SCRAMBLED SETS OF CONTINUOUS MAPS OF 1-DIMENSIONAL POLYHEDRA

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Abstract. Let \( K \) be a 1-dimensional simplicial complex in \( \mathbb{R}^3 \) without isolated vertexes, \( X = |K| \) be the polyhedron of \( K \) with the metric \( d_K \) induced by \( K \), and \( f : X \to X \) be a continuous map. In this paper we prove that if \( K \) is finite, then the interior of every scrambled set of \( f \) in \( X \) is empty. We also show that if \( K \) is an infinite complex, then there exist continuous maps from \( X \) to itself having scrambled sets with nonempty interiors, and if \( X = \mathbb{R} \) or \( \mathbb{R}_+ \), then there exist \( C^\infty \) maps of \( X \) with the whole space \( X \) being a scrambled set.

1. Introduction

Chaotic behavior is a manifestation of the complexity of nonlinear dynamical systems. There are some distinct definitions given by different authors. The following definition of chaos mainly stems from Li and Yorke [11].

**Definition 1.1.** Let \((X, d)\) be a metric space, and \( f : X \to X \) be a continuous map. A subset \( S \) of \( X \) containing at least two points is called a **scrambled set** of \( f \) if for any \( x, y \in S \) with \( x \neq y \),

\[
\limsup_{n \to \infty} d(f^n(x), f^n(y)) > 0, \tag{1.1}
\]

and

\[
\liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0. \tag{1.2}
\]

\( f \) is said to be **chaotic (in the sense of Li and Yorke)** if \( f \) has an uncountable scrambled set.

**Remark 1.2.** Let \( P(f) \) denote the set of all periodic points of \( f \). In Definition 1.1, we do not insist that

\[
\limsup_{n \to \infty} d(f^n(x), f^n(p)) > 0 \tag{1.3}
\]

holds for any \( x \in S \) and any \( p \in P(f) \) because condition (1.3) is not important. In fact, if (1.1) and (1.2) hold for any \( x, y \in S \) with \( x \neq y \), then the set

\[
\{x \in S : \limsup_{n \to \infty} d(f^n(x), f^n(p)) = 0 \text{ for some } p \in P(f)\} \tag{1.4}
\]

is uncountable.

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353
contains at most one point. Also we do not insist that \( S \cap P(f) = \emptyset \) because if (1.2) holds, then \( S \cap P(f) \) also contains at most one point (see (ii) of Lemma 2.1 below).

Let \( I \) be a compact interval. For the case \( X = I \), Li and Yorke in [11] first showed that if \( f : I \to I \) has a periodic point of period 3, then it is chaotic, i.e. “period three implies chaos”. Kuchta and Smital in [9] indicated that if \( f : I \to I \) has a two point scrambled set, then it has an uncountable scrambled set. In [5]–[8], [12], [13] and [15], scrambled sets of some maps were further discussed from the point of view of measure.

In this paper we will consider the case of \( X \) being a 1-dimensional polyhedron. The tree is a particular kind of 1-dimensional polyhedron. In [1], [2], [3] and [10], the sets of periods of periodic orbits and the topological entropies of tree maps were discussed. Now we will study scrambled sets of continuous maps for general 1-dimensional polyhedra from the point of view of topology. Our main result is the following theorem.

**Theorem A.** Let \( K \) be a finite 1-dimensional simplicial complex in \( \mathbb{R}^3 \) without isolated vertexes, and let \( X = |K| \) be the polyhedron of \( K \). Suppose \( f : X \to X \) is a continuous map. Then the interior of any scrambled set of \( f \) in \( X \) is empty.

In addition, we will show that if \( K \) is an infinite 1-dimensional complex, then Theorem A is not true. Particularly, if \( K \) is a triangulation of \( \mathbb{R} \) or \( \mathbb{R}^+ \), i.e., if \( X \) is the real line \( \mathbb{R} \) or the real half-line \( \mathbb{R}^+ \), then there exist \( C^\infty \) maps from \( X \) to itself with the whole space \( X \) being a scrambled set.

### 2. Some elementary properties of scrambled sets

Let \((X,d)\) be a metric space, and \( f : X \to X \) be a continuous map. A point \( x \in X \) is called an eventually periodic point of \( f \) if there are integers \( n > m \geq 0 \) such that \( f^n(x) = f^m(x) \). If \( m = 0 \), i.e. \( f^n(x) = x \), then \( x \) is called a periodic point.

The following lemma will be useful, of which the proof is easy and is omitted.

**Lemma 2.1.** Let \( S \) be a scrambled set of \( f : X \to X \). Then

- (i) \( f(S) \) is an injection.
- (ii) There is at most one eventually periodic point of \( f \) in \( S \).
- (iii) For any integer \( n \geq 0 \), \( f^n(S) \) is also a scrambled set of \( f \).
- (iv) Let \( S' \subset X \) contain at least two points. If \( f(S') \subset S \) and \( f|S' \) is injective, then \( S' \) is also a scrambled set of \( f \).
- (v) If \( f \) is uniformly continuous, then \( S \) is also a scrambled set of \( f^n \) for any integer \( n > 0 \).

**Definition 2.2.** Let \((X,d)\) and \((X',d')\) be two metric spaces, and \( h : X \to X' \) be a homeomorphism. \( h \) is called a uniform homeomorphism if both \( h \) and \( h^{-1} \) are uniformly continuous. \( X \) and \( X' \) are said to be uniformly homeomorphic if there exists a uniform homeomorphism \( h : X \to X' \).

Obviously, every homeomorphism between two compact metric spaces must be a uniform homeomorphism, and we have

**Lemma 2.3.** Let \( h : X \to Y \) be a uniform homeomorphism, and \( S \subset X \). \( T = h(S) \). Suppose \( f : X \to X \) is a continuous map, and \( g = h \circ f \circ h^{-1} \). Then \( T \) is a scrambled set of \( g \) if and only if \( S \) is a scrambled set of \( f \).
3. Scrambled sets of continuous maps of 1-dimensional polyhedra

Let $K$ be a 1-dimensional simplicial complex in $R^3$. Every 0-dimensional simplex of $K$ is called a vertex, and every 1-dimensional simplex of $K$ is called an edge. Denote by $K_0$ the set of all vertices of $K$. Let $X = |K|$ be the polyhedron of $K$ (see [4]). Define the metric $d_K$ on $X$ as follows:

(i) If points $x$ and $y$ lie on the same edge $E$, the two vertexes of $E$ are $u$ and $v$, and $x = ru + (1 - r)v$, $y = su + (1 - s)v$ for some $r, s \in [0, 1]$, then $d_K(x, y) = |r - s|$.

(ii) If there is a connected subcomplex $K'$ of $K$ such that $\{x, y\} \subset |K'|$, then

$$d_K(x, y) = \min \left\{ \sum_{i=1}^{n} d_K(x_{i-1}, x_i) : (x_0, x_1, ..., x_n) \text{ is a sequence of points in } X \text{ with } x_0 = x, \ x_n = y, \text{ and } x_{i-1} \text{ and } x_i \text{ lying on the same edge of } K \text{ for } i = 1, ..., n \right\}.$$

(iii) If there is no connected subcomplex $K'$ of $K$ such that $\{x, y\} \subset |K'|$, then $d_K(x, y) = \infty$.

Remark 3.1. In order to avoid the case $d_K(x, y) = \infty$ arises, we can give another metric $d'_K$ on $X$ by

$$d'_K(x, y) = \begin{cases} \arctg d_K(x, y), & \text{if } d_K(x, y) < \infty; \\ 1, & \text{if } d_K(x, y) = \infty. \end{cases}$$

However, it is easy to see that the identical map $id : (X, d_K) \rightarrow (X, d'_K)$ is a uniform homeomorphism. Thus, for convenience, we use $d_K$ rather than $d'_K$.

Remark 3.2. If for any bounded subset $B$ of $R^3$, the number of the simplexes of $K$ intersecting $B$ is finite, then the topology on $X$ induced by $d_K$ coincides with that as subspace of the Euclidean space $R^3$.

Recall that an arc is a space homeomorphic to the unit interval $[0,1]$. Let $A \subset X$ be an arc. Denote by $\partial A$ the two endpoints of $A$, and write $A = A - \partial A$. Let $x$ and $y$ be two points on arc $A$. Denote by $A[x, y]$ the subarc of $A$ from $x$ to $y$. If $A[x, y]$ is a straight line segment, then it is simply written as $[x, y]$, and put $(x, y) = [x, y] - \{x\}$, $(x, y) = [x, y] - \{y\}$. Let $u$ and $v$ be the two endpoints of $A$. We denote by $(A; u, v)$ the directed arc $A$ from $u$ to $v$. In addition, we denote by $l(A)$ the length of arc $A$ under metric $d_K$.

Lemma 3.3. Let $(A; u, v)$ be a directed arc on $X = |K|$. Suppose $\{u, v\} \cap K_0 = \emptyset$. Then there is a unique sequence $(w_0, w_1, \cdots, w_n)$ of vertexes of $K$ with $n \geq 1$ satisfying the following four conditions:

(i) For $i = 1, 2, \cdots, n$, $[w_{i-1}, w_i]$ is an edge of $K$. 
(ii) $u \in [w_0, w_1)$, $v \in (w_{n-1}, w_n]$. 
(iii) If $n = 1$, then $u \in [w_0, v)$ and $A = [u, v]$. If $n > 1$, then 

$$A = [u, w_1] \cup \left( \bigcup_{i=2}^{n-1} [w_{i-1}, w_i]\right) \cup [w_{n-1}, v].$$

(iv) $w_i \neq w_j$ for $1 \leq i < j \leq n - 1$. 

Lemma 3.3 is evident. The sequence \((w_0, w_1, \ldots, w_n)\) in Lemma 3.3 will be called the carrier sequence of the directed arc \((A; u, v)\) and we write \(CS(A; u, v) = (w_0, w_1, \ldots, w_n)\).

**Theorem A.** Let \(K\) be a finite 1-dimensional simplicial complex in \(R^3\) without isolated vertexes, and let \(X = |K|\) be the polyhedron of \(K\). Suppose \(f : X \to X\) is a continuous map. Then the interior of any scrambled set of \(f\) in \(X\) is empty.

**Proof.** If not, there is a scrambled set \(S\) of \(f\) having a nonempty interior in \(X\). Then \(S\) contains an arc \(L\). By (ii) of Lemma 2.1, we may assume that \(L\) contains no eventually periodic points of \(f\). By (i) of Lemma 2.1, \(f^k(L)\) is also an arc in \(X(k = 1, 2, \ldots)\). Let the two endpoints of \(L\) be \(x'\) and \(y'\). By Definition 1.1, we have \(\lim_{k \to \infty} d(f^k(x'), f^k(y')) > 0\). Thus \(\sum_{k=0}^{\infty} l(f^k(L)) = \infty\). This implies that \(f^\mu(L) \cap f^m(L) \neq \emptyset\) for some integers \(\mu > m \geq 0\). Write \(A' = f^m(L)\). Take \(u, v \in A'\) such that \(v = f^\mu - m(u)\) and \(l(A'[u, v])\) achieves the minimum. Let \(A = A'[u, v]\), and \(g = f^\mu - m\). By (iii) and (v) of Lemma 2.1 we know that \(A \subset A'\) is a scrambled set of \(g\). Write \(A_k = g^k(A)\) and \(u_k = g^k(u)\) for \(k = 0, 1, \ldots\). Then \(v = g(u) = u_1\). It follows from (i) of Lemma 2.1 that \(A_k\) is an arc and the two endpoints of \(A_k\) are \(u_k\) and \(u_{k+1}\). Since \(A \cap P(g) = \emptyset\), we have

\[
A_i \not\subset A_j, \quad \text{for any nonnegative integers } i \neq j.
\]

From (3.1) we get the following

**Claim 1.** Let \(k \geq 0\). If there exist edges \(E\) and \(E'\) of \(K\) such that \(A_k \subset E\) and \(A_{k+1} \subset E'\), then \(A_k \cap A_{k+1} = \{u_{k+1}\}\), and \(E = E'\) if \(u_{k+1} \not\in K_0\).

Since \(K_0\) is a finite set and \(A\) contains \(E\) and \(E'\) of \(K\) such that \(A_k \subset E\) and \(A_{k+1} \subset E'\), then \(A_k \cap A_{k+1} = \{u_{k+1}\}\), and \(E = E'\) if \(u_{k+1} \not\in K_0\).

**Claim 2.** Write \(Z_0 = \{k : k \geq k_0 + 2, \text{ and } A_k \cap K_0 \neq \emptyset\}\). Then \(Z_0\) is an infinite set.

By (iv) of Lemma 3.3, the number of carrier sequences of all directed arcs in \(X\) is finite. Hence there exist integers \(a\) and \(b\) in \(Z_0\) with \(|a - b| \geq 3\) such that

\[
CS(A_a; u_a, u_{a+1}) = CS(A_b; u_b, u_{b+1}),
\]

\[
CS(A_{a+1}; u_{a+1}, u_{a+2}) = CS(A_{b+1}; u_{b+1}, u_{b+2}).
\]

Suppose the carrier sequence \(CS(A_a; u_a, u_{a+1})\) is \((v_0, v_1, \ldots, v_n)\). Then \(n \geq 2\). By (3.2), we have \(u_a \in (v_0, u_b)\) or \(u_b \in (v_0, u_a)\). By the symmetry, we may assume that

\[
u_a \in (v_0, u_b).
\]

It follows from (3.4), (3.2) and (3.1) that \(u_b \in (u_a, v_1)\), and \(u_{a+1} \in (v_{n-1}, u_{b+1})\), \(u_{b+1} \in (u_{a+1}, v_n)\). We now claim

\[
g([u_a, u_b]) = [u_{a+1}, u_{b+1}].
\]

In fact, if (3.5) does not hold, then \(g([u_a, u_b])\) is an arc in \(X\) with endpoints \(u_{a+1}\) and \(u_{b+1}\) which does not intersect \((u_{a+1}, u_{b+1})\). Noting \(A_{a+1} = g(A_a) = g([u_a, u_b]) \cup g(A[a, u_{a+1}])\) is an arc and \(g|A_a\) is injective, we have

\[
g(A_a[u_b, u_{a+1}]) \subset [u_{b+1}, u_{a+1}], \quad \text{and } u_{a+2} \in (u_{b+1}, u_{a+1}).
\]
Define a projection $S$ by

$$S = \{v_n, v_{n-1}, w_1, \ldots, w_m, v_n, v_{n-1}\}, \text{ for some } \{w_1, \ldots, w_m\} \subset K_0.$$

(see Fig. 3.1). This implies that there is a point $x \in (u_{a+1}, u_{b+1})$ such that $g([u_{a+1}, x]) = [u_{a+2}, v_{n-1}]$ and $g(x) = v_{n-1}$, and hence there is a fixed point $p$ of $g$ in $(u_{a+1}, x)$. However, $(u_{a+1}, x) \subset [u_{a+1}, u_{b+1}] \subset A_{b+1}$, which contains no fixed points of $g$. This leads to a contradiction. Thus (3.5) must hold.

From (3.5) and (i) of Lemma 2.1 it is easy to see that, for sufficiently small $\varepsilon > 0$, $g(u_b + \varepsilon(v_1 - u_b)) \in (u_{a+1}, v_n) - A_b$. Thus we have $u_b + \varepsilon(v_1 - u_b) \notin A_{b-1}$ and hence

$$[u_a, u_b] \subset A_{b-1}.$$

There are two cases to consider:

**Case 1.** $b > a$. In this case, let $Q = \bigcup_{k=1}^{b-1} A_k$. By (3.5) and (3.7) we can easily verify that $g(Q) = Q$. Let $S^1 = \{e^{2\pi it} : t \in R\}$ be the unit circle in the complex plane, and $d$ be the usual metric on $S^1$. Take a sequence $t_a < t_{a+1} < \cdots < t_b$ of real numbers such that $t_{b-1} < t_a + 1 < t_b < t_{a+1} + 1$. Put $z_k = e^{2\pi it_k}$ for $k = a, a+1, \ldots, b$. Let $C_k = \{e^{2\pi it} : t_k \leq t \leq t_{k+1}\}$ for $k = a+1, \ldots, b-2, b-1,$ and let $C_b = \{e^{2\pi it} : t_b \leq t \leq t_{a+1} + 1\}$ (see Fig. 3.2). Then $C_j$ is an arc on $S^1 (j = a+1, \ldots, b)$. For $k = a+1, \ldots, b-1$, choose a homeomorphism $h_k : C_k \to A_k$ such that $h_k(z_k) = u_k$, $h_k(z_{k+1}) = u_{k+1}$, and $h_{b-1}(z_a) = u_a$. Choose again a homeomorphism $h_b : C_b \to A_b[u_a, u_{a+1}]$ such that $h_b(z_b) = u_b$, $h_b(z_{a+1}) = u_{a+1}$. Define a projection $h : S^1 \to Q$ by $h(C_k) = h_k$ for $k = a+1, \ldots, b$. Then $h$ is continuous, and $h(S^1) = Q$. Define $\varphi : S^1 \to S^1$ by

$$\varphi|C_k = h_{k+1}^{-1} \circ g \circ h_k, \text{ for } k = a+1, \ldots, b-2;$$

$$\varphi|C_{b-1}[z_{b-1}, z_a] = h_{b-1}^{-1} \circ g \circ h_{b-1}[C_{b-1}[z_{b-1}, z_a];$$

$$\varphi|C_{b-1}[z_a, z_b] = h_{a+1}^{-1} \circ g \circ h_{b-1}[C_{b-1}[z_a, z_b];$$

$$\varphi|C_b = h_{a+1}^{-1} \circ g \circ h_b.$$
Then \( \varphi \) is also continuous, and \( h \circ \varphi = g \circ h \). We say that \( \varphi \) is the lift of \( g|Q \) relative to the projection \( h \), or relative to the sequence \( (h_{a+1}, h_{a+2}, \ldots, h_b) \) of homeomorphisms. Note that \( \varphi : S^1 \rightarrow S^1 \) is both injective and surjective. Thus \( \varphi \) is a homeomorphism.

\( \varphi \) has no periodic points because \( g|Q \) has no periodic points and \( h \circ \varphi = g \circ h \).

This implies that the rotation number of \( \varphi \) is irrational. If \( \varphi \) has wandering points, then we know (for example, see [14, Chap.1]) that the wandering set \( W(\varphi) \) of \( \varphi \) is an open set dense in \( S^1 \). Let \( z \) and \( w \) be two different points on the same connected component of \( W(\varphi) \cap C_{a+1} \). Then \( \lim_{k \to \infty} d(\varphi^k(z), \varphi^k(w)) = 0 \) (see [14]). Since \( S^1 \) is compact, \( h : S^1 \rightarrow Q \) is uniformly continuous. Thus

\[
\lim_{k \to \infty} d_K(g^k(h(z)), h^k(h(w))) = \lim_{k \to \infty} d_K(h(\varphi^k(z)), h(\varphi^k(w))) = 0.
\]

This implies that the points \( h(z) \) and \( h(w)(\neq h(z)) \) of \( A_{a+1} \) can not lie in the same scrambled set of \( g \). However, as indicated above, \( A \) is a scrambled set of \( g \), and hence \( A_{a+1} = g^{a+1}(A) \) is also a scrambled set. This reduces to a contradiction.

Therefore, \( \varphi \) has no wandering points. Thus \( \varphi \) is topologically conjugate to an irrational rotation of \( S^1 \), i.e. there exist an irrational number \( c \) and an orientation preserving homeomorphism \( \eta : S^1 \rightarrow S^1 \) such that

\[
(3.8) \quad \eta^{-1} \varphi \eta(e^{2\pi it}) = e^{2\pi i(t+c)}, \quad \text{for any} \ t \in R.
\]

Let \( \psi = \eta^{-1} \varphi \eta : S^1 \rightarrow S^1 \), and \( \xi = h \eta : S^1 \rightarrow Q \). Then \( \xi \) is a continuous surjection, and

\[
(3.9) \quad \xi \circ \psi = g \circ \xi.
\]

For any given positive number \( r \), if there exists \( s \in R \) such that \( \xi(e^{2\pi i(s+r)}) = \xi(e^{2\pi is}) \), then it follows from (3.8) and (3.9) that

\[
(3.10) \quad \xi(e^{2\pi i(s+kc+r)}) = \eta \psi^k(e^{2\pi i(s+r)}) = g^k \xi(e^{2\pi i(s+r)}) = g^k \xi(e^{2\pi is}) = \xi(\psi^k(e^{2\pi is})) = \xi(e^{2\pi i(s+kc)})
\]
holds for all \(k = 0, 1, \ldots\). Since the point set \(\{e^{2\pi i(s+kc)} : k = 0, 1, \ldots\}\) is dense in \(S^1\), by (3.10) and the continuity of \(\xi\) we have
\[
(3.11) \quad \xi(e^{2\pi i(t+r)}) = \xi(e^{2\pi it}), \quad \text{for any } t \in R.
\]

Let \(T = \{r : r \in (0, 1], \text{ and there exists } s = s(r) \in R \text{ such that } \xi(e^{2\pi i(s+r)}) = \xi(e^{2\pi it})\}. \text{ Then } T \text{ is evidently a nonempty closed set in } (0, 1]. \text{ Let } r_0 = \inf T. \text{ Then } r_0 > 0 \text{ since } \xi|_{\eta^{-1}(C_{a+1})} \text{ is a homeomorphism from } \eta^{-1}(C_{a+1}) \text{ to } A_{a+1} \subset Q.

It is easy to see that there is an integer \(q \geq 1\) such that \(r_0 = 1/q\). By (3.11) and the definition of \(r_0\) we know that, for any \(t, t' \in R, \xi(e^{2\pi it}) = \xi(e^{2\pi it'}) \text{ if and only if } q(t' - t) \text{ is an integer. Thus we can define } \zeta : S^1 \to Q \text{ by }
\[
\zeta(e^{2\pi it}) = \xi(e^{2\pi it/q}), \quad \text{for any } t \in R.
\]

Obviously, this \(\zeta\) is injective, surjective and continuous. Hence \(\zeta\) is a homeomorphism. Define \(\omega : S^1 \to S^1\) by
\[
\omega(e^{2\pi it}) = e^{2\pi iqt}, \quad \text{for any } t \in R.
\]

Then \(\zeta \circ \omega = \xi\). Define \(\Psi : S^1 \to S^1\) by
\[
\Psi(e^{2\pi it}) = e^{2\pi i(t+qc)}, \quad \text{for any } t \in R.
\]

Then \(\Psi \circ \omega = \omega \circ \psi\). Therefore, we have the following commutative diagram.

\[
\begin{array}{ccc}
S^1 & \xrightarrow{\Psi} & S^1 \\
\downarrow{\xi} & & \downarrow{\xi} \\
S^1 & \xrightarrow{\psi} & S^1 \\
\downarrow{\eta} & & \downarrow{\eta} \\
Q & \xrightarrow{h(g)} & Q \\
\end{array}
\]

Thus \(\zeta \Psi \omega = \zeta \omega \psi = \xi \psi = g \xi = g \zeta \omega\), and hence \(\zeta \circ \Psi = g \circ \zeta\) because \(\omega\) is a surjection. This implies that \(g|Q\) and \(\Psi\) are topologically conjugate. Since the irrational rotation \(\Psi\) has no scrambled set, by Lemma 2.3, \(g|Q\) also has no scrambled set. However, as indicated above, \(A_{a+1} \subset G\) is a scrambled set of \(g\). This is still a contradiction.

Case 2. \(a > b\). Analogous to Case 1, Case 2 also leads to a contradiction.

Thus, the interior of any scrambled set of \(f\) in \(X\) must be empty. Theorem A is proven.

If complex \(K\) is not finite, then Theorem A is not true. In fact, we have

**Theorem 3.1.** Let \(K\) be an infinite 1-dimensional simplicial complex in \(R^3\), and let \(X = |K|\) be the polyhedron of \(K\) with the metric \(d_K\). Then there exists a continuous map \(f : X \to X\) which has a scrambled set containing a nonempty interior in \(X\).

**Proof.** If \(K\) has isolated vertexes, take an edge \(E\) of \(K\), and let \(f_0 : E \to E\) be a continuous map which has an uncountable scrambled set \(S_0\). Suppose the set
of isolated vertexes of $K$ is $V$. Choose a continuous map $g : X \to E$ such that $g(x) = x$ for all $x \in E, g(V) \subseteq S_0$, and $g|V$ is injective. Let $f = f_0 \circ g$, and let $S = S_0 \cup V - g(V)$. Then $S$ is an uncountable scrambled set of $f$, the interior of $S$ in $X$ contains $V$, and is nonempty.

Now we assume that $K$ has no isolated vertexes. Then $K$ has infinitely many edges. Take countably infinitely many edges $E_0, E_1, E_2, \cdots$ of $K$ with $E_i \neq E_j$ for $i \neq j$ such that one of the following two conditions holds:

(C.1) If $K$ has infinitely many connected components, then for any $0 \leq i < j < \infty$, $E_i$ and $E_j$ belong to different components of $K$.

(C.2) If $K$ has only finitely many connected components, then all of $E_0, E_1, E_2, \cdots$ belong to the same component of $K$.

For $n = 0, 1, 2, \cdots$, suppose $\partial E_n = \{v_n, w_n\}$. Let $x_n = (2v_n + w_n)/3, y_n = (v_n + 2w_n)/3$, and $A_n = [x_n, y_n]$. Denote by $J$ the open interval $(0, 1)$. For any rational number $r \in J$, write $u_n(r) = (1 - r)x_n + ry_n$. Put

$$W = \{(a, b, r, s) : a, b, r, s \text{ are all rational numbers, and } 0 < a < b < 1, 0 < r < s < 1\}.$$ 

Then $W$ is a countable set in $J^4(\subset R^4)$. Arrange all points in $W$ to be an infinite sequence. Assume the sequence is 

$$W = \{(a_n, b_n, r_n, s_n) : n = 0, 1, 2, \cdots\}.$$ 

For $n = 1, 2, 3, \cdots$, choose a homeomorphism $g_n : A_0 \to A_n$ such that $g_n(x_0) = x_n, g_n(y_0) = y_n, g_n(u_0(a_n)) = u_n(r_n)$, and $g_n(u_0(b_n)) = u_n(s_n)$, and then define the homeomorphism $h_n : A_n \to A_{n+1}$ by $h_n = g_{n+1} \circ g_n^{-1}$. Let $h_0 = g_1 : A_0 \to A_1$. Put $X_0 = \bigcup_{n=0}^\infty A_n$. Define $f_0 : X_0 \to X_0$ by $f_0|A_n = h_n$ for $n = 0, 1, 2, \cdots$. Then it is easy to see that $A_0$ is a scrambled set of $f_0$.

If condition (C.1) holds, suppose the connected component of $K$ containing $E_n$ is $K^{(n)}$, and $Y_n = |K^{(n)}|, (n = 0, 1, 2, \cdots)$. Obviously, we can construct a continuous map $f : X \to X$ such that $f|X_0 = f_0, f(Y_n) = E_{n+1}$ for $n = 0, 1, 2, \cdots$, and $f(z) = z$ for any $z \in X - \bigcup_{n=0}^\infty Y_n$. Clearly, $A_0$ is still a scrambled set of $f$.

If condition (C.2) holds, put $X_1 = X - \bigcup_{n=0}^\infty E_n$, and $X_2 = X_0 \cup X_1$. Define $f_2 : X_2 \to X_2$ by $f_2|X_0 = f_0$, and $f_2(z) = z$ for any $z \in X_1$. Then $f_2$ is continuous. Evidently, $f_2$ can be extended to be a continuous map $f : X \to X$, and $A_0$ is also a scrambled set of this $f$.

Note that $A_0$ is a nonempty open set in $X$. Theorem 3.1 is proven. □

4. Totally chaotic continuous maps of the real line

In the proof of Theorem 3.1 we construct a continuous map $f : X \to X$, which has a scrambled set with a nonempty interior in $X$. But this $f$ is not a totally chaotic map defined as follows.

**Definition 4.1.** Let $(X, d)$ be a metric space. A continuous map $f : X \to X$ is called totally chaotic if the whole space $X$ is a scrambled set of $f$.

What metric space $X$ can admit a totally chaotic map? This is an interesting problem. In this section we will consider the real line $R$ and the real half-line $R_+ = [0, \infty)$, which can be regarded as the polyhadra of the infinite 1-dimensional...
simplicial complexes $K(R)$ and $K(R_+)$, respectively, where
\[
K(R) = \{ n, [n, n+1] : n \text{ is an integer} \},
\]
\[
K(R_+) = \{ n, [n, n+1] : n \text{ is a nonnegative integer} \}.
\]

We have the following theorem.

**Theorem 4.2.** There exists a totally chaotic continuous map $f : R \to R$, which satisfies that $f(R_+) \subset R_+$, and $f|_R$ is also a totally chaotic map.

**Proof.** Let $c_0, c_1, c_2, \cdots$ be a given infinite sequence of positive numbers with $c_k \leq 1/2$ for every even number $k \geq 0$. We first choose a $C^\infty$ map $f_0 : R \to R$ such that
1. $f_0(0) = 1$, $f_0(x) > x$ for any $x \leq 0$, and $\lim_{x \to -\infty} f_0(x) = -\infty$;
2. $f_0'(x) \equiv 1 - c_0$ for any $x \geq 1$ and $f_0'(x) > 0$ for all $x \in R$, where $f_0'(x)$ is the derivative of $f_0(x)$;
3. $1 - c_0 < f_0'(x) < 1$ for $0 < x < 1$.

The equation $f_0(x) = x$ has a unique root. Suppose this root is $a_0$. Then $a_0 > 2$. Obviously, there is an integer $a_0 \geq 2$ such that $f_0^{a_0}([-1, 1]) \subset [a_0 - 1, a_0]$.

Let $\varepsilon_0 = a_0 - f_0^{a_0}(1)$. Then $0 < \varepsilon_0 < 1$.

Next, we take a $C^\infty$ map $f_1 : R \to R$ such that
1. $f_1(x) = f_0(x)$ for any $x \leq a_0 - \varepsilon_0$, and $f_1(x) > x$ for all $x \in R$;
2. the derivative $f_1'(x) \equiv 1 + c_1$ for any $x \geq a_0$;
3. $f_1'(x) < f_1(x) < 1 + c_1$ for $a_0 - \varepsilon_0 < x < a_0$.

Evidently, there is an integer $n_1 > n_0$ such that $f_1^{n_1}(-a_0) > 2a_0$, and the derivative $d(f_1^{n_1}(x))/dx > 2$ for all $x \in [-a_0, a_0]$. Let $a_1 = f_1^{n_1}(a_0)$. Then $a_1 > 2a_0$.

Put $\varepsilon_1 = \varepsilon_0$.

Now we choose again a $C^\infty$ map $f_2 : R \to R$ such that
1. $f_2(x) = f_1(x)$ for any $x \leq a_1 + 1$;
2. $f_2'(x) \equiv 1 - c_2$ for any $x \geq a_1 + 2$;
3. $1 - c_2 < f_2'(x) < 1 + c_1$ for $a_1 + 1 < x < a_1 + 2$.

It is easy to see that the equation $f_2(x) = x$ has a unique root. Suppose this root is $a_2$. Then $a_2 > a_1 + 1$. Clearly, there is an integer $n_2 > n_1$ such that $f_2^{n_2}([-a_2, a_2]) \subset [a_2 - 2^{-2}, a_2]$. Let $\varepsilon_2 = a_2 - f_2^{n_2}(a_2)$. Then $0 < \varepsilon_2 < 2^{-2}$.

Continuing this process, for every positive integer $k = 1, 2, 3, \cdots$, we can choose a $C^\infty$ map $f_k : R \to R$ and take a positive integer $n_k$ and two positive numbers $a_k, \varepsilon_k$ satisfying the following conditions:

(a) If $k \geq 1$ is odd, then
   1. $f_k(x) = f_{k-1}(x)$ for all $x \leq a_{k-1} - \varepsilon_{k-1}$, and $f_k(x) > x$ for all $x \in R$;
   2. the derivative $f_k'(x) \equiv 1 + c_k$ for all $x \geq a_{k-1}$;
   3. $f_k'(x) < f_k(x) < 1 + c_k$ for $a_{k-1} - \varepsilon_{k-1} < x < a_{k-1}$;
   4. $n_k > n_{k-1}$, $f_k^{n_k}(-a_{k-1}) > 2a_{k-1}$, and the derivative $d(f_k^{n_k}(x))/dx > 2^k$ for all $x \in [-a_{k-1}, a_{k-1}]$;
   5. $a_k = f_k^{n_k}(a_{k-1}) > 2a_{k-1}$, and $\varepsilon_k = \varepsilon_{k-1}$.
(b) If $k \geq 2$ is even, then
   1. $f_k(x) = f_k(x)$ for all $x \leq a_{k-1} + 1$;
   2. $f_k'(x) \equiv 1 - c_k$ for all $x \geq a_{k-1} + 2$;
   3. $1 - c_k < f_k'(x) < 1 + c_k$ for $a_{k-1} + 1 < x < a_{k-1} + 2$;
   4. the equation $f_k(x) = x$ has a unique root, and $a_k > a_{k-1} + 1$;
   5. $n_k > n_{k-1}$, and $f_k^{n_k}([-a_{k-1}, a_{k-1}]) \subset [a_k - 2^{-k}, a_k]$;
(k.6.b) \( \varepsilon_k = a_k - f^n_k(a_{k-1}) \in (0, 2^{-k}] \).

From these conditions we know that there exists a limit function \( f = \lim_{k \to \infty} f_k \) with \( f(x) = f_k(x) \) for \( x \leq a_k - \varepsilon_k \). Thus \( f : R \to R \) is a \( C^\infty \) map. For any given \( u, v \in R \) with \( u \neq v \), take a positive integer \( j \geq 1 \) such that \( \{u, v\} \subset [-a_j, a_j] \).

Then by (k.4.a) we have \( |f^{n_k}(u) - f^{n_k}(v)| > 2^k|u - v| \) for every odd \( k > j \), and by (k.5.b) we have \( |f^{n_k}(u) - f^{n_k}(v)| < 2^{-k} \) for every even \( k > j \). This implies that

\[
\liminf_{k \to \infty} |f^k(u) - f^k(v)| = 0, \quad \text{and} \quad \limsup_{k \to \infty} |f^k(u) - f^k(v)| = \infty.
\]

Hence \( f \) is totally chaotic.

Noting that \( f(x) > x \) for any \( x \in R \), we have \( f(R_+) \subset R_+ \). Therefore, \( f|R_+ \) is also totally chaotic. Theorem 4.2 is proven. \( \square \)

**Remark 4.3.** Let \( f : R \to R \) be as in the proof of Theorem 4.2. Then \( f \) is a \( C^\infty \) diffeomorphism. For any \( n \geq 2 \), define \( F_n : R^n \to R^n \) by

\[
F_n(x_1, x_2, \cdots, x_n) = (f(x_1), f(x_2), \cdots, f(x_n)), \quad \text{for any} \ (x_1, x_2, \cdots, x_n) \in R^n.
\]

It is easy to see that \( F_n \) is also a \( C^\infty \) diffeomorphism, and \( F_n \) is totally chaotic.

**References**


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