \mathbb{C}1$, and we have $a_j a_k = 0$ and, for $j \neq k$, $a_j^* a_k = 0$, whereas $a_j^* a_j = a_k^* a_k$. But this is an irreducible representation of the cone \mathbf{CM}_n , and thus the pull-back of φ of evaluation at $t = 1 + \beta^{1/2}$.

We have shown that all irreducible representations of the universal C^* -algebra B are pull-backs of φ , which proves that φ is an isomorphism.

3.10. Proposition. *The C^* -algebra $B = \mathbf{T}(\mathbb{C}^n, \mathbb{M}_n)$, that is*

$$B = \{f \in C_0([0, 2], \mathbb{M}_n) \mid t \leq 1 \Rightarrow f(t) \text{ is diagonal}\}$$

is projective.

Proof. Let Q denote the C^* -subalgebra of B consisting of diagonal functions on $]0, 2]$ which are constant on $[1, 2]$, and let P denote the closed ideal of functions vanishing on $]0, 1]$. Then P is isomorphic to \mathbf{CM}_n (identifying $]0, 1]$ with $]1, 2]$) and B is the split extension of P by Q .

Given a morphism $\varphi : B \rightarrow A/I$, we can find a lift $\psi_2 : Q \rightarrow A$ of $\varphi|_Q$, because Q , being isomorphic to \mathbf{CC}^n , is projective. Let $r(t) = t \wedge 1$ and $s(t) = (1 - t) \vee 0$ as before, and put $h_j = \psi_2(r \otimes e_{jj})$, $1 \leq j \leq n$, so that h_1, \dots, h_n is a set of mutually orthogonal, positive contractions in A . Define

$$A_j = \{x \in A \mid h_j x = x h_j = x\} \quad (1 \leq j \leq n)$$

to obtain mutually orthogonal hereditary C^* -subalgebras of A , and note that $\varphi(s \otimes e_{jj}) \in \pi(A_j)$ for all j . By Theorem 3.7 there is a lift $\psi_1 : P \rightarrow A$ of $\varphi|_P$ such that $\psi_1(s \otimes e_{jj}) \in A_j$ for all j . With δ the Kronecker symbol this implies that

$$\psi_1(s \otimes e_{jj})\psi_2(r \otimes e_{ii}) = \psi_1(s \otimes e_{jj})\delta_{ij},$$

whence

$$\psi_1(s \otimes e_{jj})\psi_2(f \otimes e_{ii}) = \psi_1(s \otimes e_{jj})f(1)\delta_{ij}$$

for every f in $C_0([0, 2])$, constant on $[1, 2]$. In the general case, where $f = \sum f_j \otimes e_{jj}$ is in Q and $g = \sum g_{ij} \otimes e_{ij}$ is in P , we get

$$\begin{aligned} \psi_1(g)\psi_2(f) &= \sum \psi_1(g_{ij} \otimes e_{ij})\psi_2(f_j \otimes e_{jj}) \\ &= \sum \psi_1(g_{ij} \otimes e_{ij})f_j(1) \\ &= \psi_1(gf(1)). \end{aligned}$$

By Corollary 2.5 the pair (ψ_1, ψ_2) defines a morphism $\psi : B \rightarrow A$, which is clearly a lift of φ , since $B = P + Q$. □

3.11. Example. Let B denote the universal C^* -algebra generated by contractions x_1, \dots, x_n satisfying the relations

$$(*) \quad \begin{aligned} x_i^* x_i &= x_j^* x_j, & \text{for all } i, j, \\ x_i^* x_j &= 0, & \text{if } i \neq j. \end{aligned}$$

If we add the relations $x_1 \geq 0$ (whence $x_1 = (x_j^*x_j)^{1/2}$ for $2 \leq j \leq n$) and $x_i x_j = 0$ if $i \neq j$, we have the relations for \mathbf{CM}_n ; cf. (*) in the proof of Theorem 3.5. This means that there is a surjective morphism of B onto \mathbf{CM}_n . If, on the other hand, we add the relations $x_i^* x_i = 1$, $1 \leq i \leq n$ (or just $x_1^* x_1 = 1$), we have the relations for Cuntz's C^* -algebra O_n (cf. [6]). There is therefore also a surjective morphism of B onto O_n . Now the surprise:

3.12. Proposition. *The C^* -algebra B defined above is projective.*

Proof. Given a morphism $\varphi : B \rightarrow A/I$, we can find, using Proposition 3.4, orthogonal, positive contractions h_1, \dots, h_n in A , such that $\pi(h_j) = \varphi((x_j x_j^*)^{1/2})$. Applying Theorem 3.3 recursively, we find that there are elements y_1, \dots, y_n in A with $\pi(y_i) = x_j$, such that $y_j y_j^* \leq h_j^2$ and $y_{j+1}^* y_{j+1} \leq y_j^* y_j$ for all j . The renormalization

$$z_j = \lim y_j \left(\frac{1}{n} 1 + y_j^* y_j \right)^{-1/2} (y_n^* y_n)^{1/2},$$

which exists in A by Lemma 3.2, produces elements z_1, \dots, z_n in A that satisfy conditions (*) in Example 3.11, and thus we have a morphism $\psi : B \rightarrow A$. That $\pi \circ \psi = \varphi$ follows from the fact that $\pi(z_j) = \varphi(x_j)$ for all j . This, in turn, follows from (iv) in Lemma 3.2. \square

4. TRANSITIVITY THEOREMS

Recall the following version of the Glimm–Kadison transitivity theorem (cf. [13] or [21, 2.7.5]).

4.1. Theorem. *Let $\pi : A \rightarrow \mathbb{B}(\mathfrak{H})$ be an irreducible representation of a C^* -algebra A . If q is a finite-dimensional projection on \mathfrak{H} and x is a self-adjoint contraction in $\mathbb{B}(q\mathfrak{H}) = q\mathbb{B}(\mathfrak{H})q$, then there is a self-adjoint contraction a in A such that $\pi(a)q = x$ (and so also $x = q\pi(a)$).*

In fact, if x is positive or unitary (but being an exponential, necessarily), or a product of these (this covers all contractions in $\mathbb{B}(q\mathfrak{H})$), then a can be chosen of the same type. Using the Glimm–Kadison result, we have the following *Projective Transitivity Theorem* which subsumes the earlier versions.

4.2. Theorem. *Let P be a projective C^* -algebra and $\pi : A \rightarrow \mathbb{B}(\mathfrak{H})$ an irreducible representation. If q is a finite-dimensional projection on \mathfrak{H} and $\theta : P \rightarrow \mathbb{B}(q\mathfrak{H})$ is a representation of P , there is a morphism $\varphi : P \rightarrow A$ such that*

$$\pi(\varphi(x))q = \theta(x), \quad (x \in P).$$

(and so also $\theta(x) = q\pi(\varphi(x))$).

Proof. Let $\{x_g \in P \mid g \in G\}$ be a generating set of self-adjoint contractions for P . By the Glimm–Kadison transitivity theorem there is for each g in G a self-adjoint contraction a_g in A with $\pi(a_g)q = \theta(x_g)$. Consider now the universal C^* -algebra

$$B = C^*\langle G \mid \|g\| \leq \|a_g\|, g \in G \rangle,$$

cf. Section 2. By universality there exists three morphisms

$$\alpha : B \rightarrow \mathbb{B}(q\mathfrak{H}) \oplus \mathbb{B}((1-q)\mathfrak{H}),$$

$$\beta : B \rightarrow P,$$

$$\gamma : B \rightarrow A,$$

given by

$$\begin{aligned} \alpha(g) &= \pi(a_g), \\ \beta(g) &= x_g, \\ \gamma(g) &= a_g. \end{aligned}$$

With $\rho(x \oplus y) = x$ we then have a commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi} & \mathbb{B}(\mathfrak{H}) \\ \uparrow \gamma & & \uparrow \iota \\ B & \xrightarrow{\alpha} & \mathbb{B}(q\mathfrak{H}) \oplus \mathbb{B}((1-q)\mathfrak{H}) \\ \downarrow \beta & & \downarrow \rho \\ P & \xrightarrow{\theta} & \mathbb{B}(q\mathfrak{H}) \end{array}$$

Since P is projective, there is a lifting morphism $\sigma : P \rightarrow B$, with $\beta \circ \sigma = \text{id}$. Define $\varphi = \gamma \circ \sigma$. Then for each g in G

$$\pi(\varphi(x_g)) = \pi \circ \gamma \circ \sigma(x_g) = \alpha(\sigma(x_g)),$$

which must commute with q . Furthermore,

$$\pi(\varphi(x_g))q = \alpha(\sigma(x_g))q = \rho \circ \alpha \circ \sigma(x_g) = \theta(x_g).$$

This completes the proof, since the x_g 's generate P . □

With $P = \mathbb{C}\mathbb{C}$ ($= C_0([0, 1])$) we recover the version of Theorem 4.1 where x is assumed to be positive and a is required to be positive. With $P = \mathbb{C}\mathbb{M}_2$ we recover a result of Glimm which states that if ξ and η are orthogonal unit vectors in \mathfrak{H} , and $\pi : A \rightarrow \mathbb{B}(\mathfrak{H})$ is irreducible, there is an a in A with $a^2 = 0$, $\|a\| = 1$ and $\pi(a)\xi = \eta$. (Recall that $\mathbb{C}\mathbb{M}_2$ is the universal C^* -algebra for the relation $x^2 = 0$; cf. Section 2.) Glimm used this result to produce a sufficient supply of nilpotents of order two inside an antiliminary C^* -algebra. So shall we.

The following notation will be used in the sequel: if x and y are positive elements in a C^* -algebra, we write $x \ll y$ if $xy = x$. Necessarily this means that x and y commute, and in the function algebra $C^*(x, y)$ every character γ with $\gamma(x) \neq 0$ must have $\gamma(y) = 1$. The concept $x \ll y$ is implicit in many of our previous proofs (notably in 3.9 and 3.10), and it is worth noting that the relation $x \ll y$ is one of the few liftable relations that preserve commutativity.

4.3. Lemma. *If A is an antiliminary C^* -algebra, there are elements x, h in A such that*

$$\begin{aligned} \|x\| &= \|h\| = 1, \\ x^2 &= 0, \\ h &\geq 0, \\ x^*x &\gg h. \end{aligned}$$

Proof. Being antiliminary, A has an irreducible representation $\pi : A \rightarrow \mathbb{B}(\mathfrak{H})$ with $\dim(\mathfrak{H}) = \infty$. Take a two-dimensional projection q on \mathfrak{H} and define $\theta : \mathbb{C}\mathbb{M}_2 \rightarrow$

$\mathbb{B}(q\mathfrak{H})$ by $\theta(f) = f(1)$ (identifying \mathbb{M}_2 with $\mathbb{B}(q\mathfrak{H})$). By the Projective Transitivity Theorem there is a morphism $\varphi : \mathbf{CM}_2 \rightarrow A$ such that

$$\pi(\varphi(g \otimes e_{ij}))q = \theta(g \otimes e_{ij}) = g(1) \otimes e_{ij}$$

for all g in $C_0([0, 1])$ and all matrix units e_{ij} . Let

$$\begin{aligned} g_1(t) = 2t, \quad g_2(t) = 0 & \quad \text{for } 0 < t \leq \frac{1}{2}, \\ g_1(t) = 1, \quad g_2(t) = 2t - 1 & \quad \text{for } \frac{1}{2} \leq t \leq 1. \end{aligned}$$

The two desired elements are then defined by

$$x = \varphi(g_1 \otimes e_{21}), \quad h = \varphi(g_2 \otimes e_{11}).$$

□

Evidently the lemma above uses the antilimilarity of A only in a rather superficial way, i.e. A having at least one irreducible representation which is not a character. However, in applications the lemma will be applied to arbitrarily small hereditary C^* -subalgebras of A , and the full force of antilimilarity will be needed.

We shall later remark on the benefits of replacing \mathbf{CM}_2 by \mathbf{CM}_n , $n > 2$. For now, we give a short proof of a result which is also, at least implicitly, due to Glimm.

4.4. Proposition. *If A is an antiliminary C^* -algebra, there exists a sequence (x_n) in A such that, for all n ,*

$$\begin{aligned} \|x_n\| &= 1, \\ x_n^2 &= 0, \\ x_n^*x_n &\gg x_{n+1}^*x_{n+1}, \\ x_n^*x_n &\gg x_{n+1}x_{n+1}^*. \end{aligned}$$

Proof. By Lemma 4.3 we have x_1 and h_1 in A of norm one, with

$$x_1^2 = 0, \quad h_1 \geq 0, \quad x_1^*x_1 \gg h_1.$$

But $x_1^*x_1 \gg h_1$ and $x_2 \in \overline{h_1Ah_1}$ imply that $x_1^*x_1 \gg x_2^*x_2$ and $x_1^*x_1 \gg x_2x_2^*$. An iterative process now produces the sequence (x_n) (together with the auxiliary sequence (h_n)). □

We now only have to discover what kind of C^* -algebra is generated by a sequence of nilpotents as above, and we will have recovered Glimm's celebrated result that the Fermion algebra is a quotient of some C^* -subalgebra of A , with the advantage that we will now know something about the form of the subalgebra.

4.5. Proposition. *The universal C^* -algebra B_n generated by elements x_1, \dots, x_n , subject to the relations*

$$\begin{aligned} \|x_j\| &\leq 1, & (1 \leq j \leq n), \\ x_j^2 &= 0 & (1 \leq j \leq n), \\ (*) \quad x_j^*x_j &\gg x_k^*x_k, \\ x_j^*x_j &\gg x_kx_k^*, & (j < k), \end{aligned}$$

is isomorphic to the mapping telescope $\mathbf{T}(\mathbb{M}_2, \mathbb{M}_4, \dots, \mathbb{M}_{2^n})$. Regarding the telescope as a subalgebra of $C_0([0, n]) \otimes \mathbb{M}_2 \otimes \mathbb{M}_2 \otimes \dots \otimes \mathbb{M}_2$, the isomorphism is given

by

$$x_j \mapsto f_j \otimes \underbrace{e_{11} \otimes \cdots \otimes e_{11}}_{j-1} \otimes e_{21} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j},$$

where

$$f_j(t) = ((t + 1 - j) \vee 0) \wedge 1 = \begin{cases} 0, & 0 < t \leq j - 1, \\ t + 1 - j, & j - 1 \leq t \leq j, \\ 1, & j \leq t \leq n. \end{cases}$$

Proof. That we obtain a surjective morphism from B_n onto the finite telescope follows from universality; cf. Section 2. To prove injectivity it suffices to show that every irreducible representation of B_n is the pull-back of an irreducible representation of $\mathbf{T}(\mathbb{M}_2, \dots, \mathbb{M}_{2^n})$. Equivalently, we show by induction that the only irreducible representations of the relations (*) are those of the form

$$x_j = \alpha_j \underbrace{e_{11} \otimes \cdots \otimes e_{11}}_{j-1} \otimes e_{21} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j}$$

in $\mathbb{B}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2)$, where the scalars α_j satisfy

$$\begin{aligned} \alpha_1 &= \alpha_2 = \cdots = \alpha_{n-1} = 1, \\ 0 &< \alpha_k < 1, \\ \alpha_{n+1} &= \cdots = \alpha_n = 0, \end{aligned}$$

for some k .

The case $n = 1$ is just a restatement of the fact that \mathbf{CM}_2 is universal for the relations $x^2 = 0, \|x\| \leq 1$. Now suppose that we have proved the assertion for all $(n - 1)$ -tuples, and take x_1, \dots, x_n in $\mathbb{B}(\mathfrak{H})$ satisfying (*), such that

$$C^*(x_1, \dots, x_n)' = \mathbb{C}1.$$

The operator $x_1^*x_1 + x_1x_1^*$ commutes with all the x_j 's, hence with $C^*(x_1, \dots, x_n)$, and so $x_1^*x_1 + x_1x_1^* = \alpha 1$, where $0 \leq \alpha \leq 1$. If $\alpha < 1$, the relations

$$1 > x_1^*x_1 \gg x_j^*x_j \quad \text{for } 2 \leq j \leq n$$

force $x_2 = x_3 = \cdots = x_n = 0$. This reduces to the known case, $n = 1$. If $\alpha = 1$, we let

$$e_{11} = x_1^*x_1, \quad e_{12} = x_1^*, \quad e_{21} = x_1, \quad e_{22} = x_1x_1^*.$$

These elements act like matrix units, and $x_j = e_{11}x_je_{11}$ for all $j > 1$. Working in matrix notation, we write

$$x_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad x_j = \begin{pmatrix} y_j & 0 \\ 0 & 0 \end{pmatrix}, \quad (2 \leq j \leq n).$$

Then the elements y_2, \dots, y_n are seen to be an irreducible representation of the relations (*) on $e_{11}\mathfrak{H}$ for $n - 1$. By induction the y_j 's, and thus also x_1, \dots, x_n , are of the desired form. □

We denote by $\mathbf{T}(\mathbb{M}_{2^\infty})$ the mapping telescope corresponding to the inductive system $\mathbb{M}_2 \subset \mathbb{M}_4 \subset \cdots \subset \mathbb{M}_{2^n} \subset \cdots \subset \mathbb{M}_{2^\infty}$.

4.6. Theorem. *The telescope $\mathbf{T}(\mathbb{M}_{2^\infty})$ is the universal C^* -algebra generated by elements x_1, x_2, \dots , subject to the relations*

$$\begin{aligned} \|x_n\| &\leq 1, \\ x_n^2 &= 0, \\ x_n^* x_n &\gg x_{n+1}^* x_{n+1}, \\ x_n^* x_n &\gg x_{n+1} x_{n+1}^*. \end{aligned}$$

Proof. Clearly the universal C^* -algebra is the inductive limit of the algebras B_n from the previous proposition, whereas the telescope $\mathbf{T}(\mathbb{M}_{2^\infty})$ is the inductive limit of the finite telescopes (cf. Section 2). The result is therefore immediate from Proposition 4.5. \square

Since the Fermion algebra \mathbb{M}_{2^∞} is simple, the primitive spectrum of $\mathbf{T}(\mathbb{M}_{2^\infty})$ is homeomorphic to $]0, \infty]$ (where the point ∞ supports all the irreducible representations of \mathbb{M}_{2^∞} !). In particular, the quotients of $\mathbf{T}(\mathbb{M}_{2^\infty})$ are found by restriction to a closed subset Z of $]0, \infty]$. Only if $\infty \in Z$ will all the generators have norm one. So Proposition 4.4 in conjunction with Theorem 4.6 yield our version of Glimm's theorem:

4.7. Theorem. *If A is an antiliminary C^* -algebra, it contains a C^* -subalgebra isomorphic to*

$$\{f \in C_0(Z, \mathbb{M}_{2^\infty}) \mid f(t) \in \mathbb{M}_{2^n} \text{ if } t \in Z \cap]0, n]\}$$

for some closed subset Z of $]0, \infty]$ containing ∞ .

4.8. Remark. In [20] (cf. [21, 6.7.3]), Glimm's theorem was extended from the Fermion algebra to arbitrary UHF -algebras. We wish to point out that our method also covers this generalization. Choose a sequence $m(1), m(2), \dots$ of natural numbers greater than one, and put $m(n)! = \prod_{k=1}^n m(k)$. Then consider the inductive system

$$\mathbb{M}_{m(1)!} \subset \mathbb{M}_{m(2)!} \subset \dots,$$

where the embeddings are given by writing

$$\mathbb{M}_{m(n)!} = \mathbb{M}_{m(n-1)!} \otimes \mathbb{M}_{m(n)}.$$

The inductive limit is the UHF -algebra (or Glimm algebra) $\mathbb{M}_{m(\infty)!}$. Corresponding to each UHF -algebra we have a mapping telescope $\mathbf{T}(\mathbb{M}_{m(\infty)!})$.

It is straightforward to show – mimicking the proof of Proposition 4.5 – that $\mathbf{T}(\mathbb{M}_{m(\infty)!})$ is the universal C^* -algebra for a sequence of generators $x_{j,m(n)}$, $2 \leq j \leq m(n)$, $n \in \mathbb{N}$ (a sequence of finite sets, really), subject to the relations

$$\begin{aligned} \|x_{j,m(n)}\| &\leq 1, \\ x_{j,m(n)} x_{k,m(n)} &= 0, & 2 \leq j, k \leq m(n), \\ x_{j,m(n)}^* x_{j,m(n)} &= x_{k,m(n)}^* x_{k,m(n)}, & 2 \leq j, k \leq m(n), \\ x_{j,m(n)}^* x_{j,m(n)} &\gg x_{k,m(n+1)}^* x_{k,m(n+1)}, & \text{for all } j, k, n, \\ x_{j,m(n)}^* x_{j,m(n)} &\gg x_{k,m(n+1)} x_{k,m(n+1)}^*, & \text{for all } j, k, n \\ x_{j,m(n)}^* x_{k,m(n)} &= 0, & \text{if } j \neq k. \end{aligned}$$

The notation and the implications can be found in [21, 6.6], but it suffices to note that for each n , the elements

$$x_{2,m(n)}, x_{3,m(n)}, \dots, x_{m(n),m(n)}$$

are (multiples of) the first column in $\mathbb{M}_{m(n)}$, except for the deleted $(1, 1)$ -element.

To extend Theorem 4.7 from \mathbb{M}_{2^∞} to $\mathbb{M}_{m(n)}$ we just have to extend Lemma 4.3 (Proposition 4.4 will apply, mutatis mutandis) and show that every antiliminary C^* -algebra A contains elements x_2, x_3, \dots, x_m and h such that

$$\begin{aligned} \|x_j\| = \|h\| = 1, & \quad 2 \leq j \leq m, \\ x_j x_k = 0, & \quad \text{all } j, k, \\ x_j^* x_j = x_k^* x_k, & \quad \text{all } j, k, \\ x_j^* x_j \gg h, & \quad \text{all } j, \\ x_j x_k = 0, & \quad \text{if } j \neq k. \end{aligned}$$

This is done by replacing \mathbf{CM}_2 by \mathbf{CM}_m in the proof of Lemma 4.3.

4.9. Remark. We will show later, in Theorem 7.2, that every AF-telescope is projective. This means that in Theorem 4.7 (and its generalization hinted at in Remark 4.8) we may replace the condition that A is antiliminary, by the weaker condition that A is not of type I. For in the latter case A has a non-zero antiliminary quotient.

The presentation of $\mathbf{T}(\mathbb{M}_{2^\infty})$ given in Theorem 4.6 is not related to the Fermion picture of \mathbb{M}_{2^∞} . There is such a presentation, and though we have no immediate use for it, we display it for its elegance. For completeness we first state the following well-known fact.

4.10. Lemma. *The universal C^* -algebra with generators $\{a_j \mid j \leq n\}$, where $1 \leq n \leq \infty$, subject to the relations*

$$\begin{aligned} a_j a_k + a_k a_j = 0, & \quad \text{all } j, k, \\ a_j^* a_k + a_k a_j^* = \delta_{jk} 1, & \quad \text{all } j, k, \end{aligned}$$

is isomorphic to $\mathbb{M}_{2^n} = \mathbb{M}_2 \otimes \dots \otimes \mathbb{M}_2$ via the map

$$a_j \mapsto \underbrace{v \otimes \dots \otimes v}_{j-1} \otimes e_{21} \otimes \underbrace{1 \otimes \dots \otimes 1}_{n-j},$$

where $v = e_{22} - e_{11}$.

Proof. For $n < \infty$ this is pure (linear) algebra. For $n = \infty$ it follows from the description of the Fermion algebra \mathbb{M}_{2^∞} as the inductive limit of the matrix algebras \mathbb{M}_{2^n} . □

4.11. Theorem. *The Fermion telescope $\mathbf{T}(\mathbb{M}_{2^\infty})$ is the universal C^* -algebra generated by a sequence (a_n) of contractions, subject to the relations*

$$\begin{aligned} a_m a_n + a_n a_m = 0, & \quad \text{all } n, m, \\ a_m^* a_n + a_n a_m^* = 0, & \quad \text{if } n \neq m, \\ a_m^* a_m + a_m a_m^* \gg a_n^* a_n + a_n a_n^*, & \quad \text{if } m < n. \end{aligned}$$

Proof. Let B denote the universal C^* -algebra generated by the sequence (a_n) . Note now that each element

$$h_n = a_n^* a_n + a_n a_n^*$$

commutes with all a_m , and thus is central in B . Moreover,

$$0 \leq h_n \leq 1$$

and

$$h_m \gg h_n \quad \text{if } m < n.$$

Therefore, if $\pi : B \rightarrow \mathbb{B}(\mathfrak{H})$ is an irreducible representation of B , and if we put $b_n = \pi(a_n)$ and $k_n = \pi(h_n)$, there are two cases: either for some n and some α in $]0, 1[$ we have

$$\begin{aligned} k_m &= 1 && \text{for } m < n, \\ k_n &= \alpha 1, \\ k_m &= 0 && \text{for } m > n, \end{aligned}$$

or else we have $k_n = 1$ for all n .

In the first case the relations force $b_j = 0$ for $j > n$, and the elements

$$b_1, b_2, \dots, b_{n-1}, \alpha^{-1/2} b_n$$

now satisfy the standard Fermion relations, which by Lemma 4.10 means that $\pi(B) = \mathbb{M}_{2^n}$. If $\varphi : B \rightarrow \mathbf{T}(\mathbb{M}_{2^\infty})$ is the surjective morphism determined by

$$\varphi(a_j) = f_j \otimes \underbrace{v \otimes \cdots \otimes v}_{j-1} \otimes e_{21} \otimes 1 \otimes \cdots,$$

where $f_j(t) = ((t + 1 - j) \vee 0) \wedge 1$, then we see from Lemma 4.10 that π is the pull-back of φ by evaluation at $t = n - 1 + \alpha^{1/2}$.

In the second case the sequence (b_n) satisfies the Fermion relations, and, again from Lemma 4.10 we see that π is the pull-back of a subrepresentation of φ by evaluation at $t = \infty$.

We have shown that every irreducible representation of B is a pull-back of φ , which proves that φ is an isomorphism. \square

5. MULTIPLIER REALIZATION THEOREMS

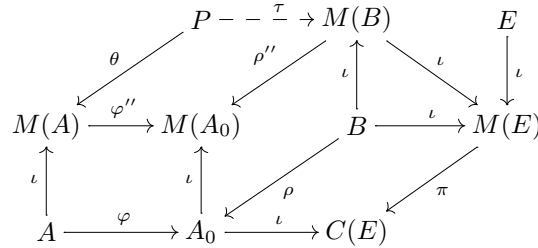
Recall from [21, 3.12] that every non-unital C^* -algebra E is embedded as an essential ideal in its *multiplier algebra* $M(E)$, and $M(E)$ is the universal C^* -algebra with this property, being the non-commutative analogue of the Stone-Ćech compactification. The *corona algebra* $C(E) = M(E)/E$ has many exciting properties (the *SAW**-property, the asymptotically abelian, countable Riesz separation property, etc.; see [19] or [21]) which facilitate liftings from $C(E)$ to $M(E)$. The fact – already mentioned – that a C^* -algebra P is projective if it is merely corona projective makes these properties important for our study.

5.1. Theorem. *Let A and E be σ -unital C^* -algebras and $\varphi : A \rightarrow C(E)$ a morphism of A into the corona algebra of E . If P is a projective C^* -algebra, then for every morphism $\theta : P \rightarrow M(A)$ there is a morphism $\psi : P \rightarrow M(E)$ such that for all p in P and a in A*

$$\pi(\psi(p))\varphi(a) = \varphi(\theta(p)a),$$

where $\pi : M(E) \rightarrow C(E)$ is the quotient map.

Proof. Consider the following diagram:



Here $A_0 = \varphi(A)$, so that $\iota : A_0 \rightarrow C(E)$ is an inclusion. Moreover, since A is σ -unital, there is, by [22] and [10], a canonical surjective morphism $\varphi'' : M(A) \rightarrow M(A_0)$, extending φ . Let $B = \pi^{-1}(A_0)$, so that we have a surjective morphism $\rho : B \rightarrow A_0$, making the part of the diagram involving $A_0, B, M(E)$ and $C(E)$ commutative. Since $E \subset B \subset M(E)$, so that E is an essential ideal in B , we have an embedding $\iota : M(B) \rightarrow M(E)$.

Note that since both A and E are σ -unital, so is B . In fact, any element $h + k$ will be strictly positive for B if h is strictly positive for E and $\rho(k)$ is strictly positive for A_0 . Since ρ is a surjective morphism there is therefore a canonical surjective morphism $\rho'' : M(B) \rightarrow M(A_0)$ extending ρ . Consider now the morphism $\varphi'' \circ \theta : P \rightarrow M(A_0)$. Since P is projective, this lifts to a morphism $\tau : P \rightarrow M(B)$, such that $\rho'' \circ \tau = \varphi'' \circ \theta$. Define $\psi = \iota \circ \tau$ as a morphism from P into $M(E)$. If $p \in P$ and $a \in A$, then since $\varphi(a) = \rho(b)$ for some b in B we get, by diagram chasing,

$$\begin{aligned}
 \pi(\psi(p))\varphi(a) &= (\pi \circ \iota \circ \tau(p))\rho(b) = \pi(\iota \circ \tau(p)\iota(b)) \\
 &= \pi(\tau(p)b) = \rho(\tau(p)b) = \rho''(\tau(p))\rho(b) \\
 &= \varphi''(\theta(p))\varphi(a) = \varphi(\theta(p)a),
 \end{aligned}$$

as desired. □

5.2. Corollary. *If A is the split extension of P by Q , where P is σ -unital, and if Q is projective, then every morphism φ of P into a corona algebra $C(E)$ of a σ -unital C^* -algebra E extends to a morphism $\tilde{\varphi} : A \rightarrow C(E)$.*

Proof. Let $\theta : Q \rightarrow M(P)$ be the morphism that determines A (cf. Proposition 2.1), and apply Theorem 5.1 to obtain a morphism $\varphi_0 : Q \rightarrow C(E)$ such that $\varphi_0(q)\varphi(p) = \varphi(\theta(q)p)$ for all q in Q and p in P . By Proposition 2.2 the pair (φ, φ_0) determines a morphism $\tilde{\varphi} : A \rightarrow C(E)$. □

5.3. Theorem. *If A is a projective C^* -algebra which is written as a (split) extension $P \rightarrow A \rightarrow Q$, where also Q is projective, then P is projective.*

Proof. Let $\varphi : P \rightarrow C(E)$ be a morphism of P into the corona algebra of some σ -unital C^* -algebra E . By Corollary 5.2 we have an extension $\tilde{\varphi} : A \rightarrow C(E)$, and since A is projective this lifts to a morphism $\tilde{\psi} : A \rightarrow M(E)$ such that $\pi \circ \tilde{\psi} = \tilde{\varphi}$. Now let $\psi = \tilde{\psi}|_P$ to obtain the desired lifting of φ . □

For later use (in section 7) we give an application of the previous theorem.

5.4. Proposition. *Let $A = \overline{\bigcup A_n}$ be an inductive limit of C^* -algebras, and, with \sim denoting (forced) unitization, consider \tilde{A} as the inductive limit $\bigcup \tilde{A}_n$. If $\mathbf{T}(\tilde{A})$ is projective then so is $\mathbf{T}(A)$.*

Proof. We leave it to the reader to establish that there is an exact sequence

$$0 \rightarrow \mathbf{T}(A) \rightarrow \mathbf{T}(\tilde{A}) \rightarrow \mathbf{T}(\mathbb{C}) \rightarrow 0.$$

Here $T(\mathbb{C})$ means the telescope for the system

$$\mathbb{C} \xrightarrow{\text{id}} \mathbb{C} \xrightarrow{\text{id}} \dots$$

However, $\mathbf{T}(\mathbb{C}) \cong \mathbf{CC}$; so, again, Theorem 5.3 applies. \square

At this stage the reader must have asked — and possibly solved — the question whether (split) extensions of projective C^* -algebras are again projective. Certainly our telescopic examples support this hypothesis. Unfortunately it is not true.

5.5. Example. Let X be the closure of the set

$$G = \{(t, \sin t^{-1}) \mid 0 < t \leq 1\}$$

in \mathbb{R}^2 with the point $(0, 1)$ removed. The intersection of X with the Y -axis is a closed set F , homeomorphic with $]0, 1]$. The rest is $X \setminus F = G$, which is also homeomorphic to $]0, 1]$ (projecting on the X -axis). We have therefore a short exact sequence

$$C_0(G) \rightarrow C_0(X) \rightarrow C_0(F),$$

where both the ideal and the quotient are projective, being isomorphic to \mathbf{CC} . But $C_0(X)$ is not projective, because X is not an absolute retract. If it were, there would be a continuous map $f : \hat{X} \rightarrow X$, where \hat{X} denotes the cone of X , such that $f(x, 1) = x$ for each x in X . (This corresponds to making $C_0(X)$ a subalgebra of $\mathbf{CC}_0(X)$, lifting the morphism $g \rightarrow g(1)$ of $\mathbf{CC}_0(X)$ onto $C_0(X)$.) The definition $f_t(x) = f(x, t)$ shows that X is contractible, and in particular arcwise connected. But X is the standard example of a (connected) not arcwise connected set.

In the positive direction one can show, rather easily, that an extension of projective C^* -algebras is residually finite-dimensional. A C^* -algebra is residually finite-dimensional if it embeds into $\prod M_{n_k}$ for some sequence of natural numbers n_k . If a C^* -algebra embeds into $\prod M_{n_k} / \bigoplus M_{n_k}$, then it is an MF algebra. This is not the definition of an MF , or matricial field, algebra, but it is equivalent by [3, Theorem 3.2.2].

5.6. Theorem. *If I is an MF algebra and P is projective, and A is an extension of I by P , then A is an MF algebra.*

Proof. By the assumption on I , there exists an injective morphism $\varphi_0 : I \rightarrow C(E)$ where $E = \bigoplus M_{n_k}$. Corollary 5.2 implies that there is an extension of φ_0 to a morphism $\varphi : A \rightarrow C(E)$. \square

As noted in the introduction, the projectivity of P implies that it is residually finite-dimensional. Taking an infinite family of finite-dimensional representations, each occurring infinitely often, we obtain an embedding $\psi_0 : P \rightarrow C(F)$, where $F = \bigoplus M_{r_k}$. Using a splitting $\lambda : P \rightarrow A$ we obtain a morphism $\psi : A \rightarrow C(F)$, so that now $\varphi \oplus \psi : A \rightarrow C(E) \oplus C(F)$ is injective. Since

$$C(E) \oplus C(F) \subseteq C\left(\bigoplus M_{n_k+r_k}\right),$$

we have proven that A is also MF .

6. PRESENTATIONS OF TELESCOPES

As a prerequisite for showing that every AF -telescope is a projective C^* -algebra we need a detailed study of its presentations. A glance at the problem will explain why. To lift a morphism from an infinite telescope, one must work inductively and lift the cone \mathbf{CA}_n , having already lifted the finite telescope $\mathbf{T}(A_1, \dots, A_{n-1})$ (cf. Proposition 2.4). This means lifting an ideal, given the constraints of having lifted the quotient! Only the most careful labeling of the embeddings $A_n \hookrightarrow A_{n+1}$ will make this process possible. The model we present uses graph-theoretic language instead of numbers, and the groupoid of paths in the diagram will be our chosen object.

Let S be a finite set of cardinality $\#S$, equipped with an equivalence relation \sim . Choose a set $[S]$ of representatives for the equivalence classes, and let $e \mapsto [e]$ denote the selection function (so that $e \sim [e] \in [S]$). Finally, let $\#[e]$ denote the cardinality of the equivalence class represented by $[e]$.

6.1. Lemma. *With notations as above, the mapping cone for the C^* -algebra $\bigoplus_{[S]} \mathbb{M}_{\#[e]}$, i.e. the algebra $\bigoplus_{[S]} \mathbf{CM}_{\#[e]}$, is the universal C^* -algebra with generators $\{x_e \mid e \in S\}$, subject to the relations*

- (i) $\|x_e\| \leq 1$ for all e in S ,
- (ii) $x_e x_f = 0$ if $[e] \neq [f]$,
- (iii) $x_e^* x_f = 0$ if $e \neq f$,
- (iv) $x_e x_f^* = 0$ if $[e] \neq [f]$,
- (v) $x_e^* x_e = x_f^* x_f$ if $[e] = [f]$,
- (vi) $x_{[e]} = (x_{[e]}^* x_{[e]})^{1/2}$ for all $[e]$ in $[S]$.

Proof. For a single equivalence class this is just a reformulation of [16, 2.7]. The general case follows from the usual properties of direct sums.

To describe in detail the isomorphism of the universal C^* -algebra generated by the x_e 's with the mapping cone, note first that $\bigoplus \mathbb{M}_{\#[e]}$ is linearly spanned by generalized matrix units

$$\{v_{e,f} \mid (e, f) \in S^2 \text{ and } [e] = [f]\}$$

satisfying the rules

$$\begin{aligned} v_{e,f}^* &= v_{f,e}, \\ v_{e,f} v_{g,h} &= \delta_{f,g} v_{e,h}. \end{aligned}$$

Identifying $\mathbf{C}(\bigoplus \mathbb{M}_{\#[e]})$ with $C_0([0, 1]) \otimes (\bigoplus \mathbb{M}_{\#[e]})$ as usual, the isomorphism of the universal C^* -algebra to the mapping cone is given by

$$x_e \mapsto \text{id} \otimes v_{e,[e]}.$$

The inverse map is defined on a set of elements with dense span by

$$\gamma(t^2)t^2 v_{e,f} \mapsto x_e \gamma(x_e^* x_e) x_f^* \quad (\gamma \in C_0([0, 1])).$$

□

6.2. Remark. With just one set of extra relations:

$$x_{[e]}^2 = x_{[e]} \quad ([e] \in [S]),$$

the lemma above becomes a presentation of $\bigoplus \mathbb{M}_{\#|e|}$, because now all the x_e become partial isometries. Note also that we have deliberately introduced the “unnecessary” generators $x_{|e|}$. This small redundancy in our model is amply compensated for by its elegance.

We now, for the rest of the section, fix a Bratteli diagram (with multiple embeddings represented by multiple edges) corresponding to a *unital* AF system

$$\mathbb{C}1 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots .$$

We shall obtain a presentation for

$$\mathbf{T}(A) = \mathbf{T}(A_1, A_2, A_3, \dots).$$

The algebra $\mathbb{C}1$ is included in the AF system so that in the Bratteli diagram we have a single vertex at the zeroth level, which we call a *root* vertex. A key observation is that the *weight* of a vertex (= the size of the corresponding matrix algebra) equals the number of paths from the root to that vertex.

By the n th level of vertices we mean those connected to the root by a length- n path, so these are the vertices that represent the factors of A_n . The total edge set we denote $E = \bigcup E_n$, and $e \in E_n$ means that e is an edge that connect a level- $(n-1)$ vertex $s(e)$ to a level- n vertex $r(e)$. We call $s(e)$ the source of e and $r(e)$ the range of e . (Ranges below; sources above.)

We shall write $\bar{e} = e_1 e_2 \cdots e_n$ for a downward (= away from the root, sadly) path along adjacent edges. We let $r(\bar{e}) = r(e_n)$ and $s(\bar{e}) = s(e_1)$, and shall define the composition of two paths \bar{e} and \bar{f} (denoted $\bar{e}\bar{f}$) if $r(\bar{e}) = s(\bar{f})$.

With the above notation, the algebra A_n is linearly spanned by the generalized matrix units $v_{\bar{e}, \bar{f}}$, where \bar{e} and \bar{f} range over all paths from the root to the n th level satisfying $r(\bar{e}) = r(\bar{f})$. The rules for adjoints and multiplication are simple:

$$v_{\bar{e}, \bar{f}}^* = v_{\bar{f}, \bar{e}}$$

and

$$v_{\bar{e}, \bar{f}} v_{\bar{g}, \bar{h}} = \delta_{\bar{f}, \bar{g}} v_{\bar{e}, \bar{h}}.$$

The embedding $A_n \rightarrow A_{n+1}$ is determined by

$$v_{\bar{e}, \bar{f}} \mapsto \sum v_{\bar{e}e_n, \bar{f}e_n},$$

where the summation is over all edges e_n in E_{n+1} with $s(e_n) = r(\bar{e})$ ($= r(\bar{f})$).

If we identify each $v_{\bar{e}, \bar{f}}$ with its image in the AF-algebra $A = \overline{\bigcup A_n}$, we can define multiplication between matrix units in different subalgebras. A moment's reflection shows that the rules are

$$v_{\bar{e}, \bar{f}} v_{\bar{g}, \bar{h}} = 0$$

unless $\bar{g} = \bar{f}\bar{d}$ or $\bar{f} = \bar{g}\bar{d}$ for some path \bar{d} , in which case

$$v_{\bar{e}, \bar{f}} v_{\bar{f}\bar{d}, \bar{h}} = v_{\bar{e}\bar{d}, \bar{h}},$$

$$v_{\bar{e}, \bar{g}\bar{d}} v_{\bar{g}, \bar{h}} = v_{\bar{e}, \bar{h}\bar{d}}$$

(of course, \bar{d} might be the zero path).

Because of these multiplication rules, if we have a subset of the $v_{\bar{e}, \bar{f}}$'s that generates A_{n-1} , we need only add enough extra elements to account for the edges in E_n , in order to create a generating set for A_n . In fact, a minimal generating set for A can be associated with the edges of the Bratteli diagram. (Well, almost minimal.)

The construction is non-canonical, though, so we now mark some edges as being the “preferred” way back to the root.

More precisely, choose a subset $[E_n]$ of E_n that contains exactly one edge adjacent to each vertex “below”. Thus from each level- n vertex there is exactly one way back through $[E_n]$ to some level- $n - 1$ vertex. Let $e \mapsto [e]$ denote the (selection) function which sends an edge e to the preferred edge in $[E] = \bigcup [E_n]$ that leads down to the same vertex ($r(e) = r([e])$).

With the choice of $[E]$, each vertex determines a unique path through elements from $[E]$ from the root to that vertex. Therefore, each edge e in E_n is associated with two canonical paths $p(e)$ and $q(e)$, where $p(e)$ is the path from the root to $r(e)$ through $[E]$ and $q(e)$ is the path from the root to $r(e)$ of the form $\bar{e}e$, where \bar{e} is a path through $[E]$.

6.3. Lemma. *With notations as above, the C^* -algebra A_n is generated by the set*

$$\{v_{q(e),p(e)} \mid e \in E_1 \cup E_2 \cup \dots \cup E_n\}.$$

Proof. For $n = 1$ we have for each e in E_1 that $p(e) = [e]$ and $q(e) = e$. That the set $v_{e,[e]}$, $e \in E_1$, generates A_1 follows from Remark 6.2. Assume now that we have established the result for some A_n ($n \geq 1$), and consider $v_{\bar{e},\bar{f}}$ in A_{n+1} . We can write $\bar{e} = \hat{e}e$ and $\bar{f} = \hat{f}f$ for some e, f in E_{n+1} with $r(e) = r(f)$. The multiplication rules for matrix units given above show that for any paths \bar{h} and \bar{k} from the root down to the n th level with $r(\bar{h}) = s(e)$, $r(\bar{k}) = s(f)$ we have

$$v_{\bar{e},\bar{f}} = v_{\hat{e}e,\hat{f}f} = v_{\hat{e},\bar{h}}v_{\bar{h}e,\hat{f}f} = v_{\hat{e},\bar{h}}v_{\bar{h}e,\hat{f}f} = v_{\hat{e},\bar{h}}v_{\bar{h}e,\bar{k}f}v_{\bar{k},\hat{f}}.$$

Here $v_{\hat{e},\bar{h}}$ and $v_{\bar{k},\hat{f}}$ belong to A_n by assumption, and we choose \bar{h} and \bar{k} such that $\bar{h}e = q(e)$ and $\bar{k}f = q(f)$. Noting that $p(e) = p(f)$, because $r(e) = r(f)$, we see that

$$v_{\bar{h}e,\bar{k}f} = v_{q(e),p(e)}v_{p(e),q(f)} = v_{q(e),p(e)}v_{q(f),p(f)}^*.$$

Combining these equations we see that $v_{\bar{e},\bar{f}}$ is generated by matrix units of the form $v_{q(e),p(e)}$, $e \in E_{n+1}$, together with elements from A_n , and the proof is completed by induction. □

With Lemma 6.3 in mind we construct generators of $\mathbf{T}(A)$ as follows: define a sequence (α_n) of functions in $C_0([0, \infty])$ by

$$\alpha_n(t) = ((t + 1 - n) \vee 0) \wedge 1,$$

so that each α_n is “the identity map” on the relevant interval $]n - 1, n]$, and $\alpha_n\alpha_m = \alpha_m$ if $n < m$. Then consider the elements

$$\alpha_n \otimes v_{q(e),p(e)} \quad (e \in E_n, n \in \mathbb{N}).$$

With this model in mind we present the following.

6.4. Theorem. *The mapping telescope $\mathbf{T}(A)$ for the AF-algebra A is the universal C^* -algebra with generators x_e , $e \in E$, where $E = \bigcup E_n$ denotes the set of edges in the Bratteli diagram, equipped with an equivalence relation determined by a set*

$[E] = \bigcup [E_n]$ of preferred edges, subject to the relations

- (i) $\|x_e\| \leq 1$ for all e in E ,
- (ii) $x_e x_f = 0$ if $[e] \neq [f]$ and $e, f \in E_n$,
- (iii) $x_e^* x_f = 0$ if $e \neq f$ and $e, f \in E_n$,
- (iv) $x_e x_f^* = 0$ if $[e] \neq [f]$ and $e, f \in E_n$,
- (v) $x_e^* x_e = x_f^* x_f$ if $[e] = [f]$ and $e, f \in E_n$,
- (vi) $x_{[e]} = (x_{[e]}^* x_{[e]})^{1/2}$ for all e ,
- (vii) $x_{[e]} \gg x_f x_f^*$ if $e \in E_n, f \in E_{n+1}$ and $r(e) = s(f)$.

For the proof we shall need some notations and some preliminary results. We use the graph-theoretic notation explained earlier, and let S_n denote the set of paths in the Bratteli diagram from the root to the n th level. So each \bar{e} in S_n has the form

$$\bar{e} = e_1 e_2 \dots e_n \quad (e_j \in E_j).$$

The preferred paths of length n are denoted

$$[S_n] = \{\bar{e} = e_1 e_2 \dots e_n \mid e_j \in [E_j]\}.$$

For each path \bar{e} in S_n there is a unique path $[\bar{e}]$ in $[S_n]$ with $r([\bar{e}]) = r(\bar{e})$.

We combine the edge-generators x_e given in Theorem 6.4 to obtain the path-generators $x_{\bar{e}}$, defined by

$$x_{\bar{e}} = x_{e_1} x_{e_2} \dots x_{e_n}$$

if $\bar{e} = e_1 e_2 \dots e_n$.

6.5. Lemma. *With notation as above, the elements $x_{\bar{e}}, \bar{e} \in \bigcup S_n = S$, satisfy the relations*

- (i) $\|x_{\bar{e}}\| \leq 1$ for all \bar{e} in S ,
- (ii) $x_{\bar{e}} x_{\bar{f}} = 0$ if $[\bar{e}] \neq [\bar{f}]$ and $\bar{e}, \bar{f} \in S_n$,
- (iii) $x_{\bar{e}}^* x_{\bar{f}} = 0$ if $\bar{e} \neq \bar{f}$ and $\bar{e}, \bar{f} \in S_n$,
- (iv) $x_{\bar{e}} x_{\bar{f}}^* = 0$ if $[\bar{e}] \neq [\bar{f}]$ and $\bar{e}, \bar{f} \in S_n$,
- (v) $x_{\bar{e}}^* x_{\bar{e}} = x_{\bar{f}}^* x_{\bar{f}}$ if $[\bar{e}] = [\bar{f}]$ and $\bar{e}, \bar{f} \in S_n$,
- (vi) $x_{[\bar{e}]} = (x_{[\bar{e}]}^* x_{[\bar{e}]})^{1/2}$ for all $[\bar{e}]$,
- (vii) $x_{[\bar{e}]} \gg x_{\bar{f}} x_{\bar{f}}^*$ if $\bar{e} \in S_n$, and $\bar{f} = \bar{e} f$ for some f in E_{n+1} .

Proof. For $n = 1$ the conditions (i)–(vii) are given by definition. Assume they have been established for paths in $S_1 \cup \dots \cup S_n$ and consider \bar{e} and \bar{f} in S_{n+1} of the form $\bar{e} = \hat{e}e$ and $\bar{f} = \hat{f}f$ with e, f in E_{n+1} . Since $r(\hat{e}) = r([\hat{e}])$, there is a path $[\hat{e}]e$. Moreover, $[\bar{e}] = \hat{d}[e]$ for some \hat{d} in $[S_n]$. Now compute

$$x_{\bar{e}}^* x_{\bar{e}} = x_e^* x_{\hat{e}}^* x_{\hat{e}} x_e = x_e^* x_{[\hat{e}]}^2 x_e = x_e^* x_e = x_{[e]}^2,$$

using (v)–(vii) for S_n . Moreover,

$$x_{[\bar{e}]} x_{[\bar{e}]}^* = x_{\hat{d}} x_{[e]} x_{[e]}^* x_{\hat{d}}^* = x_{[e]}^2$$

by (vi) and (vii), since $\hat{d} \in [S_n]$. Combining the two computations we get (v), (vi) and (vii) for S_{n+1} .

Using also the other path \bar{f} we get

$$x_{\bar{e}}^* x_{\bar{e}} x_{\bar{f}} = x_{[e]}^2 x_{\bar{f}} x_f = x_{[e]}^2 x_{\bar{d}} x_{\bar{f}} x_f = 0$$

if $\widehat{d} \neq \widehat{f}$ by (ii), using that $x_{\bar{d}} \gg x_{[e]}$. If, on the other hand $\widehat{d} = \widehat{f}$ (so $\widehat{f} \in [S_n]$), then

$$x_{\bar{e}}^* x_{\bar{e}} x_{\bar{f}} = x_{[e]}^2 x_{\bar{f}} x_f = x_{[e]}^2 x_f = 0$$

unless $f = [e]$. Combining the computations we see that $x_{\bar{e}} x_{\bar{f}} = 0$ unless $\bar{f} = [\bar{e}]$, which gives (ii) for S_{n+1} . To prove (iii),

$$x_{\bar{e}}^* x_{\bar{f}} = x_{\bar{e}}^* x_{\bar{e}}^* x_{\bar{f}} x_f = 0$$

unless $\widehat{e} = \widehat{f}$, in which case (by (vii))

$$x_{\bar{e}}^* x_{\bar{f}} = x_{\bar{e}}^* x_f = 0$$

if $e \neq f$ by definition. Finally,

$$x_{\bar{e}}^* x_{\bar{e}} x_{\bar{f}}^* x_{\bar{f}} = x_{[e]}^2 x_{[f]}^2 = 0$$

unless $[e] = [f]$ by definition, in which case $r(e) = r(f)$ and so $[\bar{e}] = [\bar{f}]$. This proves (iv) for S_{n+1} , and the lemma follows by induction. \square

Proof of Theorem 6.4. Let B denote the universal C^* -algebra satisfying the relations (i)–(vii). It is elementary to check that these relations are also satisfied by the elements $\alpha_n \otimes v_{q(e),p(e)}$ defined previously, and since these generate the telescope $\mathbf{T}(A)$, cf. Lemma 6.3, it follows that the assignment

$$x_e \mapsto \alpha_n \otimes v_{q(e),p(e)}$$

gives a surjective morphism of B onto $\mathbf{T}(A)$.

To construct the inverse morphism note that the relations (i)–(vi) in Lemma 6.5 show that the assignment

$$\text{id}^2 \otimes v_{\bar{e},\bar{f}} \mapsto x_{\bar{e}} x_{\bar{f}}^*$$

where $\bar{e}, \bar{f} \in S_n$, and $r(\bar{e}) = r(\bar{f})$, induces a $*$ -homomorphism $\varphi_n : \mathbf{C}A_n \rightarrow B$. Indeed,

$$\varphi_n(\gamma(t^2)t^2 v_{\bar{e},\bar{f}}) = x_{\bar{e}} \gamma(x_{[\bar{e}]}^2) x_{\bar{f}}^*$$

for each γ in $C_0([0, 1])$ (cf. the proof of Lemma 6.1), so we know φ_n on a set whose closed linear span is $\mathbf{C}A_n$.

For $m < n$, if $\bar{e}, \bar{f} \in S_m$ and $\bar{g}, \bar{h} \in S_n$, then

$$\varphi_m(\gamma(t^2)t^2 v_{\bar{e},\bar{f}}) \varphi_n(\beta(t^2)t^2 v_{\bar{g},\bar{h}}) = x_{\bar{e}} \gamma(x_{[\bar{e}]}^2) x_{\bar{f}}^* x_{\bar{g}} \beta(x_{[\bar{g}]}^2) x_{\bar{h}}^* = 0,$$

by (iii), unless $\bar{g} = \bar{f}\bar{d}$ for some path \bar{d} , in which case the product above becomes

$$x_{\bar{e}} \alpha(x_{[\bar{e}]}^2) x_{[\bar{f}]}^2 x_{\bar{d}} \beta(x_{[\bar{g}]}^2) x_{\bar{h}}^*.$$

Note that $[\bar{e}] = [\bar{f}]$ and $[\bar{g}] = [\bar{h}]$, since they have the same ranges. Note also that $x_{[\bar{e}]} x_{\bar{d}} = x_{\bar{d}}$, since $s(\bar{d}) = r(\bar{f}) (= r(\bar{e}))$ by (vii), so that the product above becomes

$$\begin{aligned} x_{\bar{e}} \gamma(1) x_{\bar{d}} \beta(x_{[\bar{h}]}^2) x_{\bar{h}}^* &= x_{\bar{e}\bar{d}} \gamma(1) \beta(x_{[\bar{h}]}^2) x_{\bar{h}}^* \\ &= \varphi_n(\gamma(1) \beta(t^2) t^2 v_{\bar{e}\bar{d},\bar{h}}) \\ &= \varphi_n(\gamma(1) v_{\bar{e},\bar{f}} \beta(t^2) t^2 v_{\bar{g},\bar{h}}). \end{aligned}$$

We have shown that, for a in \mathbf{CA}_m and b in \mathbf{CA}_n ,

$$\varphi_m(a)\varphi_n(b) = \varphi_n(a(1)b),$$

where $a(1)b$ describes the action of \mathbf{CA}_m in $M(\mathbf{CA}_n)$. By Corollary 2.5 such a coherent sequence of morphisms (φ_n) determines a unique morphism φ of $\mathbf{T}(A)$ into B .

Finally, since $\varphi|_{\mathbf{CA}_n} = \varphi_n$ only after identifying $]n-1, n]$ with $]0, 1]$, we get

$$\varphi(\alpha_n^2 \otimes v_{q(e), p(e)}) = 1x_{q(e)}x_{p(e)}^*.$$

Here $q(e) = e_1e_2 \dots e_{n-1}e$, where $e_k \in [E]$ for $1 \leq k \leq n-1$, whereas $p(e) = f_1f_2 \dots f_n$ with f_k in $[E]$ for all k . Thus

$$\varphi(\alpha_n \otimes v_{q(e), p(e)}) = x_{e_1}x_{e_2} \dots x_{e_{n-1}}x_e x_{f_n} \dots x_{f_1} = x_e,$$

because the x_{e_k} 's and the x_{f_k} 's act as units against x_e by conditions (v)–(vii) in Theorem 6.4. This proves that φ is indeed the inverse of our first morphism of B onto $\mathbf{T}(A)$, and thus an isomorphism. \square

To appreciate the presentation in Theorem 6.4 note the economy: The blunt approach via matrix units would give a set of generators $\alpha_n \otimes v_{\bar{e}, \bar{f}}$, labeled by all possible paths in $\bigcup S_n$. In our presentation the generators are labeled by the edges only — taking advantage of the structure of the previous subalgebras. To illustrate the effect we offer the following simple example.

6.6. Proposition. *Let \tilde{K} denote the unitized C^* -algebra of compact operators on the Hilbert space ℓ^2 , realized as the inductive limit of matrix algebras $\mathbb{M}_n \oplus \mathbb{C}$, as usual. Then the mapping telescope $\mathbf{T}(\tilde{K})$ has a presentation with generators (x_n) and (h_n) , subject to the relations*

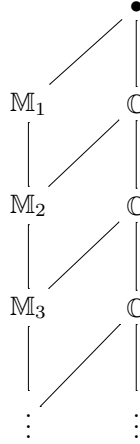
$$\begin{aligned} \|x_n\| &\leq 1, \\ 0 &\leq h_n \leq 1, \\ x_1h_1 &= h_1x_1 = 0, \\ 0 &\leq x_1, \\ x_n^2 &= h_nx_n = 0 \quad (n \geq 2), \\ x_n^*x_n &\gg x_{n+1}^*x_{n+1}, \\ h_n &\gg x_{n+1}x_{n+1}^*, \\ h_n &\gg h_{n+1}. \end{aligned}$$

Proof. With $\alpha_n(t) = ((t+1-n) \vee 0) \wedge 1$ as before, and with e_{ij} the usual matrix units in $\mathbb{B}(\ell^2)$, we define our generators by

$$\begin{aligned} x_n &= \alpha_n \otimes e_{n1}, \\ h_n &= \alpha_n \otimes \sum_{k=n+1}^{\infty} e_{kk}. \end{aligned}$$

That these elements satisfy the relations and generate $\mathbf{T}(\tilde{K})$ is immediate.

The Bratteli diagram for \tilde{K} is



so our model — labeling the generators by the edges — should really contain a third sequence of generators (y_n) . These are the redundant ones (cf. Remark 6.2) given by

$$y_n = \alpha_n \otimes e_{nn} = (x_n^* x_n)^{1/2}.$$

□

6.7. *Remark.* In Theorem 6.4, if the generators (and accordingly the relations) are restricted to

$$\{x_e \mid e \in E_1 \cup \dots \cup E_n\},$$

then the universal C^* -algebra is

$$\mathbf{T}(A_1, \dots, A_n).$$

If we add the relations

$$x_{[e]}^2 = x_{[e]} \quad (e \in E_j)$$

for all j , then the presentation becomes one for the AF algebra $A = \overline{\bigcup A_n}$, or if the generators are truncated, for A_n .

6.8. *Remark.* Let B_n be the subalgebra of A_n generated, in the path model for A_n , by matrix units corresponding to pairs of paths that have first edge in $[E_1]$. Notice that B_1 is commutative, and $B_n \subseteq B_{n+1}$. Let $B = \overline{\bigcup B_n}$. If in Theorem 6.4 the generators

$$\{x_e \mid e \in E_1 \setminus [E_1]\}$$

are omitted, the resulting C^* -algebra is

$$\mathbf{T}(B) = \mathbf{T}(B_1, B_2, \dots).$$

Since there are no relations between the dropped generators and the x_f for $f \in E_n$ when $n \geq 2$, we immediately obtain the isomorphism

$$\mathbf{T}(A_1, A_2, \dots) \cong \mathbf{C}A_1 *_{\mathbf{C}B_1} \mathbf{T}(B_1, B_2, \dots).$$

Similarly

$$A \cong A_1 *_{B_1} B,$$

which generalizes the well-known isomorphism

$$\mathbf{M}_n(B) \cong \mathbb{M}_n *_\mathbb{C} B$$

(amalgamating so that $e_{11} \leftrightarrow 1_B$), which holds for all unital \mathbb{C} -algebras, not just for unital AF algebras.

7. PROJECTIVITY OF AF-TELESCOPES

As in the previous section, we consider a fixed unital Bratteli diagram (with root and multiple edges) corresponding to a system of finite-dimensional C^* -algebras

$$\mathbb{C} \subseteq A_1 \subseteq A_2 \subseteq \dots,$$

and we put $A = \overline{\bigcup A_n}$. We denote by $E = \bigcup E_n$ the set of edges in the diagram and define $e \sim f$ if $r(e) = r(f)$ (for e, f in some E_n). Then we choose a set $[E] = \bigcup [E_n]$ of preferred edges, one from each equivalence class, and let $e \mapsto [e]$ denote the selection function. It follows from Theorem 6.4 that each of the finite telescopes $\mathbf{T}(A_1, A_2, \dots, A_{n+1})$ is the universal C^* -algebra generated by elements $x_e, e \in \bigcup_{k=1}^n E_k$, subject to the relations (i)–(vii) in that theorem.

To facilitate the lifting process we shall need the additional elements $h_e, e \in [E_n]$, defined by $h_e = \sum x_f x_f^*$, the summation being over all f in E_{n+1} with $s(f) = r(e)$. Thus $h_e \ll x_e$ for every e in $[E_n]$ by condition (vii). These additional elements come free, as we see from the next lemma.

7.1. Lemma. *With notation as in Lemma 6.1, let*

$$F = \bigoplus_{[S]} \mathbb{M}_{\#[e]}.$$

Let (i)–(vi) denote the relations (i)–(vi) given in Lemma 6.1 in the variable $\{x_e \mid e \in S\}$. With the additional variables $\{h_e \mid e \in [S]\}$ consider the relations

$$(vii) \quad 0 \leq h_e \leq 1,$$

$$(viii) \quad h_e \ll x_e \quad (e \in [S]).$$

- (1) *The universal C^* -algebra generated by the x_e and the $h_{[e]}$ subject to (i)–(viii) is $\mathbf{T}(F, F) \cong \mathbf{CF}$.*
- (2) *Suppose x_e and $h_{[e]}$ (for all $e \in S$) are elements of some quotient B/J of a C^* -algebra B satisfying (i)–(viii). Suppose further that for each $e \in S$ there is a hereditary subalgebra B_e of B with $x_e \in \pi(B_e)$ and that*

$$e \neq f \implies (B_e = B_f \text{ or } B_e \perp B_f).$$

Then there exist y_e and $k_{[e]}$ in B with $\pi(y_e) = x_e$ and $\pi(k_e) = h_e$ that satisfy (i)–(viii) and so that $y_e \in B_e$ for all e .

Proof. Evidently part (1) follows from Remark 6.7. If α and β are the functions on $[0, 2]$ given by $\alpha(t) = t \wedge 1$, $\beta(t) = (t - 1) \vee 0$, then a specific set of generators for $\mathbf{T}(F, F)$ are

$$\begin{aligned} x_e &= \alpha \otimes v_{e,[e]} \quad (e \in S), \\ h_e &= \beta \otimes v_{e,e} \quad (e \in [S]). \end{aligned}$$

Notice that $\alpha \otimes v_{e,e}$ and $\text{id} \otimes v_{e,e}$ generate the same hereditary subalgebra of \mathbf{CF} (namely $\mathbf{C}(Cv_{ee})$), and so (2) is equivalent to Theorem 3.7. \square

7.2. Theorem. *The mapping telescope of every countable inductive limit of finite-dimensional C^* -algebras (A_n) is projective.*

Proof. Assume first that the inductive limit is unital and put $A = \overline{\bigcup A_n}$. Then, with notation as above, consider the set $\{x_e \mid e \in E\}$ of generators for $\mathbf{T}(A)$.

Given a quotient map $\pi : B \rightarrow Q$ between C^* -algebras and a morphism $\varphi : \mathbf{T}(A) \rightarrow Q$, we may assume, working by induction, that for some n we have found elements

$$\left\{ y_e \mid e \in \bigcup_{k=1}^{n-1} E_k \right\} \quad \text{and} \quad \{k_e \mid e \in [E_{n-1}]\}$$

in B , such that $\pi(y_e) = \varphi(x_e)$, $\pi(k_e) = \varphi(h_e)$, and such that the y_e 's satisfy the relations (i)–(vii) in Theorem 6.4, whereas $k_e \ll y_e$ for every e in $[E_{n-1}]$. Note that for $n = 1$ no choices have been made.

For each e in $[E_{n-1}]$ let B_e denote the hereditary C^* -subalgebra of B generated by k_e . Since $\pi(k_e) = h_e$, it follows that $\varphi(x_f x_f^*) \in \pi(B_e)$ for every f in E_n with $s(f) = r(e)$. We can therefore apply Theorem 3.7 to find y_f and k_g ($g \in [E_n]$) such that (i)–(viii) hold, $k_g \ll y_g$ and $y_f y_f^* \in B_e$ if $s(f) = r(e)$. The crucial condition (vii) follows because $y_f y_f^* \in B_e$, and y_e is a unit for B_e , because $k_e \ll y_e$, $e \in [E_{n-1}]$.

Continuing by induction, we obtain a sequence $\{y_e \mid e \in E\}$ that satisfies the conditions (i)–(vii) in Theorem 6.4. (The additional elements $\{k_e \mid e \in [E]\}$ we discard.) By universality this defines a morphism $\psi : \mathbf{T}(A) \rightarrow B$; and since $\pi(y_e) = \varphi(x_e)$ for every e in E , it follows that $\pi \circ \psi = \varphi$, whence $\mathbf{T}(A)$ is projective.

The condition that all embeddings are unital is removed by Propositions 5.4, so we conclude that $\mathbf{T}(A)$ is projective for any sequence (A_n) of finite-dimensional C^* -algebras. \square

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