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THE KREIN-MILMAN THEOREM IN OPERATOR CONVEXITY

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Abstract. We generalize the Krein-Milman theorem to the setting of matrix convex sets of Effros-Winkler, extending the work of Farenick-Morenz on compact C*-convex sets of complex matrices and the matrix state spaces of C*-algebras. An essential ingredient is to prove the non-commutative analogue of the fact that a compact convex set K may be thought of as the state space of the space of continuous affine functions on K.

The Krein-Milman theorem is without doubt one of the cornerstones of functional analysis. With the rise of non-commutative functional analysis and related notions of convexity ([15], [10], [11]), the question naturally arises how to formulate a notion of extreme points for which the theorem remains true.

Such a notion exists in the case of C*-convexity, which has been studied by Loebl-Paulsen ([15]), Hopenwasser-Moore-Paulsen ([12]), and, more recently, Farenick-Morenz ([7], [8], [9], [17]). C*-convexity is the natural extension of the classical scalar-valued convex combination to include C*-algebra-valued coefficients. It therefore makes sense in a C*-algebra and, more generally, for bimodules over C*-algebras. In particular, there is a rich class of such C*-convex sets in the $n \times n$ complex matrices, $\mathbb{M}_n$. The matrix state spaces of a C*-algebra are another class of examples. In both these cases the C*-convexity version of the Krein-Milman has been proven to hold by Farenick and Morenz ([9], [17]).

The above two examples both fit in the framework of another non-commutative convexity theory, the theory of matrix convex sets, developed by Effros and the second author ([11], [21]). In this paper we develop a notion of extreme points in this context, and we prove a corresponding Krein-Milman result, including a minimality condition which shows that the result is indeed optimal. Even though the difference between extremality in C*-convexity and matrix convexity might seem minor at first, we hope to convince the reader that our approach is the natural one. Moreover, our methods are seemingly new and different, the central idea being to prove the analogue of the fact that a compact convex set K can be represented as the state space of the space of continuous affine functions on K.

We begin with a review of matrix convexity, followed by a treatment of extreme points in this context. We then prove our representation result, from which we proceed to prove the Krein-Milman theorem.

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1. Matrix convexity

All vector spaces are assumed to be complex throughout this paper. Let $M_{m,n}(V)$ denote the vector space of $m \times n$ matrices over a vector space V, and set $M_n(V) = M_{n,n}(V)$. We write $\mathbb{M}_{m,n} = M_{m,n}(\mathbb{C})$ and $\mathbb{M}_n = M_{n,n}(\mathbb{C})$, which means that we may identify $M_{m,n}(V)$ with the tensor product $\mathbb{M}_{m,n} \otimes V$. We use the standard matrix multiplication and $*$-operation for compatible scalar matrices, and $\mathbb{I}_n$ for the identity matrix in $\mathbb{M}_n$.

The multiplication of scalar matrices induces a bimodule operation of scalar matrices on $M_m(V)$ via the identification with $\mathbb{M}_m \otimes V$, i.e., for $v \in M_m(V)$ and $\alpha \in \mathbb{M}_{n,m}$, $\beta \in \mathbb{M}_{m,n}$, we define

$$\alpha v \beta = \left[\sum_{j,k} \alpha_{ij} v_{jk} \beta_{kl}\right] \in M_n(V).$$

The following definition of a non-commutative convex set was first proposed by Wittstock ([22]).

Definition 1.1. A *matrix convex set* in a vector space V is a collection $\boldsymbol{K} = (K_n)$ of subsets $K_n \subset M_n(V)$ such that

$$\sum_{i=1}^{k} \gamma_i^* v_i \gamma_i \in K_n$$

for all $v_i \in K_{n_i}$ and $\gamma_i \in \mathbb{M}_{n_i,n}$ for $i = 1, \ldots, k$ satisfying $\sum_i \gamma_i^* \gamma_i = \mathbb{I}_n$.

We shall say that $v = \sum_i \gamma_i^* v_i \gamma_i$ as above is a *matrix convex combination* of $v_1, \ldots, v_k$.

If we restrict ourselves to a fixed $n \in \mathbb{N}$ and a single set $K_n \subset M_n(V)$ satisfying the above with $n = n_1 = \cdots = n_k$, then we exactly get the definition of a C*-*convex* set over $\mathbb{M}_n$. This can easily be extended to arbitrary bimodules over unital C*-algebras (cf. [8]), but in this paper we will only consider the case of $\mathbb{M}_n$-bimodules. The examples below show in particular that the standard examples of C*-convex sets (cf. [9], [17]) come with natural matrix convexity structure.

Example 1.2. Given $a, b \in \mathbb{R} \cup \{\pm\infty\}$, the collection $[\boldsymbol{a}\,\mathbf{I}, \boldsymbol{b}\,\mathbf{I}] = ([a\mathbb{I}_n, b\mathbb{I}_n])$ of intervals

$$[a\mathbb{I}_n, b\mathbb{I}_n] = \{\alpha \in \mathbb{M}_n \mid a\mathbb{I}_n \leq \alpha \leq b\mathbb{I}_n\}$$

defines a matrix convex set in $\mathbb{C}$. It is easy to show ([11, Lemma 3.1]) that any matrix convex set $\boldsymbol{K} = (K_1)$ in $\mathbb{C}$ where K_1 is a closed convex subset of $\mathbb{R}$ is of this form.

As for more general matrix convex sets over $\mathbb{C}$, it follows from results of Arveson that any closed and bounded matrix convex set $\boldsymbol{K}$ in $\mathbb{C}$ is the set of *matrix ranges* $\boldsymbol{\mathcal{W}}(T) = (\mathcal{W}_n(T))$ of a Hilbert space operator T acting on a separable Hilbert space $\mathcal{H}$ (cf. [3], p.301, and [15, Proposition 31]). The matrix ranges of T are defined as

$$\mathcal{W}_n(T) = \{\varphi(T) \mid \varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{M}_n \text{ completely positive and } \varphi(\mathrm{I}) = \mathbb{I}_n\}.$$

As the second part of above example shows, there is already a rich class of matrix convex sets in even the simplest possible case. The study of matrix ranges has been a driving force behind the C*-convexity theory. (See [19], [20], [7] and the references therein, for example.)

Example 1.3. Consider an *operator space* $\mathcal{M}$ (i.e., a closed linear subspace of the bounded operators on a Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$). The natural inclusion

$$M_n(\mathcal{M}) \hookrightarrow M_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^n) \tag{1}$$

endows $M_n(\mathcal{M})$ with a norm using the operator norm on $\mathcal{B}(\mathcal{H}^n)$, and it is easy to check that the collection $\boldsymbol{B} = (B_n)$ of unit balls

$$B_n = \{x \in M_n(\mathcal{M}) \mid \|x\| \leq 1\}$$

is a matrix convex set in $\mathcal{M}$.

Based on Ruan's abstract characterization ([18]), operator spaces may be thought of as the non-commutative analogue of Banach spaces. In this line of thought, the above example is an analogue of a balanced convex set.

The objects of the next example are the non-commutative analogues of Kadison's *function systems* ([13]), which are closed subspaces of $C(X)$, the continuous functions on a compact set X, closed under conjugation and containing the identity function.

Example 1.4. If $\mathcal{R}$ is an *operator system* (a closed subspace of $\mathcal{B}(\mathcal{H})$, closed under the adjoint operation and containing the identity operator I), then the inclusion (1) defines an ordering on $M_n(\mathcal{R})$ via the usual ordering on $\mathcal{B}(\mathcal{H}^n)$. In this case the collection $\boldsymbol{P} = (P_n)$ of positive cones

$$P_n = \{x \in M_n(\mathcal{R}) \mid x \geq 0\}$$

form a matrix convex set in $\mathcal{R}$.

We may also consider the collection $\boldsymbol{CS}(\mathcal{R}) = (CS_n(\mathcal{R}))$ of *matrix states*

$$CS_n(\mathcal{R}) = \{\varphi : \mathcal{R} \to \mathbb{M}_n \mid \varphi \text{ completely positive}, \varphi(\mathrm{I}) = \mathbb{I}_n\},$$

where we recall that $\varphi : \mathcal{R} \to \mathbb{M}_n$ is *completely positive* if the canonical amplifications $\varphi_r : M_r(\mathcal{R}) \to M_r(\mathbb{M}_n)$ given by $\varphi_r = \mathrm{id} \otimes \varphi$ are positive for all $r \in \mathbb{N}$. Again we get a matrix convex set (in $\mathcal{R}^*$). We consider $\boldsymbol{CS}(\mathcal{R})$ the matricial version of the state space. This fits well with Example 1.2 because for $T \in \mathcal{B}(\mathcal{H})$ the matrix ranges are given by $\mathcal{W}_n(T) = \{\varphi(T) \mid \varphi \in CS_n(\mathcal{B}(\mathcal{H}))\}$.

Parallel to the abstract characterization of Kadison's function systems as complete *order unit spaces* (cf. [1, Section II 1]), operator systems have been characterized by Choi-Effros ([5, Theorem 4.4]) as those matrix ordered spaces $\mathcal{R}$, where $\mathcal{R}$ itself is a function system, and $M_n(\mathcal{R})^+$ satisfies the Archimedian property for all $n \in \mathbb{N}$. We will use this theorem in Section 3. We refer to [1] and [5] for the relevant definitions.

We should comment here, for the benefit of those unfamiliar with the theory of operator systems and operator spaces, that these examples explain why we consider matrix convex sets to be the appropriate generalization of convex sets for non-commutative functional analysis. Not only do the sets $\boldsymbol{B}$ and $\boldsymbol{P}$ play analogous roles to convex sets in the classical theory, but they are in some sense optimal. Simple examples show that one cannot replace these sets by a collection of sets in a finite number of levels and still be able to distinguish general operator spaces or operator systems using them. On the other hand no more information is needed, since Ruan's theorem and the Choi-Effros characterization ([5, Theorem 4.4]) respectively tell us that these matrix sets are sufficient to tell spaces of the appropriate type apart.

Much knowledge about matrix states and their extremal properties goes back to Arveson's seminal work [2], [3]. We shall need the following consequence of

Arveson's boundary theorem ([3, Theorem 2.1.1], [6]). (Arveson's theorem is much more general than the case below for which a simpler proof is possible.)

Proposition 1.5. *Let $\mathcal{R}$ be an operator system in $\mathbb{M}_n$, and assume that $\mathcal{R}$ is irreducible in the sense that only the trivial subspaces $\{0\}$ and $\mathbb{C}^n$ are invariant under $\mathcal{R}$. If $\psi : \mathbb{M}_n \to \mathbb{M}_n$ is a matrix state and $\psi|_{\mathcal{R}} = \mathrm{id}$, then $\psi = \mathrm{id}$.*

For a detailed account of matrix convexity we refer to [11] or [21]. By an easy translation argument the following version of the separation-type Hahn-Banach theorem follows from the generalized Bipolar theorem proved in [11].

Theorem 1.6. *Let V be a locally convex vector space. Assume that $\boldsymbol{K} = (K_r)$ is a matrix convex set in V, such that K_r is closed in the product topology in $M_r(V)$ for all $r \in \mathbb{N}$. Given $v_0 \notin K_n$ for some $n \in \mathbb{N}$, there exist a continuous linear mapping $\Phi : V \to \mathbb{M}_n$ and a self-adjoint $\alpha \in \mathbb{M}_n$ such that*

$$\operatorname{Re} \Phi_r(v) \leq \alpha \otimes \mathbb{I}_r$$

for all $r \in \mathbb{N}$, $v \in K_r$, and

$$\operatorname{Re} \Phi_n(v_0) \not\leq \alpha \otimes \mathbb{I}_n.$$

Moreover, if $0 \in K_1$, then α may be chosen to be $\mathbb{I}_n$.

2. Matrix extreme points

Inspired by the notion of extreme points in the C*-convexity case, we now introduce extreme points suitable for matrix convexity. As a natural extension of a proper scalar convex combination, we say that a matrix convex combination

$$v = \sum_{i=1}^{k} \gamma_i^* v_i \gamma_i$$

with $\gamma_i \in \mathbb{M}_{n_i,n}$ for $i = 1, \ldots, k$ satisfying $\sum_i \gamma_i^* \gamma_i = \mathbb{I}_n$ is *proper* if each γ_i has a right inverse belonging to $\mathbb{M}_{n,n_i}$, i.e., if γ_i is surjective as a linear map $\mathbb{C}^n \to \mathbb{C}^{n_i}$. In particular, we must have that $n \geq n_i$.

Definition 2.1. Suppose that $\boldsymbol{K} = (K_n)$ is a matrix convex set in V. Then $v \in K_n$ is a *matrix extreme point* if whenever v is a proper matrix convex combination of $v_i \in K_{n_i}$ for $i = 1, \ldots, k$, then each $n_i = n$ and $v = u_i^* v_i u_i$ for some unitary $u_i \in \mathbb{M}_n$.

Let ∂K_n be the (possibly empty) set of matricial extreme points in K_n and set $\boldsymbol{\partial K} = (\partial K_n)$.

Observe that for $n = 1$, a proper matrix convex combination reduces to a proper scalar convex combination of elements in V. Therefore the matrix extreme points in K_1 coincide with the usual extreme points. This also shows that if K_1 is compact, then, by Krein-Milman, ∂K_1 is non-empty. By contrast, ∂K_n might be empty for all $n > 1$, as Example 2.2 below will show.

As remarked in [15, Remark 12] the occurrence of unitary equivalence in the definition of matrix extreme points is quite natural, because if $v \in K_n$ and $u \in \mathbb{M}_n$ is unitary, then $w = u^* v u$ is a proper matrix combination of v.

We saw in the previous section how each K_n of a matrix convex set $\boldsymbol{K}$ is a C*-convex set over $\mathbb{M}_n$. Similarly, if we fix n in the above, we get the definition of a C*-*extreme point* of K_n. As we shall see in Example 2.3 the C*-extreme points and the matrix extreme points do not necessarily agree, but clearly the matrix extreme points are also C*-extreme. We shall see later (Corollary 3.6) that matrix extreme points are also extreme points in the usual sense.

In the case of a compact matrix convex set in $\mathbb{C}$, i.e., if $\boldsymbol{K} = \boldsymbol{\mathcal{W}}(T)$ for some operator $T \in \mathcal{B}(\mathcal{H})$ (cf. Example 1.2), it follows from the work of Morenz ([17, Proposition 2.2]) that matrix extreme points in K_n correspond exactly to the so-called *structural elements* of size n. Adding to our conviction that one should study the whole of $\boldsymbol{K}$ and not just K_n, is the observation that the results of the same paper are obtained by introducing structural elements in K_r for $r \leq n$ (cf. [17, Definition 2.3]).

Example 2.2. With $a, b \in \mathbb{R}$ the matrix extreme points of the matrix interval $[\boldsymbol{a}\,\mathbf{I}, \boldsymbol{b}\,\mathbf{I}]$ of Example 1.2 are just a and b, i.e.,

$$\partial[a\mathbb{I}_n, b\mathbb{I}_n] = \begin{cases} \{a,b\}, & n = 1; \\ \emptyset, & n > 1. \end{cases}$$

Since the matrix extreme points of K_1 are the classical extreme points, we have $\partial[a\mathbb{I}_1, b\mathbb{I}_1] = \partial[a,b] = \{a,b\}$. Moreover, any element $v \in [a\mathbb{I}_n, b\mathbb{I}_n]$ can be written

$$v = u^* \begin{bmatrix} v_1 & & \\ & \ddots & \\ & & v_n \end{bmatrix} u = \sum_i \gamma_i^* v_i \gamma_i$$

with $v_i \in [a,b]$ and a unitary

$$u = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix} \in \mathbb{M}_n,$$

where $\gamma_1, \ldots, \gamma_n \in \mathbb{M}_{1,n}$. Since u is unitary, $\gamma_1, \ldots, \gamma_n$ defines a proper matrix convex combination, and therefore no element in $[a\mathbb{I}_n, b\mathbb{I}_n]$ for $n > 1$ can be matrix extreme.

Based on Farenick-Morenz' description of the C*-extreme points of the matrix state spaces $CS_n(\mathcal{A})$ for a C*-algebra $\mathcal{A}$ (cf. Example 1.4), we may characterize the matrix extreme points of $\boldsymbol{CS}(\mathcal{A})$. Recall, that a completely positive map φ is *pure* if whenever ψ is a completely positive map such that $\varphi - \psi$ is completely positive, then $\psi = t\varphi$ for some $0 \leq t \leq 1$.

Example 2.3. The matrix extreme points of $\boldsymbol{CS}(\mathcal{A})$ for a C*-algebra $\mathcal{A}$ are exactly the pure matrix states, i.e.,

$$\partial CS_n(\mathcal{A}) = \{\varphi : \mathcal{A} \to \mathbb{M}_n \,|\, \varphi \text{ completely positive, pure, and } \varphi(\mathrm{I}) = \mathbb{I}_n\}.$$

Assume that $\varphi \in CS_n(\mathcal{A})$ is pure, and that $\varphi = \sum_i \gamma_i^* \varphi_i \gamma_i$ is a proper matrix convex combination of $\varphi_i \in CS_{n_i}(\mathcal{A})$ and $\gamma_i \in \mathbb{M}_{n_i,n}$. Since $\varphi - \gamma_i^* \varphi_i \gamma_i$ is completely positive, $\gamma_i^* \varphi_i \gamma_i = t_i \varphi$ for some $0 \leq t_i \leq 1$. But φ and φ_i are unital, so $\gamma_i^* \gamma_i = t_i \mathbb{I}_n$. Since γ_i is surjective, $t_i^{-1/2} \gamma_i$ implements a unitary equivalence between φ and φ_i. Therefore all pure matrix states are matrix extreme.

Moreover, by [9, Theorem 2.1] every C*-extreme point φ is unitarily equivalent to a direct sum of pure matrix states. If the direct sum contains more than just one pure matrix state, then φ is a proper matrix convex combination of smaller pure matrix states, just as in Example 2.2, and therefore not matrix extreme. Since all matrix extreme points are C*-extreme, this shows that only the pure matrix states are matrix extreme.

The above two examples illustrate the advantages of the notion of matrix extreme points. On the one hand they establish a clear-cut analogy with the commutative case: in Example 2.2, where the non-commutative aspect plays no important role (since the matrix sets are completely determined by their first levels), we find that the theory reverts to the classical theory. On the other hand they provide us with a "non-commutative" structure which gives us strictly more information about the matrix set than we could gain from the extreme points at the first level alone. Additionally, in the second example, they demonstrate a particularly clear relationship with objects of interest in C*-algebra theory.

3. Compact matrix convex sets

There is a natural correspondence between compact convex sets and function systems. On one hand, each compact convex subset K of a locally convex space determines the function system $A(K) = \{F : K \to \mathbb{C} \,|\, F \text{ continuous and affine}\}$. Conversely, if we are given a function system $\mathcal{R}$, then the state space $S(\mathcal{R}) = \{\varphi \in \mathcal{R}^* \,|\, \varphi \geq 0, \varphi(\mathrm{I}) = 1\}$ is a weakly compact convex subset of $\mathcal{R}^*$. Moreover, K is affinely homeomorphic to $S(A(K))$, and $\mathcal{R}$ and $A(S(\mathcal{R}))$ are isomorphic as function systems. (See [1, Section II 1].) The real case is usually the only one considered in the literature, but the fact that it remains true in the complex case is fundamental to this section.

With our claim that operator systems are the non-commutative analogues of function systems (cf. Example 1.4), it is only natural to demand that a "compact matrix convex set" should satisfy a similar correspondence. Establishing this is the main purpose of this section.

Definition 3.1. We define a *compact matrix convex set* to be a matrix convex subset $\boldsymbol{K} = (K_n)$ of a locally convex vector space V such that each K_n is compact in the product topology in $M_n(V)$.

We remark that it is a consequence of [16, Theorems 3.1 and 3.2] that compactness of K_n is not necessarily implied by compactness of K_1.

Example 3.2. The matrix intervals $[\boldsymbol{a}\,\mathbf{I}, \boldsymbol{b}\,\mathbf{I}]$ for $a, b \in \mathbb{R}$ and the matrix ranges $\boldsymbol{\mathcal{W}}(T)$ for $T \in \mathcal{B}(\mathcal{H})$ as defined in Example 1.2, are both compact matrix convex sets in $\mathbb{C}$. Conversely, any compact matrix convex set in $\mathbb{C}$ is of this form, as already pointed out in Example 1.2.

Example 3.3. If $\boldsymbol{CS}(\mathcal{R})$ are the matrix state spaces of an operator system $\mathcal{R}$, as defined in Example 1.4, then it is straightforward to see that $\boldsymbol{CS}(\mathcal{R})$ is a compact matrix convex set in $\mathcal{R}^*$, equipped with the weak* topology.

We recall that the adjoint or $*$-operation in $\mathcal{R}$ induces an adjoint operation in $\mathcal{R}^*$, in which the self-adjoint elements correspond to linear functionals mapping self-adjoint elements of $\mathcal{R}$ into $\mathbb{R}$. This, in turn, induces a $*$-operation in $M_n(\mathcal{R}^*)$, and under the canonical identification of $M_n(\mathcal{R}^*)$ with the weakly continuous linear maps $\mathcal{R} \to \mathbb{M}_n$, the self-adjoint elements of $M_n(\mathcal{R}^*)$ correspond to the maps that send self-adjoint elements in $\mathcal{R}$ to self-adjoint elements in $\mathbb{M}_n$. This shows in particular that $\boldsymbol{CS}(\mathcal{R})$ is closed under this $*$-operation.

Given a compact matrix convex set $\boldsymbol{K}$, we know from the commutative case that K_1 and $S(A(K_1))$ are affinely homeomorphic. We wish to define an operator system structure on $A(K_1)$ that extends this to K_n and $CS_n(A(K_1))$. At the same time we also demand that if $\boldsymbol{K} = \boldsymbol{CS}(\mathcal{R})$ for an operator system $\mathcal{R}$, then the corresponding operator system structure on $A(S(\mathcal{R}))$ coincides with $\mathcal{R}$.

To this end we apply the Choi-Effros' abstract characterization of operator systems (cf. Example 1.4), but first we introduce the following natural notion of a morphism on a matrix convex set.

Definition 3.4. A *matrix affine* mapping on a matrix convex subset $\boldsymbol{K} = (K_n)$ of a vector space V is a sequence $\boldsymbol{\theta} = (\theta_n)$ of mappings $\theta_n : K_n \to M_n(W)$ for some vector space W, such that

$$\theta_n(\sum_{i=1}^{k} \gamma_i^* v_i \gamma_i) = \sum_{i=1}^{k} \gamma_i^* \theta_{n_i}(v_i)\gamma_i,$$

for all $v_i \in K_{n_i}$ and $\gamma_i \in \mathbb{M}_{n_i,n}$ for $i = 1, \dots, k$ satisfying $\sum_i \gamma_i^* \gamma_i = \mathbb{I}_n$.

If $\varphi : V \to W$ is a linear map and $w_0 \in W$, then $\theta_n = \varphi_n|_{K_n} + \mathbb{I}_n \otimes w_0$ defines a matrix affine map. The converse is not true in general, even in the classical case: consider for instance $z \to \operatorname{Re} z$ with $V = W = \mathbb{C}$, $K = [0,1] \times [0,1]$ and $L = \{0\} \times [0,1]$ (identifying $\mathbb{C}$ with $\mathbb{R}^2$). The obstruction disappears if V and W are $*$-vector spaces and K_1 is self-adjoint, but we have not included the proof of this fact, as it will not be needed in this paper.

If V and W are locally convex spaces, then we say that $\boldsymbol{\theta}$ as above is a *matrix affine homeomorphism* if each θ_n is a homeomorphism. Note that in this case (θ_n^{-1}) is automatically matrix affine, and that it suffices to prove continuity of θ_n if $\boldsymbol{K}$ is compact.

Given a compact matrix convex set $\boldsymbol{K}$, we define $A(\boldsymbol{K}, \mathbb{M}_r)$ to be the set of all matrix affine mappings $\boldsymbol{F} = (F_n) : \boldsymbol{K} \to \mathbb{M}_r$, such that F_1 is continuous. Using the linear structure and the adjoint operation in $M_n(\mathbb{M}_r)$, $A(\boldsymbol{K}, \mathbb{M}_r)$ becomes a vector space with a $*$-operation under pointwise operations. Similarly, the order structure in $M_n(\mathbb{M}_r)$ defines a positive cone in $A(\boldsymbol{K}, \mathbb{M}_r)$, where $\boldsymbol{F} \geq 0$ in $A(\boldsymbol{K}, \mathbb{M}_r)$ if $F_n(v) \geq 0$ for all $n \in \mathbb{N}$ and $v \in K_n$.

Observe that if we define $\mathbf{I} = (\mathrm{I}_n)$ in $A(\boldsymbol{K}, \mathbb{C})$ by $\mathrm{I}_n(v) = \mathbb{I}_n$ for $v \in K_n$, then we can define a unital order preserving bijection Ω of $A(\boldsymbol{K}, \mathbb{C})$ onto $A(K_1)$ by mapping $\boldsymbol{F} = (F_n)$ to F_1. Indeed, this is clearly a positive unital map. Moreover, for a self-adjoint $\boldsymbol{F} = (F_n)$, F_n is completely determined by F_1 via

$$\langle F_n(v)\xi \,|\, \xi \rangle = \xi^* F_n(v) \xi = F_1(\xi^* v \xi)$$

for any unit vector $\xi \in \mathbb{C}^n$, considered as a row matrix, and $v \in K_n$. This formula shows that Ω is injective, and it may in turn be used to define an order preserving inverse. By the characterization of function systems as complete order unit spaces, this means that $A(\boldsymbol{K}, \mathbb{C})$ and $A(K_1)$ are isomorphic as function systems. We remark that the above formula also shows that each F_n is continuous.

It is now possible to identify $M_r(A(\boldsymbol{K}, \mathbb{C}))$ and $A(\boldsymbol{K}, \mathbb{M}_r)$, and we may thus use the ordering on $A(\boldsymbol{K}, \mathbb{M}_r)$ to define a positive cone in $M_r(A(\boldsymbol{K}, \mathbb{C}))$. In this manner $A(\boldsymbol{K}, \mathbb{C})$ becomes a matrix ordered space. Using the identification of $A(\boldsymbol{K}, \mathbb{C})$ and $A(K_1)$, it is now straightforward to check that $A(\boldsymbol{K}, \mathbb{C})$ satisfies the Choi-Effros axioms ([5]) for an operator system with the order unit $\mathbf{I}$. We will simply denote the corresponding operator system by $A(\boldsymbol{K})$.

Proposition 3.5.

1. *If $\mathcal{R}$ is an operator system, then $\boldsymbol{CS}(\mathcal{R})$ is a self-adjoint compact matrix convex set in $\mathcal{R}^*$, equipped with the weak* topology, and $A(\boldsymbol{CS}(\mathcal{R}))$ and $\mathcal{R}$ are isomorphic as operator systems.*
2. *If $\boldsymbol{K}$ is a compact matrix convex set in a locally convex space V, then $A(\boldsymbol{K})$ is an operator system, and $\boldsymbol{K}$ and $\boldsymbol{CS}(A(\boldsymbol{K}))$ are matrix affinely homeomorphic.*

Proof. (1). Set $\boldsymbol{K} = \boldsymbol{CS}(\mathcal{R})$. We need to show that there exists a unital matrix order preserving bijection between $\mathcal{R}$ and $A(\boldsymbol{K})$. We know that $\mathcal{R}$ and $A(K_1) \simeq A(\boldsymbol{K})$ are isomorphic as function systems via the usual embedding, mapping $x \in \mathcal{R}$ to $\varphi \mapsto \varphi(x)$ for $\varphi \in K_1$ (cf. [1, Section II 1]). It therefore suffices to check that the matrix orderings are preserved. On the level of matrices, this map sends $x \in M_r(\mathcal{R})$ to $\boldsymbol{F} \in M_r(A(\boldsymbol{K})) \simeq A(\boldsymbol{K}, M_r)$ given by $F_n(\varphi) = \varphi_r(x)$ for $\varphi \in K_n$. This shows the claim, since $x \geq 0$ if and only if $\varphi_r(x) \geq 0$ for all $\varphi \in CS_n(\mathcal{R})$, $n \in \mathbb{N}$ by [5, p. 178].

(2). The usual evaluation map of K_1 onto $S(A(K_1))$ extends to a mapping $\theta_n : K_n \to CS_n(A(\boldsymbol{K}))$ mapping $v \in K_n$ to $\boldsymbol{F} \mapsto F_n(v)$. We claim that $\boldsymbol{\theta} = (\theta_n)$ is a matrix affine homeomorphism of $\boldsymbol{K}$ onto $\boldsymbol{CS}(A(\boldsymbol{K}))$.

It is straightforward to check that $\boldsymbol{\theta}$ is a matrix affine map into $\boldsymbol{CS}(A(\boldsymbol{K}))$, and that each θ_n is continuous using the weak* topology in $A(\boldsymbol{K})^*$.

To see injectivity, let V' be the continuous dual of V, and then observe that $f \in V'$ defines an element in $A(\boldsymbol{K})$ determined by the linear map $v \in V \mapsto f(v) \in \mathbb{C}$. If $\theta_n(v) = \theta_n(w)$ for $v, w \in K_n$, then in particular $f_n(v) = f_n(w)$ for all $f \in V'$, which again implies that $v = w$, since V' separates points in V.

It remains to show surjectivity. Assume that $\varphi_0 \in CS_n(A(\boldsymbol{K})) \backslash \theta_n(K_n)$. By the matricial separation theorem, Theorem 1.6, applied to $A(\boldsymbol{K})^*$, equipped with the weak* topology, and the weakly closed matrix convex set $\boldsymbol{\theta}(\boldsymbol{K})$ in $A(\boldsymbol{K})^*$, there exist a weakly continuous linear map $\Phi : A(\boldsymbol{K})^* \to \mathbb{M}_n$ and a self-adjoint $\alpha \in \mathbb{M}_n$, such that

$$\operatorname{Re} \Phi_r(\theta_r(v)) \leq \alpha \otimes \mathbb{I}_r$$

for all $r \in \mathbb{N}$, $v \in K_r$, and

$$\operatorname{Re} \Phi_n(\varphi_0) \not\leq \alpha \otimes \mathbb{I}_n.$$

Identifying Φ with $\boldsymbol{F} \in M_n(A(\boldsymbol{K})) \simeq A(\boldsymbol{K}, \mathbb{M}_n)$, this means that

$$\operatorname{Re} F_r(v) \leq \alpha \otimes \mathbb{I}_r$$

for all $v \in K_r$, $r \in \mathbb{N}$, and

$$\operatorname{Re}(\varphi_0)_n(\boldsymbol{F}) \not\leq \alpha \otimes \mathbb{I}_n.$$

But the first inequality says that $\operatorname{Re} \boldsymbol{F} \leq \alpha \otimes \mathbf{I}$ in $M_n(A(\boldsymbol{K}))$, and since φ_0 is completely positive and unital,

$$\operatorname{Re}(\varphi_0)_n(\boldsymbol{F}) \leq (\varphi_0)_n(\alpha \otimes \mathbf{I}) = \alpha \otimes \varphi_0(\mathbf{I}) = \alpha \otimes \mathbb{I}_n,$$

a contradiction. Hence θ_n is also onto. ☐

The above proposition shows that we can always think of a compact matrix convex set $\boldsymbol{K}$ as the matrix state spaces of an operator system $\mathcal{R}$. This will be crucial in our approach to the Krein-Milman theorem, but there are other benefits obtained from this result. The corollary below shows that matrix extreme points are also classical extreme points, adapting the proofs of [9, Proposition 1.1] and [15, Proposition 23].

Corollary 3.6. *Let $\boldsymbol{K} = (K_n)$ be a compact matrix convex set in a locally convex space V. If v is a matrix extreme point in K_n, then v is also an extreme point in K_n.*

Proof. By Proposition 3.5 it suffices to consider the case where $\boldsymbol{K} = \boldsymbol{CS}(\mathcal{R})$ for some operator system $\mathcal{R}$, since both matrix extreme and extreme points are preserved under matrix affine homeomorphisms.

Assume that $\varphi \in CS_n(\mathcal{R})$ is a matrix extreme point, and that we are given a proper convex combination $\varphi = t\varphi_1 + (1-t)\varphi_2$ with $0 < t < 1$ and $\varphi_1, \varphi_2 \in CS_n(\mathcal{R})$. Then φ is unitarily equivalent to φ_1 and φ_2, i.e., for any $x \in \mathcal{R}$, $\varphi(x)$ is written as a proper convex combination of elements from its unitary orbit in $\mathbb{M}_n$. By [14], this implies that $\varphi(x) = \varphi_1(x) = \varphi_2(x)$. Hence φ is extreme. ☐

4. The Krein-Milman theorem for matrix convex sets

Given a collection $\boldsymbol{S} = (S_n)$ of subsets $S_n \subset M_n(V)$ for some locally convex vector space V, we define the *closed matrix convex hull* $\overline{\mathrm{co}}(\boldsymbol{S})$ to be the smallest closed matrix convex set containing $\boldsymbol{S}$. We can either describe $\overline{\mathrm{co}}(\boldsymbol{S})$ as the intersection of all closed matrix convex sets containing $\boldsymbol{S}$, or, more explicitly, as the closure of all elements $v \in M_n(V)$ of the form

$$v = \sum_{i=1}^{k} \gamma_i^* v_i \gamma_i$$

where $v_i \in S_{n_i}$ and $\gamma_i \in M_{n_i,n}$ for $i = 1, \ldots, k$ satisfying $\sum_i \gamma_i^* \gamma_i = \mathbb{I}_n$. The latter description relies on the easy fact that the closure of a matrix convex set is again a matrix convex set.

We remark in passing that using the notion of *matrix polar* as defined in [11], it follows easily from Theorem 1.6 that if $\boldsymbol{S}$ contains the origin, then the double matrix polar of $\boldsymbol{S}$ coincides with $\overline{\mathrm{co}}(\boldsymbol{S})$.

Example 4.1. We saw in Example 2.2 that the matrix extreme points of the matrix interval $[\boldsymbol{a\,\mathrm{I}}, \boldsymbol{b\,\mathrm{I}}]$ with $a, b \in \mathbb{R}$ are a and b, i.e.,

$$\partial[a\mathbb{I}_n, b\mathbb{I}_n] = \begin{cases} \{a, b\}, & n = 1; \\ \emptyset, & n > 1. \end{cases}$$

It follows that $[\boldsymbol{a\,\mathrm{I}}, \boldsymbol{b\,\mathrm{I}}] = \overline{\mathrm{co}}(\boldsymbol{\partial}[\boldsymbol{a\,\mathrm{I}}, \boldsymbol{b\,\mathrm{I}}])$, since $\overline{\mathrm{co}}(\boldsymbol{\partial}[\boldsymbol{a\,\mathrm{I}}, \boldsymbol{b\,\mathrm{I}}])_1 = [a, b]$, and this determines $\overline{\mathrm{co}}(\boldsymbol{\partial}[\boldsymbol{a\,\mathrm{I}}, \boldsymbol{b\,\mathrm{I}}])$ uniquely by Example 1.2.

The above is clearly an example of a Krein-Milman type result. The work of Farenick-Morenz establishes a similar statement in the case of the matrix state spaces on a C*-algebra.

Example 4.2. In [9, Theorem 3.5] it is shown that for a C*-algebra $\mathcal{A}$, $CS_n(\mathcal{A})$ is the closed C*-convex hull of the the set of C*-extreme points in $CS_n(\mathcal{A})$. In Example 2.3 we observed that any C*-extreme point of $CS_n(\mathcal{A})$ is a matrix convex combination of matrix extreme points in $\boldsymbol{CS}(\mathcal{A})$. Since any C*-convex combination is also a matrix convex combination, the closed C*-convex hull of the C*-extreme points coincides with the closed matrix convex hull of the matrix extreme points, i.e.,

$$\boldsymbol{CS}(\mathcal{A}) = \overline{\mathrm{co}}(\boldsymbol{\partial CS}(\mathcal{A})).$$

The above two examples are special cases of the following generalized version of the Krein-Milman theorem.

Theorem 4.3. *Let $\boldsymbol{K}$ be a compact matrix convex set in a locally convex space V, and let $\boldsymbol{\partial K} = (\partial K_n)$ denote the collection of matrix extreme points ∂K_n of K_n. Then $\boldsymbol{\partial K}$ is non-empty, and*

$$\boldsymbol{K} = \overline{\mathrm{co}}(\boldsymbol{\partial K}).$$

Before embarking on the proof, we shall introduce an auxiliary convex set $\Delta_n(\boldsymbol{K})$, which is an essential tool in the reduction to the classical Krein-Milman. For a collection $\boldsymbol{K} = (K_r)$ of subsets $K_r \subset M_r(V)$ and a fixed $n \in \mathbb{N}$, we define the subset $\Delta_n(\boldsymbol{K})$ of $M_n(V)$ by

$$\Delta_n(\boldsymbol{K}) = \{\xi^* v \xi \mid v \in K_r, \xi \in \mathbb{M}_{r,n}, \|\xi\|_2 = 1, r \in \mathbb{N}\}, \tag{2}$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. We observe that if $\boldsymbol{K}$ is matrix convex, then $\Delta_n(\boldsymbol{K})$ is convex. Indeed, given $0 \le t \le 1$ and $\xi^* v\xi$, $\eta^* w \eta \in \Delta_n(\boldsymbol{K})$ with $v \in K_r$, $w \in K_s$ and $\xi \in \mathbb{M}_{r,n}$, $\eta \in \mathbb{M}_{s,n}$ satisfying $\|\xi\|_2, \|\eta\|_2 = 1$, then

$$t\xi^* v\xi + (1-t)\eta^* w\eta = \begin{bmatrix} t^{1/2}\xi^* & (1-t)^{1/2}\eta^* \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \begin{bmatrix} t^{1/2}\xi \\ (1-t)^{1/2}\eta \end{bmatrix}$$

where

$$\begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} = \begin{bmatrix} \mathbb{I}_r \\ 0 \end{bmatrix} v \begin{bmatrix} \mathbb{I}_r & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbb{I}_s \end{bmatrix} w \begin{bmatrix} 0 & \mathbb{I}_s \end{bmatrix} \in K_{r+s},$$

and

$$\left\| \begin{bmatrix} t^{1/2}\xi \\ (1-t)^{1/2}\eta \end{bmatrix} \right\|_2^2 = t\|\xi\|_2^2 + (1-t)\|\eta\|_2^2 = 1.$$

Moreover, in (2) we may always choose $\xi \in \mathbb{M}_{r,n}$ such that ξ has a right inverse (and in particular $r \le n$). To see this, let $v \in K_r$ and $\xi \in \mathbb{M}_{r,n}$ with $\|\xi\|_2 = 1$ be given, and let s be the dimension of the range of ξ. Letting $\nu \in \mathbb{M}_{r,s}$ be an isometry of $\mathbb{C}^s$ onto the range of ξ, we have that

$$\xi^* v \xi = (\nu^*\xi)^*(\nu^* v \nu)(\nu^*\xi) \tag{3}$$

is the desired decomposition. In particular,

$$\Delta_n(\boldsymbol{K}) = \{\xi^* v \xi \mid v \in K_r, \xi \in \mathbb{M}_{r,n}, \|\xi\|_2 = 1, r \le n\}, \tag{4}$$

from which it follows that $\Delta_n(\boldsymbol{K})$ is compact whenever K_n is. Observe that this only relies on the fact that $\boldsymbol{K}$ is closed under isometries.

One of the important features of $\Delta_n(\boldsymbol{K})$ is that there is a good description of the extreme points in terms of the matrix extreme points. We begin with the case of the matrix state spaces of an operator system.

Lemma 4.4. *Let $\mathcal{R}$ be an operator system, and let $\Delta_n(\boldsymbol{CS}(\mathcal{R}))$ be defined as above. If $\bar{\varphi}$ is an extreme point of $\Delta_n(\boldsymbol{CS}(\mathcal{R}))$, then there exist a matrix extreme point $\varphi \in CS_r(\mathcal{R})$ for some $r \in \mathbb{N}$ and a right invertible element $\xi \in \mathbb{M}_{r,n}$ with $\|\xi\|_2 = 1$ such that*

$$\bar{\varphi} = \xi^* \varphi \xi.$$

Proof. Assume that $\bar{\varphi}$ is an extreme point of $\Delta_n(\boldsymbol{CS}(\mathcal{R}))$. By (3), we may write $\bar{\varphi} = \xi^*\varphi\xi$ for some $\varphi \in CS_r(\mathcal{R})$ and $\xi \in \mathbb{M}_{r,n}$, where $\|\xi\|_2 = 1$ and ξ has a right inverse.

We claim that φ is a matrix extreme point. To see this, assume that φ is written as a proper matrix convex combination $\varphi = \sum_i \gamma_i^*\varphi_i\gamma_i$ with $\varphi_i \in CS_{r_i}(\mathcal{R})$ and $\gamma_i \in \mathbb{M}_{r_i,r}$ for $i = 1, \ldots, k$. Set $t_i = \|\gamma_i\xi\|_2^2$, and observe that

$$\sum_i t_i = \sum_i \|\gamma_i\xi\|_2^2 = \sum_i \operatorname{Tr}(\xi^*\gamma_i^*\gamma_i\xi) = \operatorname{Tr}(\xi^*\xi) = \|\xi\|_2^2 = 1,$$

and that $t_i \neq 0$, since both γ_i and ξ have right inverses. Thus we can write $\bar{\varphi}$ as the proper convex combination

$$\bar{\varphi} = \xi^*\varphi\xi = \sum_i \xi^*\gamma_i^*\varphi_i\gamma_i\xi = \sum_i t_i \frac{(\gamma_i\xi)^*}{\|\gamma_i\xi\|_2}\,\varphi_i\,\frac{(\gamma_i\xi)}{\|\gamma_i\xi\|_2}.$$

Since $\bar{\varphi}$ is extreme, this means that $\xi^*\varphi\xi = \|\gamma_i\xi\|_2^{-2}(\gamma_i\xi)^*\,\varphi_i\,(\gamma_i\xi)$, and using that ξ has a right inverse, we get that

$$\varphi\|\gamma_i\xi\|_2^2 = \gamma_i^*\varphi_i\gamma_i$$

for $i = 1, \ldots, k$. Since φ and φ_i are unital, this in particular implies that

$$\mathbb{I}_r\|\gamma_i\xi\|_2^2 = \gamma_i^*\gamma_i.$$

Therefore $\gamma_i\|\gamma_i\xi\|_2^{-1}$ is an isometry, and since γ_i is known to be surjective, we have that $r = r_1 = \cdots = r_k$, and $\gamma_i\|\gamma_i\xi\|_2^{-1}$ is a unitary implementing a unitary equivalence between φ and φ_i. Hence φ is a matrix extreme point. $\square$

Using the representation theorem of the previous section we may extend the above result to general compact matrix convex sets.

Lemma 4.5. *Let $\boldsymbol{K} = (K_n)$ be a compact matrix convex set in a locally convex space V. If $\bar{v}$ is an extreme point of $\Delta_n(\boldsymbol{K})$, then there exist a matrix extreme point $v \in K_r$ for some $r \in \mathbb{N}$ and a right invertible element $\xi \in \mathbb{M}_{r,n}$ with $\|\xi\|_2 = 1$ such that*

$$\bar{v} = \xi^* v \xi.$$

Proof. By Proposition 3.5 (2) there exists an operator system $\mathcal{R}$ and a matrix affine homeomorphism $\boldsymbol{\theta} = (\theta_n)$ of $\boldsymbol{CS}(\mathcal{R})$ onto $\boldsymbol{K}$. It suffices to show that $\Gamma : \Delta_n(\boldsymbol{CS}(\mathcal{R})) \to \Delta_n(\boldsymbol{K})$ given by

$$\Gamma(\xi^*\varphi\xi) = \xi^*\theta_r(\varphi)\xi$$

for $\varphi \in CS_r(\mathcal{R})$ and $\xi \in \mathbb{M}_{r,n}$ satisfying $\|\xi\|_2 = 1$, is a well-defined continuous affine surjection. If this is so, and $\bar{v} \in \Delta_n(\boldsymbol{K})$ is an extreme point, then $\Gamma^{-1}(\bar{v})$ is a compact face of $\Delta_n(\boldsymbol{CS}(\mathcal{R}))$. By Krein-Milman, this set has an extreme point, which is also an extreme point of $\Delta_n(\boldsymbol{CS}(\mathcal{R}))$. The conclusion now follows by applying Lemma 4.4, and the observation that $\boldsymbol{\theta}$ preserves matrix extreme points.

To see that Γ is well-defined, first observe that if ν is an isometry chosen as in (3), then

$$\xi^*\theta_r(\varphi)\xi = \xi^*\nu\nu^*\theta_r(\varphi)\nu\nu^*\xi = \xi^*\nu\theta_s(\nu^*\varphi\nu)\nu^*\xi.$$

We may therefore assume without loss of generality that ξ is right invertible. If $\xi^*\varphi\xi = \eta^*\psi\eta$ with $\psi \in CS_t(\mathcal{R})$ and $\eta \in \mathbb{M}_{t,n}$, then using that φ and ψ are unital we see that $\eta\xi^{-1}$ is an isometry. Thus

$$\theta_r(\varphi) = \theta_r((\eta\xi^{-1})^*\psi(\eta\xi^{-1})) = (\eta\xi^{-1})^*\theta_t(\psi)(\eta\xi^{-1}),$$

or $\xi^*\theta_r(\varphi)\xi = \eta^*\theta_t(\psi)\eta$, which shows that Γ is well-defined.

It is immediate that Γ is affine and surjective. To see that Γ is also continuous, consider a convergent net

$$\xi_\alpha^*\varphi_\alpha\xi_\alpha \to \xi^*\varphi\xi$$

in $\Delta(\boldsymbol{CS}(\mathcal{R}))$ with $\varphi_\alpha \in CS_{r_\alpha}(\mathcal{R})$ and $\xi_\alpha \in M_{r_\alpha,n}$. Set $\eta_\alpha = \xi_\alpha\xi^{-1} \in \mathbb{M}_{r_\alpha,r}$, and observe that since all maps are unital,

$$\eta_\alpha^*\eta_\alpha \to \mathbb{I}_r.$$

If $\eta_\alpha = \nu_\alpha|\eta_\alpha|$ is the polar decomposition of η_α, this means that $|\eta_\alpha|$ is surjective from some step, and hence that ν_α is an isometry. Moreover, as $\nu_\alpha - \eta_\alpha \to 0$,

$$\nu_\alpha^*\varphi_\alpha\nu_\alpha = \eta_\alpha^*\varphi_\alpha\eta_\alpha + (\nu_\alpha - \eta_\alpha)^*\varphi_\alpha\nu_\alpha + \eta_\alpha^*\varphi_\alpha(\nu_\alpha - \eta_\alpha) \to \varphi.$$

By the continuity of θ_r,

$$\eta_\alpha^*\theta_{r_\alpha}(\varphi_\alpha)\eta_\alpha = |\eta_\alpha|\theta_r(\nu_\alpha^*\varphi_\alpha\nu_\alpha)|\eta_\alpha| \to \mathbb{I}_r\theta_r(\varphi)\mathbb{I}_r = \theta_r(\varphi),$$

or, equivalently,

$$\Gamma(\xi_\alpha^*\varphi_\alpha\xi_\alpha) = \xi_\alpha^*\theta_{r_\alpha}(\varphi_\alpha)\xi_\alpha \to \xi^*\theta_r(\varphi)\xi = \Gamma(\xi^*\varphi\xi),$$

and we are done. □

It would be tempting to assume that $\Delta_n(\boldsymbol{K})$ is preserved under matrix affine homeomorphisms of $\boldsymbol{K}$, but we see from the proof above that the situation is not that simple.

Proof of Theorem 4.3. Let $\boldsymbol{K} = (K_n)$ be a compact matrix convex set, and let $\boldsymbol{\partial K} = (\partial K_n)$ be the collection of matrix extreme points. Since ∂K_1 coincides with the usual extreme points, $\boldsymbol{\partial K}$ is non-empty, and we clearly have $\overline{\text{co}}(\boldsymbol{\partial K}) \subset \boldsymbol{K}$. We may assume that $0 \in \overline{\text{co}}(\boldsymbol{\partial K})$ without loss of generality by translating K_n by $v_0 \otimes \mathbb{I}_n$ for some $v_0 \in \partial K_1$.

For the converse inclusion, assume that there exists $v_0 \in K_n \backslash \overline{\text{co}}(\boldsymbol{\partial K})_n$. By the matricial separation theorem, Theorem 1.6, there exists a continuous linear mapping $\Phi : V \to \mathbb{M}_n$ such that

$$\operatorname{Re}\Phi_r(v) \leq \mathbb{I}_n \otimes \mathbb{I}_r \tag{5}$$

for all $v \in \overline{\text{co}}(\boldsymbol{\partial K})_r$ and $r \in \mathbb{N}$, and

$$\operatorname{Re}\Phi_n(v_0) \not\leq \mathbb{I}_n \otimes \mathbb{I}_n. \tag{6}$$

Φ induces a continuous linear functional $F : M_n(V) \to \mathbb{C}$ satisfying

$$F(\eta^*v\xi) = \langle \Phi_r(v)\xi \,|\, \eta\rangle$$

for all $v \in M_r(V)$ and $\xi, \eta \in \mathbb{M}_{r,n}$, simultaneously considered as vectors in $\mathbb{C}^{rn}$. If $\bar{v}$ is an extreme point of $\Delta_n(\boldsymbol{K})$, then, by Lemma 4.5, we may write $\bar{v} = \xi^*v\xi$

where $v \in \partial K_r$ and $\xi \in \mathbb{M}_{r,n}$ with $\|\xi\|_2 = 1$ and $r \leq n$. By (5), we therefore get that

$$\begin{aligned}\operatorname{Re} F(\bar{v}) &= \operatorname{Re} F(\xi^* v \xi) = \operatorname{Re}\langle \Phi_r(v)\xi \,|\, \xi\rangle \\ &\leq \langle \mathbb{I}_n \otimes \mathbb{I}_r \xi \,|\, \xi\rangle = \|\xi\|_2^2 = 1.\end{aligned}$$

for all extreme points $\bar{v}$ of $\Delta_n(\boldsymbol{K})$. Since $\Delta_n(\boldsymbol{K})$ is compact by (4), the Krein-Milman theorem implies that

$$\operatorname{Re} F(\Delta_n(\boldsymbol{K})) \leq 1.$$

This, in turn, implies that for any unit vector $\xi \in \mathbb{C}^{rn}$ and $v \in K_r$,

$$\operatorname{Re}\langle \Phi_r(v)\xi \,|\, \xi\rangle = \operatorname{Re} F(\xi^* v \xi) \leq 1,$$

i.e., $\operatorname{Re} \Phi_r(v) \leq \mathbb{I}_n \otimes \mathbb{I}_r$, contradicting (6). Hence $\boldsymbol{K} = \overline{\operatorname{co}}(\boldsymbol{\partial K})$. □

We remark that an inspection of the above proof reveals that only matrix extreme points in K_r for $r \leq n$ are necessary to generate K_n.

The key idea of the above proof is to use the matricial separation theorem and the correspondence between linear functionals on $M_n(V)$ and the linear mappings $V \to \mathbb{M}_n$ to reduce the matricial problem to a scalar one in $M_n(V)$. This naturally leads to introducing the convex set $\Delta_n(\boldsymbol{K})$ and establishing a connection between the matrix extreme points of $\boldsymbol{K}$ and the extreme points of $\Delta_n(\boldsymbol{K})$, which allows us to use the classical Krein-Milman theorem.

The converse result, which says that the extreme points are contained in any closed set with closed convex hull equal to the compact convex set in question, is usually considered an integral part of the classical Krein-Milman theorem. Morenz proved a similar condition for his structural elements in the C*-convexity case in $\mathbb{M}_n$ ([17, Theorem 4.5]). We present a similar condition in the matrix convexity case to document that the situation in our Krein-Milman theorem is actually optimal.

Theorem 4.6. *Let $\boldsymbol{K}$ be a compact matrix convex set in a locally convex space V, and let $\boldsymbol{S} = (S_n)$ be a collection of closed subsets $S_n \subset K_n$ such that $\nu^* S_m \nu \subset S_n$ for all isometries $\nu \in \mathbb{M}_{m,n}$. If $\overline{\operatorname{co}}(\boldsymbol{S}) = \boldsymbol{K}$, then*

$$\boldsymbol{\partial K} \subset \boldsymbol{S}.$$

We note that the condition that $\boldsymbol{S}$ be closed under isometries is actually necessary. In the case, say, where $\boldsymbol{K} = \boldsymbol{CS}(\mathcal{A})$ for some C*-algebra $\mathcal{A}$, we saw in Example 2.3 that $\boldsymbol{\partial K}$ consists of all pure matrix states. Using the fact that the minimal Stinespring representation of a pure matrix state is irreducible (cf. [2]), it easily follows that the pure matrix states are closed under isometries. One may therefore remove elements from the isometric orbit of a pure matrix state and still have them generate the whole matrix state space, but there is no canonical way of choosing which pure states to exclude.

The proof of the above theorem follows more or less by reversing the proof of Theorem 4.3. In this respect the lemma below is the converse to Lemma 4.4

Lemma 4.7. *Let $\mathcal{R}$ be an operator system. Given a matrix extreme point $\varphi \in CS_n(\mathcal{R})$ and an invertible element $\xi \in \mathbb{M}_n$ satisfying $\|\xi\|_2 = 1$, then*

$$\bar{\varphi} = \xi^* \varphi \xi$$

is an extreme point in $\Delta_n(\boldsymbol{CS}(\mathcal{R}))$.

Proof. Assume that $\varphi \in CS_n(\mathcal{R})$ is a matrix extreme point, and that $\xi \in \mathbb{M}_n$ satisfying $\|\xi\|_2 = 1$ is invertible. We wish to prove that $\bar{\varphi} = \xi^* \varphi \xi$ is an extreme point in $\Delta_n(\boldsymbol{CS}(\mathcal{R}))$. Given a proper convex combination

$$\xi^* \varphi \xi = t \xi_1^* \varphi_1 \xi_1 + (1-t) \xi_2^* \varphi_2 \xi_2$$

with $\varphi_1 \in CS_r(\mathcal{R})$, $\varphi_2 \in CS_s(\mathcal{R})$, right invertible elements $\xi_1 \in \mathbb{M}_{r,n}$, $\xi_2 \in \mathbb{M}_{s,n}$ satisfying $\|\xi_1\|_2$, $\|\xi_2\|_2 = 1$, and $0 < t < 1$, then

$$\varphi = t(\xi_1 \xi^{-1})^* \varphi_1 (\xi_1 \xi^{-1}) + (1-t)(\xi_2 \xi^{-1})^* \varphi_2 (\xi_2 \xi^{-1}).$$

This is a proper matrix convex combination since φ, φ_1, and φ_2 are unital, ξ_1, ξ_2 are right invertible, and ξ is invertible. Hence $n = r = s$, and we have unitaries u_1, $u_2 \in \mathbb{M}_n$ such that $\varphi_1 = u_1^* \varphi u_1$ and $\varphi_2 = u_2^* \varphi u_2$, i.e.,

$$\varphi = t(u_1 \xi_1 \xi^{-1})^* \varphi (u_1 \xi_1 \xi^{-1}) + (1-t)(u_2 \xi_2 \xi^{-1})^* \varphi (u_2 \xi_2 \xi^{-1}).$$

If we define the matrix state $\psi : \mathbb{M}_n \to \mathbb{M}_n$ by

$$\psi(\alpha) = t(u_1 \xi_1 \xi^{-1})^* \alpha (u_1 \xi_1 \xi^{-1}) + (1-t)(u_2 \xi_2 \xi^{-1})^* \alpha (u_2 \xi_2 \xi^{-1})$$

for $\alpha \in \mathbb{M}_n$, then the above equation says that

$$\varphi = \psi \circ \varphi.$$

We claim that this implies that $\psi = \mathrm{id}$. The operator system $\varphi(\mathcal{R})$ in $\mathbb{M}_n$ is irreducible, because otherwise φ is unitarily equivalent to a diagonal matrix of matrix states, contradicting that φ is matrix extreme. The claim now follows from Proposition 1.5.

Since $\psi = \mathrm{id}$, the uniqueness part of Choi's description of completely positive maps ([4, Remark 4]) implies that

$$\sqrt{t}(u_1 \xi_1 \xi^{-1}) = \sqrt{s} \lambda_1 \mathbb{I}_n, \qquad \sqrt{1-t}(u_2 \xi_2 \xi^{-1}) = \sqrt{1-s} \lambda_2 \mathbb{I}_n$$

for $0 \le s \le 1$ and λ_1, $\lambda_2 \in \mathbb{C}$ satisfying $|\lambda_1|$, $|\lambda_2| = 1$. But $t = s$ since

$$t = t\|\xi_1\|_2^2 = \mathrm{Tr}((\sqrt{t} u_1 \xi_1)^* (\sqrt{t} u_1 \xi_1)) = \mathrm{Tr}(s(\lambda_1 \xi)^* (\lambda_1 \xi)) = s\|\xi\|_2^2 = s.$$

Hence $u_1 \xi_1 \xi^{-1} = \lambda_1 \mathbb{I}_n$, and so

$$\xi_1^* \varphi_1 \xi_1 = \xi_1^* u_1^* \varphi u_1 \xi_1 = (\lambda_1 \xi)^* \varphi (\lambda_1 \xi) = \xi^* \varphi \xi.$$

Similarly, $\xi_2^* \varphi_2 \xi_2 = \xi^* \varphi \xi$, and we are done. □

As extreme points are not necessarily preserved under affine surjections the method of Lemma 4.5 does not lead to an extension of the above lemma to general compact matrix convex sets. Luckily, Lemma 4.7 is all we need.

Proof of Theorem 4.6. By Proposition 3.5 (2) we may assume that $\boldsymbol{K} = \boldsymbol{CS}(\mathcal{R})$ for some operator system $\mathcal{R}$, as the statement of the theorem is preserved under matrix affine homeomorphism.

We begin by proving that $\Delta_n(\boldsymbol{CS}(\mathcal{R}))$ is the closed convex hull L of $\Delta_n(\boldsymbol{S})$ by reversing the argument in the proof of Theorem 4.3. Clearly $L \subset \Delta_n(\boldsymbol{CS}(\mathcal{R}))$.

For the converse, assume that $\bar{\varphi}_0 \in \Delta_n(\boldsymbol{CS}(\mathcal{R})) \backslash L$. Then there exist a weakly continuous linear functional $F : M_n(\mathcal{R}^*) \to \mathbb{C}$ and $\lambda \in \mathbb{R}$ such that

$$\mathrm{Re}\, F(\bar{\varphi}) \le \lambda < \mathrm{Re}\, F(\bar{\varphi}_0)$$

for all $\bar{\varphi} \in \Delta_n(\boldsymbol{S})$. If F corresponds to the weakly continuous linear mapping $\Phi : \mathcal{R}^* \to \mathbb{M}_n$, as in the proof of Theorem 4.3, and if we write $\bar{\varphi}_0 = \xi_0^* \varphi_0 \xi_0$ with $\varphi_0 \in K_{r_0}$ and $\xi_0 \in \mathbb{M}_{r_0,n}$ satisfying $\|\xi_0\|_2 = 1$, then the above is equivalent to

$$\mathrm{Re}\langle \Phi_r(\varphi)\xi \mid \xi\rangle \leq \lambda < \mathrm{Re}\langle \Phi_{r_0}(\varphi_0)\xi_0 \mid \xi_0\rangle$$

for all $r \in \mathbb{N}$, $\varphi \in S_r$, and $\xi \in \mathbb{M}_{r,n}$ satisfying $\|\xi\|_2 = 1$. Since $\boldsymbol{K} = \boldsymbol{CS}(\mathcal{R}) = \overline{\mathrm{co}}(\boldsymbol{S})$, this implies that

$$\mathrm{Re}\,\Phi_r(K_r) \leq \lambda \mathbb{I}_{rn}$$

for all $r \in \mathbb{N}$, contradicting that $\varphi_0 \in K_{r_0}$.

Hence $\Delta_n(\boldsymbol{CS}(\mathcal{R}))$ is the closed convex hull of $\Delta_n(\boldsymbol{S})$. By Krein-Milman, this implies that the extreme points of $\Delta_n(\boldsymbol{CS}(\mathcal{R}))$ are contained in the closure of $\Delta_n(\boldsymbol{S})$. Since $\boldsymbol{S}$ is closed under isometries, (4) also holds for $\Delta_n(\boldsymbol{S})$. Hence $\Delta_n(\boldsymbol{S})$ is also closed. From this we claim to be able to show that $\boldsymbol{\partial CS}(\mathcal{R}) \subset \boldsymbol{S}$.

Let $\varphi \in \partial CS_n(\mathcal{R})$. Choosing an arbitrary invertible element $\xi \in \mathbb{M}_n$ satisfying $\|\xi\|_2 = 1$, Lemma 4.7 shows that $\bar{\varphi} = \xi^* \varphi \xi$ is an extreme point in $\Delta_n(\boldsymbol{CS}(\mathcal{R}))$. By the above, we may find $\eta^* \psi \eta \in \Delta_n(\boldsymbol{S})$ such that $\bar{\varphi} = \eta^* \psi \eta$ with $\psi \in S_r$, i.e.,

$$\varphi = (\eta\xi^{-1})^* \psi (\eta\xi^{-1}).$$

In particular, $(\eta\xi^{-1})^*(\eta\xi^{-1}) = \mathbb{I}_n$. Since $\boldsymbol{S}$ is closed under isometries, this shows that $\varphi \in S_n$. □

We wish to conclude with a few remarks about the often mentioned connections with C*-convexity. It is important to observe that the C*-convexity Krein-Milman theorems of Farenick and Morenz do not follow immediately from our work. In both cases additional technical results from their papers are needed. If $\boldsymbol{K} = (K_n)$ is a compact matrix convex set, then we know that any element in K_n can be approximated by matrix convex combinations $\sum_i \gamma_i^* v_i \gamma_i$ of matrix extreme points $v_i \in K_{n_i}$ for $i = 1, \ldots, n$ and $n_i \leq n$. Even though $v_1, \ldots, v_n$ are C*-extreme, they need not lie in K_n. We therefore need to alter the matrix convex combination to include C*-extreme points in K_n, i.e., we wish to write

$$\gamma_i^* v_i \gamma_i = \begin{bmatrix} \gamma_i^* & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & * \end{bmatrix} \begin{bmatrix} \gamma_i \\ 0 \end{bmatrix}$$

such that

$$\begin{bmatrix} v & 0 \\ 0 & * \end{bmatrix} \in K_n$$

is C*-extreme. In the case of a compact matrix convex set in $\mathbb{C}$, [17, Corollary 5.3] shows how to choose the missing entry, whereas for the matrix state spaces of a C*-algebra the choice is given by [9, Theorem 3.3]. With these additional results the corresponding C*-convexity versions of Krein-Milman follow.

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