EIGENFUNCTIONS OF THE LAPLACIAN ON ROTATIONALLY SYMMETRIC MANIFOLDS

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ABSTRACT. Eigenfunctions of the Laplacian on a negatively curved, rotationally symmetric manifold \( M = (\mathbb{R}^n, ds^2), \) \( ds^2 = dr^2 + f(r)^2d\theta^2, \) are constructed explicitly under the assumption that an integral of \( f(r) \) converges. This integral is the same one which gives the existence of nonconstant harmonic functions on \( M. \)

1. Introduction

Let us equip \( \mathbb{R}^n, n \geq 2, \) with a Riemannian metric which can be written in polar coordinates as

\[
ds^2 = dr^2 + f(r)^2d\theta^2,
\]

\[f(0) = 0, \quad f'(0) = 1, \quad \text{and} \quad f(r) > 0, \quad \forall r > 0.
\]

\( \mathbb{R}^n \) with the metric (1.1) becomes a rotationally symmetric manifold \( M. \) \( M \) is called a weak model in [9] and a Ricci model in [1]. The Laplace operator in these coordinates is

\[
\Delta = f(r)^2\Delta_\theta + \partial_r^2 + (n-1)f(r)^{-1}f'(r)\partial_r,
\]

where \( \Delta_\theta \) is the Laplace operator of the unit sphere \( S_{n-1}(0,1) = S. \) In this paper we assume that the radial curvature \( k(r) = -f''(r)f(r)^{-1} \) is negative.

In this paper we give a method of constructing eigenfunctions of the Laplacian \( \Delta \) with eigenvalue \( \lambda > 0, \) provided that

\[
J(f) := \int_1^\infty f(r)^{n-3}dr \int_r^\infty f(\rho)^{1-n}d\rho < +\infty.
\]

The integral \( J(f) \) is the same which gives the existence of nonconstant bounded harmonic functions on \( M. \) In [12], March, using a probabilistic approach, proved the following alternative: If \( J(f) < \infty, \) then there exist nonconstant bounded harmonic functions on \( M, \) while if \( J(f) = \infty, \) there are none such.

The integral \( J(f) \) is closely related to the radial curvature. Indeed, if \( c_2 = 1, \) \( c_n = 1/2, n \geq 3, \) then, [12], under the assumption that \( k(r) \) is nonpositive, it follows that

\[
J(f) < +\infty, \quad \text{if} \quad k(r) \leq \frac{-c}{r^2\log r}, \quad \text{for} \quad c > c_n \quad \text{and large} \ r,
\]
Proof. The radial part of $\Delta$ is given by $g$

Lemma 1. For any $\mu$ measure eigenfunction $g(\theta)$ bounded harmonic functions on $\Omega$ is proved that $J(f) < +\infty$ is a sufficient condition for the existence of nonconstant bounded harmonic functions on $M$.

Under the assumption (1.2), eigenfunctions $W_{\lambda}$, $\lambda > 0$, of $\Delta$ with eigenvalue $\lambda$, are constructed explicitly in Section 3 by means of the heat kernel $K_S$ of the unit sphere $S$ and a subordination measure $\mu(\lambda, r, t)dt$, $t > 0$. More precisely, in Section 2, we show that there exists a positive and nondecreasing $\theta$-independent eigenfunction $g(\lambda, r)$, $r > R$, with eigenvalue $\lambda > 0$, such that $g(\lambda, R) = 1$. The measure $\mu(\lambda, r, t)$ is a solution of the parabolic equation $\partial_t \mu = B_{r,\lambda} \mu$, where

$$B_{r,\lambda} = f(r)^2 \partial_r^2 + \{(n - 1)f(r)f'(r) + 2f(r)^2g'(\lambda, r)g(\lambda, r)^{-1}\} \partial_r.$$ 

The measure $\mu(\lambda, r, t)$ is in fact the law of the first hitting time of the boundaries of $(R, \infty)$ by the diffusion associated with the operator $B_{r,\lambda}$. This gives the splitting $\mu = \mu_1 + \mu_2$, where $\mu_1$ (resp. $\mu_2$) is related to the first hitting time of $\infty$ (resp. $R$).

In Section 3, Theorem 9, we show that if $J(f) < \infty$, then for any $h_1, h_2 \in C_\infty(S)$ and $\lambda > 0$,

$$W_{\lambda}(r, \theta) := g(\lambda, r) \sum_{j=1,2} \int_0^\infty \int_S K_S(t, \theta, \theta') h_j(\theta') \mu_j(\lambda, r, t) d\theta' dt,$$

is an eigenfunction of $\Delta$ with eigenvalue $\lambda$, outside the ball $B(0, R)$, such that

$$\lim_{r \to \infty} g(\lambda, r)^{-1} W_{\lambda}(r, \theta) = h_1(\theta) \quad \text{and} \quad \lim_{r \to R} g(\lambda, r)^{-1} W_{\lambda}(r, \theta) = h_2(\theta).$$

In a second step, using the probabilistic interpretation of harmonic functions, it is proved, Theorem 11, that an eigenfunction on $B(0, R)^c$ which satisfies (1.4) is given by (1.3). This allow us to extend uniquely to the whole of $M$ the eigenfunction $W_{\lambda}(r, \theta)$ given by (1.3), Theorem 12.

Finally, in Section 4, we treat the case of the Euclidean upper-half space $\mathbb{R}^{n+1}_+$. The Poisson kernels which give rise to the eigenfunctions with eigenvalue $\lambda > 0$ are computed explicitly and reduce to the classical one as $\lambda \to 0$.

2. The subordination measures

In this section we construct a pair of measures $\mu_j(\lambda, r, t)dt$, $j = 1, 2$, which enable one to write down explicitly eigenfunctions of the Laplacian with eigenvalue $\lambda > 0$.

We start with the existence of a $\theta$-independent eigenfunction $g(\lambda, r)$.

Lemma 1. For any $\lambda > 0$, there exists a positive and nondecreasing $\theta$-independent eigenfunction $g(\lambda, r)$, $r > 0$, with eigenvalue $\lambda > 0$, such that $g(\lambda, 0) = 1$.

Proof. The radial part of $\Delta$ is given by $\partial_r^2 + (n - 1)f(r)^{-1}f'(r)\partial_r$ and $f(r) \sim r$, for $r$ small enough. So, if

$$B(r) = \int_1^r (n - 1)f(s)f(s)^{-1} ds = (n - 1) \log f(r)f(1)^{-1},$$

then

$$e^{B(r)} = cf(r)^{n-1} \sim r^{n-1}, \quad \text{for } r \text{ small enough}.$$
The integral tests as in [5], p. 408, give that 0 is an entrance boundary and the result follows. 

Let \( W_\lambda(r, \theta) \) be an eigenfunction of \( \Delta \) with eigenvalue \( \lambda > 0 \). If we set

\[
w_\lambda(r, \theta) = g(\lambda, r)^{-1}W_\lambda(r, \theta),
\]

then it is straightforward to check that \( w_\lambda(r, \theta) \) is a harmonic function for the operator \( \Delta_\theta + B_{r,\lambda} \), where

\[
B_{r,\lambda} = f(r)^2 \partial_r^2 + \left\{ (n-1)f(r)f'(r) + 2f(r)^2g'(\lambda, r)g(\lambda, r)^{-1}\right\} \partial_r
\]

\[
= a(r)\partial_r^2 + b(r)\partial_r.
\]

**Lemma 2.** If \( J(f) < \infty \), then for any \( \lambda > 0 \), the origin is an entrance boundary for the operator \( B_{r,\lambda} \), while \( +\infty \) is an exit boundary.

**Proof.** As in Lemma 1 we set

\[
W(r) = \int_1^r b(s)a(s)^{-1}ds = (n-1) \log f(r)f(1)^{-1} + 2 \log g(\lambda, r)g(\lambda, 1)^{-1}.
\]

So,

\[
e^{W(r)} = cf(r)^{n-1}g(\lambda, r)^2 \sim r^{n-1},
\]

for \( r \) small enough, since \( g(\lambda, r) \sim 1 \), as \( r \to 0 \), by Lemma 1. It follows ([7], p.515), that 0 is an entrance boundary.

On the other hand, since the radial curvature is negative, we get that \( f(r) \geq cr \) ([9], p.36). This yields

\[
\int_1^\infty e^{W(r)} \, dr = c \int_1^\infty f(r)^{n-1}g(\lambda, r)^2 \, dr \geq cg(\lambda, 1)^2 \int_1^\infty r^{n-1} \, dr = +\infty,
\]

since \( g(\lambda, r) \) is nondecreasing by Lemma 1. Therefore, \( +\infty \) is not a regular boundary. But

\[
\int_1^\infty e^{-W(r)} \, dr \int_1^r e^{W(s)}a(s)^{-1}ds
\]

\[
= c \int_1^\infty f(r)^{1-n}g(\lambda, r)^{-2} \, dr \int_1^r f(s)^{n-3}g(\lambda, s)^2 \, ds
\]

\[
\leq c \int_1^\infty f(r)^{1-n} \, dr \int_1^r f(s)^{n-3} \, ds := I(f),
\]

since \( g(\lambda, r) \) is nondecreasing. From Fubini’s theorem we get that \( I(f) = J(f) < +\infty \), so \( +\infty \) is an exit boundary. 

From now on we fix a positive \( R \). As in Lemma 2, one can show that \( R \) is a regular boundary for \( B_{r,\lambda} \). Therefore the situation is the same as in [11], Section 3. Following the method developed in [11], one can prove the statements below, which we give without proof.

**Lemma 3.** If \( J(f) < \infty \), then for any \( \lambda > 0 \) and \( k > 0 \), there exists a positive and nondecreasing solution \( \varphi(\lambda, k, r) \) of the equation

\[
(2.1) \quad B_{r,\lambda}\varphi(\lambda, k, r) = k\varphi(\lambda, k, r), \quad r \in (R, \infty),
\]

such that \( \varphi(\lambda, k, R) = 0 \) and \( \varphi(\lambda, k, r) \to 1 \), as \( r \to \infty \).
The function $\varphi(\lambda, k, r)$ appearing in Lemma 3 has the following probabilistic interpretation. Let us denote by $r^\lambda_t$ the diffusion on $(R, \infty)$ associated with the operator $B_{r, \lambda}$, and let $\sigma$ (resp. $\tau$) be the first hitting time of $\infty$ (resp. $R$) by $r^\lambda_t$ starting at $r > R$. The explosion time of $r^\lambda_t$ is then given by $\zeta = \sigma \wedge \tau$. Let us also denote by $P_{\lambda, r}$ the probability attached to the motion $r^\lambda_t$ and by $E_{\lambda, r}$ the corresponding expectation.

**Lemma 4.** If $J(f) < \infty$, then for any $\lambda > 0$,

$$P_{\lambda, r}(\zeta < +\infty) = 1, \ \forall r \in (R, \infty),$$

and

$$(2.2) \quad \varphi(\lambda, k, r) = E^r_{\lambda, r}(e^{-k\sigma}), \ \forall r \in (R, \infty),$$

where $P^r_{\lambda, r}$ is the restriction of $P_{\lambda, r}$ to the set $\Omega_1$ of the paths $r^\lambda_t$ such that $r^\lambda_\sigma = \infty$.

As in [11], Section 3, using (2.2), one can show the following

**Proposition 5.** If $J(f) < \infty$, then for any $\lambda > 0$ and $r \in (R, \infty)$, there exists a positive, bounded and $C^\infty$ function $t \to \mu_1(\lambda, r, t), t > 0$, such that

$$(2.3) \quad \varphi(\lambda, k, r) = \int_0^\infty e^{-kt} \mu_1(\lambda, r, t) \, dt,$$

$$(2.4) \quad \partial_t \mu_1 = B_{r, \lambda} \mu_1,$$

$$(2.5) \quad \mu_1(\lambda, r, t) = o(t^m), \ \forall m > 0, \ \text{as } t \to 0, \ \text{and } \lim_{t \to \infty} \mu_1(\lambda, r, t) = 0,$$

$$(2.6) \quad \int_0^\infty \mu_1(\lambda, r, t) \, dt < 1, \ \text{for any } \lambda > 0, \ \text{and } r \in (R, \infty).$$

**Remark 6.** From (2.3) of Proposition 5 and the boundary values of $\varphi(\lambda, k, r)$, we deduce that

$$(2.7) \quad \mu_1(\lambda, r, t) \longrightarrow \delta_0(t), \ \text{as } r \to \infty,$$

in the sense of distributions, and

$$(2.8) \quad \mu_1(\lambda, r, t) \longrightarrow 0, \ \text{as } r \to R.$$

**Remark 7.** The subordination measure $\mu_1$ admits a companion $\mu_2$ which, like $\mu_1$, is related to the first hitting time $\tau$ of $R$. Indeed, since $\infty$ is an exit boundary and $R$ is a regular one, there exists a positive and nonincreasing solution $\psi(\lambda, k, r)$ of (2.1) such that $\psi(\lambda, k, R) = 1$ and $\psi(\lambda, k, r) \longrightarrow 0$, as $r \to \infty$. As in the case of $\varphi$, the subordination measure $\mu_2$ associated with $\psi$ satisfies the properties stated for $\mu_1$ in Proposition 5. From the boundary values of $\psi$ we get that

$$(2.9) \quad \mu_1(\lambda, r, t) \longrightarrow \delta_0(t), \ \text{as } r \to R,$$

and

$$(2.10) \quad \mu_1(\lambda, r, t) \longrightarrow 0, \ \text{as } r \to \infty.$$

**Remark 8.** For any $\lambda > 0$ and $r \in (R, \infty)$, we set $\mu = \mu_1 + \mu_2$. The measure $\mu$ is the law of the explosion time $\zeta = \sigma \wedge \tau$ of the diffusion $r^\lambda_t$. Indeed, as in Lemma 4, we have
\[E_{\lambda,r}(e^{-k\xi}) = E_{\lambda,r}(e^{-k\xi}\chi_{\Omega_1}) + E_{\lambda,r}(e^{-k\xi}\chi_{\Omega_1})\]
\[= E_{\lambda,r}(e^{-k\xi}\chi_{\Omega_1}) + E_{\lambda,r}(e^{-k\tau}\chi_{\Omega_1}) = \varphi(\lambda,k,r) + \psi(\lambda,k,r)\]
\[= \int_0^{\infty} e^{-kt} \mu(\lambda,r,t) dt.\]

3. A CLASS OF EIGENFUNCTIONS

In this section, using the subordination measures, we construct, in a first step, eigenfunctions of \(\Delta\) outside the ball \(B(0,R)\), with eigenvalue \(\lambda > 0\). In a second step, we show that they can be extended uniquely to the whole of \(M\).

**Theorem 9.** If \(J(f) < \infty\), then for any \(\lambda > 0\) and \(h_1, h_2 \in C^\infty(S)\),

\[W_\lambda(r, \theta) := g(\lambda, r) \sum_{j=1,2} \int_0^\infty \int_S K_S(t, \theta, \theta') h_j(\theta') \mu_j(\lambda, r,t) d\theta' dt,
\]
is an eigenfunction of \(\Delta\) outside the ball \(B(0,R)\), with eigenvalue \(\lambda > 0\), such that

\[\lim_{r \to \infty} g(\lambda, r)^{-1} W_\lambda(r, \theta) = h_1(\theta)\]
and

\[\lim_{r \to R} g(\lambda, r)^{-1} W_\lambda(r, \theta) = h_2(\theta).
\]

Let us set

\[W^1_j(r, \theta) := g(\lambda, r) \int_0^{\infty} \int_S K_S(t, \theta, \theta') h_j(\theta') \mu_j(\lambda, r, t) dt, \quad j = 1,2.
\]

According to the boundary behaviour of the measures \(\mu_1, \mu_2\), given in Remark 6 and Remark 7 of Section 2, for the proof of the Theorem 9, it suffices to show that \(W^1_j(r, \theta)\) are eigenfunctions of \(\Delta\) outside the ball \(B(0,R)\), with eigenvalue \(\lambda > 0\), such that

\[\lim_{r \to \infty} g(\lambda, r)^{-1} W^1_\lambda(r, \theta) = h_1(\theta)\]
and

\[\lim_{r \to R} g(\lambda, r)^{-1} W^1_\lambda(r, \theta) = 0,
\]

and

\[\lim_{r \to \infty} g(\lambda, r)^{-1} W^2_\lambda(r, \theta) = 0\]
and

\[\lim_{r \to R} g(\lambda, r)^{-1} W^2_\lambda(r, \theta) = h_2(\theta).
\]

Below, we treat the case of \(W^1_\lambda(r, \theta)\). The treatment of \(W^2_\lambda(r, \theta)\) is similar.

Let us set \(w^1_\lambda(r, \theta) = g(\lambda, r)^{-1} W^1_\lambda(r, \theta)\). For the proof of Theorem 9, it suffices to show that \(w^1_\lambda\) satisfies the following Dirichlet problem:

\[\left(\Delta_\theta + B_{r,\lambda}\right) w^1_\lambda(r, \theta) = 0, \quad r \in (R,\infty), \quad \theta \in S,
\]

\[\lim_{r \to \infty} w^1_\lambda(r, \theta) = h_1(\theta), \quad \text{and} \quad \lim_{r \to R} w^1_\lambda(r, \theta) = 0.
\]

In general, the proof of (3.6) is probabilistic, [12], [11]. Here, we give an analytic proof since estimates of \(\partial_t K_S\) are available. Indeed, since \(S\) is a complete manifold with strictly positive Ricci curvature, then

\[|\partial_t K_S(t, \theta, \theta')| \leq c V(\theta, \sqrt{t})^{-1/2} (1 + d^2/t)^{\frac{1}{2} + \frac{2}{d}} e^{-d^2/4t},\]

for all \(t > 0, \forall \theta, \theta' \in S\), where \(V(\theta, r)\) is the volume of the ball \(B(\theta, r)\) and \(d = d(\theta, \theta')\) is the distance of \(\theta, \theta'\). (See for instance the survey article [15], Theorem 4.2 combined with Example 1.)
Lemma 10. If \( J(f) < \infty \), then for any \( \lambda > 0 \) and \( h_1 \in C^\infty(S) \):

(i) The integral

\[
w_1^\lambda(r, \theta) := \int_0^\infty \int_S K_S(t, \theta, \theta') h_1(\theta') \mu_1(\lambda, r, t) \, d\theta' \, dt
\]

is absolutely convergent and satisfies

\[
\lim_{r \to \infty} w_1^\lambda(r, \theta) = h_1(\theta), \quad \text{and} \quad \lim_{r \to R} w_1^\lambda(r, \theta) = 0.
\]

(ii) The integral

\[
\int_0^\infty \int_S \partial_i K_S(t, \theta, \theta') h_1(\theta') \mu_1(\lambda, r, t) \, d\theta' \, dt,
\]

is absolutely convergent.

Proof.

(i) For any \( h_1 \in C^\infty(S) \),

\[
\int_0^\infty \int_S K_S(t, \theta, \theta') |h_1(\theta')| \mu_1(\lambda, r, t) \, d\theta' \, dt \leq \|h_1\|_\infty \int_0^\infty \mu_1(\lambda, r, t) \, dt \leq \|h_1\|_\infty,
\]

since \( \int_S K_S(t, \theta, \theta') \, d\theta' = 1 \), by (16), and \( \int_0^\infty \mu_1(\lambda, r, t) \, dt < 1 \), by (2.6).

So, \( t \to \int_S K_S(t, \theta, \theta') h_1(\theta') \, d\theta' \) belongs in \( L^1(\mu_1) \). Thus \( \lim_{r \to \infty} w_1^\lambda(r, \theta) = h_1(\theta) \) by (2.7) since \( C_0^\infty \) is dense in \( L^1(\mu_1) \). From (2.8) we get that \( \lim_{r \to R} w_1^\lambda(r, \theta) = 0 \).

(ii) Suppose for instance that \( d \leq 1 \). Bearing in mind that \( V(\theta, T) > c \), \( \forall \theta \in S \), \( \forall T \geq 1 \), the estimates (3.7), (3.8) and (3.9) of \( \partial_i K_S \) give

\[
\int_0^\infty |\partial_i K_S(t, \theta, \theta')| \mu_1(\lambda, r, t) \, dt
\]

\[
\leq c \int_0^{d^2} t^{-1/2} (1 + d^2/t)^{1+\frac{d}{2}} e^{-\kappa/d} \mu_1(\lambda, r, t) \, dt
\]

\[
+ c \int_{d^2}^t \frac{t^{1-\frac{d}{2}}}{d^2} e^{-\kappa/d} \mu_1(\lambda, r, t) \, dt + c \mu_0(S)^{-1} \int_1^\infty t^{-1} \mu_1(\lambda, r, t) \, dt
\]

\[
\leq c d^{n+1} \int_0^1 t^{-n-2} \mu_1(\lambda, r, t) \, dt
\]

\[
+ c \int_0^1 t^{-\frac{d}{2}-1} \mu_1(\lambda, r, t) \, dt + c \int_1^\infty t^{-1} \mu_1(\lambda, r, t) \, dt \leq c,
\]

since \( t \to \mu_1(\lambda, r, t) \) is infinitely flat near \( t = 0 \) by (2.5), and
\[ \int_{1}^{\infty} t^{-1} \mu_{1}(\lambda, r, t) \, dt \leq \left( \int_{1}^{\infty} t^{-2} \, dt \right)^{\frac{1}{2}} \left( \int_{1}^{\infty} \mu_{1}(\lambda, r, t)^{2} \, dt \right)^{\frac{1}{2}} \]

\[ \leq c \| \mu \|_{\infty}^{\frac{1}{2}} \left( \int_{1}^{\infty} \mu_{1}(\lambda, r, t) \, dt \right)^{\frac{1}{2}} \leq c, \]

by (2.6). This gives that

\[ \int_{0}^{\infty} \int_{S} |\partial_{t} K_{S}(t, \theta, \theta') h_{1}(\theta') \mu_{1}(\lambda, r, t) d\theta' dt \leq c \| h_{1} \|_{1}, \]

if \( d \leq 1 \). The case \( d > 1 \) is similar. \( \square \)

**End of the proof of Theorem 9.** It remains to show that

\((\Delta_{\theta} + B_{r, \lambda}) w_{1}^{\lambda}(r, \theta) = 0, \ r \in (R, \infty), \ \theta \in S.\)

From Lemma 10 we get

\[ \Delta_{\theta} w_{1}^{\lambda}(r, \theta) = \int_{0}^{\infty} \int_{S} \Delta_{\theta} K_{S}(t, \theta, \theta') h_{1}(\theta') \mu_{1}(\lambda, r, t) d\theta' dt \]

\[ = - \int_{0}^{\infty} \int_{S} K_{S}(t, \theta, \theta') h_{1}(\theta') \partial_{t} \mu_{1}(\lambda, r, t) d\theta' dt = -B_{r, \lambda} w_{1}^{\lambda}(r, \theta), \]

since \( \partial_{t} \mu_{1} = B_{r, \lambda} \mu_{1} \), by (2.4). \( \square \)

**Theorem 11.** If \( J(f) < \infty \), then for any \( \lambda > 0 \) the eigenfunction \( W_{\lambda}(r, \theta) \) defined by (3.1) is the unique eigenfunction satisfying (3.2).

**Proof.** The proof of the theorem is probabilistic and is based on the integral representation of bounded harmonic functions.

Let \( U_{\lambda}(r, \theta) \) be an eigenfunction of \( \Delta \) outside the ball \( B(0, R) \), with eigenvalue \( \lambda > 0 \), satisfying (3.2). Then \( u_{\lambda}(r, \theta) = g(\lambda, r)^{-1} U_{\lambda}(r, \theta) \) satisfies the Dirichlet problem (3.6).

Let us denote by \( B_{t} \) the Brownian motion on \( S \). The diffusion on \( B(0, R)^{c} \) associated with the operator \( \Delta_{\theta} + B_{r, \lambda} \) is \( (B_{t}, r^{\lambda}_{t}) \). We choose \( r^{\lambda}_{t} \) independent of \( B_{t} \). Let us also denote by \( P_{t} \) the probability attached to the Brownian motion \( B_{t} \). From the independence of the motions \( B_{t} \), \( r^{\lambda}_{t} \), it follows that the probability \( P_{t}(r, \theta) \) attached to the motion \( (B_{t}, r^{\lambda}_{t}) \) splits to the product \( P_{t} \times P_{t}^{\lambda} \). Finally, from the fact that \( B_{t} \) does not explode, we get that the explosion time of the product motion \( (B_{t}, r^{\lambda}_{t}) \) is in fact the explosion time \( \zeta = \sigma \wedge \tau \) of \( r^{\lambda}_{t} \), which satisfies

\[ (3.10) \quad P_{t}(r, \theta) \{ \zeta < +\infty \} = P_{r}^{\lambda} \{ \zeta < +\infty \} = 1, \]

by Lemma 4 of Section 2.

From (3.10) and [8], Theorem 2.1, p.127, and Remark 2, p.130, it follows that every bounded harmonic function \( u_{\lambda}(r, \theta) \) for the operator \( \Delta_{\theta} + B_{r, \lambda} \) on \( B(0, R)^{c} \) has the following integral representation:

\[ (3.11) \quad u_{\lambda}(r, \theta) = E_{\theta} E_{r}^{\lambda} \{ \Psi(B_{\zeta}, r^{\lambda}_{\zeta}) \}, \]

where \( \Psi \) is the boundary value of \( u_{\lambda}(r, \theta) \). But, the boundary of \( B(0, R)^{c} \) is \( S(0, R) \cup S_{\infty} \), where the "sphere at infinity" \( S_{\infty} \) is isomorphic to the unit sphere \( S \), [4].

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Therefore

\[ \Psi(r, \theta) = \begin{cases} h_1(\theta), & \text{if } r = +\infty, \\ h_2(\theta), & \text{if } r = R. \end{cases} \] (3.12)

From (3.11), (3.12) and the fact that \( \mu(\lambda, r, t) \) is the law of the explosion time \( \zeta \), we obtain that

\[ u_\lambda(r, \theta) = \sum_{j=1,2} \int_0^\infty \int_S K_S(t, \theta, \theta') h_j(\theta') \mu_j(\lambda, r, t) d\theta' dt, \]

and the result follows. \( \square \)

**Theorem 12.** The eigenfunction \( W_\lambda(r, \theta) \) defined by (3.1) extends uniquely to the whole of \( M \).

**Proof.** By (1.1) and Lemma 1, the operator \( \Delta_\theta + B_{r, \lambda} \) has bounded coefficients on the bounded domain \( B(0, R) \). Further, its boundary \( S(0, R) \), is regular. So, by [8], Theorem 3.1, p.142, the Dirichlet problem

\[ \begin{cases} \Delta_\theta + B_{r, \lambda} u = 0, \forall (r, \theta) \in B(0, R), \\ u(r, \theta) \to h_2(\theta), \text{ as } r \to R, \end{cases} \]

admits a unique solution which is given by

\[ u(r, \theta) = E_\theta E^\lambda_r \{ h_2(\lambda_t) \}, \]

where \( T = \inf \{ t > 0 \mid r_t = R \} \) is the first hitting time of \( S(0, R) \). \( \square \)

4. The Euclidean upper half-space

In this section we treat the case of the Euclidean upper half-space \( \mathbb{R}^{n+1}_+ \). The Poisson kernels which give rise to the eigenfunctions with eigenvalue \( \lambda > 0 \) are computed explicitly and reduce to the classical one as \( \lambda \to 0 \).

For any \( \lambda > 0 \), \( g(\lambda, y) = e^{-y\sqrt{\lambda}} \), \( y > 0 \), is an \( x \)-independent eigenfunction of the Laplacian \( \Delta = \Delta_x + \partial^2_y \) with eigenvalue \( \lambda \). So, if \( U_\lambda(x, y) \) is an eigenfunction of \( \Delta \) with eigenvalue \( \lambda \), then \( u_\lambda(x, y) = g(\lambda, y)^{-1} U_\lambda(x, y) \) is a harmonic function for the operator \( \Delta_x + \partial^2_y - 2\sqrt{\lambda} \partial_y \). Therefore, the subordination measure \( \nu(\lambda, y, t) dt \) is a solution of the parabolic equation

\[ \partial_t \nu = \partial^2_y \nu - 2\sqrt{\lambda} \partial_y \nu, \quad y > 0, \quad t > 0, \]

which satisfies \( \nu(\lambda, y, t) \to \delta_0(t) \), as \( y \to 0 \), and \( \nu(\lambda, y, t) \to 0 \), as \( y \to \infty \).

Passing to the Laplace transform variables, we get that \( \nu(\lambda, y, t) \) is the Laplace transform of the function

\[ \varphi(\lambda, y, k) = e^{-y\sqrt{\lambda}} e^{-y\sqrt{\lambda}+k}. \]

But, \( e^{-y\sqrt{\lambda}} \) is the Laplace transform of

\[ (4\pi)^{-1/2} y t^{-3/2} e^{-y^2/4t}, \]

so \( k \to \varphi(\lambda, y, k) \) is the Laplace transform of

\[ \nu(\lambda, y, t) = \frac{ye^{-y\sqrt{\lambda}} e^{-\lambda_t e^{-y^2/4t}}}{2\sqrt{\pi}} t^{3/2}. \]
The Poisson kernels $Q_\lambda(x, y, x')$ which give rise to the eigenfunctions with eigenvalue $\lambda > 0$ are obtained by

$$Q_\lambda(x, y, x') = g(\lambda, y) \int_0^\infty K_{R^n}(t, x, x') \nu(\lambda, y, t) dt$$

$$= \frac{ye^{-2y\sqrt{\lambda}}}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-||x-x'||^2/4t} e^{-\lambda t} e^{-y^2/4t}}{t^{3/2}} dt$$

$$= 2y^{\frac{n+1}{2}} e^{-2y\sqrt{\lambda}} \frac{\Gamma(n+1)}{\pi^{n/2}} \left( \frac{2}{x'} \right)^{\frac{n+1}{2}} \Gamma \left( \frac{n+1}{2} \right) \frac{y}{\left( ||x-x'||^2 + y^2 \right)^{\frac{n+1}{2}}},$$

as $\lambda \to 0$, i.e. the classical Poisson kernel of the Euclidean upper-half space.

**References**


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