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TURNPIKE PROPERTY FOR EXTREMALS OF VARIATIONAL PROBLEMS WITH VECTOR-VALUED FUNCTIONS

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Abstract. In this paper we study the structure of extremals $\nu\colon [0,T] \to R^n$ of variational problems with large enough T, fixed end points and an integrand f from a complete metric space of functions. We will establish the turnpike property for a generic integrand f. Namely, we will show that for a generic integrand f, any small $\varepsilon > 0$ and an extremal $\nu\colon [0,T] \to R^n$ of the variational problem with large enough T, fixed end points and the integrand f, for each $\tau \in [L_1, T - L_1]$ the set $\{\nu(t)\colon t \in [\tau, \tau + L_2]\}$ is equal to a set $H(f)$ up to ε in the Hausdorff metric. Here $H(f) \subset R^n$ is a compact set depending only on the integrand f and $L_1 > L_2 > 0$ are constants which depend only on ε and $|\nu(0)|$, $|\nu(T)|$.

1. Introduction

In this paper we analyse the structure of optimal solutions of the variational problem

$$\text{(P)} \qquad \int_0^T f(z(t), z'(t))\,dt \to \min, \quad z(0) = x,\ z(T) = y,$$
$$z\colon [0,T] \to R^n \text{ is an absolutely continuous function}$$

where $T > 0$, $x, y \in R^n$ and $f\colon R^{2n} \to R^1$ is an integrand.

An optimal solution $\nu\colon [0,T] \to R^n$ of the variational problem (P) always depends on the integrand f and on x, y, T. We say that the integrand f has the *turnpike property* if for large enough T the dependence on x, y, T is not essential. Namely, for any $\varepsilon > 0$ there exist constants $L_1 > L_2 > 0$ which depend only on $|x|$, $|y|$, ε such that for each $\tau \in [L_1, T - L_1]$ the set

$$\{\nu(t)\colon t \in [\tau, \tau + L_2]\}$$

is equal to a set $H(f)$ up to ε in the Hausdorff metric where $H(f) \subset R^n$ is a compact set depending only on the integrand f.

More formally we say that an integrand $f = f(x,u) \in C(R^{2n})$ has the *turnpike property* if there exists a compact set $H(f) \subset R^n$ such that for each bounded set $K \subset R^n$ and each $\varepsilon > 0$ there exist numbers $L_1 > L_2 > 0$ such that for each

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$T \geq 2L_1$, each $x, y \in K$ and an optimal solution $\nu\colon [0,T] \to R^n$ for the variational problem (P) the relation

$$\operatorname{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau + L_2]\}) \leq \varepsilon$$

holds for each $\tau \in [L_1, T - L_1]$. (Here $\operatorname{dist}(\cdot,\cdot)$ is the Hausdorff metric.)

The turnpike property is well known in mathematical economics. It was studied by many authors for optimal trajectories of a von Neumann-Gale model determined by a superlinear set-valued mapping (see Makarov and Rubinov [14] and the survey [16]) and for optimal trajectories of convex autonomous systems (see Carlson, Haurie and Leizarowitz [7, Ch. 4.6].) In the control theory the turnpike property was established by Artstein and Leizarowitz [1] for a tracking periodic problem. In all these cases we have an optimal control problem with a convex cost function and a convex set of trajectories. Asymptotic turnpike properties for optimal control problems with infinite time horizon were studied by Brock and Haurie [4], Carlson [5], Carlson, Haurie and Jabrane [6], Leizarowitz [10] and Zaslavski [19].

Our goal is to show that the turnpike property is a general phenomenon which holds for a large class of variational problems with vector-valued functions. We consider the complete metric space of integrands $\overline{\mathfrak{M}}_k$ (k is a nonnegative integer) described below and establish the existence of a set $\mathcal{F} \subset \overline{\mathfrak{M}}_k$ which is a countable intersection of open everywhere dense sets in $\overline{\mathfrak{M}}_k$ and such that each integrand $f \in \mathcal{F}$ has the turnpike property.

Moreover we show that the turnpike property holds for approximate solutions of variational problems with a generic integrand f and that the turnpike phenomenon is stable under small perturbations of a generic integrand f.

A better understanding of the general nature of the turnpike phenomenon was achieved by our recent study of discrete-time control systems [17, 18] for which we established a weak version of the turnpike property. More recently in Zaslavski [20] employing the reduction to finite rewards by Leizarowitz [11, 12] and the representation formula by Leizarowitz and Mizel [13] an analogous result was established for optimal solutions of the variational problem (P) with $x, y \in R^n$, large enough T and a generic integrand f belonging to the space of functions $\mathfrak{A}$ described below.

In the weak version of the turnpike property established in [20] for an optimal solution of the problem (P) with $x, y \in R^n$, large enough T and a generic integrand $f \in \mathfrak{A}$ the relation

$$\operatorname{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau + L_2]\}) \leq \varepsilon$$

with L_2 which depends on ε and $|x|$, $|y|$ and a compact set $H(f) \subset R^n$ depending only on the integrand f, holds for each $\tau \in [0,T] \setminus E$ where $E \subset [0,T]$ is a measurable subset such that the Lebesgue measure of E does not exceed a constant which depends on ε and $|x|$, $|y|$.

The turnpike property which will be established in the present work guarantees that we may take $E = [0, L_1] \cup [T - L_1, T]$ where $L_1 > 0$ is a constant which depends on ε and $|x|$, $|y|$.

Denote by $|\cdot|$ the Euclidean norm in R^n and denote by $\mathfrak{A}$ the set of continuous functions $f\colon R^n \times R^n \to R^1$ which satisfy the following assumptions:

(A)(i) *for each* $x \in R^n$ *the function* $f(x, \cdot)\colon R^n \to R^1$ *is convex*;

(ii) $f(x,u) \geq \sup\{\psi(|x|), \psi(|u|)|u|\} - a$ *for each* $(x,u) \in R^n \times R^n$ *where* $a > 0$ *is a constant and* $\psi\colon [0,\infty) \to [0,\infty)$ *is an increasing function such that* $\psi(t) \to +\infty$ *as* $t \to \infty$ (*here* a *and* ψ *are independent on* f);

(iii) *for each* $M, \varepsilon > 0$ *there exist* $\Gamma, \delta > 0$ *such that*

$$|f(x_1, u_1) - f(x_2, u_2)| \le \varepsilon \sup\{f(x_1, u_1), f(x_2, u_2)\}$$

for each $u_1, u_2, x_1, x_2 \in R^n$ *which satisfy*

$$|x_i| \le M,\ |u_i| \ge \Gamma\ (i = 1, 2), \quad \sup\{|x_1 - x_2|, |u_1 - u_2|\} \le \delta.$$

It is an elementary exercise to show that an integrand $f = f(x, u) \in C^1(R^{2n})$ belongs to $\mathfrak{A}$ if f satisfies assumptions (Ai), (Aii) with a constant $a > 0$ and a function $\psi\colon [0, \infty) \to [0, \infty)$ and there exists an increasing function $\psi_0\colon [0, \infty) \to [0, \infty)$ such that for each $x, u \in R^n$

$$\sup\{|\partial f/\partial x(x, u)|, |\partial f/\partial u(x, u)|\} \le \psi_0(|x|)(1 + \psi(|u|)|u|).$$

For the set $\mathfrak{A}$ we consider the uniformity which is determined by the following base

$$E(N, \varepsilon, \lambda) = \{(f, g) \in \mathfrak{A} \times \mathfrak{A}\colon |f(x, u) - g(x, u)| \le \varepsilon\ (u, x \in R^n, |x|, |u| \le N), \\ (|f(x, u)| + 1)(|g(x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda]\ (x, u \in R^n, |x| \le N)\}$$

where $N > 0$, $\varepsilon > 0$, $\lambda > 1$ (see Kelley [9]).

It was shown in Zaslavski [20] that the uniform space $\mathfrak{A}$ is metrizable and complete. We consider functionals of the form

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t))\, dt \tag{1.1}$$

where $f \in \mathfrak{A}$, $0 \le T_1 \le T_2 < +\infty$ and $x\colon [T_1, T_2] \to R^n$ is an absolutely continuous (a.c.) function.

For $f \in \mathfrak{A}$, $y, z \in R^n$ and numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ we set

$$U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x)\colon x\colon [T_1, T_2] \to R^n \text{ is an a.c. function} \\ \text{satisfying } x(T_1) = y,\ x(T_2) = z\}. \tag{1.2}$$

It is easy to see that $-\infty < U^f(T_1, T_2, y, z) < +\infty$ for each $f \in \mathfrak{A}$, each $y, z \in R^n$ and all numbers T_1, T_2 satisfying $0 \le T_1 < T_2$.

Let $f \in \mathfrak{A}$. For any a.c. function $x\colon [0, \infty) \to R^n$ we set

$$J(x) = \liminf_{T \to \infty} T^{-1} I^f(0, T, x). \tag{1.3}$$

Of special interest is the *minimal long-run average cost growth rate*

$$\mu(f) = \inf\{J(x)\colon x\colon [0, \infty) \to R^n \text{ is an a.c. function}\}. \tag{1.4}$$

Clearly $-\infty < \mu(f) < +\infty$. By a simple modification of the proof of Proposition 4.4 in Leizarowitz and Mizel [13] (see [20, Theorems 8.1, 8.2]) we established the representation formula

$$U^f(0, T, x, y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta^f_T(x, y), \\ x, y \in R^n, \quad T \in (0, \infty), \tag{1.5}$$

where $\pi^f\colon R^n \to R^1$ is a continuous function and $(T,x,y) \to \theta^f_T(x,y) \in R^1$ is a continuous nonnegative function defined for $T > 0$, $x, y \in R^n$,

$$(1.6)\quad \pi^f(x) = \inf\Big\{ \liminf_{T\to+\infty} [I^f(0,T,\nu) - \mu(f)T] \colon \nu\colon [0,\infty) \to R^n \text{ is an a.c. function satisfying } \nu(0) = x \Big\},\quad x \in R^n,$$

and for every $T > 0$, every $x \in R^n$ there is $y \in R^n$ satisfying $\theta^f_T(x,y) = 0$.

Here we follow Leizarowitz [11] in defining "good functions" for the infinite horizon variational problem with the integrand f.

An a.c. function $x\colon [0,\infty) \to R^n$ is called an (f)-*good function* if the function $\Phi^f_x\colon T \to I^f(0,T,x) - \mu(f)T$, $T \in (0,\infty)$ is bounded. In [20] we showed that for each $f \in \mathfrak{A}$ and each $z \in R^n$ there exists an (f)-good function $\nu\colon [0,\infty) \to R^n$ satisfying $\nu(0) = z$.

Propositions 1.1 and 3.2 in Zaslavski [20] imply the following result.

Proposition 1.1. *For any a.c. function $x\colon [0,\infty) \to R^n$ either*

$$I^f(0,T,x) - T\mu(f) \to +\infty \text{ as } T \to \infty$$

or

$$\sup\{|I^f(0,T,x) - T\mu(f)| \colon T \in (0,\infty)\} < \infty.$$

Moreover any (f)-good function $x\colon [0,\infty) \to R^n$ is bounded.

We denote $d(x,B) = \inf\{|x-y| \colon y \in B\}$ for $x \in R^n$, $B \subset R^n$. Denote by $\mathrm{dist}(A,B)$ the Hausdorff metric for two sets $A \subset R^n$, $B \subset R^n$. For every bounded a.c. function $x\colon [0,\infty) \to R^n$ define

$$(1.7)\quad \Omega(x) = \{y \in R^n \colon \text{there exists a sequence } \{t_i\}_{i=0}^\infty \subset (0,\infty) \text{ for which } t_i \to \infty,\ x(t_i) \to y \quad \text{as} \quad i \to \infty\}.$$

We say that an integrand $f \in \mathfrak{A}$ has Property B if $\Omega(\nu_2) = \Omega(\nu_1)$ for all (f)-good functions $\nu_i\colon [0,\infty) \to R^n$, $i = 1, 2$.

In Zaslavski [20, Theorem 2.1] we establish the following result which describes the limit behaviour of (f)-good functions for a generic $f \in \mathfrak{A}$.

Theorem 1.1. *There exists a set $\mathcal{F} \subset \mathfrak{A}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$ and such that each $f \in \mathcal{F}$ has Property B.*

By Proposition 1.1 for each integrand $f \in \mathfrak{A}$ which has Property B there exists a compact set $H(f) \subset R^n$ such that $\Omega(\nu) = H(f)$ for each (f)-good function $\nu\colon [0,\infty) \to R^n$.

Denote by $\mathfrak{M}$ the set of all functions $f \in C^1(R^{2n})$ satisfying the following assumptions which ensure that each solution of (P) belongs to $C^2([0,T];R^n)$:

$$\partial f/\partial u_i \in C^1(R^{2n}) \quad \text{for } i = 1, \dots, n;$$

the matrix $(\partial^2 f/\partial u_i \partial u_j)(x,u)$, $i, j = 1, \dots, n$, is positive definite for all $(x,u) \in R^{2n}$;

$$f(x,u) \geq \sup\{\psi(|x|), \psi(|u|)|u|\} - a \quad \text{for all } (x,u) \in R^n \times R^n;$$

there exist a number $c_0 > 1$ and monotone increasing functions $\phi_i\colon [0,\infty) \to [0,\infty)$, $i = 0, 1, 2$, such that

$$\phi_0(t)t^{-1} \to +\infty \text{ as } t \to +\infty,\ f(x,u) \geq \phi_0(c_0|u|) - \phi_1(|x|), \quad x, u \in R^n;$$

$$\sup\{|\partial f/\partial x_i(x,u)|, |\partial f/\partial u_i(x,u)|\} \leq \phi_2(|x|)(1 + \phi_0(|u|)), \qquad x, u \in R^n,\ i = 1, \dots, n.$$

It is easy to see that $\mathfrak{M} \subset \mathfrak{A}$. We will establish the following result.

Theorem 1.2. *Assume that an integrand $f \in \mathfrak{M}$ has Property B and $\varepsilon, K > 0$. Then there exists a neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_0$ and each a.c. function $\nu\colon [0,T] \to R^n$ which satisfies*

$$|\nu(0)|, |\nu(T)| \leq K, \quad I^g(0,T,\nu) \leq U^g(0,T,\nu(0),\nu(T)) + \delta$$

the relation $|\nu(t)| \leq M$ holds for all $t \in [0,T]$ and

$$\operatorname{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau + l]\}) \leq \varepsilon \tag{1.8}$$

for each $\tau \in [l_0, T - l_0]$. Moreover if $d(\nu(0), H(f)) \leq \delta$, then (1.8) holds for each $\tau \in [0, T - l_0]$ and if $d(\nu(T), H(f)) \leq \delta$, then (1.8) holds for each $\tau \in [l_0, T - l]$.

Let $k \geq 1$ be an integer. Denote by $\mathfrak{A}_k$ the set of all integrands $f \in \mathfrak{A} \cap C^k(R^{2n})$. For $p = (p_1, \dots, p_{2n}) \in \{0, \dots, k\}^{2n}$ and $f \in C^k(R^{2n})$ we set

$$|p| = \sum_{i=1}^{2n} p_i, \quad D^p f = \partial^{|p|} f / \partial y_1^{p_1} \dots \partial y_{2n}^{p_{2n}}.$$

For the set $\mathfrak{A}_k$ we consider the uniformity which is determined by the following base.

$$\begin{aligned} E(N,\varepsilon,\lambda) = \{(f,g) \in \mathfrak{A}_k \times \mathfrak{A}_k\colon & |D^p f(x,u) - D^p g(x,u)| \leq \varepsilon \\ & (u, x \in R^n, |x|, |u| \leq N, p \in \{0, \dots, k\}^{2n}, |p| \leq k), \\ & |f(x,u) - g(x,u)| \leq \varepsilon \ (u, x \in R^n, |x|, |u| \leq N), \\ & (|f(x,u)| + 1)(|g(x,u)| + 1)^{-1} \in [\lambda^{-1}, \lambda](x, u \in R^n, |x| \leq N)\} \end{aligned}$$

where $N > 0$, $\varepsilon > 0$, $\lambda > 1$ (see Kelley [9]). It is easy to verify that the uniform space $\mathfrak{A}_k$ is metrizable and complete (see [20], Section 2).

For each integer $k \geq 1$ we define $\mathfrak{M}_k = \mathfrak{M} \cap \mathfrak{A}_k$. Set

$$\mathfrak{A}_0 = \mathfrak{A}, \quad \mathfrak{M}_0 = \mathfrak{M}.$$

Let $k \geq 0$ be an integer. Denote by $\overline{\mathfrak{M}}_k$ the closure of $\mathfrak{M}_k$ in $\mathfrak{A}_k$ and consider the topological subspace $\overline{\mathfrak{M}}_k \subset \mathfrak{A}_k$ with the relative topology. We will establish the following result.

Theorem 1.3. *Let $q \geq 0$ be an integer. Then there exists a set $\mathcal{F}_q \subset \overline{\mathfrak{M}}_q$ which is a countable intersection of open everywhere dense subsets of $\overline{\mathfrak{M}}_q$ and such that each $f \in \mathcal{F}_q$ has Property B and the following property*:

For each $\varepsilon, K > 0$ there exist a neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_0$ and each a.c. function $\nu\colon [0,T] \to R^n$ which satisfies

$$|\nu(0)|,\ |\nu(t)| \leq K, \quad I^g(0,T,\nu) \leq U^g(0,T,\nu(0),\nu(T)) + \delta$$

the relation $|\nu(t)| \le M$ *holds for all* $t \in [0, T]$ *and*

$$\operatorname{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau + l]\}) \le \varepsilon \tag{1.9}$$

for each $\tau \in [l_0, T - l_0]$. *Moreover if* $d(\nu(0), H(f)) \le \delta$, *then* (1.9) *holds for each* $\tau \in [0, T - l_0]$ *and if* $d(\nu(T), H(f)) \le \delta$, *then* (1.9) *holds for each* $\tau \in [l_0, T - l]$.

Theorems 1.2 and 1.3 are extensions to the class of variational problems with vector-valued functions of the main result in Zaslavski [21] established for a class of one-dimensional variational problems arising in continuum mechanics which was discussed in Leizarowitz and Mizel [13] and Coleman, Marcus and Mizel [8]. In the approach used in [21] the following property of this class of one-dimensional variational problems established in Leizarowitz and Mizel [13] played a crucial role.

Property C. In the space of integrands there exists an everywhere dense subset E such that for each $f \in E$ there exists an (f)-good periodic trajectory.

It is not clear whether Property C holds in general. In Zaslavski [20] and in the present paper we develop a more general approach based on the idea that the validity of Property B implies the weak version of the turnpike property for an integrand $f \in \mathfrak{A}$ and implies the turnpike property for an integrand $f \in \mathfrak{M}$.

2. Auxiliary results

In [20] we established the following results.

Proposition 2.1 ([20, Proposition 3.1]). *For each* $f \in \mathfrak{A}$ *there exists a neighborhood* $\mathcal{U}$ *of* f *in* $\mathfrak{A}$ *and a number* $M > 0$ *such that for each* $g \in \mathcal{U}$ *and each* (g)*-good function* $x\colon [0, \infty) \to R^n$

$$\limsup_{t \to \infty} |x(t)| < M.$$

Proposition 2.2 ([20, Proposition 3.2]). *Let* $f \in \mathfrak{A}$ *and* $M_1, M_2, c > 0$. *Then there exist a neighborhood* $\mathcal{U}$ *of* f *in* $\mathfrak{A}$ *and* $S > 0$ *such that for each* $g \in \mathcal{U}$, *each* $T_1 \in [0, \infty)$ *and each* $T_2 \in [T_1 + c, \infty)$ *the following property holds.*

For each $x, y \in R^n$ *satisfying* $|x|, |y| \le M_1$ *and each a.c. function* $\nu\colon [T_1, T_2] \to R^n$ *satisfying* $\nu(T_1) = x$, $\nu(T_2) = y$, $I^g(T_1, T_2, \nu) \le U^g(T_1, T_2, x, y) + M_2$ *the following relation holds:* $|\nu(t)| \le S$ $(t \in [T_1, T_2])$.

Proposition 2.3 ([20, Proposition 3.8]). *Let* $f \in \mathfrak{A}$, $0 < c_1 < c_2 < \infty$, $D, \varepsilon > 0$. *Then there is a neighborhood* V *of* f *in* $\mathfrak{A}$ *such that for each* $g \in V$, *each* $T_1, T_2 \ge 0$ *satisfying* $T_2 - T_1 \in [c_1, c_2]$ *and each a.c. function* $x\colon [T_1, T_2] \to R^n$ *satisfying* $\inf\{I^f(T_1, T_2, x), I^g(T_1, T_2, x)\} \le D$ *the relation* $|I^f(T_1, T_2, x) - I^g(T_1, T_2, x)| \le \varepsilon$ *holds.*

Proposition 2.4 ([20, Proposition 3.9]). *Let* $f \in \mathfrak{A}$, $0 < c_1 < c_2 < \infty$, $c_3, \varepsilon > 0$. *Then there exists a neighborhood* V *of* f *in* $\mathfrak{A}$ *such that for each* $g \in V$, *each* $T_1, T_2 \ge 0$ *satisfying* $T_2 - T_1 \in [c_1, c_2]$ *and each* $z, y \in R^n$ *satisfying* $|y|, |z| \le c_3$ *the relation* $|U^f(T_1, T_2, y, z) - U^g(T_1, T_2, y, z)| \le \varepsilon$ *holds.*

Proposition 2.5 ([20, Theorem 5.1]). *Assume that* $f \in \mathfrak{A}$ *and there exists a compact set* $H(f) \subset R^n$ *such that* $\Omega(\nu) = H(f)$ *for each* (f)*-good function* $\nu\colon [0, \infty) \to R^n$. *Let* ε *be a positive number. Then there exists an integer* $L \ge 1$ *such that for each* (f)*-good function* $\nu\colon [0, \infty) \to R^n$

$$\operatorname{dist}(H(f), \{\nu(t)\colon t \in [T, T + L]\}) \le \varepsilon \quad \text{for all large } T.$$

Proposition 2.6 ([20, Theorem 6.1]). *Assume that* $f \in \mathfrak{A}$. *Then the mapping* $(T_1, T_2, x, y) \to U^f(T_1, T_2, x, y)$ *is continuous for* $T_1 \in [0, \infty)$, $T_2 \in (T_1, \infty)$, $x, y \in R^n$.

Proposition 2.7 ([20, Theorem 2.3]). *Assume that* $f \in \mathfrak{A}$ *and there exists a compact set* $H(f) \subset R^n$ *such that* $\Omega(\nu) = H(f)$ *for each* (f)*-good function* $\nu\colon [0, \infty) \to R^n$. *Let* ε *be a positive number. Then there exist an integer* $L \geq 1$ *and a neighborhood* $\mathcal{U}$ *of* f *in* $\mathfrak{A}$ *such that for each* $g \in \mathcal{U}$ *and each* (g)*-good function* $\nu\colon [0, \infty) \to R^n$

$$\mathrm{dist}(H(f), \{\nu(t)\colon t \in [T, T+L]\}) \leq \varepsilon \quad \textit{for all large } T.$$

Proposition 2.8 ([20, Lemma 10.2]). *Assume that* $f \in \mathfrak{A}$ *and* $H(f) \subset R^n$ *is a compact set such that* $\Omega(\nu) = H(f)$ *for each* (f)*-good function* $\nu\colon [0, \infty) \to R^n$. *Let* $\varepsilon_0 \in (0, 1)$, $K_0, M_0 > 0$ *and let* l *be a positive integer such that for each* (f)*-good function* $x\colon [0, \infty) \to R^n$

$$\mathrm{dist}(H(f), \{x(t)\colon t \in [T, T+l]\}) \leq 8^{-1}\varepsilon_0$$

for all large T (*the existence of* l *follows from Proposition 2.5*). *Then there exist an integer* $N \geq 10$ *and a neighborhood* $\mathcal{U}$ *of* f *in* $\mathfrak{A}$ *such that for each* $g \in \mathcal{U}$, *each* $S \in [0, \infty)$ *and each a.c. function* $x\colon [S, S + Nl] \to R^n$ *satisfying*

$$|x(S)|,\ |x(S + Nl)| \leq K_0,$$
$$I^g(S, S + Nl, x) \leq U^g(S, S + Nl, x(S), x(S + Nl)) + M_0$$

there exists an integer $i_0 \in [0, N-8]$ *such that for all* $T \in [S + i_0 l, S + (i_0 + 7)l]$

$$\mathrm{dist}(H(f), \{x(t)\colon t \in [T, T+l]\}) \leq \varepsilon_0.$$

Proposition 2.9 ([20, Lemma 10.3]). *Assume that* $f \in \mathfrak{A}$ *and* $H(f) \subset R^n$ *is a compact set such that* $\Omega(\nu) = H(f)$ *for each* (f)*-good function* $\nu\colon [0, \infty) \to R^n$. *Let* $\varepsilon > 0$. *Then there exists* $\delta > 0$ *such that for each* $x_1, x_2 \in R^n$ *which satisfy* $d(x_i, H(f)) \leq \delta$, $i = 1, 2$ *there exists an a.c. function* $\nu\colon [0, T] \to R^n$ *for which*

$$T \geq 1,\ \nu(0) = x_1,\ \nu(T) = x_2, \quad I^f(0, T, \nu) - \pi^f(x_1) + \pi^f(x_2) - T\mu(f) \leq \varepsilon.$$

Proposition 2.10 ([20, Lemma 10.4]). *Assume that* $f \in \mathfrak{A}$ *and* $H(f) \subset R^n$ *is a compact set such that* $\Omega(\nu) = H(f)$ *for each* (f)*-good function* $\nu\colon [0, \infty) \to R^n$. *Let* $\varepsilon \in (0, 1)$ *and let* L *be a positive integer such that for each* (f)*-good function* $\nu\colon [0, \infty) \to R^n$

$$\mathrm{dist}(H(f), \{\nu(t)\colon t \in [S, S+L]\}) \leq \varepsilon \tag{2.1}$$

for all large S (*the existence of* L *follows from Proposition 2.5*).

Then there exists $\delta > 0$ *such that for each* $T \in [L, \infty)$ *and each a.c. function* $\nu\colon [0, T] \to R^n$ *which satisfies*

$$d(\nu(0), H(f)) \leq \delta,\ d(\nu(T), H(f)) \leq \delta,$$
$$I^f(0, T, \nu) - T\mu(f) - \pi^f(\nu(0)) + \pi^f(\nu(T)) \leq \delta$$

relation (2.1) *holds for every* $S \in [0, T-L]$.

Proposition 2.11 ([20, Lemma 9.1]). *Assume that* $f \in \mathfrak{A}$. *Then there exists a compact set* $H^* \subset R^n$ *which has the following properties*:

there exists an (f)*-good function* $u\colon [0, \infty) \to R^n$ *such that* $\Omega(u) = H^*$;

for each (f)*-good function* $\nu\colon [0, \infty) \to R^n$ *either* $\Omega(\nu) = H^*$ *or* $\Omega(\nu) \backslash H^* \neq \varnothing$.

Proposition 2.12 ([20, Lemmas 9.3, 9.4])**.** *Let $f \in \mathfrak{A}$ and let H^* be as guaranteed in Proposition 2.11. Assume that $\phi\colon R^n \to [0,\infty)$ is a continuous bounded function such that $H^* = \{x \in R^n : \phi(x) = 0\}$. For $r \in (0,1]$ we set*

$$f_r(x,u) = f(x,u) + r\phi(x), \ \ x, u \in R^n.$$

Then $f_r \in \mathfrak{A}$, $r \in (0,1]$ and for each $r \in (0,1]$ and each (f_r)-good function $\nu\colon [0,\infty) \to R^n$

$$\Omega(\nu) = H^*.$$

Moreover for any neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ there exists $r_0 \in (0,1)$ such that $f_r \in \mathfrak{A}$ for every $r \in (0, r_0)$.

Proposition 2.13 ([20, Proposition 3.4])**.** *Assume that $f \in \mathfrak{A}$, $M_1 > 0$, $0 \le T_1 < T_2$, $x_i\colon [T_1,T_2] \to R^n$, $i = 1,2,\dots$ is a sequence of a.c. functions such that $I^f(T_1,T_2,x_i) \le M_1$, $i = 1,2,\dots$. Then there exist a subsequence $\{x_{i_k}\}_{k=1}^\infty$ and an a.c. function $x\colon [T_1,T_2] \to R^n$ such that $I^f(T_1,T_2,x) \le M_1$, $x_{i_k}(t) \to x(t)$ as $k \to \infty$ uniformly in $[T_1,T_2]$ and $x'_{i_k} \to x'$ as $k \to \infty$ weakly in $L^1(R^n;(T_1,T_2))$.*

Theorem 8.1, (8.2) and Proposition 7.3 in [20] imply the following result.

Proposition 2.14. *Let $f \in \mathfrak{A}$. Then $\pi^f(x) \to +\infty$ as $|x| \to \infty$.*

The following result was established in Aubin and Ekeland [2, Ch. 2, Sec. 3].

Proposition 2.15. *Let Ω be a closed subset of R^q. Then there exists a bounded nonnegative function $\phi \in C^\infty(R^q)$ such that $\Omega = \{x \in R^q : \phi(x) = 0\}$ and for each sequence of nonnegative integers $p_1,\dots,p_q$ the function $\partial^{|p|}\phi/\partial x_1^{p_1},\dots,\partial x_q^{p_q}\colon R^q \to R^1$ is bounded where $|p| = \sum_{i=1}^q p_i$.*

We can establish the following proposition which is a higher dimensional version of a well known result (Ball and Mizel [3], Morrey [15]).

Proposition 2.16. *Suppose that $f \in \mathfrak{M}$, $x, y \in R^n$, $T_1 \in [0,\infty)$, $T_2 > T_1$ and $w\colon [T_1,T_2] \to R^n$ is an a.c. function such that*

$$w(T_1) = x, \quad w(T_2) = y, \quad I^f(T_1,T_2,w) = U^f(T_1,T_2,x,y).$$

Then $w \in C^2([T_1,T_2];R^n)$.

3. Structure of the proof of Theorem 1.2

Assume that $f \in \mathfrak{M}$ and $H(f) \subset R^n$ is a compact set such that $\Omega(\nu) = H(f)$ for each (f)-good function $\nu\colon [0,\infty) \to R^n$.

We will describe briefly the proof of Lemma 4.4 which is established in Section 4 and which plays a crucial role in our discussion of Theorem 1.2.

For each a.c. function $u\colon [\tau_1,\tau_2] \to R^n$ where $\tau_1 \ge 0$, $\tau_2 > \tau_1$ and each $r_1, r_2 \in [\tau_1,\tau_2]$ satisfying $r_1 < r_2$ we set

$$\sigma(r_1,r_2,u) = I^f(r_1,r_2,u) - \pi^f(u(r_1)) + \pi^f(u(r_2)) - (r_2 - r_1)\mu(f).$$

Let $\varepsilon > 0$. To prove Lemma 4.4 we need to show that there is a number $q \ge 8$ such that for each $h_1, h_2 \in H(f)$ there exists an a.c. function $\nu\colon [0,q] \to R^n$ which satisfies

$$\nu(0) = h_1, \quad \nu(q) = h_2, \quad \sigma(0,q,\nu) \le \varepsilon. \tag{3.1}$$

By Proposition 2.6 there exists a sequence of positive numbers $\{\delta_i\}_{i=0}^{\infty}$ such that

$$\delta_0 \in (0, 8^{-1}\varepsilon), \quad \delta_{i+1} < \delta_i, \quad i = 0, 1, \dots . \tag{3.2}$$

and for each integer $i \geq 0$, each $x_1, x_2, y_1, y_2 \in H(f)$ which satisfy $|x_j - y_j| \leq \delta_i$, $j = 1, 2$ the following relations hold:

$$\begin{aligned} |U^f(0,1,x_1,x_2) - U^f(0,1,y_1,y_2)| \leq 2^{-i-8}\varepsilon, \\ |\pi^f(x_j) - \pi^f(y_j)| \leq 2^{-i-8}\varepsilon, \quad j = 1, 2. \end{aligned} \tag{3.3}$$

We will show that there exists an (f)-good function $\nu_*\colon [0,\infty) \to H(f)$ such that $\sigma(T_1, T_2, \nu_*) = 0$ for each $T_1 \geq 0$, $T_2 > T_1$. Then we will define a function $\phi\colon [0,\infty) \to R^1$ as

$$\phi(\tau) = I^f(0, 1, \nu_* + P_\tau) - \mu(f) - \pi^f(\nu_*(0)) + \pi^f(\nu_*(\tau)), \quad \tau \in [0, \infty), \tag{3.4}$$

where

$$P_\tau(t) = t(\nu_*(\tau) - \nu_*(1)), \quad t \in R^1, \quad \tau \in [0,\infty), \tag{3.5}$$

and verify that

$$\phi \in C^1([0,\infty); R^1),\ \phi(1) = 0,\ \phi(t) \geq 0,\ t \in [0,\infty). \tag{3.6}$$

We can find a number $L \geq 10$ and a sequence of numbers $\{T_p\}_{p=1}^{\infty}$ such that

$$\mathrm{dist}(H(f), \{\nu_*(t)\colon t \in [T, T+L]\}) \leq 4^{-1}\delta_0 \text{ for all } T \in [0,\infty), \tag{3.7}$$

$$T_p \geq 2L + 8,\ |\nu_*(0) - \nu_*(T_p)| \leq 2^{-8}\delta_p, \quad p = 1, 2, \dots . \tag{3.8}$$

Fix a positive number $\varepsilon_0 < 2^{-8}L^{-1}\varepsilon$. By (3.6) there exists a positive number Δ such that

$$\Delta < 2^{-8},\ |\phi'(t)| \leq 2^{-1}\varepsilon_0,\ t \in [1-\Delta, 1+\Delta]. \tag{3.9}$$

Choose an integer $N > 64(L+1)\Delta^{-1}$ and set $q = \sum_{i=1}^{N} T_i + 8L + 8$.

Let $h_1, h_1 \in H(f)$. We will construct an a.c. function $\nu\colon [0,q] \to R^n$ satisfying (3.1). By the definition of L there exists numbers t_1, t_2 such that

$$t_1 \in [0, L],\ t_2 \in [8, L+8],\ |h_j - \nu_*(t_j)| \leq 4^{-1}\delta_0, \quad j = 1, 2. \tag{3.10}$$

We set $\Delta_0 = (N-1)^{-1}(8L+8-(t_2-t_1))$ and verify that $\Delta_0 \in (0, \Delta)$. By using (3.8), (3.10) and the definition of $\{\delta_i\}_{i=0}^{\infty}$ we can construct functions $w_0\colon [0, T_1 - t_1] \to R^n$ and $u_0\colon [0, t_2] \to R^n$ such that

$$\begin{aligned} w_0(0) = h_1, \quad w_0(T_1 - t_1) = \nu_*(0), \quad \sigma(0, T_1 - t_1, w_0) \leq 2^{-6}\varepsilon, \\ u_0(0) = \nu_*(0), \quad u_0(t_2) = h_2, \quad \sigma(0, t_2, u_0) \leq 2^{-7}\varepsilon. \end{aligned} \tag{3.11}$$

For each integer $k \geq 1$ there exists an a.c. function $w_k\colon [0, \Delta_0 + T_{k+1}] \to R^n$ such that

$$w_k(t) = \nu_*(t) + P_{1-\Delta_0}(t),\ t \in [0,1], \quad w_k(t) = \nu_*(t - \Delta_0),\ t \in [1, \Delta_0 + T_{k+1} - 1],$$
$$w_k(\Delta_0 + T_{k+1}) = \nu_*(0),$$
$$I^f(\Delta_0 + T_{k+1} - 1, \Delta_0 + T_{k+1}, w_k) = U^f(0, 1, w_k(\Delta_0 + T_{k+1} - 1), w_k(\Delta_0 + T_{k+1})).$$

By using (3.4)–(3.6), (3.9), (3.8) and the definition of $\{\delta_i\}_{i=0}^{\infty}$ we show that

$$\sigma(0, T_{k+1} + \Delta_0, w_k) \leq 2^{-1}\varepsilon_0\Delta_0 + 2^{-k-8}\varepsilon, \qquad k = 1, 2, \dots .$$

We will finally define $\nu\colon [0,q] \to R^n$ as a concatenation of the functions w_k, $k = 0, \dots, N-1$, u_0 and show that (3.1) holds.

The structure of the proof of Theorem 1.2. For simplicity we will only sketch the proof of the turnpike property for the integrand f and will not discuss the stability of the turnpike phenomenon under small perturbations of f. We choose a small enough number $\delta > 0$ and large enough numbers $l_0 > l > 0$ depending on ε, K.

Assume that $T \geq 2l_0$ and an a.c. function $\nu\colon [0,T] \to R^n$ satisfies

$$|\nu(0)|,\ |\nu(T)| \leq K, \quad I^f(0,T,\nu) \leq U^f(0,T,\nu(0),\nu(T)) + \delta. \tag{3.12}$$

We will show that for each $\tau \in [l_0, T-l_0]$

$$\mathrm{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau+l]\}) \leq \varepsilon. \tag{3.13}$$

Assume the contrary. Then there is a number $\tau \in [l_0, T-l_0]$ for which

$$\mathrm{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau+l]\}) > \varepsilon. \tag{3.14}$$

By Proposition 2.8 there are numbers $S_1, S_2 \in [0,T]$ such that

$$d(\nu(S_i), H(f)) \leq \delta, \quad i = 1,2, \quad S_2 - \tau,\ \tau - S_1 \in [c_1, c_2], \tag{3.15}$$

where c_1, c_2 are some positive constants depending on ε, K.

It follows from (3.14), (3.15) and Proposition 2.10 that

$$\sigma(S_1, S_2, \nu) > \delta_0 \tag{3.16}$$

where $\delta_0 > 8\delta$ is some constant depending on ε, K. By using (3.15) and Lemma 4.4 we show that there exists an a.c. function $u\colon [0,T] \to R^n$ such that

$$u(t) = \nu(t), \quad t \in [0, S_1] \cup [S_2, T], \quad \sigma(S_1, S_2, u) < \delta_0 - \delta. \tag{3.17}$$

It follows from (3.12), (3.16), (3.17) that

$$\delta \geq I^f(0,T,\nu) - I^f(0,T,u) = \sigma(S_1,S_2,\nu) - \sigma(S_1,S_2,u) > \delta.$$

The obtained contradiction proves that (3.13) holds for each $\tau \in [l_0, T-l_0]$.

4. Proof of Theorem 1.2

Assume that $f \in \mathfrak{M}$ and $H(f) \subset R^n$ is a compact set such that $\Omega(\nu) = H(f)$ for each (f)-good function $\nu\colon [0,\infty) \to R^n$.

Lemma 4.1. *Let $h \in H(f)$. Then there exists an (f)-good function $\nu\colon [0,\infty) \to H(f)$ such that $\nu(0) = h$ and*

$$I^f(T_1, T_2, \nu) = \mu(f)(T_2 - T_1) + \pi^f(\nu(T_1)) - \pi^f(\nu(T_2)) \tag{4.1}$$

for each $T_1 \geq 0$, $T_2 > T_1$.

Proof. Consider any (f)-good function $w\colon [0,\infty) \to R^n$. Then

$$\Omega(w) = H(f).$$

By Proposition 2.1 the function w is bounded. It is easy to see that the following property holds:

(a) for each $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that for each $T_1 \geq T(\varepsilon)$, $T_2 > T_1$

$$I^f(T_1, T_2, w) - \mu(f)(T_2 - T_1) - \pi^f(w(T_1)) + \pi^f(w(T_2)) \leq \varepsilon.$$

There exists a sequence of numbers $\{T_p\}_{p=0}^{\infty} \subset [0,\infty)$ such that

$$T_{p+1} \geq T_p + 1, \quad p = 0,1,\dots,\ w(T_p) \to h \quad \text{as} \quad p \to \infty. \tag{4.2}$$

For every integer $p \geq 1$ we set

$$\nu_p(t) = w(t + T_p), \quad t \in [0,\infty). \tag{4.3}$$

By Proposition 2.13, the boundness of w, (4.3) and property (a) there exists a subsequence $\{\nu_{p_j}\}_{j=1}^{\infty}$ and an a.c. function $\nu\colon [0,\infty) \to R^n$ such that for each integer $N \geq 1$

$$\nu_{p_j}(t) \to \nu(t) \quad \text{as} \quad j \to \infty \text{ uniformly in } [0,N], \tag{4.4}$$
$$I^f(0,N,\nu) \leq \liminf_{j\to\infty} I^f(0,N,\nu_{p_j}).$$

(4.2)–(4.4) imply that $\nu(0) = h$ and $\nu(t) \in H(f)$, $t \in [0,\infty)$. It follows from property (a) and (4.3), (4.4) that (4.1) holds for each $T_1 \geq 0$, $T_2 > T_1$. The lemma is proved.

By Lemma 4.1 there exists an (f)-good function $\nu_*\colon [0,\infty) \to H(f)$ such that

$$I^f(T_1,T_2,\nu_*) = \mu(f)(T_2 - T_1) + \pi^f(\nu_*(T_1)) - \pi^f(\nu_*(T_2)) \tag{4.5}$$

for each $T_1 \geq 0$, $T_2 > T_1$.

It follows from Proposition 2.16 that

$$\nu_* \in C^2([0,\infty);R^n). \tag{4.6}$$

Lemma 4.2. *The function* $\pi^f \cdot \nu_* \in C^1([0,\infty);R^1)$.

Proof. By (4.5) for each $T \geq 0$

$$\pi^f(\nu_*(T)) = -I^f(0,T,\nu_*) + \mu(f)T + \pi^f(\nu_*(0)).$$

Together with (4.6) this implies the assertion of the lemma.

For each $\tau \in [0,\infty)$ we define

$$P_\tau(t) = t(\nu_*(\tau) - \nu_*(1)), \quad t \in R^1, \quad \psi(\tau) = I^f(0,1,\nu_* + P_\tau). \tag{4.7}$$

Lemma 4.3. $\psi \in C^1([0,\infty);R^1)$.

Proof. For $\lambda, t \in [0,\infty)$ we set

$$B(\lambda,t) = (\nu_*(t) + P_\lambda(t), \nu_*'(t) + P_\lambda'(t)). \tag{4.8}$$

Let $\tau, h \in [0,\infty)$, $\tau \neq h$ and $t \in [0,1]$. By (4.7), (4.8) there exists $\lambda_h(t) \in [\inf\{h,\tau\}, \sup\{h,\tau\}]$ such that

$$(h-\tau)^{-1}[f(B(h,t)) - f(B(\tau,t))] = \partial f/\partial x(B(\lambda_h(t),t))t\nu_*'(\lambda_h(t))$$
$$+ \partial f/\partial u(B(\lambda_h(t),t))\nu_*'(\lambda_h(t)) \to \partial f/\partial x(B(\tau,t))t\nu_*'(\tau) + \partial f/\partial u(B(\tau,t))\nu_*'(\tau)$$

as $h \to \tau$ uniformly for all $t \in [0,1]$. This implies that $\psi \in C^1([0,\infty);R^1)$. The lemma is proved.

Lemma 4.4. *Let* $\varepsilon > 0$. *Then there exists a number* $q \geq 8$ *such that for each* $h_1, h_2 \in H(f)$ *there exists an a.c. function* $\nu\colon [0,q] \to R^n$ *which satisfies*

$$\nu(0) = h_1, \quad v(q) = h_2, \tag{4.9}$$

$$I^f(0,q,\nu) \leq q\mu(f) + \pi^f(\nu(0)) - \pi^f(\nu(q)) + \varepsilon. \tag{4.10}$$

Proof. Define a function $\phi\colon [0,\infty) \to R^1$ as follows:

$$\phi(t) = \psi(t) - \mu(f) - \pi^f(\nu_*(0)) + \pi^f(\nu_*(t)), \quad t \in [0,\infty). \tag{4.11}$$

It follows from (4.11), (4.7), Lemmas 4.2, 4.3, (4.5) and the representation formula (see (1.5), (1.6)) that

$$\phi \in C^1([0,\infty);R^1), \quad \phi(1) = 0, \quad \phi(t) \geq 0, \quad t \in [0,\infty). \tag{4.12}$$

By Proposition 2.6 there exists a sequence of positive numbers $\{\delta_i\}_{i=0}^{\infty}$ such that

$$\delta_0 \in (0, 8^{-1}\varepsilon), \quad \delta_{i+1} < \delta_i, \quad i = 0, 1, \dots . \tag{4.13}$$

and for each integer $i \geq 0$, each $x_1, x_2, y_1, y_2 \in H(f)$ which satisfy $|x_j - y_j| \leq \delta_i$, $j = 1, 2$ the following relations hold:

$$\begin{gathered} |U^f(0,1,x_1,x_2) - U^f(0,1,y_1,y_2)| \leq 2^{-i-8}\varepsilon, \\ |\pi^f(x_j) - \pi^f(y_j)| \leq 2^{-i-8}\varepsilon, \quad j = 1, 2. \end{gathered} \tag{4.14}$$

By the definition of ν, Proposition 2.5 there exists an integer $L \geq 10$ such that

$$\mathrm{dist}(H(f), \{\nu_*(t) \colon t \in [T, T+L]\} \leq 4^{-1}\delta_0 \tag{4.15}$$

for all $T \in [0, \infty)$.

Since $\Omega(\nu_*) = H(f)$ and $\nu_*(0) \in H(f)$ there exists a sequence of numbers $\{T_p\}_{p=1}^{\infty}$ such that

$$T_p \geq 2L + 8, \quad |\nu_*(0) - \nu_*(T_p)| \leq 2^{-8}\delta_p, \quad p = 1, 2, \dots . \tag{4.16}$$

Fix a positive number ε_0 for which

$$\varepsilon_0 < 2^{-8}L^{-1}\varepsilon. \tag{4.17}$$

It follows from (4.12) that there exists a positive number Δ such that

$$\Delta < 2^{-8}, \quad |\phi'(t)| \leq 2^{-1}\varepsilon_0, \quad t \in [1-\Delta, 1+\Delta]. \tag{4.18}$$

Choose an integer

$$N > 64(L+1)\Delta^{-1} \tag{4.19}$$

and set

$$q = \sum_{i=1}^{N} T_i + 8L + 8. \tag{4.20}$$

Let $h_1, h_1 \in H(f)$. We will construct an a.c. function $\nu \colon [0, q] \to R^n$ satisfying (4.9), (4.10). It follows from (4.15) which holds for each $T \in [0, \infty)$ that there exists numbers t_1, t_2 such that

$$t_1 \in [0, L], \quad t_2 \in [8, L+8], \quad |h_j - \nu_*(t_j)| \leq 4^{-1}\delta_0, \quad j = 1, 2. \tag{4.21}$$

Set

$$\Delta_0 = (N-1)^{-1}(8L + 8 - (t_2 - t_1)). \tag{4.22}$$

(4.22), (4.21), (4.19), (4.18) imply that

$$\Delta_0 \in (0, \Delta). \tag{4.23}$$

For each a.c. function $u \colon [\tau_1, \tau_2] \to R^n$ where $\tau_1 \geq 0$, $\tau_2 > \tau_1$ and each $r_1, r_2 \in [\tau_1, \tau_2]$ satisfying $r_1 < r_2$ we set

$$\sigma(r_1, r_2, u) = I^f(r_1, r_2, u) - \pi^f(u(r_1)) + \pi^f(u(r_2)) - (r_2 - r_1)\mu(f). \tag{4.24}$$

It follows from (4.16), (4.21) and Proposition 2.13 that there exists an a.c. function $w_0 \colon [0, T_1 - t_1] \to R^n$ such that

(4.25)

$$\begin{gathered} w_0(0) = h_1, \quad w_0(t) = \nu_*(t_1 + t), \quad t \in [1, T_1 - t_1 - 1], \quad w_0(T_1 - t_1) = \nu_*(0), \\ I^f(\tau, \tau+1, w_0) = U^f(0, 1, w_0(\tau), w_0(\tau+1)), \qquad \tau = 0, T_1 - t_1 - 1. \end{gathered}$$

By (4.24), the definition of ν_*, (4.5), (4.25), (4.21), (4.16) and the definition of $\{\delta_j\}_{j=0}^{\infty}$

$$\begin{aligned}
&\sigma(0,T_1-t_1,w_0)\\
&\quad=\sigma(0,1,w_0)+\sigma(T_1-t_1-1,T_1-t_1,w_0)\\
&\quad=U^f(0,1,h_1,\nu_*(t_1+1))-\pi^f(h_1)+\pi^f(\nu_*(t_1+1))-\mu(f)\\
&\qquad+U^f(0,1,\nu_*(T_1-1),\nu_*(0))-\pi^f(\nu_*(T_1-1))+\pi^f(\nu_*(0))-\mu(f)\\
&\quad\le 4\cdot 2^{-8}\varepsilon+U^f(0,1,\nu_*(t_1),\nu_*(t_1+1))-\pi^f(\nu_*(t_1))\\
&\qquad+\pi^f(\nu_*(t_1+1))-\mu(f)+U^f(0,1,\nu_*(T_1-1),\nu_*(T_1))\\
&\qquad-\pi^f(\nu_*(T_1-1))+\pi^f(\nu_*(T_1))-\mu(f)\le 2^{-6}\varepsilon.
\end{aligned}\tag{4.26}$$

Let $k\ge 1$ be an integer. By (4.16), (4.7), (4.23), (4.18) and Proposition 2.13 there exists an a.c. function $w_k\colon[0,\Delta_0+T_{k+1}]\to R^n$ such that

(4.27)
$$\begin{gathered}
w_k(t)=\nu_*(t)+P_{1-\Delta_0}(t),\ t\in[0,1],\ w_k(t)=\nu_*(t-\Delta_0),\ t\in[1,\Delta_0+T_{k+1}-1],\\
w_k(\Delta_0+T_{k+1})=\nu_*(0),\\
I^f(\Delta_0+T_{k+1}-1,\Delta_0+T_{k+1},w_k)=U^f(0,1,w_k(\Delta_0+T_{k+1}-1),w_k(\Delta_0+T_{k+1})).
\end{gathered}$$

By (4.27), (4.7)

$$w_k(0)=\nu_*(0).\tag{4.28}$$

We will estimate $\sigma(0,T_{k+1}+\Delta_0,w_k)$. It follows from (4.27), (4.24), (4.5) that

$$\sigma(0,T_{k+1}+\Delta_0,w_k)=\sigma(0,1,w_k)+\sigma(T_{k+1}+\Delta_0-1,T_{k+1}+\Delta_0,w_k).\tag{4.29}$$

(4.27), (4.24), (4.7), (4.11) imply that

$$\sigma(0,1,w_k)=\phi(1-\Delta_0).\tag{4.30}$$

It follows from (4.30), (4.23), (4.18), (4.12) that

$$\sigma(0,1,w_k)\le 2^{-1}\Delta_0\varepsilon_0.\tag{4.31}$$

By (4.27), (4.24), (4.16), the definition of ν_*, (4.5), the definition of $\{\delta_i\}_{i=0}^{\infty}$ (see (4.13), (4.14))

$$\begin{aligned}
&\sigma(T_{k+1}+\Delta_0-1,T_{k+1}+\Delta_0,w_k)\\
&\quad=U^f(0,1,\nu_*(T_{k+1}-1),\nu_*(0))-\pi^f(\nu_*(T_{k+1}-1))\\
&\qquad+\pi^f(\nu_*(0))-\mu(f)\\
&\quad\le U^f(0,1,\nu_*(T_{k+1}-1),\nu_*(T_{k+1}))-\pi^f(\nu_*(T_{k+1}-1))\\
&\qquad+\pi^f(\nu_*(T_{k+1}))-\mu(f)+2\cdot 2^{-k-9}\varepsilon=2^{-k-8}\varepsilon.
\end{aligned}\tag{4.32}$$

Combining (4.29), (4.31), (4.32) we obtain that

$$\sigma(0,T_{k+1}+\Delta_0,w_k)\le 2^{-1}\varepsilon_0\Delta_0+2^{-k-8}\varepsilon.\tag{4.33}$$

By Proposition 2.13 there exists an a.c. function $u_0\colon[0,t_2]\to R^n$ such that

$$\begin{gathered}
u_0(t)=\nu_*(t),\ t\in[0,t_2-1],\ u_0(t_2)=h_2,\\
I^f(t_2-1,t_2,u_0)=U^f(0,1,u_0(t_2-1),u_0(t_2)).
\end{gathered}\tag{4.34}$$

It follows from (4.34), (4.24), (4.21), the definition of $\{\delta_i\}_{i=0}^{\infty}$ (see (4.13), (4.14)), the definition of ν_*, (4.5) that

$$\begin{aligned}\sigma(0,t_2,u_0) &= \sigma(t_2-1,t_2,u_0)\\ &= U^f(0,1,\nu_*(t_2-1),h_2) - \pi^f(\nu_*(t_2-1)) + \pi^f(h_2) - \mu(f)\\ &\le U^f(0,1,\nu_*(t_2-1),\nu_*(t_2)) - \pi^f(\nu_*(t_2-1))\\ &\quad + \pi^f(\nu_*(t_2)) - \mu(f) + 2^{-7}\varepsilon \le 2^{-7}\varepsilon.\end{aligned} \tag{4.35}$$

(4.22), (4.20) imply that

$$T_1 - t_1 + \sum_{k=1}^{N-1}(\Delta_0 + T_{k+1}) + t_2 = q. \tag{4.36}$$

By (4.36), (4.25), (4.27), (4.28), (4.34) there exists an a.c. function $\nu\colon [0,q] \to R^n$ such that

$$\begin{aligned}&\nu(t) = w_0(t),\ t\in[0,T_1-t_1],\quad \nu(t) = w_k\left(t - \left(\sum_{i=1}^k T_i + (k-1)\Delta_0 - t_1\right)\right),\\ &t \in \left[\sum_{i=1}^k T_i + (k-1)\Delta_0 - t_1, \sum_{i=1}^{k+1} T_i + k\Delta_0 - t_1\right],\quad k = 1,\dots,N-1,\\ &\nu(t) = u_0\left(t - \left(\sum_{i=1}^N T_i + (N-1)\Delta_0 - t_1\right)\right),\quad t\in\left[\sum_{i=1}^N T_i + (N-1)\Delta_0 - t_1, q\right].\end{aligned} \tag{4.37}$$

(4.37), (4.25), (4.36), (4.34) imply that

$$\nu(0) = h_1,\quad \nu(q) = h_2.$$

It follows from (4.24), (4.37), (4.26), (4.33), (4.35), (4.22), (4.21), (4.17) that

$$\begin{aligned}&I^f(0,q,\nu) - \pi^f(\nu(0)) + \pi^f(\nu(q)) - q\mu(f)\\ &\quad = \sigma(0,T_1-t_1,w_0) + \sum_{k=1}^{N-1}\sigma(0,T_{k+1}+\Delta_0,w_k) + \sigma(0,t_2,u_0)\\ &\quad \le 2^{-6}\varepsilon + \sum_{k=1}^{N-1}(2^{-1}\varepsilon_0\Delta_0 + 2^{-k-8}\varepsilon) + 2^{-7}\varepsilon \le 2^{-5}\varepsilon + 2^{-1}(N-1)\varepsilon_0\Delta_0\\ &\quad \le 2^{-5}\varepsilon + 2^{-1}(9L+16)\varepsilon_0 \le 2^{-1}\varepsilon.\end{aligned}$$

This completes the proof of the lemma.

Proof of Theorem 1.2. Let $\varepsilon, K > 0$. We may assume that

$$\varepsilon < 1,\ K > \sup\{|h|\colon h\in H(f)\} + 4.$$

By Proposition 2.2 there exist a neighborhood $\mathcal{U}_1$ of f in $\mathfrak{A}$ and a number $M > K$ such that for each $g\in\mathcal{U}_1$, each $T_1 \ge 0$, $T_2 \ge T_1 + 1$ and each a.c. function $\nu\colon [T_1,T_2]\to R^n$ which satisfies

$$|\nu(T_i)| \le 2K+4,\ i=1,2,\quad I^g(T_1,T_2,\nu) \le U^g(T_1,T_2,\nu(T_1),\nu(T_2)) + 2 \tag{4.38}$$

the following relation holds:

$$|\nu(t)| \le M\ (t\in[T_1,T_2]). \tag{4.39}$$

By Proposition 2.6 there exists

$$\delta_1 \in (0, 8^{-1}\varepsilon) \tag{4.40}$$

such that for each $x_1, x_2, y_1, y_2 \in R^n$ which satisfy

$$|x_i|, |y_i| \le 2M + 4 + 2\sup\{|h| \colon h \in H(f)\}, \quad |x_i - y_i| \le 4\delta_1, \quad i = 1, 2 \tag{4.41}$$

the following relations hold:

$$\begin{gathered} |U^f(0, 1, x_1, x_2) - U^f(0, 1, y_1, y_2)| \le 2^{-8}\varepsilon, \\ |\pi^f(x_i) - \pi^f(y_i)| \le 2^{-8}\varepsilon, \quad i = 1, 2. \end{gathered} \tag{4.42}$$

By Proposition 2.5 there exists an integer $l \ge 1$ such that for each (f)-good function $\nu \colon [0, \infty) \to R^n$

$$\operatorname{dist}(H(f), \{\nu(t) \colon t \in [T, T + l]\}) \le \varepsilon \tag{4.43}$$

for all large T. By Proposition 2.10 there exists

$$\delta_0 \in (0, 2^{-1}\delta_1) \tag{4.44}$$

such that for each $T \in [l, \infty)$ and each a.c. function $\nu \colon [0, T] \to R^n$ which satisfies

$$d(\nu(0), H(f)) \le \delta_0, \quad d(\nu(T), H(f)) \le \delta_0, \tag{4.45}$$

$$I^f(0, T, \nu) - T\mu(f) - \pi^f(\nu(0)) + \pi^f(\nu(T)) \le \delta_0 \tag{4.46}$$

the relation

$$\operatorname{dist}(H(f), \{\nu(t) \colon t \in [S, S + l]\}) \le \varepsilon \tag{4.47}$$

holds for every $S \in [0, T - l]$. By Proposition 2.6 there exists

$$\delta \in (0, 32^{-1}\delta_0) \tag{4.48}$$

such that for each $x_1, x_2, y_1, y_2 \in R^n$ which satisfy

$$|x_i|, |y_i| \le 2M + 4 + 2\sup\{|h| \colon h \in H(f)\}, \quad |x_i - y_i| \le 4\delta, \quad i = 1, 2, \tag{4.49}$$

the following relations hold:

$$\begin{gathered} |U^f(0, 1, x_1, x_2) - U^f(0, 1, y_1, y_2)| \le 2^{-8}\delta_0, \\ |\pi^f(x_i) - \pi^f(y_i)| \le 2^{-8}\delta_0, \quad i = 1, 2. \end{gathered} \tag{4.50}$$

By Proposition 2.5 there exists an integer $L \ge 1$ such that for each (f)-good function $\nu \colon [0, \infty) \to R^n$

$$\operatorname{dist}(H(f), \{\nu(t) \colon t \in [T, T + L]\}) \le 8^{-1}\delta \tag{4.51}$$

for all large T.

By Proposition 2.8 there exists an integer $N \ge 10$ and a neighborhood $\mathcal{U}_2$ of f in $\mathfrak{A}$ such that for each $g \in \mathcal{U}_2$, each $S \in [0, \infty)$ and each a.c. function $x \colon [S, S+NL] \to R^n$ which satisfies

$$\begin{gathered} |x(S)|, |x(S + NL)| \le 2M + 2, \\ I^g(S, S + NL, x) \le U^g(S, S + NL, x(S), x(S + NL)) + 4 \end{gathered} \tag{4.52}$$

there exists an integer $i_0 \in [0, N - 8]$ such that for all $T \in [S + i_0 L, S + (i_0 + 7)L]$

$$\operatorname{dist}(H(f), \{x(t) \colon t \in [T, T + L]\}) \le \delta. \tag{4.53}$$

By Lemma 4.4 there exists a number $q \geq 8$ such that for each $h_1, h_2 \in H(f)$ there exists an a.c. function $\nu\colon [0,q] \to R^n$ which satisfies

$$\nu(0) = h_1, \quad v(q) = h_2, \quad I^f(0,q,\nu) - q\mu(f) - \pi^f(\nu(0)) + \pi^f(\nu(q)) \leq 8^{-1}\delta. \tag{4.54}$$

By Proposition 2.3 there exists a neighborhood $\mathcal{U}_3$ of f in $\mathfrak{A}$ such that for each $g \in \mathcal{U}_3$, each $T_1 \geq 0$, $T_2 \in [T_1 + 8^{-1}, T_1 + 6N(q+l+L)]$ and each a.c. function $x\colon [T_1,T_2] \to R^n$ which satisfies

$$\begin{aligned} &\inf\{I^f(T_1,T_2,x), I^g(T_1,T_2,x)\} \\ &\quad \leq 4 + 2\sup\{|\pi^f(h)|\colon h \in R^n, |h| \leq \sup\{|z|\colon z \in H(f)\} + 4\} \\ &\quad\quad + 6|\mu(f)|N(q+l+L) \end{aligned} \tag{4.55}$$

the following relation holds

$$|I^f(T_1,T_2,x) - I^g(T_1,T_2,x)| \leq 4^{-1}\delta. \tag{4.56}$$

By Proposition 2.4 there exists a neighborhood $\mathcal{U}_4$ of f in $\mathfrak{A}$ such that for each $g \in \mathcal{U}_4$, each $x_1, x_2 \in R^n$ which satisfy

$$|x_1|, |x_2| \leq 2M + 4 + 2\sup\{|z|\colon z \in H(f)\}$$

the following relation holds:

$$|U^f(0,1,x_1,x_2) - U^g(0,1,x_1,x_2)| \leq 2^{-8}\delta. \tag{4.57}$$

Set

$$l_0 = 2l + 2q + 2NL + 6, \tag{4.58}$$

$$\mathcal{U} = \bigcap_{i=1}^{4} \mathcal{U}_i. \tag{4.59}$$

Assume that $g \in \mathcal{U}$, $T \geq 2l_0$ and an a.c. function $\nu\colon [0,T] \to R^n$ satisfies

$$|\nu(0)|, |\nu(T)| \leq K, \quad I^g(0,T,\nu) \leq U^g(0,T,\nu(0),\nu(T)) + \delta. \tag{4.60}$$

It follows from the definition of $\mathcal{U}_1$ (see (4.38), (4.39)) and (4.60) that

$$|\nu(t)| \leq M, \quad t \in [0,T]. \tag{4.61}$$

Assume that there exist numbers $S_1, S_2 \in [0,T]$ such that

$$d(\nu(S_i), H(f)) \leq \delta, \quad i = 1,2, \quad S_2 - S_1 \in [1 + l + q, 5N(L + l + q)]. \tag{4.62}$$

We will show that for each $\tau \in [S_1, S_2 - l]$

$$\operatorname{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau + l]\}) \leq \varepsilon. \tag{4.63}$$

Let us assume the converse. Then there exists a number τ such that

$$\tau \in [S_1, S_2 - l], \quad \operatorname{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau + l]\}) > \varepsilon. \tag{4.64}$$

It follows from (4.62), (4.64), (4.48) and the definition of δ_0 (see (4.44)–(4.47)) that

$$I^f(S_1,S_2,\nu) - (S_2 - S_1)\mu(f) - \pi^f(\nu(S_1)) + \pi^f(\nu(S_2)) > \delta_0. \tag{4.65}$$

We will show that

$$I^g(S_1,S_2,\nu) - (S_2 - S_1)\mu(f) - \pi^f(\nu(S_1)) + \pi^f(\nu(S_2)) \geq 2^{-1}\delta_0. \tag{4.66}$$

Assume the contrary. Then by (4.62)

$$I^g(S_1,S_2,\nu) \leq 2\sup\{|\pi^f(z)|\colon z \in R^n, d(z, H(f)) \leq 1\} + |\mu(f)|(S_2 - S_1) + 1.$$

It follows from this relation, (4.62) and the definition of $\mathcal{U}_3$ (see (4.55), (4.56)) that

$$|I^f(S_1, S_2, \nu) - I^g(S_1, S_2, \nu)| \leq 4^{-1}\delta.$$

Together with (4.65) this implies (4.66). The obtained contradiction proves that (4.66) holds.

By (4.62) there exist $h_1, h_2 \in H(f)$ such that

$$|\nu(S_i) - h_i| \leq \delta, \quad i = 1, 2. \tag{4.67}$$

By Lemma 4.1 there exists an (f)-good function

$$w_0\colon [0, \infty) \to H(f) \quad \text{such that} \quad w_0(0) = h_1, \tag{4.68}$$

$$I^f(t_1, t_2, w_0) - (t_2 - t_1)\mu(f) - \pi^f(w_0(t_1)) + \pi^f(w_0(t_2)) = 0 \tag{4.69}$$

for each $t_1 \geq 0$, $t_2 > t_1$.

It follows from the definition of q (see (4.54), (4.62), (4.68)) that there exists an a.c. function $w_1\colon [0, q] \to R^n$ such that

$$\begin{gathered} w_1(0) = w_0(S_2 - S_1 - q), \quad w_1(q) = h_2, \\ I^f(0, q, w_1) - q\mu(f) - \pi^f(w_1(0)) + \pi^f(w_1(q)) \leq 8^{-1}\delta. \end{gathered} \tag{4.70}$$

By (4.62), Proposition 2.13, (4.68), (4.70) there exists an a.c. function $u\colon [0, T] \to R^n$ such that

$$\begin{gathered} u(t) = \nu(t), \quad t \in [0, S_1] \cup [S_2, T], \\ u(t) = w_0(t - S_1), \quad t \in [S_1 + 1, S_2 - q], \\ u(t) = w_1(t - (S_2 - q)), \quad t \in [S_2 - q, S_2 - 1], \\ I^g(r, r+1, u) = U^g(0, 1, u(r), u(r+1)), \quad r = S_1, S_2 - 1. \end{gathered} \tag{4.71}$$

For each a.c. function $y\colon [a, b] \to R^n$ where $a \geq 0$, $b > a$ and each $r_1, r_2 \in [a, b]$ satisfying $r_1 \leq r_2$ we set

$$\sigma(r_1, r_2, y) = I^g(r_1, r_2, y) - \pi^f(y(r_1)) + \pi^f(y(r_2)) - (r_2 - r_1)\mu(f). \tag{4.72}$$

It follows from (4.60), (4.71), (4.72) that

$$\begin{aligned} \delta &\geq I^g(0, T, \nu) - I^g(0, T, u) \\ &= \sigma(0, T, \nu) - \sigma(0, T, u) = \sigma(S_1, S_2, \nu) - \sigma(S_1, S_2, u). \end{aligned} \tag{4.73}$$

By (4.70), (4.68) and the definition of M (see (4.38), (4.39))

$$|w_1(t)| \leq M, \quad t \in [0, q]. \tag{4.74}$$

By (4.70) there exists an a.c. function $\tilde{w}\colon [S_1, S_2] \to R^n$ such that

$$\begin{gathered} \tilde{w}(t) = w_0(t - S_1), \quad t \in [S_1, S_2 - q], \\ \tilde{w}(t) = w_1(t - (S_2 - q)), \quad t \in [S_2 - q, S_2]. \end{gathered} \tag{4.75}$$

(4.73), (4.72), (4.66), (4.71), (4.75) imply that

$$\begin{aligned} \delta \geq 2^{-1}\delta_0 - \sigma(S_1, S_2, u) &= 2^{-1}\delta_0 - \sigma(S_1, S_2, \tilde{w}) \\ &+ [\sigma(S_1, S_1 + 1, \tilde{w}) - \sigma(S_1, S_1 + 1, u)] \\ &+ [\sigma(S_2 - 1, S_2, \tilde{w}) - \sigma(S_2 - 1, S_2, u)]. \end{aligned} \tag{4.76}$$

We will estimate $\sigma(S_1, S_2, \tilde{w})$ and $\sigma(h, h+1, \tilde{w}) - \sigma(h, h+1, u)$, $h = S_1, S_2 - 1$.

Let $h \in \{S_1, S_2 - 1\}$. (4.70), (4.75), (4.71), (4.74), (4.68), (4.61), (4.62), (4.67) imply that

$$|\tilde{w}(h)|, |\tilde{w}(h+1)|, |u(h)|, |u(h+1)| \le M + \sup\{|z| \colon z \in H(f)\},$$
$$|\tilde{w}(h) - u(h)|, |\tilde{w}(h+1) - u(h+1)| \le \delta.$$

It follows from these relations, (4.71), (4.72), the definition of $\mathcal{U}_4$ (see (4.57)) and δ (see (4.49), (4.50), (4.48)) that

$$\begin{aligned} &\sigma(h, h+1, \tilde{w}) - \sigma(h, h+1, u) \\ &\quad \ge U^g(0, 1, \tilde{w}(h), \tilde{w}(h+1)) - \pi^f(\tilde{w}(h)) + \pi^f(\tilde{w}(h+1)) \\ &\qquad - [U^g(0, 1, u(h), u(h+1)) - \pi^f(u(h)) + \pi^f(u(h+1))] \\ &\quad \ge U^f(0, 1, \tilde{w}(h), \tilde{w}(h+1)) - \pi^f(\tilde{w}(h)) + \pi^f(\tilde{w}(h+1)) \\ &\qquad - [U^f(0, 1, u(h), u(h+1)) - \pi^f(u(h)) + \pi^f(u(h+1))] - 2^{-7}\delta \\ &\quad \ge -2^{-6}\delta_0, \quad h \in \{S_1, S_2 - 1\}. \end{aligned} \tag{4.77}$$

We will estimate $\sigma(S_1, S_2, \tilde{w})$. It follows from (4.69), (4.70), (4.75) that

$$I^f(S_1, S_2, \tilde{w}) - \pi^f(\tilde{w}(S_1)) + \pi^f(\tilde{w}(S_2)) - (S_2 - S_1)\mu(f) \le 8^{-1}\delta. \tag{4.78}$$

By this relation, (4.62), (4.75), (4.68), (4.70) and the definition of $\mathcal{U}_3$ (see (4.55), (4.56))

$$|I^f(S_1, S_2, \tilde{w}) - I^g(S_1, S_2, \tilde{w})| \le 4^{-1}\delta.$$

Together with (4.78), (4.72) this implies that

$$\sigma(S_1, S_2, \tilde{w}) \le 3 \cdot 8^{-1}\delta.$$

It follows from this relation, (4.76), (4.77) that

$$\delta \ge 2^{-1}\delta_0 - 3 \cdot 8^{-1}\delta - 2^{-5}\delta_0.$$

This is contradictory to (4.48). The obtained contradiction proves that (4.63) holds for each $\tau \in [S_1, S_2 - l]$. Therefore we have shown that the following property holds:

Property D. For each $S_1, S_2 \in [0, T]$ which satisfy (4.62) relation (4.63) holds for each $\tau \in [S_1, S_2 - l]$.

It follows from (4.60), (4.61) and the definition of $\mathcal{U}_2, N$ (see (4.52), (4.53)) that for each $r_0 \in [0, T - (1 + l + q + L(N+2))]$ there exists a number r_1 such that

$$r_1 - r_0 \in [1 + l + q + 2L, 1 + l + q + L(N+2)], \quad d(\nu(r_1), H(f)) \le \delta.$$

This implies that there exists a finite sequence of numbers $\{S_i\}_{i=0}^{Q} \subset [0, T]$ such that

$$S_0 = 0, S_{i+1} - S_i \in [1 + l + q + 2L, 1 + l + q + L(N+2)], \quad i = 0, \dots, Q-1,$$
$$T - S_Q \le 1 + l + q + L(N+2), \quad d(\nu(S_i), H(f)) \le \delta, \quad i = 1, \dots, Q.$$

The assertion of the theorem follows from these relations and Property D.

5. Proof of Theorem 1.3

Lemma 5.1. *Assume that an integrand $f \in \mathfrak{M}$ has Property B and $\varepsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ such that for each $g \in \mathcal{U}$ and each (g)-good function $\nu\colon [0,\infty) \to R^n$*

$$\operatorname{dist}(\Omega(\nu), H(f)) \le \varepsilon.$$

Proof. By Proposition 2.1 there exist a neighborhood $\mathcal{U}_1$ of f in $\mathfrak{A}$ and a number $K > 0$ such that for each $g \in \mathcal{U}_1$ and each (g)-good function $\nu\colon [0,\infty) \to R^n$

$$\limsup_{t\to\infty} |\nu(t)| < K.$$

By Theorem 1.2 there exist a neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ which satisfies $\mathcal{U} \subset \mathcal{U}_1$ and numbers $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \ge 2l_0$ and each a.c. function $\nu\colon [0,T] \to R^n$ which satisfies

$$|\nu(0)|, |\nu(T)| \le K, \quad I^g(0,T,\nu) \le U^g(0,T,\nu(0),\nu(T)) + \delta$$

the relation $\operatorname{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau + l]\}) \le \varepsilon$ holds for each $\tau \in [l_0, T - l_0]$.

Assume that $g \in \mathcal{U}$ and $v\colon [0,\infty) \to R^n$ is a (g)-good function. It follows from the definition of $\mathcal{U}$, $\mathcal{U}_1$, K and Proposition 1.1 that there exists a number $T_0 \ge 0$ such that

$$|\nu(t)| \le K, \quad t \in [T_0, \infty),$$
$$I^g(t_1,t_2,\nu) \le U^g(t_1,t_2,\nu(t_1),\nu(t_2)) + \delta \quad \text{for each } t_1 \ge T_0,\ t_2 > t_1.$$

It follows from these relations and the definition of $\mathcal{U}$, l_0, l, δ that

$$\operatorname{dist}(H(f), \Omega(\nu)) \le \varepsilon.$$

The lemma is proved.

Construction of the set $\mathcal{F}_q$. Suppose that q is a nonnegative integer. By Propositions 2.11, 2.12, 2.15 there exists a set $E_q \subset \mathfrak{M}_q$ which is an everywhere dense subset of $\overline{\mathfrak{M}}_q$ and such that each integrand $f \in E_q$ has Property B. Therefore for each $f \in E_q$ there exists a compact set $H(f) \subset R^n$ such that $\Omega(\nu) = H(f)$ for each (f)-good function $\nu\colon [0,\infty) \to R^n$.

By Theorem 1.2 and Lemma 5.1 for each $f \in E_q$ and each integer $p \ge 1$ there exist an open neighborhood $\mathcal{U}(f,p)$ of f in $\mathfrak{A}$ and numbers $M(f,p) > p$, $l_0(f,p) > l(f,p) > 0$, $\delta(f,p) \in (0, p^{-1})$ such that for each $g \in \mathcal{U}(f,p)$ and each (g)-good function $\nu\colon [0,\infty) \to R^n$

$$\operatorname{dist}(H(f), \Omega(\nu)) \le 4^{-1}\delta(f,p);$$

for each $g \in \mathcal{U}(f,p)$, each $T \ge 2l_0(f,p)$ and each a.c. function $\nu\colon [0,T] \to R^n$ which satisfies

$$|\nu(0)|, |\nu(T)| \le p, \quad I^g(0,T,\nu) \le U^g(0,T,\nu(0),\nu(T)) + \delta(f,p) \tag{5.1}$$

the relation $|\nu(t)| \le M(f,p)$ holds for all $t \in [0,T]$ and the following properties hold:

(i) for each $\tau \in [l_0(f,p), T - l_0(f,p)]$

$$\operatorname{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau + l(f,p)]\}) \le p^{-1}; \tag{5.2}$$

(ii) if $d(\nu(0), H(f)) \le \delta(f,p)$, then (5.2) holds for each $\tau \in [0, T - l_0(f,p)]$;
(iii) if $d(\nu(T), H(f)) \le \delta(f,p)$, then (5.2) holds for each $\tau \in [l_0(f,p), T - l(f,p)]$.

We define

$$\mathcal{F}_q = \left[\bigcap_{p=1}^{\infty} \cup\{\mathcal{U}(f,p)\colon f \in E_q\}\right] \cap \overline{\mathfrak{M}}_q. \tag{5.3}$$

Clearly $\mathcal{F}_q$ is a countable intersection of open everywhere dense subsets of $\overline{\mathfrak{M}}_q$.

Assume that $f \in \mathcal{F}_q$, $\varepsilon, K > 0$. Fix a natural number p such that

$$p > 2K + 4 + 8\varepsilon^{-1}. \tag{5.4}$$

There exists $G \in E_q$ such that

$$f \in \mathcal{U}(G,p). \tag{5.5}$$

It follows from (5.4), (5.5) and the definition of $\mathcal{U}(G,p)$, $\delta(G,p)$ that for each (f)-good function $\nu\colon [0,\infty) \to R^n$

$$\mathrm{dist}(H(G), \Omega(\nu)) \le 4^{-1}\delta(G,p) < (4p)^{-1} < 8^{-1}\varepsilon. \tag{5.6}$$

This implies that for each (f)-good function $\nu_i\colon [0,\infty) \to R^n$, $i = 1,2$,

$$\mathrm{dist}(\Omega(\nu_1), \Omega(\nu_2)) \le \varepsilon.$$

Since ε is any positive number we conclude that f has Property B and there exists a compact set $H(f) \subset R^n$ such that $\Omega(w) = H(f)$ for each (f)-good function $w\colon [0,\infty) \to R^n$. It follows from (5.6) that

$$\mathrm{dist}(H(G), H(f)) \le 4^{-1}\delta(G,p). \tag{5.7}$$

Set

$$\begin{aligned} &\mathcal{U} = \mathcal{U}(G,p), \quad M = M(G,p),\\ &l_0 = l_0(G,p), \quad l = l(G,p), \quad \delta = 8^{-1}\delta(G,p). \end{aligned} \tag{5.8}$$

Assume that $g \in \mathcal{U}$, $T \ge 2l_0$ and an a.c. function $\nu\colon [0,T] \to R^n$ satisfies

$$|\nu(0)|, |\nu(T)| \le K, \quad I^g(0,T,\nu) \le U^g(0,T,\nu(0),\nu(T)) + \delta. \tag{5.9}$$

It follows from (5.9), (5.8), (5.5), (5.4) and the definition of $\mathcal{U}(G,p)$, $M(G,p)$, $l_0(G,p)$, $l(G,p)$, $\delta(G,p)$ that

$$|\nu(t)| \le M, \quad t \in [0,T], \tag{5.10}$$

and properties (i)–(iii) hold with $f = G$. Together with (5.7), (5.8), (5.4) this implies that

$$\mathrm{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau + l]\}) \le \varepsilon \tag{5.11}$$

for each $\tau \in [l_0, T - l_0]$; if $d(\nu(0), H(f)) \le \delta$, then (5.11) holds for each $\tau \in [0, T - l_0]$. If $d(\nu(T), H(f)) \le \delta$, then (5.11) holds for each $\tau \in [l_0, T - l]$. This completes the proof of the theorem.

6. Examples

Fix a constant $a > 0$ and set $\psi(t) = t$, $t \in [0,\infty)$. Consider the complete metric space $\mathfrak{A}$ of integrands $f\colon R^n \times R^n \to R^1$ defined in Section 1.

Example 1. Consider an integrand $f(x,u) = |x|^2 + |u|^2$, $x, u \in R^n$. It is easy to see that $f \in \mathfrak{M}_q$ for each integer $q \ge 0$ if the constant a is large enough. We can show (see [20, Section 14]) that $\Omega(\nu) = \{0\}$ for every (f)-good function $\nu\colon [0,\infty) \to R^n$. Therefore Theorem 1.2 holds with the integrand f.

Example 2. Fix a number $q > 0$ and consider an integrand

$$g(x,u) = q|x|^2|x-e|^2 + |u|^2, \quad x, u \in R^n,$$

where $e = (1, 1, \ldots, 1)$. It is easy to see that $g \in \mathfrak{M}$ if the constant a is large enough. Clearly f does not have the turnpike property.

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