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THE STABLE HOMOTOPY TYPES OF STUNTED LENS SPACES MOD 4

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Abstract. Let L_n^{n+k} be the mod 4 stunted lens space L^{n+k}/L^{n-1}. Let $\nu(m)$ denote the exponent of 2 in m, and $\phi(k)$ the number of integers j satisfying $j \equiv 0, 1, 2, 4 \pmod 8$, and $0 < j \le k$. In this paper we complete the classification of the stable homotopy types of mod 4 stunted lens spaces. The main result (Theorem 1.3 (i)) is that, under some appropriate conditions, L_n^{n+k} and L_m^{m+k} are stably equivalent iff $\nu(n-m) \ge \phi(k) + \delta$, where $\delta = -1, 0$ or 1.

1. Introduction

Let $L_n^{n+k} = L^{n+k}/L^{n-1}$ and $P_n^{n+k} = P^{n+k}/P^{n-1}$ be respectively the mod 4 stunted lens space and the stunted real projective space. Let $\nu(m)$ denote the exponent of 2 in m, and $\phi(k)$ the number of integers j satisfying $j \equiv 0, 1, 2, 4 \pmod 8$, and $0 < j \le k$. Two stunted lens spaces L_n^{n+k} and L_m^{m+k} are said to be stably equivalent, denoted by $L_n^{n+k} \sim L_m^{m+k}$, if there exists a homotopy equivalence $\Sigma^N L_n^{n+k} \to \Sigma^{N+n-m} L_m^{m+k}$ for some N.

Classification of the stable homotopy types of stunted real projective spaces was begun by Feder, Gitler, and Mahowald in 1977 ([7]), and finished by Davis and Mahowald in 1986 ([4]). For an odd prime p, the classification of mod p stunted lens spaces was recently finished by Gonzalez ([9]). The classification of mod 2^r stunted lens spaces, in particular the mod 4 stunted lens spaces, has been a favorite subject of some mathematicians for many years ([8], [11], [13], [14], [20]).

Let $\delta(n,k)$ and $\epsilon(n,k)$ be functions defined by the mod 4 values of n and the mod 8 values of k, as given by the following tables:

$n \equiv$ \ $k \equiv$	4	5	6	7
0	3	3	3	3
1	3	3	3	4
2	3	3	4	4
3	3	4	4	4

Function $\epsilon(n,k)$

$n \equiv$ \ $k \equiv$	0	1	2	3	4	5	6	7
0	0	−1	−1	−1	−1	−1	0	0
1	0	0	−1	0	−1	0	0	1
2	0	−1	0	0	−1	−1	1	1
3	0	0	0	0	−1	1	1	1

Function $\delta(n,k)$

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Recall that L_n^{n+k} is S-coreducible if there exists a map $f:\Sigma^N L_n^{n+k}\to S^{N+n}$ for some $N\geq 0$ such that the composite $S^{N+n}\xrightarrow{i}\Sigma^N L_n^{n+k}\xrightarrow{f}S^{N+n}$ is of degree one, where i is the inclusion to the bottom cell. While L_n^{n+k} is S-reducible if there is a map $g:S^{N+n+k}\to\Sigma^N L_n^{n+k}$ for some $N\geq 0$ such that $S^{N+n+k}\xrightarrow{g}\Sigma^N L_n^{n+k}\xrightarrow{p}S^{N+n+k}$ is of degree one, where p is the standard projection. It is known that the S-dual of L_n^{n+k} is ΣL_{-n-k-1}^{-n-1}. Thus it is clear that L_n^{n+k} is S-coreducible iff its S-dual DL_n^{n+k} is S-reducible.

In this paper we complete the classification of the stable homotopy types of mod 4 stunted lens spaces by giving proofs to the following Theorems 1.1-1.3. Here Theorem 1.1 follows from the order of $J(\lambda_k-2)$ in $J(L^k)$, the J-group of L^k. In Theorem 1.3, results for $n\equiv 2\ (\mathrm{mod}\,4)$ and $k\equiv 2,3,6,7\ (\mathrm{mod}\,8)$; $n\equiv 3\ (\mathrm{mod}\,4)$ and $k\equiv 2,5,6\ (\mathrm{mod}\,8)$; can be found in [8, Theorem 1.6] and [13, Theorem 1].

Theorem 1.1. (i) *Let* $0<k<4$. *Then* $L_n^{n+k}\sim L_m^{m+k}$ *iff* $\nu(n-m)\geq 1$ *when* $k=1$, *and* 2 *when* $k=2,3$.

(ii) *Let* $k\geq 4$. *Then* L_n^{n+k} *is* S*-coreducible* (*resp.* S*-reducible*) *iff*

$$\nu(n)\ (\textit{resp. } \nu(n+k+1))\geq\begin{cases}\phi(k)+1 & \textit{when } k\equiv 0,1,4,5,6,7\ (\mathrm{mod}\,8),\\ \phi(k) & \textit{when } k\equiv 2,3\ (\mathrm{mod}\,8).\end{cases}$$

(iii) *Let* $k\geq 4$. *Suppose either* L_n^{n+k} *or* L_m^{m+k} *is* S*-coreducible* (*or* S*-reducible*). *Then* $L_n^{n+k}\sim L_m^{m+k}$ *iff*

$$\nu(n-m)\geq\begin{cases}\phi(k)+1 & \textit{when } k\equiv 0,1,4,5,6,7\ (\mathrm{mod}\,8),\\ \phi(k) & \textit{when } k\equiv 2,3\ (\mathrm{mod}\,8).\end{cases}$$

Theorem 1.2. *Let* $4\leq k<8$. *Suppose neither of* L_n^{n+k}, L_m^{m+k} *is* S*-coreducible nor* S*-reducible. Then* $L_n^{n+k}\sim L_m^{m+k}$ *iff* $\nu(n-m)\geq\epsilon(n,k)$.

Theorem 1.3. *Let* $k\geq 8$. *Suppose neither of* L_n^{n+k}, L_m^{m+k} *is* S*-coreducible nor* S*-reducible.*

(i) *If none of* $\{\nu(n),\nu(m),\nu(n+k+1),\nu(m+k+1)\}$ *is*

$$\begin{cases}4b+2 & \textit{when } k=8b+4,5;\\ 4b & \textit{when } k=8b+1;\end{cases}$$

then $L_n^{n+k}\sim L_m^{m+k}$ *iff* $\nu(n-m)\geq\phi(k)+\delta(n,k)$.

(ii) *If one of* $\{\nu(n),\nu(m),\nu(n+k+1),\nu(m+k+1)\}$ *is* $4b+2$ *when* $k=8b+4,5$, *or is* $4b$ *when* $k=8b+1$, *then* $L_n^{n+k}\sim L_m^{m+k}$ *iff* $\nu(n-m)\geq 4b+3$ *when* $k=8b+4,5$, *or* $\nu(n-m)\geq 4b+1$ *when* $k=8b+1$.

The paper is organized as follows. In section 2, we study the Adams operation Ψ^3 on KO-homology. Theorems 1.1-1.3 are proved in section 3 assuming sections 4-6. Results in section 4 on J-homology and coextension will be used in section 6. In section 6, we study the triviality of the composite

$$\tilde{\beta}_t: S^{2n+1+t}\xrightarrow{a\beta_t}S^{2n+1}\to S^{2n+1}\vee L_{2n-b}^{2n}\to L_{2n-b}^{2n+t+c}, \tag{1.4}$$

where a,c,t are positive integers, and $b\geq 0$ is an appropriate integer such that L_{2n-b}^{2n+1} is S-reducible, while β_t is the generator of ImJ on a t-stem. In section 5, we study the Adams filtration of $\tilde{\beta}_{8b-1}$. As in [4], the determination of whether the element $\tilde{\beta}_t$ is null-homotopic will be crucial. J-homology, Adams spectral sequence (ASS), and Mahowald's table 8.1 in [16], will be used actively.

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2. KO-homology

Lemma 2.1. *Let $f : L_n^{n+k} \to \Sigma^{n-m} L_m^{m+k}$ be a stable equivalence and $\nu(n-m) \geq 3$.*

(i) *Let $\omega \in KO^{-4j}(\Sigma^{n-m} L_m^{m+k})$ be of order 2^e. Suppose $\Psi^3(\omega) = 3^{2j+(n-m)/2}\omega$ and $\Psi^3(f^*(\omega)) = 3^{2j} f^*(\omega)$. Then $\nu(n-m) \geq e-1$.*

(ii) *Let $\omega \in KO_{4j-1}(L_n^{n+k})$ be of order 2^e. Suppose $\Psi^3(\omega) = 3^{-2j}\omega$ and $\Psi^3 f_*(\omega) = 3^{-2j-(n-m)/2} f_*(\omega)$. Then $\nu(n-m) \geq e-1$.*

Proof. Notice that (ii) is dual to (i). For (i), we have $(3^{-(n-m)/2} - 1)\omega = 0$ since $\Psi^3 f^*(\omega) = f^*(\Psi^3(\omega))$. Thus $\nu(n-m) \geq e-1$ by [1, Lemma 8.1]. □

Let L^∞ be the mod 4 infinite lens space with a CW-structure as indicated in [21, p. 91]. It is known that $H^*(L^\infty; \mathbf{Z}_2) \approx \mathbf{Z}_2[u,v]/(u^2)$, where $\deg u = 1$ and $\deg v = 2$, satisfying

$$\mathrm{Sq}^{2i}(v^j) = \binom{j}{i} v^{i+j}, \quad \mathrm{Sq}^{2i}(uv^j) = \binom{j}{i} uv^{i+j}, \quad \mathrm{Sq}^{2i+1}(-) = 0.$$

Let $\rho : P^\infty \to L^\infty$ be the covering map. Then ρ induces a stable map $P_n^{n+k} \to L_n^{n+k}$ denoted also by ρ. The following lemma is immediate.

Lemma 2.2. (i) *The map $\rho : P_m^{m+k} \to L_m^{m+k}$ is of odd degree on even dimensional cells, and even degree on odd dimensional cells.*

(ii) *If $m \leq 2n+1 \leq m+k$, then $\rho_* : H_{2n+1}(P_m^{m+k}; \mathbf{Z}) \to H_{2n+1}(L_m^{m+k}; \mathbf{Z})$ is a monomorphism $\mathbf{Z}_2 \to \mathbf{Z}_4$, and $\rho_* : H_{2n+1}(P_m^{m+k}; \mathbf{Z}_2) \to H_{2n+1}(L_m^{m+k}; \mathbf{Z}_2)$ is trivial, while $\rho_* : H_{2n}(P_m^{m+k}; \mathbf{Z}_2) \to H_{2n}(L_m^{m+k}; \mathbf{Z}_2)$ is an isomorphism.*

(iii) *If $m \leq 2n \leq m+k$, then $\rho^* : H^{2n}(L_m^{m+k}; \mathbf{Z}) \to H^{2n}(P_m^{m+k}; \mathbf{Z})$ is an epimorphism $\mathbf{Z}_4 \to \mathbf{Z}_2$, and $\rho^* : H^{2n}(L_m^{m+k}; \mathbf{Z}_2) \to H^{2n}(P_m^{m+k}; \mathbf{Z}_2)$ is an isomorphism, while $\rho^* : H^{2n+1}(L_m^{m+k}; \mathbf{Z}_2) \to H^{2n+1}(P_m^{m+k}; \mathbf{Z}_2)$ is trivial.*

Lemma 2.3 ([20, 2.5])**.** *Let $\{E_2^{p,q}, d_r\}$ be the Atiyah-Hirzebruch spectral sequence (AHSS) for $KO^*(L_m^n)$ with $E_2^{p,q} = H^p(L_m^n; KO^q(pt))$. Then differentials $d_2^{i,8k} : E_2^{i,8k} \to E_2^{i+2,8k-1}$, $d_2^{i,8k-1} : E_2^{i,8k-1} \to E_2^{i+2,8k-2}$ and $d_3^{i,8k-2} : E_3^{i,8k-2} \to E_3^{i+3,8k-4}$ are given by $\mathrm{Sq}^2 \rho_2$, Sq^2 and $\beta_2 \mathrm{Sq}^2$ respectively, where ρ_2 is the mod 2 reduction, and β_2 is the Bockstein operation associated with the exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0$.*

Let $\bar\eta \in \widetilde{KO}(-)$ (or $\widetilde{K}(-)$) denote the reduction of a vector bundle of η. Let λ_k be the realification of the complex Hopf bundle ξ_k over L^k. The next lemma is from [14, Prop. 3.3 (2)] and [8, Theorem 2.5].

Lemma 2.4. (i) *The order of $\bar\xi_k$ in $\widetilde{K}(L^k)$ is $2^{[k/2]+1}$.*

(ii)*If $k \geq 4$, then the order of $(\bar\lambda_k)^i$ in $\widetilde{KO}(L^k)$ is*

$$\begin{cases} 2^{2+[k/2]-2i+1} & \text{if } k \equiv 1 \pmod 4 \text{ or } k \equiv 4 \pmod 8, \\ 2^{2+[k/2]-2i} & \text{otherwise.} \end{cases}$$

For a given k, let $F^n(X)$ be the subgroup of $KO_k(X)$ (or $KO^k(X)$) of elements of CW-filtrations $\geq n$ in AHSS.

Lemma 2.5. $\widetilde{KO}(L^2) \approx \widetilde{KO}(L^3) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Proof. By AHSS, we see that $\widetilde{KO}(L^2) \approx \widetilde{KO}(L^3)$, and $\widetilde{KO}(L^2)$ is of order 4. Thus it suffices to show $2x = 0$ for $x \in \widetilde{KO}(L^2)$. The left square

$$\begin{array}{ccccc} S^1 & \xrightarrow{4} & S^1 & \rightarrow & L^2 \\ \Big\downarrow 2 & & \Big\downarrow 1 & & \\ S^1 & \xrightarrow{2} & S^1 & \rightarrow & P^2 \end{array}$$

commutes. Thus there exists a stable map $f : L^2 \rightarrow P^2$ whose degrees on the top and bottom cells are respectively 2 and 1. It is known that $\widetilde{KO}(P^2) \approx \mathbf{Z}_4$. If $\widetilde{KO}(L^2) \approx \mathbf{Z}_4$, then any element of $\widetilde{KO}(L^2)$ from $F^1(L^2)$ would be a generator, and $f^* : \widetilde{KO}(P^2) \rightarrow \widetilde{KO}(L^2)$ would be an isomorphism because f is of degree 1 on the bottom cell. But it is not since f is of degree 2 on the top cell. $\square$

We assume in the following that L^n_m is the base point if $m > n$. Suppose p is either the composite $L^k \rightarrow L^k_{8b} = S^{8b} \vee L^k_{8b+1} \rightarrow S^{8b}$ when $8b \leq k \leq 8b+3$, or $L^k \rightarrow L^k_{8b+4} = S^{8b+4} \vee L^k_{8b+5} \rightarrow S^{8b+4}$ when $8b+4 \leq k \leq 8b+7$.

Theorem 2.6. *Let g_1, g_2 generate $\widetilde{KO}(S^{8b})$, $\widetilde{KO}(S^{8b+4})$ respectively.*

(i) *If $k = 8b$, $8b+1$, then there exists an odd integer a_1 and a stable vector bundle η_1 over S^{8b} such that $\eta_1 = a_1 g_1$, and $2^{4b-1}\bar{\lambda}_k = p^*(2\eta_1)$. Moreover, $2^{4b-1}\bar{\lambda}_k \in \widetilde{KO}(L^k)$ is of order 2 (resp. 4) if $k = 8b$ (resp. $8b+1$).*

(ii) *If $k = 8b+2$, $8b+3$, then there exists an odd integer a_2 and a stable vector bundle η_2 over S^{8b} such that $\eta_2 = a_2 g_1$ in $\widetilde{KO}(S^{8b})$, and $2^{4b}\bar{\lambda}_k = p^*(4\eta_2)$. Moreover, $2^{4b}\bar{\lambda}_k \in \widetilde{KO}(L^k)$ is of order 2.*

(iii) *If $8b+4 \leq k \leq 8b+7$, then there exists an odd integer a_3 and a stable vector bundle η_3 over S^{8b+4} such that $\eta_3 = a_3 g_2$, and $2^{4b+1}\bar{\lambda}_k = p^*(\eta_3)$. Moreover, $2^{4b+1}\bar{\lambda}_k \in \widetilde{KO}(L^k)$ is of order 4.*

Proof. The orders of the claimed elements follows from Lemma 2.4.

Let $\{E^{p,q}_r(i), d^{(i)}_r\}$, $i = 1, 2$, be respectively the AHSSs for $K^*(L^k)$ and $KO^*(L^k)$. By [20, p. 240, (1)], elements of $E^{p,q}_2(i)$ survive to E_∞ if $p + q \equiv 0 \pmod 4$.

Consider (i). By Lemma 2.4, $2^{4b-1}\lambda_k$ is stably trivial over L^{8b-1}. Thus there is a stable vector bundle ω_1 over L^k_{8b} such that $2^{4b-1}\bar{\lambda}_k = p_1^*(\omega_1)$, where $p_1 : L^k \rightarrow L^k_{8b}$ is the standard projection. In this case $L^k_{8b} = S^{8b} \vee L^k_{8b+1}$ and $\widetilde{KO}(L^k_{8b+1})$ is either trivial or $\mathbf{Z}_2$. Since $2^{4b}\bar{\lambda}_k$ is trivial over L^{8b} and is of order 2 over L^{8b+1}, we see that $\widetilde{KO}(S^{8b+1}) \longrightarrow \widetilde{KO}(L^{8b+1})$ maps the generator to $2^{4b}\bar{\lambda}_k$. Thus we can ignore the part L^k_{8b+1} and let ω_2 be the restriction of ω_1 on S^{8b}. Then ω_2 is not divisible by 4, otherwise $p^*(\omega_2)$ is trivial over L^{8b} and $p^*(\omega_1) \in \widetilde{KO}(L^{8b+1})$ is of order ≤ 2. This is contrary to Lemma 2.4. Next we claim that $\omega_2 \in \widetilde{KO}(S^{8b})$ is divisible by 2, which will imply (i). Note that $2^{4b-1}\bar{\lambda}_k \in F^{8b}$. If ω_2 is not divisible by 2, then $2^{4b-1}\bar{\lambda}_k$ generates $F^{8b}/F^{8b+1} \approx \mathbf{Z}_4$. This means $2^{4b}\bar{\lambda}_k$ is of order 2 in F^{8b}/F^{8b+1}. However, we will show that $2^{4b}\bar{\lambda}_k$ is trivial in F^{8b}/F^{8b+1}.

Recall that $2^{4b}\lambda_k$ is the realification of $2^{4b}\xi_k$, where ξ_k is the complex Hopf bundle over L^k, and $2^{4b}\xi_k$ is stably trivial over L^{8b-1} by Lemma 2.4. Since $2^{4b+1}\xi_{8b+1}$ is stably trivial over L^{8b+1}, $2^{4b}\bar{\xi}_k$ corresponds to an element of $E^{8b,-8b}_2(1)$ of order 2.

By [19, p. 304] the realification $\pi_{8j}(K)(\approx \mathbf{Z})\to\pi_{8j}(KO)(\approx \mathbf{Z})$ is a multiplication by 2, so is the morphism $E_2^{8b,-8b}(1)(\approx \mathbf{Z}_4)\longrightarrow E_2^{8b,-8b}(2)(\approx \mathbf{Z}_4)$. Therefore $2^{4b}\bar{\lambda}_k$ is trivial in F^{8b}/F^{8b+1}.

Consider (ii). As in (i), we have an element $\omega_3 \in \widetilde{KO}(L_{8b}^k = S^{8b}\vee L_{8b+1}^k)$ such that $p^*(\omega_3) = 2^{4b-1}\bar{\lambda}_k$ and the restriction of ω_3 on S^{8b} is $2a_2g_1 \in \widetilde{KO}(S^{8b})$ for some odd integer a_2. By Lemma 2.5, $2\omega_3$ is trivial over L_{8b+1}^k, so we can choose the desired η_2 and a_2.

For (iii), notice that by Lemma 2.4, the bundle $2^{4b+1}\lambda_k$ is stably trivial over L^{8b+3}, and is of order 4 in $\widetilde{KO}(L^{8b+4})$. It follows by a similar argument and the fact that $L_{8b+4}^k = S^{8b+4}\vee L_{8b+5}^k$. □

Theorem 2.7. *Suppose* $n \not\equiv 0 \pmod 4$, *and* $m \geq n+4$.

(i) *The Adams operation* Ψ^3 *satisfies* $\Psi^3(x) = 3^{2j}x$ *for* $x \in KO^{-4j}(L_n^m)$.

(ii) $KO^{-4j}(L_n^m) \approx \mathbf{Z}/2^{a(m+4j,n+4j-1)} \oplus \mathbf{Z}/2^{b(m+4j,n+4j-1)}$
where $h(u) = [u/4] + [(u+4)/8] + [(u+7)/8]$, *while* $a(u,v)$ *and* $b(u,v)$ *are defined by*

$$a(u,v) = h(u) - [(v+1)/4] - [(v+6)/8] - [(v+1)/8],$$
$$b(u,v) = [u/8] + [(u+6)/8] - [(v+7)/8] - [(v+5)/8],$$

satisfying $a(u,v) \geq b(u,v)$.

(iii) *There is an element in* $KO^{4k}(L_{4k-1}^{4k+t})$ *of order* $2^{c(t)}$ (*the maximum*), *where*

$$c(t) = \begin{cases} 4l+2 & \text{if } t = 8l; \\ 4l+3 & \text{if } t = 8l+1, 8l+2, 8l+3; \\ 4l+5 & \text{if } t = 8l+4, 8l+5, 8l+6, 8l+7. \end{cases}$$

Proof. Consider (i). Let $\{E_r^{p,q}, d_r\}$ be the AHSS for $KO^*(L_m^{m+k})$. Since each element of $E_2^{p,q}$ survives to E_∞ when $p+q \equiv 0 \pmod 4$, the morphism

$$KO^{-4j}(L_n^m)\to KO^{-4j}(L^m)$$

is injective when $n \not\equiv 0 \pmod 4$ and $m > n$. Thus (i) follows from [14, Lemma 4.2].

Part (ii) is from [14, Theorem 2 (1)]. Here note that $a(m+8,n+8) = a(m,n)$, $b(m+8,n+8) = b(m,n)$, and $b(m+4j,n+4j-1) \geq 0$ when $m \geq n+4$. The formula is even true for integer $j < 0$.

Consider (iii). First we have

$$h(u) = \begin{cases} u/2 & \text{if } u \equiv 0,2,6 \pmod 8, \\ (u+1)/2 & \text{if } u \equiv 1,5 \pmod 8, \\ (u-1)/2 & \text{if } u \equiv 3,7 \pmod 8, \\ u/2+1 & \text{if } u \equiv 4 \pmod 8. \end{cases}$$

$$= \begin{cases} \phi(u) & \text{if } u \not\equiv 2,3 \pmod 8, \\ \phi(u)-1 & \text{otherwise.} \end{cases}$$

By (ii), there is an element of order $2^{a(t,-2)}$ in $KO^{4k}(L_{4k-1}^{4k+t})$. Let $c(t) = a(t,-2)$. Then

$$c(t) = h(t)+2 = \begin{cases} \phi(t)+2 & \text{if } t \not\equiv 2,3 \pmod 8, \\ \phi(t)+1 & \text{otherwise.} \end{cases}$$

□

Lemma 2.8. *Let $4j+1 \geq 4n+2k$.*

(i) *The Adams operation on $KO_{4j+1}(L_{4n+1}^{4n+2k})$ satisfies $(\Psi^3-1)(g)=4g$. If $2k-1<8$, then $4g=0$ for $g \in KO_{4j+1}(L_{4n+1}^{4n+2k})$. If $2k-1 \geq 8$, then*

$$KO_{4j+1}(L_{4n+1}^{4n+2k}) \approx \mathbf{Z}/2^{[(2k+3)/4]} \oplus A \oplus B,$$

where $A \approx \mathbf{Z}_2$ if $j-n$ even, and 0 otherwise; and

$$B = \begin{cases} \mathbf{Z}_2 & \text{if } 4(j-n)-2k \equiv 0,2 \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *Suppose $k \geq 5$. Then each element $x \in KO_{4j+1}(L_{4n+1}^{4n+2k})$ is divisible by 4 if x is in F^{4n+1} and is in the image of the projection*

$$p_* : KO_{4j+1}(L_{-\infty}^{4n+2k}) \to KO_{4j+1}(L_{4n+1}^{4n+2k}).$$

Proof. Part (i) is from [14, Theorem 2 (2) and (5.10) (1)], where the summand A is from the bottom cell, and B is from the cells near the top. For (ii), consider the composite $L_{-\infty}^{4n+2k} \longrightarrow L_{4n'+1}^{4n+2k} \longrightarrow L_{4n+1}^{4n+2k}$, $n' < n$. With an appropriate n' and using (i), we can assume that the image of

$$KO_{4j+1}(L_{-\infty}^{4n+2k}) \longrightarrow KO_{4j+1}(L_{4n+1}^{4n+2k})$$

is in $\mathbf{Z}/2^{[(2k+3)/4]} \oplus B$. Since $k \geq 5$, $[(2k+3)/4] \geq 3$. Then (ii) follows from the fact that x is of CW-filtration $4n+1$. □

3. Proofs of theorems

It is known that if $J((A-B)\bar{\lambda}_k)=0$ over L^k, then L_{2A}^{2A+k} and L_{2B}^{2B+k} are Thom spaces of J-equivalent vector bundles, therefore a stable equivalence exists. We call a stable equivalence obtained in this way an equivalence (5), to distinguish from those four equivalences in Theorem 3.2.

Proof of Theorem 1.1. When $k>0$, L_n^{n+k} is not S-coreducible if n is odd, and L_n^{n+k} is not S-reducible if $n+k$ is even. So (i) is immediate when $k=1$. By Steenrod operations, and noting that by Lemma 2.5 the order of λ_k is 2 when $k=2,3$, we see that $L_n^{n+k} \sim L_m^{m+k}$ iff $\nu(n-m) \geq 2$ when $k=2,3$.

Consider (ii) and (iii). Assume that L_n^{n+k} is S-coreducible (if S-reducible use S-duality). By [3, Proposition 2.8], L_n^{n+k} is S-coreducible iff $J(\frac{n}{2}\bar{\lambda}_k)=0$, where J is the standard homomorphism $\widetilde{KO}(-) \to J(-)$. By [8, Theorem 2.1], the order of $J(\bar{\lambda}_k)$ is

$$\begin{cases} 4l & \text{if } k=8l; \\ 4l+1 & \text{if } k=8l+1, 8l+2, 8l+3; \\ 4l+3 & \text{if } k=8l+4, 8l+5, 8l+6, 8l+7. \end{cases}$$

Hence we have (ii). For (iii), if $L_n^{n+k} \sim L_m^{m+k}$, then L_n^{n+k} is S-coreducible iff L_m^{m+k} is S-coreducible, so the inequalities follow from (ii). Conversely, if the inequalities in (iii) are satisfied, then an equivalence (5) exists. □

By Adams operation Ψ^5 on $K^*(-)$, we have

Lemma 3.1 ([11, Theorem 1.1 and p. 292])**.** *If $L_n^{n+k} \sim L_m^{m+k}$, then*

$$\nu(n-m) \geq [(n+k)/2] - [n/2].$$

Proof of the necessity in Theorems 1.2 and 1.3. The necessity in Theorem 1.3 (ii) follows from Lemma 5.11 (ii) (or [13, Theorem 2 (2), p. 699]). By Steenrod operations, we have $\nu(n-m) \geq 3$ when $4 \leq k \leq 7$. The proof of the necessity in Theorem 1.3 (i) follows from the following observations.

Suppose $n \equiv 0 \pmod 4$. By Lemma 3.1, $\nu(n-m) \geq 4l+3$ when $k-8l = 7,6$; $\nu(n-m) \geq 4l+2$ when $k-8l = 5,4$; $\nu(n-m) \geq 4l+1$ when $k-8l = 3,2$; $\nu(n-m) \geq 4l$ when $k-8l = 1,0$.

Suppose $n \equiv 1 \pmod 4$. By Lemma 3.1, $\nu(n-m) \geq 4l+4$ when $k-8l = 7$; $\nu(n-m) \geq 4l+3$ when $k-8l = 6,5$; $\nu(n-m) \geq 4l+2$ when $k-8l = 4,3$; $\nu(n-m) \geq 4l+1$ when $k-8l = 2,1$; $\nu(n-m) \geq 4l$ when $k = 8l$.

Suppose $n \equiv 2 \pmod 4$. By Lemma 2.1, Theorem 2.7 (i), (iii), and using S-duality, we have $\nu(n-m) \geq 4l+4$ when $k-8l = 6$, thus when $k-8l = 7$; $\nu(n-m) \geq 4l+2$ when $k-8l = 2$, thus when $k-8l = 5,4,3$. By Lemma 3.1, $\nu(n-m) \geq 4l$ when $k-8l = 1,0$.

Suppose $n \equiv 3 \pmod 4$. By Lemma 2.1, Theorem 2.7 (i), (iii), we have $\nu(n-m) \geq 4l+4$ when $k-8l = 5$, thus when $k-8l = 7,6$; $\nu(n-m) \geq 4l+2$ when $k-8l = 2$, thus when $k-8l = 3,4$. By Lemma 3.1, $\nu(n-m) \geq 4l+1$, $4l$ when $k-8l = 1$, 0. □

Theorem 3.2. *Suppose $b \geq 0$ in (1), and $b \geq 1$ in (2)-(4). Assume none of the stunted lens spaces in question is S-coreducible or S-reducible.*

(1) $L_{4n}^{4n+8b+7} \sim L_{4m}^{4m+8b+7}$ *if* $\nu(4n-4m) \geq 4b+3$;

(2) $L_{4n}^{4n+8b+5} \sim L_{4m}^{4m+8b+5}$ *if* $\nu(4n-4m) \geq 4b+2$ (*or* $4b+3$ *when Theorem 1.3* (ii) *is satisfied*)*;*

(3) $L_{4n}^{4n+8b+3} \sim L_{4m}^{4m+8b+3}$ *if* $\nu(4n-4m) \geq 4b+1$;

(4) $L_{4n}^{4n+8b+1} \sim L_{4m}^{4m+8b+1}$ *if* $\nu(4n-4m) \geq 4b$ (*or* $4b+1$ *when Theorem* 1.3 (ii) *is satisfied*).

We call a stable equivalence given by Theorem 3.2 (j), an equivalence (j). Assuming the above theorem, we can complete the sufficiency in Theorems 1.2, 1.3.

Proof of the sufficiency in Theorem 1.2. (a) $n \equiv 0 \pmod 4$. If $k = 7$, use an equivalence (1). By removing from both sides of an equivalence $L_{4A}^{4A+7} \sim L_{4B}^{4B+7}$, the top cell, the top two cells, top three cells respectively, we have the desired equivalences for $k = 6,5,4$.

(b) $n \equiv 1 \pmod 4$. If $k = 7$, use an equivalence (4) to get an equivalence $L_{4A}^{4A+9} \sim L_{4B}^{4B+9}$, then remove the top cell and the bottom cell from both sides of that equivalence. If $k = 6$, remove the bottom cell from both sides of an equivalence $L_{4A}^{4A+7} \sim L_{4B}^{4B+7}$ given by an equivalence (1); if $k = 5,4$, remove respectively the top cell, the top two cells from both sides of an equivalence $L_{4A+1}^{4A+7} \sim L_{4B+1}^{4B+7}$.

(c) $n \equiv 2 \pmod 4$. If $k = 7,6$, use an equivalence (5); if $k = 5$, remove the bottom two cells from both sides of an equivalence $L_{4A}^{4A+7} \sim L_{4B}^{4B+7}$; if $k = 4$, remove the top cell from both sides of an equivalence $L_{4A+2}^{4A+7} \sim L_{4B+2}^{4B+7}$.

(d) $n \equiv 3 \pmod 4$. If $k = 7$, use an equivalence (4) and S-duality to get an equivalence $L_{4A+2}^{4A+11} \sim L_{4B+2}^{4B+11}$, then remove the bottom cell and the top cell; if $k = 6,5$, remove respectively the top cell, the top two cells from both sides of an equivalence $L_{4A+3}^{4A+10} \sim L_{4B+3}^{4B+10}$; if $k = 4$, remove the bottom three cells from both sides of an equivalence $L_{4A}^{4A+7} \sim L_{4B}^{4B+7}$ given by an equivalence (1). □

Proof of the sufficiency in Theorem 1.3. (a) $n \equiv 0 \pmod 4$. If $k \equiv 7,6 \pmod 8$, use equivalences (1); if $k \equiv 5,4 \pmod 8$, use equivalences (2); if $k \equiv 3,2 \pmod 8$, use equivalences (3); if $k \equiv 1,0 \pmod 8$, use equivalences (4).

(b) $n \equiv 1 \pmod 4$. If $k \equiv 7 \pmod 8$, use an equivalence (4), then remove the bottom cell from both sides of that equivalence; if $k \equiv 6 \pmod 8$, use an equivalence (1) and S-duality; if $k \equiv 5 \pmod 8$, remove the top cell from both sides of an equivalence given by the case $k \equiv 6 \pmod 8$; if $k \equiv 4 \pmod 8$, use an equivalence (2), then remove the bottom cell from both sides of that equivalence; if $k \equiv 3 \pmod 8$, remove the top cell from both sides of an equivalence given by the case $k \equiv 4 \pmod 8$; if $k \equiv 2 \pmod 8$, use an equivalence (3) and S-duality; if $k \equiv 1 \pmod 8$, remove the top cell from both sides of an equivalence given by the case $k \equiv 2 \pmod 8$; if $k \equiv 0 \pmod 8$, use an equivalence (4) and remove the bottom cell from both sides of that equivalence.

(c) $n \equiv 2 \pmod 4$. If $k \equiv 7, 6 \pmod 8$, use equivalences (5); if $k \equiv 5, 4 \pmod 8$, use equivalences (2) and S-duality; if $k \equiv 3, 2 \pmod 8$, use equivalences (5); if $k \equiv 1, 0 \pmod 8$, use equivalences (4) and S-duality.

(d) $n \equiv 3 \pmod 4$. If $k \equiv 7 \pmod 8$, use an equivalence (4) and S-duality, then remove the bottom cell from both sides of that equivalence; if $k \equiv 6, 5 \pmod 8$, remove respectively the top cell, the top two cells from both sides of an equivalence given by the case $k \equiv 7 \pmod 8$; if $k \equiv 4 \pmod 8$, use an equivalence (2) and S-duality; if $k \equiv 3 \pmod 8$, remove the top cell from both sides of an equivalence given by the case $k \equiv 4 \pmod 8$; if $k \equiv 2 \pmod 8$, use an equivalence (5); if $k \equiv 1 \pmod 8$, use an equivalence (3) and S-duality, then remove the bottom cell from both sides of that equivalence; if $k \equiv 0 \pmod 8$, use an equivalence (4) and S-duality. $\square$

Proof of Theorem 3.2. Maps will be stable here.

(1) $L_{4n}^{4n+8b+7} \sim L_{4m}^{4m+8b+7}$ when $\nu(4n-4m) \geq 4b+3$.

Let $\lambda = \lambda_{8b+7}$. Note that $L_{4n}^{4n+8b+7} = T(2n\lambda)$, $L_{4n-4m}^{4n-4m+8b+7} = T(2(n-m)\lambda)$ and $L_{4m}^{4m+8b+7} = T(2m\lambda)$. Since $\Delta^*(2(n-m)\lambda \times 2m\lambda) = 2n\lambda$, where $\Delta : L^{8b+7} \to L^{8b+7} \times L^{8b+7}$ is the diagonal map, we have a map

$$T(2n\lambda) \to T(2(n-m)\lambda \times 2m\lambda) = T(2(n-m)\lambda) \wedge T(2m\lambda),$$

namely a map $f_1 : L_{4n}^{4n+8b+7} \to L_{4n-4m}^{4n-4m+8b+7} \wedge L_{4m}^{4m+8b+7}$. We may assume $4(n-m) \geq 0$. Since $\nu(4n-4m) \geq 4b+3$, by Theorem 2.6 (iii), there is a stable vector bundle η over S^{8b+4} such that $p^*(2\eta) = 2(n-m)\lambda$, where p is the projection $L^{8b+7} \to L_{8b+4}^{8b+7} = S^{8b+4} \vee L_{8b+5}^{8b+7} \to S^{8b+4}$. Thus we have a map

$$f_2 : L_{4n-4m}^{4n-4m+8b+7} = T(2(n-m)\lambda) \to T(2\eta) = S^{4n-4m} \cup_{2\beta} e^{4n-4m+8b+4},$$

where β is the image of $\bar{\eta}$ under the J-homomorphism $\pi_{8b+4}(BO) \to \pi_{8b+3}^s$, thus $\beta = a\beta_{8b+3}$ for some integer a. Let f_3 be the composite

$$L_{4n}^{4n+8b+7} \xrightarrow{f_1} L_{4n-4m}^{4n-4m+8b+7} \wedge L_{4m}^{4m+8b+7} \xrightarrow{f_2 \wedge 1} (S^{4n-4m} \cup_{2\beta} e^{4n-4m+8b+4}) \wedge L_{4m}^{4m+8b+7}.$$

Taking a CW-approximation of f_3, we get a map

$$f_4 : L_{4n}^{4n+8b+7} \to \Sigma^{4(n-m)}(L_{4m}^{4m+8b+7} \cup_{2\beta \vee 2\beta \vee 2\beta} (e^{4m+8b+4} \vee CM \vee e^{4m+8b+7}))$$

where CM is the cone on the mod 4 Moore space $M = S^{4m+8b+4} \cup_4 e^{4m+8b+5}$. Since $L_{4m}^{4m+8b+7}$ is not S-reducible, by Theorem 1.1, we have $\nu(4m+8b+8) \leq 4b+3$. Thus by Lemma 6.2, the top part of the 3-part wedge splits off. By Lemma 5.7, the next to top part CM also splits off. Consequently we have a projection

$$r : \Sigma^{4(n-m)}(L_{4m}^{4m+8b+7} \cup_{2\beta \vee 2\beta \vee 2\beta} (e^{4m+8b+4} \vee CM \vee e^{4m+8b+7}))$$
$$\to \Sigma^{4(n-m)}(L_{4m}^{4m+8b+7} \cup_{2\beta} e^{4m+8b+4}),$$

which is the identity on the part $L_{4m}^{4m+8b+7}$. Let

$$f_5 = rf_4 : L_{4n}^{4n+8b+7} \longrightarrow \Sigma^{4(n-m)}(L_{4m}^{4m+8b+7} \cup_{2\beta} e^{4m+8b+4}).$$

Taking the S-dual of f_5, we have a map

$$g : X = (S^{-4m-8b-5} \vee L_{-4m-8b-8}^{-4m-2}) \cup_{(2\beta\vee\alpha)} e^{-4m-1} \rightarrow \Sigma^{4(n-m)} L_{-4n-8b-8}^{-4n-1}.$$

Since $L_{4n}^{4n+8b+7}$ is not S-coreducible, $\nu(4n) \leq 4b+3$. By Lemma 6.2, the composite

$$S^{-4m-2} \xrightarrow{2\beta} S^{-4m-8b-5} \xrightarrow{g} \Sigma^{4(n-m)} L_{-4n-8b-8}^{-4n-1}$$

is null, so $g : L_{-4m-8b-8}^{-4m-2} \rightarrow \Sigma^{4(n-m)} L_{-4n-8b-8}^{-4n-1}$ extends to

$$L_{-4m-8b-8}^{-4m-1} \rightarrow \Sigma^{4(n-m)} L_{-4n-8b-8}^{-4n-1},$$

which is an equivalence below the top cell. We claim it is an equivalence.

If $\nu(4n) = \nu(4m)$, then it follows from Lemma 5.11 (ii). Since $\nu(4n-4m) \geq 4b+3$, the only possible case that $\nu(4n) \neq \nu(4m)$ was when

$$\nu(4n) > \nu(4m) \geq 4b+3 \quad \text{or} \quad \nu(4m) > \nu(4n) \geq 4b+3,$$

which would imply one of the original stunted lens spaces is S-coreducible.

(2) $L_{4n}^{4n+8b+5} \sim L_{4m}^{4m+8b+5}$ when $\nu(4n-4m) \geq 4b+2$.

If both $\nu(4n)$ and $\nu(4m) \geq 4b+3$, then an equivalence follows from case (1). Suppose $\nu(4n) \leq 4b+2$. By the diagonal map and then a CW-approximation, we have a map

$$f_1 : L_{4n}^{4n+8b+5} \rightarrow \Sigma^{4n-4m}(L_{4m}^{4m+8b+5} \cup_{\beta\vee\beta} e^{4m+8b+4} \vee e^{4m+8b+5})$$

where $\beta = a\beta_{8b+3}$ for some integer a. By Lemma 6.1, the top part of the 2-part wedge splits off, and we have a map $f_2 : L_{4n}^{4n+8b+5} \rightarrow \Sigma^{4n-4m}(L_{4m}^{4m+8b+5} \cup_\beta e^{4m+8b+4})$. Let

$$g : (S^{-4m-8b-5} \vee L_{-4m-8b-6}^{-4m-2}) \cup_{\beta\vee\alpha} e^{-4m-1} \rightarrow \Sigma^{4(n-m)} L_{-4n-8b-6}^{-4n-1}$$

be the dual of f_2. By Lemmas 6.4 (iii), 6.5 (ii), and with an argument as in the previous case, we see that $g : L_{-4m-8b-6}^{-4m-2} \rightarrow \Sigma^{4(n-m)} L_{-4n-8b-6}^{-4n-1}$ extends to a map $g_5 : L_{-4m-8b-6}^{-4m-1} \rightarrow \Sigma^{4(n-m)} L_{-4n-8b-6}^{-4n-1}$ which is an equivalence below the top. If $\nu(4n) = \nu(4m)$, then an equivalence follows from Lemma 5.11 (ii). Since $\nu(4n-4m) \geq 4b+2$, the only possible case that $\nu(4n) \neq \nu(4m)$ is when

$$\nu(4n) > \nu(4m) \geq 4b+2 \quad \text{or} \quad \nu(4m) > \nu(4n) \geq 4b+2. \tag{3.3}$$

If $\nu(4n)$ and $\nu(4m)$ both $> 4b+2$, then the one of the original stunted lens spaces was S-coreducible.

If $\nu(4n)$ (or $\nu(4m)$) is $4b+2$, then the condition of (2) requires $\nu(4n-4m) \geq 4b+3$. So (3.3) is ruled out.

(4) $L_{4n}^{4n+8b+1} \sim L_{4m}^{4m+8b+1}$ when $\nu(4n-4m) \geq 4b$.

First assume $b > 1$. By taking the diagonal map and then a CW-approximation, we have a map

$$f_1 : L_{4n}^{4n+8b+1} \rightarrow \Sigma^{4n-4m}(L_{4m}^{4m+8b+1} \cup_{2\beta\vee 2\beta\vee\beta'} (e^{4m+8b} \vee e_1^{4m+8b+1} \vee e_2^{4m+8b+1}))$$

where β, β' are respectively multiples of β_{8b-1}, β_{8b}. By Lemma 6.1, the top part $e_1^{4n+8b+1}$ splits off, thus we have a map

$$f_2 : L_{4n}^{4n+8b+1} \rightarrow \Sigma^{4n-4m}(L_{4m}^{4m+8b+1} \cup_{2\beta\vee\beta'} (e^{4m+8b} \vee e^{4m+8b+1})).$$

Let $f_3 : L_{4n}^{4n+8b-1} \to \Sigma^{4n-4m} L_{4m}^{4m+8b+1}$ and $f''' : L_{4n}^{4n+8b-1} \to \Sigma^{4n-4m} L_{4m}^{4m+8b}$ be the appropriate restrictions of f_2.

We can view $L_{4n}^{4n+8b+1}$ as the complex

$$L_{4n}^{4n+8b-1} \cup_{\alpha_1 \vee \alpha_2} (e^{4n+8b} \vee e^{4n+8b+1}).$$

Let ξ be the real Hopf bundle over P^{8b}. Since $\nu(4n-4m) \geq 4b$, both $4(n-m)\xi$ and $2(n-m)\lambda_{8b-1}$ are trivial. Thus we have an equivalence $f : P_{4n}^{4n+8b} \to \Sigma^{4(n-m)} P_{4m}^{4m+8b}$ by the S-coreducibility of $P_{4(n-m)}^{4(n-m)+8b}$ and the map

$$P_{4n}^{4n+8b} \to P_{4(n-m)}^{4(n-m)+8b} \wedge P_{4m}^{4m+8b} \to \Sigma^{4n-4m} P_{4m}^{4m+8b}.$$

Consider the diagram

$$\begin{array}{ccccccc}
S^{4n+8b-1} & \xrightarrow{\alpha'} & P_{4n}^{4n+8b-1} & \xrightarrow{1} & P_{4n}^{4n+8b-1} & \xrightarrow{\rho} & L_{4n}^{4n+8b-1} \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'''} \\
S^{4n+8b-1} & \xrightarrow{\alpha''} & \Sigma^{4(n-m)} P_{4m}^{4m+8b-1} & \to & \Sigma^{4(n-m)} P_{4m}^{4m+8b} & \xrightarrow{\rho} & \Sigma^{4(n-m)} L_{4m}^{4m+8b}
\end{array}$$

where α', α'' are the attaching maps for the top cells of P_{4n}^{4n+8b} and $\Sigma^{4(n-m)} P_{4m}^{4m+8b}$, and f', f'' are restrictions of f. Let α_1' be the attaching map for the top cell of $\Sigma^{4(n-m)} L_{4m}^{4m+8b}$. Then both α_1 and α_1' are the attaching maps for the even dimensional cells, by Lemma 2.2 (i) or the CW-structure given by [21, p. 91], we have $\alpha_1 = \rho\alpha'$ and $\alpha_1' = \rho\alpha''$. Thus the composite of the bottom row is homotopic to

$$S^{4n+8b-1} \xrightarrow{\alpha_1} L_{4n}^{4n+8b-1} \xrightarrow{f'''} \Sigma^{4(n-m)} L_{4m}^{4m+8b},$$

which is null-homotopic because f'' extends to f. Here the commutativity of the middle and the right squares is immediate, while the left follows from the existence of f. Hence f_3 extends to a map $f_4 : L_{4n}^{4n+8b} \to \Sigma^{4n-4m} L_{4m}^{4m+8b+1}$, which is an equivalence on $(4n+8b)$-skeletons, by Sq^2 and Lemma 5.11 (i).

On the other hand, it is well known that the first morphism in the exact sequence

$$\pi_{4n+8b}(S^{4n+8b-1}) \xrightarrow{2\beta} \pi_{4n+8b}(\Sigma^{4(n-m)}(L_{4m}^{4m+8b+1} \cup_{\beta'} e^{4m+8b+1}))$$

$$\xrightarrow{i_*} \pi_{4n+8b}(\Sigma^{4(n-m)}(L_{4m}^{4m+8b+1} \cup_{2\beta \vee \beta'} (e^{4m+8b}) \vee e^{4m+8b+1}))$$

is 0 since β is a multiple of β_{8b-1}.

Now that $i_*(f_4\alpha_2) = f_2\alpha_2 = 0$, we conclude that $f_4\alpha_2$ is null-homotopic. Thus f_4 extends to a map

$$g : L_{4n}^{4n+8b+1} \to \Sigma^{4(n-m)}(L_{4m}^{4m+8b+1} \cup_{\beta'} e^{4m+8b+1})$$

such that g^* on $H^*(-; \mathbf{Z}_2)$ is an isomorphism on the part $\Sigma^{4(n-m)} L_{4m}^{4m+8b+1}$.

Taking the S-dual of g as in the final stage of (1) or (2), using Lemmas 5.10, 6.3 5.11 (ii), we have a desired equivalence.

Assume $b = 1$. By S-duality, it suffices to show $L_{4n+2}^{4n+11} \sim L_{4m+2}^{4m+11}$. If both $\nu(4m+12)$ and $\nu(4n+12) \geq 5$, then an equivalence (5) follows because neither of the spaces is S-reducible implies $\nu(4m+12) = \nu(4n+12) = 5$. So we may assume $2 \leq \nu(4m+12) \leq 4$.

Case (a). Suppose both n and m are odd. Taking the diagonal and then a CW-approximation, we have a map

$$f_1 : L_{4n+2}^{4n+11} \to \Sigma^{4(n-m)}(L_{4m+2}^{4m+11} \cup_{2\beta_7 \vee 2\beta_7 \vee \beta'} (e^{4m+10} \vee e_1^{4m+11} \vee e_2^{4m+11})).$$

Since $3 \leq \nu(4m+12) \leq 4$, by Lemmas 6.3 (ii), 5.10 the top parts $e_1^{4n+11} \vee e_2^{4n+11}$ split off, and we have a map

$$f_2 : L_{4n+2}^{4n+11} \to \Sigma^{4(n-m)}(L_{4m+2}^{4m+11} \cup_{2\beta_7} e^{4m+10}).$$

Just as in (1), taking the S-dual and using Lemma 6.1, we have a desired equivalence.

Case (b). Suppose both n and m are even. By S-duality, the statement that $L_{4n+2}^{4n+11} \sim L_{4m+2}^{4m+11}$ when $\nu(4(n-m)) \geq 4$ is equivalent to that $L_{8A+4}^{8A+13} \sim L_{8B+4}^{8B+13}$ when $\nu(8(A-B)) \geq 4$. By Case (a) and S-duality, we have an equivalence $g_0 : L_{8(A-B)}^{8(A-B)+9} \to \Sigma^{8(A-B)-2^5} L_{2^5}^{2^5+9}$. Since $2^3\lambda_9$ is trivial over L^7 and $L_{2^5}^{2^5+9} = T(2^4\lambda_9)$, we have a map

$$g_1 : L_{2^5}^{2^5+9} \to S^{2^5} \cup_{4\beta_7} e^{2^5+8}$$

of Thom spaces. Let g_2 be the composite

$$L_{8A+4}^{8A+13} \to L_{8(A-B)}^{8(A-B)+9} \wedge L_{8B+4}^{8B+13} \xrightarrow{g_0 \wedge 1} \Sigma^{8(A-B)-2^5} L_{2^5}^{2^5+9} \wedge L_{8B+4}^{8B+13}$$

$$\xrightarrow{g_1 \wedge 1} (S^{8(A-B)} \cup_{4\beta_7} e^{8(A-B)+8}) \wedge L_{8B+4}^{8B+13}.$$

Taking a CW-approximation, we have a map

$$g_3 : L_{8A+4}^{8A+13} \to \Sigma^{8(A-B)}(L_{8B+4}^{8B+13} \cup_{4\beta_7 \vee 4\beta_7} (e^{8B+12} \vee e^{8B+13})).$$

By Lemma 5.2, the top part of the 2-part wedge splits off, so we have map

$$g_4 : L_{8A+4}^{8A+13} \to \Sigma^{8(A-B)}(L_{8B+4}^{8B+13} \cup_{4\beta_7} e^{8B+12})$$

inducing an isomorphism on $H^*(-;\mathbf{Z}_2)$ when restricted on the part $\Sigma^{8(A-B)} L_{8B+4}^{8B+13}$. Let

$$g_5 : (S^{-8B-13} \vee L_{-8B-14}^{-8B-6}) \cup_{4\beta_7 \vee \alpha} e^{-8B-5} \to \Sigma^{8(A-B)} L_{-8A-14}^{-8A-5}$$

be the S-dual of g_4. Since $g_5|S^{-8B-13}$ is the degree one map into the $(-8B-13)$-cell of $\Sigma^{8(A-B)} L_{-8A-14}^{-8A-5}$, we have $(g_5|S^{-8B-13})_*(4\beta_7) = 0$ by Lemma 5.10. Thus $g_5|L_{-8B-14}^{-8B-6}$ extends to a map $L_{-8B-14}^{-8B-5} \to \Sigma^{8(A-B)} L_{-8A-14}^{-8A-5}$, which is an equivalence by Sq^4 and Lemma 5.11 (ii).

(3) $L_{4n}^{4n+8b+3} \sim L_{4m}^{4m+8b+3}$ when $\nu(4n-4m) \geq 4b+1$.

Suppose $b \geq 2$. By S-duality, we can assume that both n and m are even. As in (1), the diagonal map gives a map

$$f_1 : L_{4n}^{4n+8b+3} \to L_{4(n-m)}^{4(n-m)+8b+3} \wedge L_{4m}^{4m+8b+3}.$$

Since $\nu(4n-4m) \geq 4b+1$, by Theorem 2.6 (ii), there is a stable vector bundle η over S^{8b} such that $p^*(4\eta) = 2(n-m)\lambda_{8b+3}$, where p is the projection $L^{8b+3} \to L_{8b}^{8b+3} \to S^{8b}$. Thus we have a map

$$f_2 : L_{4(n-m)}^{4(n-m)+8b+3} \to T(4\eta) = S^{4(n-m)} \cup_{4\beta} e^{4(n-m)+8b},$$

where $\beta = a\beta_{8b-1}$ for some integer a. Let

$$f_3 = (f_2 \wedge 1) f_1 : L_{4n}^{4n+8b+3} \to (S^{4(n-m)} \cup_{4\beta} e^{4(n-m)+8b}) \wedge L_{4m}^{4m+8b+3}.$$

Taking a CW-approximation of f_3, we have a map

$$f_4 : L_{4n}^{4n+8b+3} \to \Sigma^{4n-4m}(L_{4m}^{4m+8b+3} \cup_{4\beta \vee 4\beta \vee 4\beta} (e^{4m+8b} \vee CM \vee e^{4m+8b+3}))$$

where CM is the cone on the mod 4 Moore space $M = S^{4m+8b} \cup_4 e^{4m+8b+1}$. By Lemma 6.3 (i), the top part of the 3-part wedge splits off. By Lemma 5.2, the next to top part also splits off. Thus we have a map

$$f_5 : L_{4n}^{4n+8b+3} \to \Sigma^{4(n-m)}(L_{4m}^{4m+8b+3} \cup_{4\beta} e^{4m+8b}).$$

View $L_{4n}^{4n+8b+3} = L_{4n}^{4n+8b-1} \cup_{\alpha_1 \vee \alpha_2 \vee \alpha_3} (e^{4n+8b} \vee CN \vee e^{4n+8b+3})$, where CN is the cone on $N = S^{4n+8n} \cup_4 e^{4n+8b+1}$. Let $f_6 : L_{4n}^{4n+8b-1} \to \Sigma^{4(n-m)} L_{4m}^{4m+8b+3}$ be the map obtained by restricting f_5 on the $(4n+8b-1)$-skeleton.

As in (4), $f_6\alpha_1 = 0$ in homotopy. Since $\eta\beta_{8b-1} = \beta_{8b}$ by [16], and $4\beta_3 = \eta^3$, we have $4\beta_3\beta_{8b-1} = \eta^3\beta_{8b-1} = 0$. So the first morphism in the exact sequence

$$[X, S^{4n+8b-1}] \xrightarrow{4\beta_{8b-1}} [X, \Sigma^{4(n-m)} L_{4m}^{4m+8b+3}] \xrightarrow{i_*} [X, \Sigma^{4(n-m)}(L_{4m}^{4m+8b+3} \cup_{4\beta} e^{4m+8b})]$$

is null for $X = S^{4n+8b+2}$. It is also null for $X = N$ by Lemma 5.2. Thus both $f_6\alpha_2$ and $f_6\alpha_3$ are null-homotopic, and f_6 extends to a map over $L_{4n}^{4n+8b+3}$ which can be required to be an equivalence by Lemma 5.11.

Suppose $b = 1$. By S-duality, we can assume that both n and m are odd. Thus both $\nu(4n+12)$ and $\nu(4m+12) \geq 3$. Using Lemma 6.3 (ii), and as for the case $b \geq 2$, we can find an equivalence $f_6 : L_{4n}^{4n+11} \to \Sigma^{4(n-m)} L_{4m}^{4m+11}$. □

4. J-homology and coextension

According to [5], the stable Adams operation $\psi^3 : bo \to bo$ is defined and $\psi^3 - 1 : bo \to bo$ lifts to $\Sigma^4 bsp$. Let $\theta : bo \to \Sigma^4 bsp$ be the lift and J its fibre. There is a long exact sequence

$$\to bsp_{*+1}(\Sigma^4 X) \xrightarrow{\omega} J_*(X) \to bo_*(X) \xrightarrow{\theta_*} bsp_*(\Sigma^4 X) \to . \tag{4.1}$$

Use the fact that $\pi_t(bo) = \pi_t(KO) \otimes \mathbf{Z}_{(2)}$ if $t \geq 0$, and is 0 otherwise; while $\pi_t(\Sigma^4 bsp) = \pi_t(KO) \otimes \mathbf{Z}_{(2)}$ if $t \geq 4$, and is 0 otherwise, we can compute $bo_*(X)$ and $bsp_*(\Sigma^4 X)$ and $\theta_* : bo_*(X) \to bsp_*(\Sigma^4 X)$ in some cases. For example, if $X = P^{2n}$, then both $bo_{4j-1}(X)$ and $bsp_{4j-1}(\Sigma^4 X)$ are cyclic and θ_* sends a generator to $2^{\nu(j)} g$, where $g \in bsp_{4j-1}(\Sigma^4 X)$ is a generator, this is because $(\psi^3 - 1)(x) = 2^{\nu(j)+2} x \pmod{2^{\nu(j)+3}}$ in $bo_{4j-1}(X)$ and the projection $bsp_{4j-1}(\Sigma^4 X) \to bo_{4j-1}(X)$ is injective with image divisible by 4.

Let $J : \pi_k(SO) \to \pi_k^s$ be the standard J-homomorphism. By [18, Theorem 1.1.13], ImJ is cyclic with 2-component $\mathbf{Z}_{(2)}/(2(k+1))$ when $k \equiv 3 \pmod 8$, and $\mathbf{Z}_2$ when $k \equiv 0, 1 \pmod 8$, and the Hurewicz map $\pi_k^s \to \pi_k(J)$ is injective on ImJ. Putting $\beta_{8b-1}, \beta_{8b+3}, \beta_{8b+1}$, generators of ImJ, into the bottom cell, we get classes in $\pi_*(L_n^{n+k})$.

Lemma 4.2. *Let $b \geq 2$* (i)*, and $b \geq 1$ in* (ii)-(iv)*.*

(i) *$2\beta_{8b-1}$ is null in $J_{4A+8b}(L_{4A+1}^{4A+10})$ but not null in $J_{4A+8b}(L_{4A+1}^{4A+9})$.*

(ii) *For odd A, β_{8b-1} is null in $J_{4A+8b-2}(P_{4A-1}^{4A+6})$ but not null in $J_{4A+8b-2}(P_{4A-1}^{4A+5})$.*

(iii) *β_{8b+3} is null in $J_{4A+8b+4}(L_{4A+1}^{4A+10})$ but not null in $J_{4A+8b+4}(L_{4A+1}^{4A+9})$.*

(iv) *If A is even, then β_{8b+1} is null in $J_{4A+8b+2}(P_{4A+1}^{4A+8})$ but not null in $J_{4A+8b+2}(P_{4A+1}^{4A+7})$.*

Proof. We just show (i) and (ii). The proofs for (iii) and (iv) are similar.

Consider (i). Since $b \geq 2$, there is a natural isomorphism $f_* : bo_{4A+8b+1}(L_{4A+1}^{4A+10}) \approx bsp_{4A+8b+1}(\Sigma^4 L_{4A+1}^{4A+10})$. By Lemma 2.8 (i), the morphism

$$\theta_* : bo_{4A+8b+1}(L_{4A+1}^{4A+10}) \to bsp_{4A+8b+1}(\Sigma^4 L_{4A+1}^{4A+10})$$

in (4.1) satisfies $\theta_*(x) = 4f_*(x) \pmod 8$. Note that $2\beta \in J_{4A+8b}(L_{4A+1}^{4A+10})$ is from an element $z \in bsp_{4A+8b+1}(\Sigma^4 L_{4A+1}^{4A+10})$, that is from the bottom cell corresponding to the order 2 element $z' \in E^2_{4A+5,8b-4}$ in the AHSS for $bsp_*(\Sigma^4 L_{4A+1}^{4A+10})$. Let p be the projection as in Lemma 2.8 (ii). Since z' survives in the AHSS for $bsp_*(\Sigma^4 L_{-\infty}^{4A+10})$, z is in the image of p_*. By Lemma 2.8 (ii), z is divisible by 4 and hence is hit by θ_*. So 2β is null in $J_{4A+8b}(L_{4A+1}^{4A+10})$. However 2β is not null in $J_{4A+8b}(L_{4A+1}^{4A+8})$ because all elements of $bsp_{4A+8b+1}(\Sigma^4 L_{4A+1}^{4A+8})$ are of order ≤ 4 by Lemma 2.8 (i), thus are not hit by θ_*. Finally the d_2-differential on the unique nontrivial element y of $E^2_{4A+9,8(b-1)} = H_{4A+9}(L_{4A+3}^{4A+9}, J_{8(b-1)}) \approx \mathbf{Z}_2$ in the AHSS for $J_*(L_{4A+3}^{4A+9})$ is not zero. Thus $J_{4A+8b+1}(L_{4A+3}^{4A+8}) \to J_{4A+8b+1}(L_{4A+3}^{4A+9})$ is surjective by exactness. This together with the fact that 2β is nontrivial in $J_{4A+8b}(L_{4A+1}^{4A+8})$ imply that the boundary $J_{4A+8b+1}(L_{4A+3}^{4A+9}) \to J_{4A+8b}(L_{4A+1}^{4A+2})$ does not hit $2\beta \in J_{4A+8b}(L_{4A+1}^{4A+2})$, and (i) follows.

Consider (ii). Note that the composite

$$bo_{4A+8b-1}(P_{4A-2}^{4A+5}) \xrightarrow{\theta_*} bsp_{4A+8b-1}(\Sigma^4 P_{4A-2}^{4A+5}) \to bo_{4A+8b-1}(P_{4A-2}^{4A+5})$$

sends x to $8x$. Thus β_{8b-1} is not null in $J_{4A+8b-2}(P_{4A-2}^{4A+5})$ because

$$bo_{4A+8b-1}(P_{4A-2}^{4A+5}) \approx KO_{4A+8b-1}(P_{4A-2}^{4A+5}) \approx \mathbf{Z}_8,$$

which implies that β_{8b-1} is not hit by θ_*. However, $bo_{4A+8b-1}(P_{4A-2}^{4A+6}) \approx \mathbf{Z}_{16}$, hence β_{8b-1} is hit by θ_*. $\square$

Let $E_r(X)$ be the E_r term in ASS for $\pi_*(X)$. Given $x \in \pi_*(X)$, let $A(x)$ be the Adams filtration of x. Let $A_k(\pi_*(X))$ be the subgroup of $\pi_*(X)$ of classes of Adams filtrations $\geq k$.

Lemma 4.3. *Let $n \geq a > b$. Let $I(n,a,b)$ be the image of the projection $J_*(L_b^n) \to J_*(L_a^n)$.*

(i) *Suppose $x : S^m \to L_a^n$ satisfies $A(x) \geq N$ and $x \in I(n,a,b)$ when put into $J_*(-)$. If the projection $A_{N+1}(\pi_{m-1}(S^{a-1})) \to J_{m-1}(S^{a-1})$ is injective, and each nontrivial element of $E_2^{s,s+m-1}(S^{a-1})$ survives to a homotopy class when $s \geq N+1$, then x coextends to a map $z : S^m \to L_{a-1}^n$ with $A(z) \geq N$.*

(ii) *If moreover the projection $A_N(\pi_m(S^{a-1})) \to J_m(S^{a-1})$ is surjective, then the z in* (i) *can be chosen to be in $I(n, a-1, b)$ when put into $J_*(-)$.*

Proof. Consider (i). Here the idea is similar to [4, Lemma 6]. First the condition implies that the boundary $\partial : \pi_m(L_a^n) \to \pi_{m-1}(S^{a-1})$ is null on x. By S-duality, it suffices to show $x_0 : L_{-n-1}^{-a-1} \to S^{-m-1}$, the dual of x, extends to a map $x' : L_{-n-1}^{-a} \to S^{-m-1}$ with $A(x') \geq N$. Let α be the attaching map for the top cell of L_{-n-1}^{-a}. Since x_0 lifts to E^N, the N-stage of the stable Adams resolution of S^{-m-1}, and $A(x_0\alpha) \geq N+1$, the only way that the composite $x_0\alpha : S^{-a-1} \to E^N$ fails to be null-homotopic is that in the ASS for $\pi_*(S^0)$ there was a nontrivial differential $E_r^{s,s+m-a+1} \to E_r^{s+r,s+r+m-a}$ with $s < N < s+r$. But this is ruled out by the condition.

Consider (ii). There exists $y \in J_m(L_b^n)$ which is a coextension of the composite $S^m \xrightarrow{x} L_a^n \to J \wedge L_a^n$. So $p(y) - z$ pulls back to $J_m(S^{a-1})$, where p is the projection $L_b^n \to L_{a-1}^n$. Pick up an element $y' \in \pi_m(S^{a-1})$ with $A(y') \geq N$, and $p(y) - z = y'$ in $J_m(S^{a-1})$. Then $z + y'$ is the desired coextension. $\square$

5. Adams filtration

By [17, Theorem 8.2], if $t \geq 11$, then β_t has both the same Adams filtration in $\pi_t(S^0)$ and $J_t(S^0)$ except when $t = 8c-1$. As indicated in [4, p. 344], the composite $S^{2N+8c-2} \xrightarrow{\beta_{8c-1}} S^{2N-1} \to P^{2N}_{2N-1}$ has Adams filtration $4c-2$ for $c \geq 2$.

Lemma 5.1. *Suppose $\nu(n+1) = i$, $4c-1 \leq i \leq 4c+2$ and $c \geq 1$. Let $a = i - 4c+1$. Suppose $e_0 : S^n \to P^n_{n-8c+2}$ is the degree one map given by the S-reducibility. Let ι generate $\pi_n(S^n)$, and $\partial : \pi_n(P^n_{n-8c+2}) \to \pi_{n-1}(P^{n-8c+1}_{n-8c-7})$ the boundary. Then $\partial(e_0) = x$, where*

$$x : S^{n-1} \xrightarrow{2^a\beta_{8c-1}} S^{n-8c} \to P^{n-8c}_{n-8c-7} \to P^{n-8c+1}_{n-8c-7}.$$

Here the third map is the inclusion, the second is given by the S-reducibility.

Proof. Choose m such that $\nu(m) = i$. Then $P^{m+8c+7}_m = T(m\xi)$, where ξ is the real Hopf bundle over P^{8c+7}. Then $m\xi$ is stably trivial over P^{8c-1}. As in Theorem 2.6 (i), there is a stable vector bundle η' over S^{8c} corresponding to a generator of $\widetilde{KO}(S^{8c}) \otimes \mathbf{Z}_{(2)}$, such that $p^*(2^a\eta') = m\bar{\xi}$, where p is the composite $P^{8c+7} \to P^{8c+7}_{8c} \to S^{8c}$ in which the last map is given by the S-coreducibility. So we have a map $g : P^{m+8c+7}_m \to S^m \cup_{2^a\beta_{8c-1}} e^{m+8c}$ of Thom spaces such that g^* is injective on $H^*(-;\mathbf{Z}_2)$. By S-duality we have a map $g : S^{n-8c} \cup_{2^a\beta_{8c-1}} e^n \to P^n_{n-8c-7}$ such that g_* is injective on $H_*(-;\mathbf{Z}_2)$. By the diagram,

$$\begin{array}{ccc} \pi_n(S^n) & \xrightarrow{\partial'} & \pi_{n-1}(S^{n-8c}) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ \pi_n(P^n_{n-8c+2}) & \xrightarrow{\partial} & \pi_{n-1}(P^{n-8c+1}_{n-8c-7}) \end{array}$$

we have $\partial(e_0) = x$ with $e_0 = g_*(\iota)$. This completes the proof. $\square$

Lemma 5.2. *Let M be the mod 4 Moore space with the bottom cell of dimension 0. Then $A_2([M,M]) = 0$, and the identity map $1 : M \to M$ is of order 4.*

Proof. Consider the exact sequence

$$[S^1, M] \xrightarrow{4} [S^1, M] \to [M, M] \to [S^0, M] \xrightarrow{4} [S^0, M]$$

derived from the cofibre sequence $S^0 \xrightarrow{4} S^0 \to M \to S^1 \xrightarrow{4} S^1$.

Then in the ASS for $\pi_*(M)$, we have $E_3^{s,s}(M) = 0$ by [18, Theorem 2.3.4, p. 63] when $s \geq 2$. So the only possible nontrivial elements of $A_2([M,M])$ are from $[S^1, M]$. By the exact sequence

$$[S^1, S^0] \xrightarrow{4} [S^1, S^0] \to [S^1, M] \to [S^1, S^1] \xrightarrow{4} [S^1, S^1],$$

we see that each nontrivial element of $E_2^{s,s+1}(M)$ supports a nontrivial d_2-differential when $s \geq 2$. This implies $A_2([M,M]) = 0$, and the order of $1 : M \to M$ is of order ≤ 4. Therefore $1 : M \to M$ must be of order 4 because $H_1(M;\mathbf{Z}) \approx \mathbf{Z}_4$. $\square$

The filtration 4 maps between real stunted projective spaces are studied in detail in [6], [15] and [17]. In the next lemma, c, i and ι are the collapse map, the inclusion and the identity.

Lemma 5.3. (i) ([6, Prop. 2.1], [15, Theorem 3.3]) *Let m be odd, n even, $m < n$. There is a stable map $g : P^{n+8}_{m+8} \to P^n_m$ such that $A(g) = 4$, the composite $P^{n+8}_m \xrightarrow{c} P^{n+8}_{m+8} \xrightarrow{g} P^n_m \xrightarrow{i} P^{n+8}_m$ is 16ι, and $g_* : KO_{8k-1}(P^{n+8}_{m+8}) \to KO_{8k-1}(P^n_m)$ is an isomorphism for all k.*

(ii) [17, Lemma 7.15] *There is a map* $g : P_{4k+7} \to P_{4k-1}$ *such that* $A(g) = 4$*, and* $g_* : E_2^{s,t}(P_{4k+7}) \to E_2^{s+4,t+4}(P_{4k-1})$ *is an isomorphism in the ASS if* $6s > t+14-4k$.

Lemma 5.4. *There exists an unstable map* $f : P_{8n}^{8(n+1)+7} \to S^{8n} \cup_{n\beta_7} e^{8(n+1)}$ *whose restriction* $S^{8n} \to S^{8n} \cup_{n\beta_7} e^{8(n+1)}$ *induces a surjection on* $\widetilde{KO}(-)$.

Proof. Since $P_8^{15} = T(8\xi_7)$ and $8\xi_7$ is trivial over P^7, we see $P_8^{15} = S^8 \vee P_9^{15}$ in the unstable case. The bundle $8\xi_{15}$ over P^{15} is trivial on P^7, so we have an 8-dimensional bundle η over P_8^{15}. Let η' be the restriction of η on S^8. The composite

$$P^{15} \longrightarrow P_8^{15} = S^8 \vee P_9^{15} \longrightarrow S^8$$

in the unstable case, induces a map

$$f : T(8n\xi_{15}) = P_{8n}^{8n+15} \longrightarrow T(n\eta') = e^{8n} \cup_{n\beta_7} e^{8(n+1)}$$

of Thom spaces, which is of degree one on the bottom cell.

Then the surjection follows from the fact that $KO^1(S^{8(n+1)}) = 0$ in the exact sequence below

$$KO^0(S^{8n} \cup_{a\beta_7} e^{8(n+1)}) \to KO^0(S^{8n}) \to KO^1(S^{8(n+1)}).$$

□

Let X be a CW-complex, X^n its n-skeleton. Let $X_{n+1}^{n+k} = X^{n+k}/X^n$.

Lemma 5.5. *Let b be an integer. Suppose both f and g make the diagram*

$$\begin{array}{ccccc} X^n & \xrightarrow{i} & X^{n+k} & \xrightarrow{p} & X_{n+1}^{n+k} \\ \downarrow b & & \downarrow f,g & & \downarrow b \\ X^n & \xrightarrow{i} & X^{n+k} & \xrightarrow{p} & X_{n+1}^{n+k} \end{array}$$

commute, where i and p are respectively the inclusion and the projection. Then $h = f-g : X^{n+k} \to X^{n+k}$ *factors as* $X^{n+k} \xrightarrow{p} X_{n+1}^{n+k} \to X^{n+k}$ *and* $X^{n+k} \to X^n \xrightarrow{i} X^{n+k}$.

Proof. The first factoring follows from the fact that the restriction of $f - g$ on X^n is null, while the second factoring is implied by $p(f - g) = 0$. □

Lemma 5.6. *There exists a map* $h : L_{2k+1}^{2k+2} \to L_{2k+1}^{2k+2}$ *factoring as* $L_{2k+1}^{2k+2} \to S^{2k+1} \to L_{2k+1}^{2k+2}$ *such that* $2 - h : L_{2k+1}^{2k+2} \to L_{2k+1}^{2k+2}$ *factors as* $L_{2k+1}^{2k+2} \to P_{2k+1}^{2k+2} \xrightarrow{\rho} L_{2k+1}^{2k+2}$.

Proof. There is a map ϵ such that the diagram

$$\begin{array}{ccccc} S^{2k+1} & \xrightarrow{4} & S^{2k+1} & \rightarrow & L_{2k+1}^{2k+2} \\ \downarrow 2 & & \downarrow 1 & & \downarrow \epsilon \\ S^{2k+1} & \xrightarrow{2} & S^{2k+1} & \rightarrow & P_{2k+1}^{2k+2} \\ \downarrow 1 & & \downarrow 2 & & \downarrow \rho \\ S^{2k+1} & \xrightarrow{4} & S^{2k+1} & \rightarrow & L_{2k+1}^{2k+2} \end{array}$$

commutes. By Lemma 5.5, the map $h = 2 - \epsilon\rho$ is the desired. □

Lemma 5.7. *Let β generate ImJ on the $(8b+3)$-stem. Let* $N = e^{4m+1} \cup_4 e^{4m+2}$. *If* $\nu(4m + 8b + 8) \leq 4b + 2$*, then the composite*

$$x : \Sigma^{8b+3} N \xrightarrow{\beta\wedge 2} N \to L_{4m}^{4m+8b+7}$$

is null-homotopic, where the second map is the inclusion to $L_{4m}^{4m+2} = S^{4m} \vee N$.

Proof. By Lemma 5.6, the map $N \xrightarrow{2} N \to L_{4m}^{4m+8b+7}$ factors as $N \to P_{4m}^{4m+8b+7} \xrightarrow{\rho} L_{4m}^{4m+8b+7}$ up to a map $N \to S^{4m+1} \to L_{4m}^{4m+8b+7}$, where the second map is the degree one map to the $(4m+1)$-cell. Thus by [4, Prop. 4] and Lemma 6.1, it is null. $\square$

Lemma 5.8. *Let $i = \nu(n+1)$.*

(i) *If $3 \leq i \leq 4b+1$, then in ASS for $\pi_*(P_{n-8b-6}^{n})$ elements in $E_2^{s,s+n-1}(P_{n-8b-6}^{n})$ of Adams filtrations $\geq 4b$ are hit by Adams differentials.*

(ii) *If $3 \leq i \leq 4b$, then in ASS for $\pi_*(P_{n-8b-6}^{n})$ elements in $E_2^{s,s+n-1}(P_{n-8b-6}^{n})$ of Adams filtrations $\geq 4b-1$ are hit by Adams differentials.*

Proof. We just show (i). The proof for (ii) is similar. In ASS for $\pi_*(P_{n-8b-6}^{n-8b+1})$ the chart for $E_2^{s,s+n-1}(P_{n-8b-6}^{n-8b+1})$ is as follows.

$E_2(P_{n-8b-6}^{n-8b+1})$

where the bottom class corresponds to $s = 4b-1$. The chart for $E_2(P_{n-8b-6}^{n-8b+1})$ can be obtained by using the pre-spectral sequence (PSS, [16, p. 26]) converging to $E_2(P_{n-8b-6}^{n-8b+1})$ with

$$E_1^{s,t} = \Sigma_{k=-1}^{6} \mathrm{Ext}_A^{s,t}(H^*(S^{n-8b-k}), \mathbf{Z}_2),$$

and repeatedly using [16, table 8.1], the Adams periodicity, [2, Lemma 2.6.1], and [18, Theorem 2.3.4, p. 63].

Case $b = 1$. Let ∂ be the boundary $\pi_n(S^n) \to \pi_{n-1}(P_{n-14}^{n-1})$. Then by Lemma 5.1, $\partial(\iota)$ is the class

$$x : S^{n-1} \xrightarrow{2^a \beta_7} S^{n-8} \to P_{n-14}^{n-8} \to P_{n-14}^{n-7}$$

when put into P_{n-14}^{n-1}, where $a = i-3$. Let $y = 2^{2-a}x$. Then y is of Adams filtration ≥ 3. Since y is of Adams filtration 4 when put into $J_{n-1}(P_{n-14}^{n-7})$, by the chart for $E_2^{s,s+n-1}(P_{n-14}^{n-7})$, y is of Adams filtration 4. This shows the case $b = 1$.

Case $b \geq 2$. Suppose (i) holds for b. We wish to show (i) holds for $b+1$.

If $3 \leq \nu(n+1) \leq 4b+1$, then it follows from the filtration 4 map $g : P_{n-8b-6}^{n+1} \to P_{n-8(b+1)-6}^{n-7}$ and the fact that the filtration $4b$ element of $E_2^{4b,4b+n-1}(P_{n-8b-6}^{n+1})$ is hit by an Adams differential. So we may assume $4b+2 \leq \nu(n+1) \leq 4(b+1)+1$.

Part A. Assume (i) for $b = 2$. Suppose $i = 4b+2$. Let $\Sigma P_m^{m+8(b+1)+6}$ be the S-dual of $P_{n-8(b+1)-6}^{n}$. By Lemma 5.4, there is an unstable map $f : P_{8b}^{8(b+1)+6} \to S^{8b} \cup_{\alpha\beta} e^{8(b+1)}$ and a stable vector bundle η over $S^{8b} \cup_{\alpha\beta} e^{8(b+1)}$ such that $f^*(\eta)$ is a 4-multiple of a generator in $\widetilde{KO}(S^{8b})$ when restricted on the bottom cell S^{8b}. So $(fp)^*(\eta) = (m/2)\xi$, where p is the projection $P^{8(b+1)+6} \to P_{8b}^{8(b+1)+6}$. Thus there is a map of Thom spaces $f_0 : P_{m/2}^{m/2+8(b+1)+6} \to T(\eta)$. Taking the diagonal and then a CW-approximation, we have a composite

$$\begin{aligned} P_m^{m+8(b+1)+6} &\xrightarrow{f_1} (P_{m/2}^{m/2+8(b+1)+6} \wedge P_{m/2}^{m/2+8(b+1)+6})^{(m+8(b+1)+6)} \\ &\xrightarrow{f_2} (T(\eta) \wedge T(\eta))^{(m+8(b+1)+6)} \end{aligned}$$

where the restriction of f_2 on $P_{m/2}^{m/2+8(b+1)+6} \wedge S^{m/2}$ or $S^{m/2} \wedge P_{m/2}^{m/2+8(b+1)+6}$ is exactly f_0 as given above, and

$$(T(\eta) \wedge T(\eta))^{(m+8(b+1)+6)}/S^m = \Sigma^{m/2}(T(\eta)/S^{m/2} \vee T(\eta)/S^{m/2})$$

since $b \geq 2$. This observation is important to (5.9).

Let

$$\partial_0 : \ \pi_n(S^n) \to \pi_{n-1}(\Sigma^{(n+1)/2} P_{(n+1)/2-8(b+1)-7}^{(n+1)/2-2})$$

and

$$\partial_1 : \ \pi_n(S^n) \to \pi_{n-1}(P_{n-8(b+1)-6}^{n-1})$$

be boundaries.

Let g_1 and g_0 be the S-duals of f_1 and f_0, $\alpha_1 = \partial_0(\iota)$. By S-duality, and as in Lemma 5.1, we have

$$\partial_1(\iota) = 2 g_{1*}(\alpha_1). \tag{5.9}$$

In general, let $a = i - 4b - 1$, and consider the composite

$$P_m^{m+8(b+1)+6} \to P_{m/2^a}^{m/2^a+8(b+1)+6} \wedge \dots \wedge P_{m/2^a}^{m/2^a+8(b+1)+6} \to T(\eta) \wedge \dots \wedge T(\eta),$$

where $P_{m/2^a}^{m/2^a+8(b+1)+6} \wedge \dots \wedge P_{m/2^a}^{m/2^a+8(b+1)+6}$ is the smash product of 2^a copies of $P_{m/2^a}^{m/2^a+8(b+1)+6}$. Taking a CW-approximation and repeating the preceding argument, we have

$$\partial_1(\iota) = 2^a g_{1*}(\alpha_1),$$

where $\alpha_1 = \partial_0(\iota)$, and $\partial_0 : \pi_n(S^n) \to \pi_{n-1}(\Sigma^{n+1-(n+1)/2^a} P_{(n+1)/2^a-8(b+1)-7}^{(n+1)/2^a-2})$.

Let $x = 2^a g_{1*}(\alpha_1)$. Then $2^{4-a}x = 2^4 g_{1*}(\alpha_1)$. By induction, α_1 is of Adams filtration $4b$ when projected to $P_{(n+1)/2^a-8b-7}^{(n+1)/2^a-2}$, so $g_{1*}(\alpha_1)$ is of Adams filtration $4b$ when projected to P_{n-8b-6}^{n-1}. Thus by the filtration 4 map again, we see that filtration $\geq 4(b+1)$ elements of $E_2^{s,s+n-1}(P_{n-8(b+1)-6}^n)$ are hit by Adams differentials. Moreover $\partial_1(\iota)$ is of Adams filtration $4(b+1)$ when $i = 4(b+1)$.

Part B. Consider (i) for $b = 2$. First assume $\nu(m) = 5$. As before we have an unstable map $f : P_8^{22} \to S^8 \cup_{\beta_7} e^{16}$ and a stable vector bundle η over $S^8 \cup_{\beta_7} e^{16}$, such that $(fp)^*(\eta) = (m/2)\xi$. Let $f_0 : P_{m/2}^{m/2+22} \to T(\eta)$ be a map of the Thom spaces. Taking the diagonal and then a CW-approximation, we have

$$P_m^{m+22} \xrightarrow{f_1} (P_{m/2}^{m/2+22} \wedge P_{m/2}^{m/2+22})^{(m+22)} \xrightarrow{f_2} (T(\eta) \wedge T(\eta))^{(m+22)}.$$

Since $\pi_7(S^0)$ is generated by β_7, $\nu(m/2) \geq 4$ and $(f_0)^*$ is injective on $H^*(-; \mathbf{Z}_2)$, we see that a' is odd in the following

$$T(\eta)/S^{m/2} = S^{m/2+8} \cup_{a'\beta_7} e^{m/2+16}$$

by Sq^8. This implies that the attaching map α' indicated below

$$(T(\eta) \wedge T(\eta))^{(m+22)}/S^m = \Sigma^{m/2}((T(\eta)/S^{m/2} \vee T(\eta)/S^{m/2}) \cup_{\alpha'} e^{m/2+16}),$$

is null. Thus

$$(T(\eta) \wedge T(\eta))^{(m+22)}/S^m = \Sigma^{m/2}(T(\eta)/S^{m/2} \vee T(\eta)/S^{m/2} \vee e^{m/2+16}).$$

So in this case time (5.9) becomes

$$\partial_1(\iota) = 2 g_{1*}(\alpha_1) + x_0,$$

where x_0 is in the image of $\pi_{n-1}(S^{n-16})\to\pi_{n-1}(P_{n-22}^{n-1})$ induced by the degree one map from S^{n-16} to the $(n-16)$-cell of P_{n-22}^{n-1} with $A(x_0)\geq 4$. Here $\alpha_1 = \partial_0(\iota)$, $\partial_0 :\ \pi_n(S^n)\to\pi_{n-1}(\Sigma^{(n+1)/2}P_{(n+1)/2-23}^{(n+1)/2-2})$ and $\partial_1 :\ \pi_n(S^n)\to\pi_{n-1}(P_{n-22}^{n-1})$ are boundaries.

For $6\leq i\leq 9$, we have

$$\partial_1(\iota) = 2^a(2g_{1*}(\alpha_1)+x_0)$$

where $a = i-5$ and $\alpha_1 = \partial_0(\iota) :\ \pi_n(S^n)\to\pi_{n-1}(\Sigma^{n+1-(n+1)/2^{a+1}}P_{(n+1)/2^{a+1}-23}^{(n+1)/2^{a+1}-2})$. Thus

$$\partial_1(2^{4-a}\iota) = 2^5 g_{1*}(\alpha_1)+2^4x_0.$$

Since each element in $\pi_{n-1}(P_{n-22}^{n-15})$ is of order $\leq 2^4$, we see $2^4x_0 = 0$. So as before, the proof follows from the filtration 4 map. Moreover if $i = 9$, then $\partial_1(\iota)$ is of Adams filtration 8. □

Lemma 5.10. *If $3\leq\nu(4n)\leq 4b$, then both composites*

$$S^{4n-2}\xrightarrow{4\beta_{8b-1}} S^{4n-8b-1}\longrightarrow L_{4n-8b-2}^{4n-1},$$

$$S^{4n-2}\xrightarrow{\beta_{8b}} S^{4n-8b-2}\longrightarrow L_{4n-8b-2}^{4n-1}$$

are null-homotopic.

Proof. Notice that the boundary $\pi_{4n-1}(S^{4n-8b})\longrightarrow\pi_{4n-2}(L_{4n-8b-2}^{4n-8b-1})$ maps β_{8b-1} to

$$S^{4n-2}\xrightarrow{\beta_{8b}\vee 4\beta_{8b-1}} S^{4n-8b-2}\vee S^{4n-8b-1} = L_{4n-8b-2}^{4n-8b-1}.$$

So in $\pi_{4n-2}(L_{4n-8b-2}^{4n-1})$ both the composites in question are equal. The lemma follows from Lemma 5.8. □

We have repeatedly used the next lemma in section 3.

Lemma 5.11. *Suppose neither the top cell of L_n^{n+k} nor of L_m^{m+k} splits off. Let $g : L_n^{n+k}\to\Sigma^{n-m}L_m^{m+k}$ be an equivalence below the top cell.*

(i) *If g is of odd degree on the top cell, then g can be adjusted to an equivalence.*

(ii) *Suppose $\nu(m+k+1)\equiv\nu(n+k+1)\equiv 0 \pmod 4$. Then g is of odd degree on the top cell* (*hence an equivalence*) $\iff$ $\nu(m+k+1) = \nu(n+k+1)$.

Proof. Consider (i). The case when $n+k$ is even is immediate. Assume $n+k$ is odd. Let α_1 and α_2 be the attaching maps for the top cells of L_n^{n+k} and $\Sigma^{n-m}L_m^{m+k}$. Then $t\alpha_2 = g_*(\alpha_1)$ for an odd t. Note that α_2 is the image of an element $\alpha_2'\in\pi_{n+k-1}(\Sigma^{n-m}L^{m+k-1})$ under the projection $\pi_*(\Sigma^{n-m}L^{m+k-1})\to\pi_*(\Sigma^{n-m}L_m^{m+k-1})$. Consider the diagram

$$\begin{array}{ccc} S^{n+k-1} & \xrightarrow{\alpha_2'} & \Sigma^{n-m}L^{m+k-1} \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle t} \\ S^{n+k-1} & \xrightarrow{t\alpha_2'} & \Sigma^{n-m}L^{m+k-1}. \end{array}$$

Since $t :\Sigma^{n-m}L^{m+k}\to\Sigma^{n-m}L^{m+k}$ is an equivalence, we can view $t\alpha_2'$ (thus $t\alpha_2$) as the attaching map for the top cell of $\Sigma^{n-m}L^{m+k}$. Write $t\alpha_2$ as α_2. Then $\alpha_2 = g_*(\alpha_1)$, and (i) follows.

Consider (ii). Let $0 \le d < k$ be an integer satisfying

$$k-d=\begin{cases} 8b & \text{if } \nu(n+k+1)=4b, \\ 8b-4 & \text{if } \nu(n+k+1)=4b-1, 4b-2, \\ 8b-7 & \text{if } \nu(n+k+1)=4b-3. \end{cases}$$

Then by Theorem 1.1, L_{n+d+1}^{n+k} is S-reducible but L_{n+d}^{n+k} is not. Consider the map

$$g' : L_{n+d}^{n+k} \rightarrow \Sigma^{n-m} L_{m+d}^{m+k}$$

induced by g, and the diagram

$$\begin{array}{ccc} \pi_{n+k}(S^{n+k}) & \xrightarrow{g'_*} & \pi_{n+k}(S^{n+k}) \\ \big\downarrow \partial_1 & & \big\downarrow \partial_2 \\ \pi_{n+k-1}(L_{n+d}^{n+k-1}) & \xrightarrow{g'_*} & \pi_{n+k-1}(\Sigma^{n-m} L_{m+d}^{m+k-1}). \end{array}$$

Suppose $\nu(m+k+1)=\nu(n+k+1)$. As in Lemma 5.1, then $\partial_i(e_0)$, $i=1,2$, and $g'_*(\partial_1(e_0))$ are the classes corresponding to $\epsilon\beta_{k-1}$ for the same ϵ. L_m^{m+k} is not S-reducible implies that $\epsilon\beta_{k-1}$ is not null in $\pi_{n+k-1}(\Sigma^{n-m} L_m^{m+k-1})$. Therefore g' (and hence g) must be of odd degree on the top cell.

If g is an equivalence, then so is g'. We have $g'_*(\partial_1(e_0)) = \partial_2(e_0)$. This implies $\nu(m+k+1)=\nu(n+k+1)$. ☐

6. Triviality of $\tilde{\beta}_t$ in (1.4)

In this section, we study under some appropriate conditions the triviality of $\tilde{\beta}_t$ in (1.4) for $\beta_t = \beta_{8b-1}$ or β_{8b+3}.

Lemma 6.1. *Let $b \ge 2$ in* (i)*, and $b \ge 1$ in* (ii)*. Let ∂_1, ∂_2 be respectively the boundaries $\pi_*(L_{4A+3}^{4A+10}) \rightarrow \pi_{*-1}(L_{4A}^{4A+2})$, $\pi_*(L_{4A+2}^{4A+10}) \rightarrow \pi_{*-1}(L_{4A}^{4A+1})$.*

(i) *Let 2^{e+1} be the order of $\beta_{8(b-1)-1}$. There is an integer i such that $e-1 \le i \le e$ and $2^i\beta_{8(b-1)-1}$ coextends to a map $x_1 : S^{4A+1+8b} \rightarrow L_{4A+3}^{4A+10}$ satisfying $A(x_1) \ge 4(b-1)-1$, and $\partial_1(x_1)=z_1$, where z_1 is the composite $S^{4A+8b} \xrightarrow{2\beta_{8b-1}} S^{4A+1} \rightarrow L_{4A}^{4A+2}$.*

(ii) *There is an integer i with $1 \le i \le 2$ such that $2^i\beta_{8b-5}$ coextends to a map $x_2 : S^{4A+8b+5} \rightarrow L_{4A+2}^{4A+10}$ satisfying $A(x_2) \ge 4b-2$, and $\partial_2(x_2)=z_2$, where z_2 is the composite*

$$S^{4A+8b+4} \xrightarrow{\beta_{8b+3}} S^{4A+1} \rightarrow L_{4A}^{4A+1}.$$

(iii) *Both composites $S^{4A+8b} \xrightarrow{z_1} L_{4A}^{4A+2} \rightarrow L_{4A}^{4A+10}$ and $S^{4A+8b+4} \xrightarrow{z_2} L_{4A}^{4A+1} \rightarrow L_{4A}^{4A+10}$ are null-homotopic.*

Proof. Part (iii) is just a corollary of (i) and (ii).

Consider (i). Let $\beta = \beta_{8b-1}$. By Lemma 4.2 (i), 2β is null in $J_{4A+8b}(L_{4A+1}^{4A+10})$ but not null in $J_{4A+8b}(L_{4A+1}^{4A+9})$. So the only possible element in $J_{4A+8b+1}(L_{4A+3}^{4A+10})$ that might hit $2\beta \in J_*(L_{4A+1}^{4A+2})$ under the boundary

$$\partial' : J_{4A+8b+1}(L_{4A+3}^{4A+10}) \rightarrow J_{4A+8b}(L_{4A+1}^{4A+2})$$

is from the top cell of L_{4A+3}^{4A+10}, that is, there is an element $x \in J_{4A+8b+1}(L_{4A+3}^{4A+10})$ that is from a nontrivial element

$$u_0 \in E^2_{4A+10,8(b-1)-1} = H_{4A+10}(L_{4A+3}^{4A+10}; J_{8(b-1)-1})$$

in AHSS satisfying $\partial'(x) = 2\beta$ in $J_{4A+8b}(L^{4A+2}_{4A+1})$. Let $2^i\beta_{8(b-1)-1} : S^{4A+8b+1} \to S^{4A+10}$ correspond to u_0, then $e-1 \le i \le e$. Denote $2^i\beta_{8(b-1)-1}$ also by u_0.

We want to coextend u_0 to a map $x_1 : S^{4A+8b+1} \to L^{4A+10}_{4A+3}$ with $A(x_1) \ge 4(b-1)-1$. By [16, table 8.1], we see that when $-1 \le t \le 5$,

$$A_{4(b-1)}(\pi_{8(b-1)+t}(S^0)) \to J_{8(b-1)+t}(S^0)$$

is injective, and each nontrivial element of $E_2^{s,s+8(b-1)+t}(S^0)$ survives to a homotopy class in the image of J if $s \ge 4(b-1)$; moreover the projection

$$A_{4(b-1)-1}(\pi_{8(b-1)+t}(S^0)) \to J_{8(b-1)+t}(S^0)$$

is surjective when $0 \le t \le 6$. By repeatedly applying Lemma 4.3 (i) and (ii), we have the desired coextension x_1.

Next we show $\partial_1(x_1) = z_1$ in $\pi_{4A+8b}(L^{4A+2}_{4A})$. Consider the Adams spectral sequences for $\pi_*(L^{4A+2}_{4A+1})$ and $\pi_*(L^{4A+2}_{4A})$ and notice that

$$E_2^{s,t}(L^{4A+2}_{4A+1}) = \sum_{k=0}^{1} \mathrm{Ext}^{s,t}(S^{4A+1+k})$$

and

$$E_2^{s,t}(L^{4A+2}_{4A}) = \sum_{0 \le k \le 2} \mathrm{Ext}^{s,t}(S^{4A+1+k}).$$

By [16, table 8.1] and [18, Theorem 2.3.4, p.63], we get the charts for $E_3^{s,t}(L^{4A+2}_{4A+1})$ and $E_3^{s,t}(L^{4A+2}_{4A})$, where $s \ge 4(b-1)$ and $4A+8b \le t-s \le 4A+8b+1$, as follows.

$E_3(L^{4A+2}_{4A+1})$ $E_3(L^{4A+2}_{4A})$

where the bottom elements u_1, u_2, v_1 and v_2 are in the same position $(t-s, s) = (4A+8b, 4(b-1)+1)$, and u_1 corresponds to β_{8b-1} into $\pi_{4A+8b}(L^{4A+2}_{4A+1})$, u_2 supports a nontrivial d_1-differential to S^{4A-1} in the ASS for $\pi_*(L^{4A+2}_{4A-2})$. Also v_2, v_3 and v_4 support nontrivial d_1-differentials to S^{4A-2} in the ASS for $\pi_*(L^{4A+2}_{4A-2})$, so those homotopy classes which v_2, v_3 and v_4 converge to, can not be in the image of ∂_1. Since $\partial'(x_1) = 2\beta$ in $J_{4A+8b}(L^{4A+2}_{4A+1})$ and $A(\partial_1(x_1)) \ge 4(b-1)$, we see that $\partial_1(x_1)$ corresponds to $2v_1$, and hence $\partial_1(x_1) = z_1$ in $\pi_{4A+8b}(L^{4A+2}_{4A})$ because $\eta(x_1) = 0$, while η is injective on the subgroup generated by classes corresponding to v_2, v_3 and v_4. Here $\eta = \beta_1$.

Consider (ii). Let $\beta = \beta_{8b+3}$. By Lemma 4.2 (iii), β is null in $J_{4A+8b+4}(L^{4A+10}_{4A+1})$ but not null in $J_{4A+8b+4}(L^{4A+9}_{4A+1})$. So the only possible element in $J_{4A+8b+4}(L^{4A+10}_{4A+3})$ that might hit $\beta \in J_*(S^{4A+1})$ under the boundary

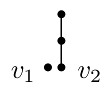
$$\partial' : J_{4A+8b+5}(L^{4A+10}_{4A+2}) \to J_{4A+8b+4}(S^{4A+1})$$

is from the top cell of L^{4A+10}_{4A+2}, and there is an element $x \in J_{4A+8b+5}(L^{4A+10}_{4A+2})$ that is from a nontrivial element $u_0 \in E^2_{4A+10,8(b-1)+3} = H_{4A+10}(L^{4A+10}_{4A+2}; J_{8(b-1)+3})$ in

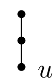
AHSS satisfying $\partial'(x) = \beta$ in $J_{4A+8b+4}(S^{4A+1})$. Let

$$2^i\beta_{8(b-1)+3} : S^{4A+8b+5} \to S^{4A+10} \to L^{4A+12}_{4A+10}$$

correspond to u_0, $1 \le i \le 2$. Consider the coextension of $2^i\beta_{8(b-1)+3}$ in a larger space L^{4A+12}_{4A+6} since in ASS the E_3 chart for L^{4A+12}_{4A+6} will be simple. Denote $2^i\beta_{8(b-1)+3}$ by u_0. By applying Lemma 4.3 (i) and (ii) repeatedly, we can coextend u_0 to a map $u_1 : S^{4A+8b+5} \to L^{4A+12}_{4A+6}$ with $A(u_1) \ge 4b-2$. In the ASS, the chart for $E_3^{s,t}(L^{4A+12}_{4A+6})$ with $s \ge 4b-2$ and $t-s = 4A+8b+5$ is as follows.



$E_3(L^{4A+12}_{4A+6})$



$E_2(L^{4A+1}_{4A})$

where both v_1 and v_2 are in the position $(t-s, s) = (4A+8b+5, 4b-2)$, v_1 corresponds to an order 2 class in homotopy that coextends to $J_{4A+8b+5}(L^{4A+12}_{4A+2})$, and v_2 corresponds to the image of $2^{d-3}\beta_{8b-1}$ under $\pi_{4A+8b+5}(S^{4A+6}) \to \pi_{4A+8b+5}(L^{4A+12}_{4A+6})$, where 2^d is the order of β_{8b-1}. If $u_1 \notin I(4A+12, 4A+6, 4A+2)$ when put into $J_*(-)$, then $u_1 = v_1 + \epsilon v_2$ for some ϵ. Here the homotopy class to which v_i survives is also denoted by v_i. Choose a new u_1 to be $u_1 - \epsilon v_2$. Then u_1 is a coextension of u_0 with $A(u_1) = 4b-2$ and $u_1 \in I(4A+12, 4A+6, 4A+2)$ when put into $J_*(-)$. Applying Lemma 4.3 (i) and (ii) repeatedly, we get the desired coextension x_2.

Finally in the ASS the charts for $E_2^{s,t}(L^{4A+1}_{4A})$ and $E_2^{s,t}(S^{4A+1})$, where $s \ge 4b-1$ and $t-s = 4A+8b+4$, coincide as the right chart above, where u is in the position $(t-s, s) = (4A+8b+4, 4b-1)$ surviving to $\beta_{8b+3} \in \pi_{4A+8b+4}(S^{4A+1})$ or $z_2 \in \pi_{4A+8b+4}(L^{4A+1}_{4A})$. Since $\partial'(x_2) = \beta$ in $J_{4A+8b+4}(S^{4A+1})$ and $A(\partial_2(x_2)) \ge 4b-1$, there must hold $\partial_2(x_2) = z_2$. $\square$

Lemma 6.2. (i) *Let x_3 be the composite $S^{4A+8b+2} \xrightarrow{\beta_{8b+3}} S^{4A-1} \to P^{4A-1}_{4A-4}$. If $\nu(4A+8b+4) \le 4b+2$, then x_3 is null-homotopic in $\pi_{4A+8b+2}(P^{4A+8b+3}_{4A-4})$.*

(ii) *Let x_4 be the composite $S^{4A+8b+2} \xrightarrow{2\beta_{8b+3}} S^{4A-1} \to L^{4A-1}_{4A-4}$. If $\nu_2(4A+8b+4) \le 4b+3$, then x_4 is null-homotopic in $\pi_{4A+8b+2}(L^{4A+8b+3}_{4A-4})$.*

Proof. Let $\beta = \beta_{8b+3}$. Consider (i). The case $\nu(4A+8b+4) < 4b+2$ is from [4, Proposition 5]. Assume $\nu(4A+8b+4) = 4b+2$. Let $DP^{4A+8b+3}_{4A-4} = \Sigma P^{n+8b+7}_n$, where $\nu(n) = 4b+2$. Since $P^{n+8b+7}_n = T(n\xi)$, where ξ is the Hopf real line bundle over P^{8b+7}, and $\nu(n) = 4b+2$, there is a stable vector bundle η over S^{8b+4} corresponding to a generator of $\widetilde{KO}(S^{8b+4}) \otimes \mathbf{Z}_{(2)}$ and $n\bar{\xi} = p^*(\eta)$, where p is the projection $P^{8b+7} \to P^{8b+7}_{8b+4} \to S^{8b+4} \vee P^{8b+7}_{8b+5} \to S^{8b+4}$. Note that $T(\eta) = S^n \cup_\beta e^{n+8b+4}$. We have a map $f : P^{n+8b+7}_n \to T(\eta)$ such that f^* is injective on $H^*(-; \mathbf{Z}_2)$, and the induced map $P^{n+8b+7}_{n+8b+4} \to T(\eta)/S^n$ is precisely the projection $S^{n+8b+4} \vee P^{n+8b+7}_{n+8b+5} \to S^{n+8b+4}$. Thus by duality we have a map $g : S^{4A-1} \cup_\beta e^{4A+8b+3} \to P^{4A+8b+3}_{4A-4}$ such that g_* is injective on $H_*(-; \mathbf{Z}_2)$ and the restriction of g on S^{4A-1} is the map $S^{4A-1} \to S^{4A-1} \vee$

$P^{4A-2}_{4A-4} = P^{4A-1}_{4A-4}$. There is a diagram

$$\begin{array}{ccccc}
\to & \pi_{4A+8b+3}(S^{4A+8b+3}) & \xrightarrow{\partial_1} & \pi_{4A+8b+2}(S^{4A-1}) & \\
& \downarrow g_* & & \downarrow g_* & \\
\to & \pi_{4A+8b+3}(P^{4A+8b+3}_{4A}) & \xrightarrow{\partial_2} & \pi_{4A+8b+2}(S^{4A-1} \vee P^{4A-2}_{4A-4}) & \\
& & & \to & \pi_{4A+8b+2}(S^{4A-1} \cup_\beta e^{4A+8b+3}) \\
& & & & \downarrow g_* \\
& & & \xrightarrow{i_*} & \pi_{4A+8b+2}(P^{4A+8b+3}_{4A-4})
\end{array}$$

whose rows are exact sequences. Then $\partial_1(\iota) = \beta$, $\partial_2(g_*(\iota)) = x_3$. This means $i_* x_3 = 0$ in $\pi_*(P^{4A+8b+3}_{4A-4})$ and (i) follows.

Consider (ii). If $\nu(4A + 8b + 4) \le 4b + 2$, then (ii) follows from (i) by the map $\rho : P^{4A+8b+3}_{4A-4} \to L^{4A+8b+3}_{4A-4}$. Suppose $\nu(4A + 8b + 4) = 4b + 3$. Let $\lambda = \lambda_{8b+7}$ and $DL^{4A+8b+7}_{4A-4} = \Sigma L^{n+8b+7}_{n}$. Then $\nu(n) = 4b+3$. Thus $\frac{n}{4}\lambda$ is stably trivial over L^{8b+3}, and $\frac{n}{2}\bar{\lambda} = p^*(2\eta)$, where η is a stable vector bundle over S^{8b+4} corresponding to a generator of $\widetilde{KO}(S^{8b+4}) \otimes \mathbf{Z}_{(2)}$. Then the proof follows just as (i). $\square$

Lemma 6.3. (i) *Let $\nu(4A + 8b) = 2$ and $b \ge 2$. The composite*

$$x_5 : S^{4A+8b-2} \xrightarrow{4\beta_{8b-1}} S^{4A-1} \to L^{4A-1}_{4A-4}$$

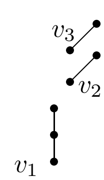
is null-homotopic in L^{4A+6}_{4A-4}.

(ii) *If $3 \le \nu(4A + 8) \le 4$, then the composite $x_5'' : S^{4A+6} \xrightarrow{2\beta_7} S^{4A-1} \to L^{4A-1}_{4A-4}$ is null-homotopic in L^{4A+7}_{4A-4}.*

Proof. Consider (i). Let ∂, ∂' be respectively the boundaries from $\pi_*(P^{4A+6}_{4A+3})$, $\pi_*(L^{4A+6}_{4A+3})$ to $\pi_{*-1}(P^{4A+2}_{4A-4})$, $\pi_{*-1}(L^{4A+2}_{4A-4})$. Let i be a suitable inclusion. Let x_5' be the composite $S^{4A+8b-2} \xrightarrow{\beta_{8b-1}} S^{4A-1} \to P^{4A-1}_{4A-4}$. Then $x_5 = 2\rho(x_5')$. Lemma 4.2 (ii) implies x_5' is null in $J_{4A+8b-2}(P^{4A+6}_{4A-4})$ but not null in $J_{4A+8b-2}(P^{4A+5}_{4A-4})$. So each element $x \in J_{4A+8b-1}(P^{4A+6}_{4A+3})$ that is from the nontrivial element

$$w_1 \in E_2 = H_{4A+6}(P^{4A+6}_{4A+3}; J_{8(b-1)+1}) \approx \mathbf{Z}_2$$

in AHSS will satisfy $\partial(x) = i_*(x_5')$ when put into $J_*(P^{4A+2}_{4A-4})$. By Lemma 4.3, it is easy to see that $w_1 : S^{4A+8b-1} \xrightarrow{\beta_{8(b-1)+1}} S^{4A+6}$ coextends to a map $w_2 : S^{4A+8b-1} \to P^{4A+6}_{4A+3}$ of Adams filtration $4(b-1)$. Thus $A(\partial(w_2)) \ge 4b-3$, so $\partial'(2\rho_*(w_2))$ is of Adams filtration $\ge 4b-2$, and $\partial'(2\rho_*(w_2)) = i_*(x_5)$ when put into $J_*(L^{4A+2}_{4A-4})$ because $\partial(w_2) = i_*(x_5')$ when put into $J_*(P^{4A+2}_{4A-4})$ and because $x_5 = 2\rho(x_5')$. In ASS the chart for $E_2^{s,t}(L^{4A+2}_{4A-4})$ with $s \ge 4b-2$ and $4A + 8b - 2 \le t - s \le 4A + 8b - 1$ is given in the below



$E_2(L^{4A+2}_{4A-4})$

where v_1 is in the position $(t-s,s)=(4A+8b-2,4b-2)$ surviving to an element in $J_*(L^{4A+2}_{4A-4})$ of Adams filtration $4b-2$. Since $A(i_*(x_5))=4b-1$, $\partial'(2\rho_*(w_2))=i_*(x_5)$ because $\eta(x_5)=0$, while η is injective on classes to which v_2 and v_3 converge.

Consider (ii). As in Lemma 6.2, if $\nu(4A+8)=3$ the composite

$$S^{4A+6}\xrightarrow{\beta_7}S^{4A-1}\to P^{4A+7}_{4A-4}$$

is null-homotopic and (ii) follows by the map ρ. The case $\nu(4A+8)=4$ can be proved in the same way by noting that $((4A+8)/2)\lambda_{11}$ is stably trivial over L^7. $\square$

Lemma 6.4. (i) *There is a map* $i_0: S^{4A-1}\cup_\eta e^{4A+1}\to P^{4A+1}_{4A-2}$ *such that* g_* *on* $H_*(-;\mathbf{Z}_2)$ *is injective.*

(ii) $S^{4A+8b+2}\xrightarrow{\beta_{8b+1}}S^{4A+1}$ *coextends to a map* $w: S^{4A+8b+2}\to S^{4A-1}\cup_\eta e^{4A+1}$ *such that* $A(w)\geq 4b$, *and* $(\rho i_0)_*(w)=z$ *in* $\pi_*(L^{4A+1}_{4A-2})$, *where* z *is the composite*

$$S^{4A+8b+2}\xrightarrow{\beta_{8b+3}}S^{4A-1}\to S^{4A-1}\vee S^{4A-2}=L^{4A-1}_{4A-2}\to L^{4A+1}_{4A-2}.$$

(iii) *If* $\nu(4A+8b+4)=2$, *then* i_0w *is null-homotopic in* $\pi_*(P^{4A+8}_{4A-2})$ (*hence* z *is null-homotopic in* $\pi_*(L^{4A+8}_{4A-2})$).

Proof. Consider (i). Let $DP^{4A+1}_{4A-2}=\Sigma P^{2B+3}_{2B}=\Sigma T(2B\xi)$, where ξ is the real Hopf bundle over P^3. There is a stable vector bundle η' over S^2 corresponding to the generator of $\widetilde{KO}(S^2)$ with $p^*(\eta')=2B\bar{\xi}$. So we have a map $f_1: P^{2B+3}_{2B}\to T(\eta')=S^{2B}\cup_\eta e^{2B+2}$ inducing a monomorphism on $H^*(-;\mathbf{Z}_2)$, where $\eta=\beta_1$. Thus (i) follows by duality.

The first half of (ii) is from [4, Lemma 6] by applying the map i_0. Consider the second half. In the ASS, the chart for $E^{s,t}_3(L^{4A+1}_{4A-2})$ with $s\geq 4b$ and $t-s=4A+8b+2$, is as follows.

$E_3(L^{4A+1}_{4A-2})$ $E_2(P^{4A+4}_{4A-2})$

where u is in the position $(t-s,s)=(4A+8b+2,4b)$ and $2u$ corresponds to z in homotopy. Since $(\rho i_0)^*$ induces zero morphism on $H^*(-;\mathbf{Z}_2)$, we have $A(\rho(i_0)(w))\geq 4b+1$. Putting $\rho i_0(w)$ into $J_*(-)$, we see $A(\rho i_0(w))=4b+1$. Thus $(\rho i_0)_*(w)$ must be z.

Consider (iii). Lemma 4.2 (iv) implies that i_0w is null in $J_{4A+8b+2}(P^{4A+8}_{4A-2})$ but not null in $J_{4A+8b+2}(P^{4A+7}_{4A-2})$. As before, each coextension $x: S^{4A+8b+3}\to P^{4A+8}_{4A+5}$ of $4\beta_{8(b-1)+3}: S^{4A+8b+3}\to S^{4A+8}$ will satisfy $\partial(x)=i_0w$ in $J_*(P^{4A+4}_{4A-2})$. By Lemma 4.3, there exists a coextension $x: S^{4A+8b+3}\to P^{4A+8}_{4A+5}$ of $4\beta_{8(b-1)+3}$ with $A(x)\geq 4(b-1)+3$, hence $A(\partial(x))\geq 4b$. Then $\partial(x)=i_0w$ follows from the chart $E^{s,t}_2(P^{4A+4}_{4A-2})$ with $s\geq 4b$ and $t-s=4A+8b+2$ as given above. Here v is in the position $(t-s,s)=(4A+8b+2,4b)$, and corresponds to i_0w when put into P^{4A+4}_{4A-2}, because $A(i_0w)=4b$. $\square$

The case $\nu(4A+8b+4)\geq 3$ in Lemma 6.4 is handled in the next lemma.

Lemma 6.5. *Let i_0 and w, z be as in Lemma* 6.4.

(i) *If $3 \leq \nu(4A+8b+4) \leq 4b+1$, then the composite*

$$x_6 : S^{4A+8b+2} \xrightarrow{i_0 w} P_{4A-2}^{4A+1} \to P_{4A-2}^{4A+8b+3}$$

is null-homotopic.

(ii) *If $3 \leq \nu(4A+8b+4) \leq 4b+2$, then z is null in $\pi_*(L_{4A-2}^{4A+8b+3})$.*

Proof. Part (i) is implied by Lemma 6.4 (ii) and Lemma 5.8. Consider (ii). Note that $z = \rho x_6$. If $3 \leq \nu(4A+8b+4) \leq 4b+1$, then z is null-homotopic in $\pi_*(L_{4A-2}^{4A+8b+3})$ by (i). If $\nu(4A+8b+4) = 4b+2$, let $\lambda = \lambda_{8b+5}$, then $\frac{n}{2}\lambda$ is stably trivial over L^{8b+3}, and (ii) follows from an argument as in the proof of Lemma 6.2. $\square$

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