THE KREIN-MILMAN THEOREM IN OPERATOR CONVEXITY

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Abstract. We generalize the Krein-Milman theorem to the setting of matrix convex sets of Effros-Winkler, extending the work of Farenick-Morenz on compact C*-convex sets of complex matrices and the matrix state spaces of C*-algebras. An essential ingredient is to prove the non-commutative analogue of the fact that a compact convex set $K$ may be thought of as the state space of the space of continuous affine functions on $K$.

The Krein-Milman theorem is without doubt one of the cornerstones of functional analysis. With the rise of non-commutative functional analysis and related notions of convexity ([15], [10], [11]), the question naturally arises how to formulate a notion of extreme points for which the theorem remains true.

Such a notion exists in the case of C*-convexity, which has been studied by Loebl-Paulsen ([15]), Hopenwasser-Moore-Paulsen ([12]), and, more recently, Farenick-Morenz ([7], [8], [9], [17]). C*-convexity is the natural extension of the classical scalar-valued convex combination to include C*-algebra-valued coefficients. It therefore makes sense in a C*-algebra and, more generally, for bimodules over C*-algebras. In particular, there is a rich class of such C*-convex sets in the $n \times n$ complex matrices, $M_n$. The matrix state spaces of a C*-algebra are another class of examples. In both these cases the C*-convexity version of the Krein-Milman has been proven to hold by Farenick and Morenz ([9], [17]).

The above two examples both fit in the framework of another non-commutative convexity theory, the theory of matrix convex sets, developed by Effros and the second author ([11], [21]). In this paper we develop a notion of extreme points in this context, and we prove a corresponding Krein-Milman result, including a minimality condition which shows that the result is indeed optimal. Even though the difference between extremality in C*-convexity and matrix convexity might seem minor at first, we hope to convince the reader that our approach is the natural one. Moreover, our methods are seemingly new and different, the central idea being to prove the analogue of the fact that a compact convex set $K$ can be represented as the state space of the space of continuous affine functions on $K$.

We begin with a review of matrix convexity, followed by a treatment of extreme points in this context. We then prove our representation result, from which we proceed to prove the Krein-Milman theorem.

We wish to thank Douglas Farenick and Phillip Morenz for making a copy of [9] available to us.
1. Matrix convexity

All vector spaces are assumed to be complex throughout this paper. Let $M_{m,n}(V)$ denote the vector space of $m \times n$ matrices over a vector space $V$, and set $M_n(V) = M_{n,n}(V)$. We write $M_{m,n} = M_{m,n}(\mathbb{C})$ and $M_n = M_{n,n}(\mathbb{C})$, which means that we may identify $M_{m,n}(V)$ with the tensor product $M_{m,n} \otimes V$. We use the standard matrix multiplication and $*$-operation for compatible scalar matrices, and $I_n$ for the identity matrix in $M_n$.

The multiplication of scalar matrices induces a bimodule operation of scalar matrices on $M_m(V)$ via the identification with $M_m \otimes V$, i.e., for $v \in M_m(V)$ and $\alpha \in M_{m,m}$, $\beta \in M_{m,n}$, we define

$$\alpha v \beta = [\sum_{j,k} \alpha_{ij} v_{jk} \beta_{kl}] \in M_n(V).$$

The following definition of a non-commutative convex set was first proposed by Wittstock ([22]).

**Definition 1.1.** A **matrix convex set** in a vector space $V$ is a collection $K = (K_n)$ of subsets $K_n \subset M_n(V)$ such that

$$\sum_{i=1}^k \gamma_i^* v_i \gamma_i \in K_n$$

for all $v_i \in K_{n_i}$ and $\gamma_i \in M_{n_i,n}$ for $i = 1, \ldots, k$ satisfying $\sum_i \gamma_i^* \gamma_i = I_n$.

We shall say that $v = \sum_i \gamma_i^* v_i \gamma_i$ as above is a **matrix convex combination** of $v_1, \ldots, v_k$.

If we restrict ourselves to a fixed $n \in \mathbb{N}$ and a single set $K_n \subset M_n(V)$ satisfying the above with $n = n_1 = \cdots = n_k$, then we exactly get the definition of a $C^*$-convex set over $M_n$. This can easily be extended to arbitrary bimodules over unital $C^*$-algebras (cf. [8]), but in this paper we will only consider the case of $M_n$-bimodules. The examples below show in particular that the standard examples of $C^*$-convex sets (cf. [9], [17]) come with natural matrix convexity structure.

**Example 1.2.** Given $a, b \in \mathbb{R} \cup \{\pm \infty\}$, the collection $[a I_n, b I_n] = ([a I_n, b I_n])$ of intervals

$$[a I_n, b I_n] = \{ \alpha \in M_n \mid a I_n \leq \alpha \leq b I_n \}$$

defines a matrix convex set in $\mathbb{C}$. It is easy to show ([11, Lemma 3.1]) that any matrix convex set $K = (K_1)$ in $\mathbb{C}$ where $K_1$ is a closed convex subset of $\mathbb{R}$ is of this form.

As for more general matrix convex sets over $\mathbb{C}$, it follows from results of Arveson that any closed and bounded matrix convex set $K$ in $\mathbb{C}$ is the set of **matrix ranges** $W(T) = (W_n(T))$ of a Hilbert space operator $T$ acting on a separable Hilbert space $\mathcal{H}$ (cf. [3], p.301, and [15, Proposition 31]). The matrix ranges of $T$ are defined as

$$W_n(T) = \{ \varphi(T) \mid \varphi : \mathcal{B}(\mathcal{H}) \to M_n \text{ completely positive and } \varphi(I) = I_n \}.$$

As the second part of above example shows, there is already a rich class of matrix convex sets in even the simplest possible case. The study of matrix ranges has been a driving force behind the $C^*$-convexity theory. (See [19], [20], [7] and the references therein, for example.)
Example 1.3. Consider an operator space $\mathcal{M}$ (i.e., a closed linear subspace of the bounded operators on a Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$). The natural inclusion

$$M_n(\mathcal{M}) \hookrightarrow M_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^n)$$

endows $M_n(\mathcal{M})$ with a norm using the operator norm on $\mathcal{B}(\mathcal{H}^n)$, and it is easy to check that the collection $\mathcal{B} = (B_n)$ of unit balls

$$B_n = \{x \in M_n(\mathcal{M}) \mid \|x\| \leq 1\}$$

is a matrix convex set in $\mathcal{M}$.

Based on Ruan’s abstract characterization ([18]), operator spaces may be thought of as the non-commutative analogue of Banach spaces. In this line of thought, the above example is an analogue of a balanced convex set.

The objects of the next example are the non-commutative analogues of Kadison’s function systems ([13]), which are closed subspaces of $C(X)$, the continuous functions on a compact set $X$, closed under conjugation and containing the identity function.

Example 1.4. If $\mathcal{R}$ is an operator system (a closed subspace of $\mathcal{B}(\mathcal{H})$, closed under the adjoint operation and containing the identity operator $I$), then the inclusion (1) defines an ordering on $M_n(\mathcal{R})$ via the usual ordering on $\mathcal{B}(\mathcal{H}^n)$. In this case the collection $\mathcal{P} = (P_n)$ of positive cones

$$P_n = \{x \in M_n(\mathcal{R}) \mid x \geq 0\}$$

form a matrix convex set in $\mathcal{R}$.

We may also consider the collection $\mathcal{CS}(\mathcal{R}) = (CS_n(\mathcal{R}))$ of matrix states

$$CS_n(\mathcal{R}) = \{\varphi : \mathcal{R} \to M_n \mid \varphi \text{ completely positive}, \varphi(I) = I_n\},$$

where we recall that $\varphi : \mathcal{R} \to M_n$ is completely positive if the canonical amplifications $\varphi_r : M_r(\mathcal{R}) \to M_r(M_n)$ given by $\varphi_r = \text{id} \otimes \varphi$ are positive for all $r \in \mathbb{N}$. Again we get a matrix convex set (in $\mathcal{R}^*$). We consider $\mathcal{CS}(\mathcal{R})$ the matricial version of the state space. This fits well with Example 1.2 because for $T \in \mathcal{B}(\mathcal{H})$ the matrix ranges are given by $W_n(T) = \{\varphi(T) \mid \varphi \in CS_n(\mathcal{B}(\mathcal{H}))\}$.

Parallel to the abstract characterization of Kadison’s function systems as complete order unit spaces (cf. [1, Section II 1]), operator systems have been characterized by Choi-Effros ([5, Theorem 4.4]) as those matrix ordered spaces $\mathcal{R}$, where $\mathcal{R}$ itself is a function system, and $M_n(\mathcal{R})^+$ satisfies the Archimedean property for all $n \in \mathbb{N}$. We will use this theorem in Section 3. We refer to [1] and [5] for the relevant definitions.

We should comment here, for the benefit of those unfamiliar with the theory of operator systems and operator spaces, that these examples explain why we consider matrix convex sets to be the appropriate generalization of convex sets for non-commutative functional analysis. Not only do the sets $\mathcal{B}$ and $\mathcal{P}$ play analogous roles to convex sets in the classical theory, but they are in some sense optimal. Simple examples show that one cannot replace these sets by a collection of sets in a finite number of levels and still be able to distinguish general operator spaces or operator systems using them. On the other hand no more information is needed, since Ruan’s theorem and the Choi-Effros characterization ([5, Theorem 4.4]) respectively tell us that these matrix sets are sufficient to tell spaces of the appropriate type apart.

Much knowledge about matrix states and their extremal properties goes back to Arveson’s seminal work [2], [3]. We shall need the following consequence of
Arveson’s boundary theorem ([3, Theorem 2.1.1], [6]). (Arveson’s theorem is much more general than the case below for which a simpler proof is possible.)

**Proposition 1.5.** Let $\mathcal{R}$ be an operator system in $M_n$, and assume that $\mathcal{R}$ is irreducible in the sense that only the trivial subspaces $\{0\}$ and $\mathbb{C}^n$ are invariant under $\mathcal{R}$. If $\psi : M_n \to M_n$ is a matrix state and $\psi|_\mathcal{R} = \text{id}$, then $\psi = \text{id}$.

For a detailed account of matrix convexity we refer to [11] or [21]. By an easy translation argument the following version of the separation-type Hahn-Banach theorem follows from the generalized Bipolar theorem proved in [11].

**Theorem 1.6.** Let $V$ be a locally convex vector space. Assume that $K = (K_r)$ is a matrix convex set in $V$, such that $K_r$ is closed in the product topology in $M_r(V)$ for all $r \in \mathbb{N}$. Given $v_0 \notin K_n$ for some $n \in \mathbb{N}$, there exist a continuous linear mapping $\Phi : V \to M_n$ and a self-adjoint $\alpha \in M_n$ such that

$$\text{Re} \Phi_r(v) \leq \alpha \otimes I_r$$

for all $r \in \mathbb{N}$, $v \in K_r$, and

$$\text{Re} \Phi_n(v_0) \leq \alpha \otimes I_n.$$

Moreover, if $0 \in K_1$, then $\alpha$ may be chosen to be $I_n$.

2. Matrix extreme points

Inspired by the notion of extreme points in the C*-convexity case, we now introduce extreme points suitable for matrix convexity. As a natural extension of a proper scalar convex combination, we say that a matrix convex combination

$$v = \sum_{i=1}^{k} \gamma_i^* v_i \gamma_i$$

with $\gamma_i \in M_{n,n}$, for $i = 1, \ldots, k$ satisfying $\sum_i \gamma_i^* \gamma_i = I_n$ is proper if each $\gamma_i$ has a right inverse belonging to $M_{n,n}$, i.e., if $\gamma_i$ is surjective as a linear map $\mathbb{C}^n \to \mathbb{C}^n$.

In particular, we must have that $n \geq n_i$.

**Definition 2.1.** Suppose that $K = (K_n)$ is a matrix convex set in $V$. Then $v \in K_n$ is a matrix extreme point if whenever $v$ is a proper matrix convex combination of $v_i \in K_n$, for $i = 1, \ldots, k$, then each $n_i = n$ and $v = u_i^* v_i u_i$ for some unitary $u_i \in M_{n_i}$.

Let $\partial K_n$ be the (possibly empty) set of matricial extreme points in $K_n$ and set $\partial K = (\partial K_n)$.

Observe that for $n = 1$, a proper matrix convex combination reduces to a proper scalar convex combination of elements in $V$. Therefore the matrix extreme points in $K_1$ coincide with the usual extreme points. This also shows that if $K_1$ is compact, then, by Krein-Milman, $\partial K_1$ is non-empty. By contrast, $\partial K_n$ might be empty for all $n > 1$, as Example 2.2 below will show.

As remarked in [15, Remark 12] the occurrence of unitary equivalence in the definition of matrix extreme points is quite natural, because if $v \in K_n$ and $u \in M_n$ is unitary, then $w = u^* vu$ is a proper matrix combination of $v$.

We saw in the previous section how each $K_n$ of a matrix convex set $K$ is a C*-convex set over $M_n$. Similarly, if we fix $n$ in the above, we get the definition of a C*-extreme point of $K_n$. As we shall see in Example 2.3 the C*-extreme points and
the matrix extreme points do not necessarily agree, but clearly the matrix extreme
points are also C*-extreme. We shall see later (Corollary 3.6) that matrix extreme
points are also extreme points in the usual sense.
In the case of a compact matrix convex set in \( \mathbb{C} \), i.e., if \( K = \mathcal{W}(T) \) for some
operator \( T \in \mathcal{B}(\mathcal{H}) \) (cf. Example 1.2), it follows from the work of Morenz ([17,
Proposition 2.2]) that matrix extreme points in \( K \) correspond exactly to the so-
called structural elements of size \( n \). Adding to our conviction that one should study
the whole of \( K \) and not just \( K_n \), is the observation that the results of the same
paper are obtained by introducing structural elements in \( K_r \) for \( r \leq n \) (cf. [17,
Definition 2.3]).

**Example 2.2.** With \( a, b \in \mathbb{R} \) the matrix extreme points of the matrix interval
\([aI, bI] \) of Example 1.2 are just \( a \) and \( b \), i.e.,
\[
\partial[aI_n, bI_n] = \begin{cases}
\{a, b\}, & n = 1; \\
\emptyset, & n > 1.
\end{cases}
\]
Since the matrix extreme points of \( K_1 \) are the classical extreme points, we have
\[
\partial[aI_1, bI_1] = \partial[a, b] = \{a, b\}.
\]
Moreover, any element \( v \in [aI_n, bI_n] \) can be written
\[
v = u^* \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_i \gamma_i^* v_i \gamma_i
\]
with \( v_i \in [a, b] \) and a unitary
\[
u = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix} \in \mathbb{M}_n,
\]
where \( \gamma_1, \ldots, \gamma_n \in \mathbb{M}_{1, n} \). Since \( u \) is unitary, \( \gamma_1, \ldots, \gamma_n \) defines a proper matrix
convex combination, and therefore no element in \([aI_n, bI_n] \) for \( n > 1 \) can be matrix
extreme.

Based on Farenick-Morenz’ description of the C*-extreme points of the matrix
state spaces \( CS_n(A) \) for a C*-algebra \( A \) (cf. Example 1.4), we may characterize
the matrix extreme points of \( CS_n(A) \). Recall, that a completely positive map \( \varphi \) is pure
if whenever \( \psi \) is a completely positive map such that \( \varphi - \psi \) is completely positive,
then \( \psi = t\varphi \) for some \( 0 \leq t \leq 1 \).

**Example 2.3.** The matrix extreme points of \( CS(A) \) for a C*-algebra \( A \) are exactly
the pure matrix states, i.e.,
\[
\partial CS_n(A) = \{ \varphi : A \to \mathbb{M}_n \mid \varphi \text{ completely positive, pure, and } \varphi(I) = I_n \}.
\]
Assume that \( \varphi \in CS_n(A) \) is pure, and that \( \varphi = \sum_i \gamma_i^* \varphi_i \gamma_i \) is a proper matrix
convex combination of \( \varphi_i \in CS_n(A) \) and \( \gamma_i \in \mathbb{M}_{n, n} \). Since \( \varphi - \gamma_i^* \varphi_i \gamma_i \) is completely
positive, \( \gamma_i^* \varphi_i \gamma_i = t_i \varphi \) for some \( 0 \leq t_i \leq 1 \). But \( \varphi \) and \( \varphi_i \) are unital, so \( \gamma_i^* \gamma_i = t_i I_n \).
Since \( \gamma_i \) is surjective, \( t_i^{-1/2} \gamma_i \) implements a unitary equivalence between \( \varphi \) and \( \varphi_i \).
Therefore all pure matrix states are matrix extreme.

Moreover, by [9, Theorem 2.1] every C*-extreme point \( \varphi \) is unitarily equivalent
to a direct sum of pure matrix states. If the direct sum contains more than just one
pure matrix state, then \( \varphi \) is a proper matrix convex combination of smaller pure
matrix states, just as in Example 2.2, and therefore not matrix extreme. Since all
matrix extreme points are $C^*$-extreme, this shows that only the pure matrix states are matrix extreme.

The above two examples illustrate the advantages of the notion of matrix extreme points. On the one hand they establish a clear-cut analogy with the commutative case: in Example 2.2, where the non-commutative aspect plays no important role (since the matrix sets are completely determined by their first levels), we find that the theory reverts to the classical theory. On the other hand they provide us with a “non-commutative” structure which gives us strictly more information about the matrix set than we could gain from the extreme points at the first level alone. Additionally, in the second example, they demonstrate a particularly clear relationship with objects of interest in $C^*$-algebra theory.

3. Compact matrix convex sets

There is a natural correspondence between compact convex sets and function systems. On one hand, each compact convex subset $K$ of a locally convex space determines the function system $A(K) = \{ F : K \to \mathbb{C} | F \text{ continuous and affine}\}$. Conversely, if we are given a function system $\mathcal{R}$, then the state space $S(\mathcal{R}) = \{ \varphi \in \mathcal{R}^* | \varphi \geq 0, \varphi(I) = 1 \}$ is a weakly compact convex subset of $\mathcal{R}^*$. Moreover, $K$ is affinely homeomorphic to $S(A(K))$, and $\mathcal{R}$ and $A(S(\mathcal{R}))$ are isomorphic as function systems. (See [1, Section II 1].) The real case is usually the only one considered in the literature, but the fact that it remains true in the complex case is fundamental to this section.

With our claim that operator systems are the non-commutative analogues of function systems (cf. Example 1.4), it is only natural to demand that a “compact matrix convex set” should satisfy a similar correspondence. Establishing this is the main purpose of this section.

Definition 3.1. We define a compact matrix convex set to be a matrix convex subset $K = (K_n)$ of a locally convex vector space $V$ such that each $K_n$ is compact in the product topology in $M_n(V)$.

We remark that it is a consequence of [16, Theorems 3.1 and 3.2] that compactness of $K_n$ is not necessarily implied by compactness of $K_1$.

Example 3.2. The matrix intervals $[a \mathbf{1}, b \mathbf{1}]$ for $a, b \in \mathbb{R}$ and the matrix ranges $\mathcal{W}(T)$ for $T \in \mathcal{B}(\mathcal{H})$ as defined in Example 1.2, are both compact matrix convex sets in $\mathbb{C}$. Conversely, any compact matrix convex set in $\mathbb{C}$ is of this form, as already pointed out in Example 1.2.

Example 3.3. If $CS(\mathcal{R})$ are the matrix state spaces of an operator system $\mathcal{R}$, as defined in Example 1.4, then it is straightforward to see that $CS(\mathcal{R})$ is a compact matrix convex set in $\mathcal{R}^*$, equipped with the weak$^*$ topology.

We recall that the adjoint or $*$-operation in $\mathcal{R}$ induces an adjoint operation in $\mathcal{R}^*$, in which the self-adjoint elements correspond to linear functionals mapping self-adjoint elements of $\mathcal{R}$ into $\mathbb{R}$. This, in turn, induces a $*$-operation in $M_n(\mathcal{R}^*)$, and under the canonical identification of $M_n(\mathcal{R}^*)$ with the weakly continuous linear maps $\mathcal{R} \to M_n$, the self-adjoint elements of $M_n(\mathcal{R}^*)$ correspond to the maps that send self-adjoint elements in $\mathcal{R}$ to self-adjoint elements in $M_n$. This shows in particular that $CS(\mathcal{R})$ is closed under this $*$-operation.
Given a compact matrix convex set $K$, we know from the commutative case that $K_1$ and $S(A(K_1))$ are affinely homeomorphic. We wish to define an operator system structure on $A(K_1)$ that extends this to $K_1$ and $CS_n(A(K_1))$. At the same time we also demand that if $K = CS(R)$ for an operator system $R$, then the corresponding operator system structure on $A(S(R))$ coincides with $R$.

To this end we apply the Choi-Effros’ abstract characterization of operator systems (cf. Example 1.4), but first we introduce the following natural notion of a morphism on a matrix convex set.

**Definition 3.4.** A matrix affine mapping on a matrix convex subset $K = (K_n)$ of a vector space $V$ is a sequence $\theta = (\theta_n)$ of mappings $\theta_n : K_n \rightarrow M_n(W)$ for some vector space $W$, such that

$$\theta_n\left(\sum_{i=1}^{k} \gamma_i^* v_i \gamma_i\right) = \sum_{i=1}^{k} \gamma_i^* \theta_n(v_i) \gamma_i,$$

for all $v_i \in K_n$ and $\gamma_i \in M_{n, n}$ for $i = 1, \ldots, k$ satisfying $\sum_i \gamma_i^* \gamma_i = \mathbb{I}_n$.

If $\varphi : V \rightarrow W$ is a linear map and $w_0 \in W$, then $\theta_n = \varphi_n|_{K_n} + \mathbb{I}_n \otimes w_0$ defines a matrix affine map. The converse is not true in general, even in the classical case: consider for instance $z \rightarrow \Re z$ with $V = W = \mathbb{C}$, $K = [0, 1] \times [0, 1]$ and $L = \{0\} \times [0, 1]$ (identifying $\mathbb{C}$ with $\mathbb{R}^2$). The obstruction disappears if $V$ and $W$ are $*$-vector spaces and $K_1$ is self-adjoint, but we have not included the proof of this fact, as it will not be needed in this paper.

If $V$ and $W$ are locally convex spaces, then we say that $\theta$ as above is a matrix affine homeomorphism if each $\theta_n$ is a homeomorphism. Note that in this case $(\theta_n^{-1})$ is automatically matrix affine, and that it suffices to prove continuity of $\theta_n$ if $K$ is compact.

Given a compact matrix convex set $K$, we define $A(K, M_r)$ to be the set of all matrix affine mappings $F = (F_n) : K \rightarrow M_r$, such that $F_1$ is continuous. Using the linear structure and the adjoint operation in $M_n(M_r)$, $A(K, M_r)$ becomes a vector space with a $*$-operation under pointwise operations. Similarly, the order structure in $M_n(M_r)$ defines a positive cone in $A(K, M_r)$, where $F \geq 0$ in $A(K, M_r)$ if $F_n(v) \geq 0$ for all $n \in \mathbb{N}$ and $v \in K_n$.

Observe that if we define $I = (I_n)$ in $A(K, \mathbb{C})$ by $I_n(v) = \mathbb{I}_n$ for $v \in K_n$, then we can define a unital order preserving bijection $\Omega$ of $A(K, \mathbb{C})$ onto $A(K_1)$ by mapping $F = (F_n)$ to $F_1$. Indeed, this is clearly a positive unital map. Moreover, for a self-adjoint $F = (F_n)$, $F_n$ is completely determined by $F_1$ via

$$(F_n(v)\xi | \xi) = \xi^* F_n(v) \xi = F_1(\xi^* v \xi)$$

for any unit vector $\xi \in \mathbb{C}^n$, considered as a row matrix, and $v \in K_n$. This formula shows that $\Omega$ is injective, and it may in turn be used to define an order preserving inverse. By the characterization of function systems as complete order unit spaces, this means that $A(K, \mathbb{C})$ and $A(K_1)$ are isomorphic as function systems. We remark that the above formula also shows that each $F_n$ is continuous.

It is now possible to identify $M_r(A(K, \mathbb{C}))$ and $A(K, M_r)$, and we may thus use the ordering on $A(K, M_r)$ to define a positive cone in $M_r(A(K, \mathbb{C}))$. In this manner $A(K, \mathbb{C})$ becomes a matrix ordered space. Using the identification of $A(K, \mathbb{C})$ and $A(K_1)$, it is now straightforward to check that $A(K, \mathbb{C})$ satisfies the Choi-Effros axioms ([5]) for an operator system with the order unit $I$. We will simply denote the corresponding operator system by $A(K)$. 

Proposition 3.5.  
1. If $\mathcal{R}$ is an operator system, then $CS(\mathcal{R})$ is a self-adjoint compact matrix convex set in $\mathcal{R}^*$, equipped with the weak$^*$ topology, and $A(CS(\mathcal{R}))$ and $\mathcal{R}$ are isomorphic as operator systems.

2. If $K$ is a compact matrix convex set in a locally convex space $V$, then $A(K)$ is an operator system, and $K$ and $CS(A(K))$ are matrix affinely homeomorphic.

Proof. (1) Set $K = CS(\mathcal{R})$. We need to show that there exists a unital matrix order preserving bijection between $\mathcal{R}$ and $A(K)$. We know that $\mathcal{R}$ and $A(K_1) \simeq A(K)$ are isomorphic as function systems via the usual embedding, mapping $x \in \mathcal{R}$ to $\varphi \mapsto \varphi(x)$ for $\varphi \in K_1$ (cf. [1, Section II 1]). It therefore suffices to check that the matrix orderings are preserved. On the level of matrices, this map sends $x \in M_r(\mathcal{R})$ to $F \in M_r(A(K)) \simeq A(K, M_r)$ given by $F_n(\varphi) = \varphi_r(x)$ for $\varphi \in K_n$. This shows the claim, since $x \geq 0$ if and only if $\varphi_r(x) \geq 0$ for all $\varphi \in CS_n(\mathcal{R})$, $n \in \mathbb{N}$ by [5, p. 178].

(2) The usual evaluation map of $K_1$ onto $S(A(K_1))$ extends to a mapping $\theta_n : K_n \to CS_n(A(K))$ mapping $v \in K_n$ to $F \mapsto F_n(v)$. We claim that $\theta = (\theta_n)$ is a matrix affine homeomorphism of $K$ onto $CS(A(K))$.

It is straightforward to check that $\theta$ is a matrix affine map into $CS(A(K))$, and that each $\theta_n$ is continuous using the weak$^*$ topology in $A(K)^*$.

To see injectivity, let $V'$ be the continuous dual of $V$, and then observe that $f \in V'$ defines an element in $A(K)$ determined by the linear map $v \in V \mapsto f(v) \in \mathbb{C}$. If $\theta_n(v) = \theta_n(w)$ for $v, w \in K_n$, then in particular $f_n(v) = f_n(w)$ for all $f \in V'$, which again implies that $v = w$, since $V'$ separates points in $V$.

It remains to show surjectivity. Assume that $\varphi_0 \in CS_n(A(K)) \setminus \theta_n(K_n)$. By the matricial separation theorem, Theorem 1.6, applied to $A(K)^*$, equipped with the weak$^*$ topology, and the weakly closed matrix convex set $\theta(K)$ in $A(K)^*$, there exist a weakly continuous linear map $\Phi : A(K)^* \to \mathcal{M}_n$ and a self-adjoint $\alpha \in \mathcal{M}_n$, such that

$$\text{Re} \, \Phi_r(\theta_r(v)) \leq \alpha \otimes I_r$$

for all $r \in \mathbb{N}$, $v \in K_r$, and

$$\text{Re} \, \Phi_n(\varphi_0) \leq \alpha \otimes I_n.$$ 

Identifying $\Phi$ with $F \in M_n(A(K)) \simeq A(K, \mathcal{M}_n)$, this means that

$$\text{Re} \, F_r(v) \leq \alpha \otimes I_r$$

for all $v \in K_r$, $r \in \mathbb{N}$, and

$$\text{Re}(\varphi_0)(F) \leq \alpha \otimes I_n.$$ 

But the first inequality says that $\text{Re} \, F \leq \alpha \otimes I$ in $M_n(A(K))$, and since $\varphi_0$ is completely positive and unital,

$$\text{Re}(\varphi_0)(F) \leq \varphi_0(\alpha \otimes I) = \alpha \otimes \varphi_0(I) = \alpha \otimes I_n,$$

a contradiction. Hence $\theta_n$ is also onto. □

The above proposition shows that we can always think of a compact matrix convex set $K$ as the matrix state spaces of an operator system $\mathcal{R}$. This will be crucial in our approach to the Krein-Milman theorem, but there are other benefits obtained from this result. The corollary below shows that matrix extreme points
are also classical extreme points, adapting the proofs of [9, Proposition 1.1] and [15, Proposition 23].

**Corollary 3.6.** Let $K = (K_n)$ be a compact matrix convex set in a locally convex space $V$. If $v$ is a matrix extreme point in $K_n$, then $v$ is also an extreme point in $K_n$.

**Proof.** By Proposition 3.5 it suffices to consider the case where $K = CS(R)$ for some operator system $R$, since both matrix extreme and extreme points are preserved under matrix affine homeomorphisms.

Assume that $\varphi \in CS_n(R)$ is a matrix extreme point, and that we are given a proper convex combination $\varphi = t\varphi_1 + (1-t)\varphi_2$ with $0 < t < 1$ and $\varphi_1, \varphi_2 \in CS_n(R)$. Then $\varphi$ is unitarily equivalent to $\varphi_1$ and $\varphi_2$, i.e., for any $x \in R$, $\varphi(x)$ is written as a proper convex combination of elements from its unitary orbit in $M_n$. By [14], this implies that $\varphi(x) = \varphi_1(x) = \varphi_2(x)$. Hence $\varphi$ is extreme. \(\square\)

### 4. The Krein-Milman theorem for matrix convex sets

Given a collection $S = (S_n)$ of subsets $S_n \subset M_n(V)$ for some locally convex vector space $V$, we define the **closed matrix convex hull** $\mathcal{C}(S)$ to be the smallest closed matrix convex set containing $S$. We can either describe $\mathcal{C}(S)$ as the intersection of all closed matrix convex sets containing $S$, or, more explicitly, as the closure of all elements $v \in M_n(V)$ of the form

$$v = \sum_{i=1}^{k} \gamma_i^* v_i \gamma_i$$

where $v_i \in S_n$ and $\gamma_i \in M_{n,n}$ for $i = 1, \ldots, k$ satisfying $\sum_i \gamma_i^* \gamma_i = I_n$. The latter description relies on the easy fact that the closure of a matrix convex set is again a matrix convex set.

We remark in passing that using the notion of **matrix polar** as defined in [11], it follows easily from Theorem 1.6 that if $S$ contains the origin, then the double matrix polar of $S$ coincides with $\mathcal{C}(S)$.

**Example 4.1.** We saw in Example 2.2 that the matrix extreme points of the matrix interval $[aI, bI]$ with $a, b \in \mathbb{R}$ are $a$ and $b$, i.e.,

$$\partial[aI, bI] = \begin{cases} \{a, b\}, & n = 1; \\ \emptyset, & n > 1. \end{cases}$$

It follows that $[aI, bI] = \mathcal{C}(\partial[aI, bI])$, since $\mathcal{C}(\partial[aI, bI]) = [a, b]$, and this determines $\mathcal{C}(\partial[aI, bI])$ uniquely by Example 1.2.

The above is clearly an example of a Krein-Milman type result. The work of Farenick-Morenz establishes a similar statement in the case of the matrix state spaces on a C*-algebra.

**Example 4.2.** In [9, Theorem 3.5] it is shown that for a C*-algebra $A$, $CS_n(A)$ is the closed C*-convex hull of the set of C*-extreme points in $CS_n(A)$. In Example 2.3 we observed that any C*-extreme point of $CS_n(A)$ is a matrix convex combination of matrix extreme points in $CS(A)$. Since any C*-convex combination is also a matrix convex combination, the closed C*-convex hull of the C*-extreme
points coincides with the closed matrix convex hull of the matrix extreme points, i.e.,
\[ CS(A) = \overline{\partial CS(A)}. \]

The above two examples are special cases of the following generalized version of the Krein-Milman theorem.

**Theorem 4.3.** Let \( K \) be a compact matrix convex set in a locally convex space \( V \), and let \( \partial K = (\partial K_n) \) denote the collection of matrix extreme points \( \partial K_n \) of \( K_n \). Then \( \partial K \) is non-empty, and
\[ K = \overline{\partial K}. \]

Before embarking on the proof, we shall introduce an auxiliary convex set \( \Delta_n(K) \), which is an essential tool in the reduction to the classical Krein-Milman. For a collection \( K = (K_r) \) of subsets \( K_r \subset M_r(V) \) and a fixed \( n \in \mathbb{N} \), we define the subset \( \Delta_n(K) \) of \( M_n(V) \) by
\[ \Delta_n(K) = \{ \xi^* v \xi \mid v \in K_r, \xi \in M_{r,n}, \|\xi\|_2 = 1, r \in \mathbb{N} \}, \]
where \( \| \cdot \|_2 \) denotes the Hilbert-Schmidt norm. We observe that if \( K \) is matrix convex, then \( \Delta_n(K) \) is convex. Indeed, given \( 0 \leq t \leq 1 \) and \( \xi^* v \xi, \eta^* w \eta \in \Delta_n(K) \) with \( v \in K_r, w \in K_s \) and \( \xi \in M_{r,n}, \eta \in M_{s,n} \) satisfying \( \|\xi\|_2, \|\eta\|_2 = 1 \), then
\[ t \xi^* v \xi + (1 - t) \eta^* w \eta = [t^{1/2} \xi^* (1 - t)^{1/2} \eta^*] \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \begin{bmatrix} t^{1/2} \xi \\ (1 - t)^{1/2} \eta \end{bmatrix}, \]
where
\[ \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} = [I_r \ 0] v [I_r \ 0] + [0 \ I_s] w [0 \ I_s] \in K_{r+s}, \]
and
\[ \left\| \begin{bmatrix} t^{1/2} \xi \\ (1 - t)^{1/2} \eta \end{bmatrix} \right\|_2^2 = t\|\xi\|_2^2 + (1 - t)\|\eta\|_2^2 = 1. \]

Moreover, in (2) we may always choose \( \xi \in M_{r,n} \) such that \( \xi \) has a right inverse (and in particular \( r \leq n \)). To see this, let \( v \in K_r \) and \( \xi \in M_{r,n} \) with \( \|\xi\|_2 = 1 \) be given, and let \( s \) be the dimension of the range of \( \xi \). Letting \( \nu \in M_{s,n} \) be an isometry of \( \mathbb{C}^s \) onto the range of \( \xi \), we have that
\[ \xi^* v \xi = (\nu^* \xi)^* (\nu^* \nu) (\nu^* \xi) \]
is the desired decomposition. In particular,
\[ \Delta_n(K) = \{ \xi^* v \xi \mid v \in K_r, \xi \in M_{r,n}, \|\xi\|_2 = 1, r \leq n \}, \]
from which it follows that \( \Delta_n(K) \) is compact whenever \( K_n \) is. Observe that this only relies on the fact that \( K \) is closed under isometries.

One of the important features of \( \Delta_n(K) \) is that there is a good description of the extreme points in terms of the matrix extreme points. We begin with the case of the matrix state spaces of an operator system.

**Lemma 4.4.** Let \( R \) be an operator system, and let \( \Delta_n(CS(R)) \) be defined as above. If \( \bar{\varphi} \) is an extreme point of \( \Delta_n(CS(R)) \), then there exist a matrix extreme point \( \varphi \in CS_r(R) \) for some \( r \in \mathbb{N} \) and a right invertible element \( \xi \in M_{r,n} \) with \( \|\xi\|_2 = 1 \) such that
\[ \bar{\varphi} = \xi^* \varphi \xi. \]
Proof. Assume that $\varphi$ is an extreme point of $\Delta_n(CS(\mathcal{R}))$. By (3), we may write $\varphi = \xi^*\varphi\xi$ for some $\varphi \in CS_r(\mathcal{R})$ and $\xi \in \mathbb{M}_{r,n}$, where $\|\xi\|_2 = 1$ and $\xi$ has a right inverse.

We claim that $\varphi$ is a matrix extreme point. To see this, assume that $\varphi$ is written as a proper matrix convex combination $\varphi = \sum \gamma_i^* \varphi_i \gamma_i$ with $\varphi_i \in CS_r(\mathcal{R})$ and $\gamma_i \in \mathbb{M}_{r,r}$ for $i = 1, \ldots, k$. Set $t_i = \|\gamma_i\xi\|_2^2$, and observe that

$$\sum t_i = \sum \|\gamma_i\xi\|_2^2 = \sum \text{Tr}(\gamma_i^* \gamma_i \xi) = \text{Tr}(\xi^* \xi) = \|\xi\|_2^2 = 1,$$

and that $t_i \neq 0$, so that both $\gamma_i$ and $\xi$ have right inverses. Thus we can write $\varphi$ as the proper convex combination

$$\varphi = \xi^* \varphi\xi = \sum \xi^* \gamma_i^* \varphi_i \gamma_i \xi = \sum t_i \frac{(\gamma_i\xi)^*}{\|\gamma_i\xi\|_2} \varphi_i \frac{(\gamma_i\xi)}{\|\gamma_i\xi\|_2}.$$

Since $\varphi$ is extreme, this means that $\xi^* \varphi\xi = \|\gamma_i\xi\|_2^{-2} (\gamma_i\xi)^* \varphi_i (\gamma_i\xi)$, and using that $\xi$ has a right inverse, we get that

$$\varphi \|\gamma_i\xi\|_2^2 = \gamma_i^* \varphi_i \gamma_i$$

for $i = 1, \ldots, k$. Since $\varphi$ and $\varphi_i$ are unital, this in particular implies that

$$\mathbb{I}_r \|\gamma_i\xi\|_2^2 = \gamma_i^* \gamma_i.$$

Therefore $\gamma_i \|\gamma_i\xi\|_2^{-1}$ is an isometry, and since $\gamma_i$ is known to be surjective, we have that $r = r_1 = \cdots = r_k$, and $\gamma_i \|\gamma_i\xi\|_2^{-1}$ is a unitary implementing a unitary equivalence between $\varphi$ and $\varphi_i$. Hence $\varphi$ is a matrix extreme point.

Using the representation theorem of the previous section we may extend the above result to general compact matrix convex sets.

**Lemma 4.5.** Let $\mathcal{K} = (K_n)$ be a compact matrix convex set in a locally convex space $V$. If $\bar{v}$ is an extreme point of $\Delta_n(\mathcal{K})$, then there exist a matrix extreme point $v \in K_r$ for some $r \in \mathbb{N}$ and a right invertible element $\xi \in \mathbb{M}_{r,n}$ with $\|\xi\|_2 = 1$ such that

$$\bar{v} = \xi^* v\xi.$$

**Proof.** By Proposition 3.5 (2) there exists an operator system $\mathcal{R}$ and a matrix affine homeomorphism $\theta = (\theta_n)$ of $CS(\mathcal{R})$ onto $\mathcal{K}$. It suffices to show that $\Gamma : \Delta_n(CS(\mathcal{R})) \to \Delta_n(\mathcal{K})$ given by

$$\Gamma(\xi^* \varphi\xi) = \xi^* \theta_r(\varphi)\xi$$

for $\varphi \in CS_r(\mathcal{R})$ and $\xi \in \mathbb{M}_{r,n}$ satisfying $\|\xi\|_2 = 1$, is a well-defined continuous affine surjection. If this is so, and $\bar{v} \in \Delta_n(\mathcal{K})$ is an extreme point, then $\Gamma^{-1}(\bar{v})$ is a compact face of $\Delta_n(CS(\mathcal{R}))$. By Krein-Milman, this set has an extreme point, which is also an extreme point of $\Delta_n(CS(\mathcal{R}))$. The conclusion now follows by applying Lemma 4.4, and the observation that $\theta$ preserves matrix extreme points.

To see that $\Gamma$ is well-defined, first observe that if $\nu$ is an isometry chosen as in (3), then

$$\xi^* \theta_r(\varphi)\xi = \xi^* \nu \nu^* \theta_r(\varphi) \nu \nu^* \xi = \xi^* \nu \theta_s(\nu^* \varphi \nu) \nu^* \xi.$$
We may therefore assume without loss of generality that \( \xi \) is right invertible. If 
\( \xi^* \varphi \xi = \eta^* \psi \eta \) with \( \psi \in \mathbb{C} S_t(\mathcal{R}) \) and \( \eta \in M_{t,n} \), then using that \( \varphi \) and \( \psi \) are unital we see that \( \eta \xi^{-1} \) is an isometry. Thus
\[
\theta_r(\varphi) = \theta_r((\eta \xi^{-1})^*(\psi(\eta \xi^{-1})) = (\eta \xi^{-1})^*\theta_t(\psi)(\eta \xi^{-1}),
\]
or \( \xi^* \theta_r(\varphi) \xi = \eta^* \theta_t(\psi) \eta \), which shows that \( \Gamma \) is well-defined.

It is immediate that \( \Gamma \) is affine and surjective. To see that \( \Gamma \) is also continuous, consider a convergent net
\[
\xi_n^* \varphi_n \xi_n \to \xi^* \varphi \xi
\]
in \( \Delta(\mathbb{C} S(\mathcal{R})) \) with \( \varphi_n \in \mathbb{C} S_{r_n}(\mathcal{R}) \) and \( \xi_n \in M_{r_n,n} \). Set \( \eta_n = \xi_n \xi^{-1} \in M_{r_n,r} \), and observe that since all maps are unital,
\[
\eta_n^* \eta_n \to \mathbb{I}_r.
\]
If \( \eta_n = \nu_n | \eta_n | \) is the polar decomposition of \( \eta_n \), this means that \( | \eta_n | \) is surjective from some step, and hence that \( \nu_n \) is an isometry. Moreover, as \( \nu_n - \eta_n \to 0 \),
\[
\nu_n^* \varphi_n \nu_n = \eta_n^* \varphi_n \eta_n + (\nu_n - \eta_n)^* \varphi_n \nu_n + \eta_n^* \varphi_n (\nu_n - \eta_n) \to \varphi.
\]
By the continuity of \( \theta_r \),
\[
\eta_n^* \theta_{r_n}(\varphi_n) \eta_n = | \eta_n | | \theta_r(\varphi_n) | \eta_n | \to \mathbb{I}_r \theta_r(\varphi) \mathbb{I}_r = \theta_r(\varphi),
\]
or, equivalently,
\[
\Gamma(\xi_n^* \varphi_n \xi_n) = \xi_n^* \theta_{r_n}(\varphi_n) \xi_n \to \xi^* \theta_r(\varphi) \xi = \Gamma(\xi^* \varphi \xi),
\]
and we are done.

It would be tempting to assume that \( \Delta_n(\mathcal{K}) \) is preserved under matrix affine homeomorphisms of \( \mathcal{K} \), but we see from the proof above that the situation is not that simple.

**Proof of Theorem 4.3.** Let \( \mathcal{K} = (K_n) \) be a compact matrix convex set, and let \( \partial \mathcal{K} = (\partial K_n) \) be the collection of matrix extreme points. Since \( \partial K_1 \) coincides with the usual extreme points, \( \partial \mathcal{K} \) is non-empty, and we clearly have \( \overline{\mathcal{K}}(\partial \mathcal{K}) \subset \mathcal{K} \).

We may assume that \( 0 \in \overline{\mathcal{K}}(\partial \mathcal{K}) \) without loss of generality by translating \( K_n \) by \( v_0 \otimes I_n \) for some \( v_0 \in \partial K_1 \).

For the converse inclusion, assume that there exists \( v_0 \in K_n \setminus \overline{\mathcal{K}}(\partial \mathcal{K})_n \). By the matricial separation theorem, Theorem 1.6, there exists a continuous linear mapping \( \Phi : V \to M_n \) such that
\[
(5) \quad \Re \Phi_r(v) \leq I_n \otimes I_r
\]
for all \( v \in \overline{\mathcal{K}}(\partial \mathcal{K})_r \) and \( r \in \mathbb{N} \), and
\[
(6) \quad \Re \Phi_n(v_0) \not\leq I_n \otimes I_n.
\]
\( \Phi \) induces a continuous linear functional \( F : M_n(V) \to \mathbb{C} \) satisfying
\[
F(\eta^* v \xi) = \langle \Phi_r(v) \xi | \eta \rangle
\]
for all \( v \in M_r(V) \) and \( \xi, \eta \in M_{r,n} \), simultaneously considered as vectors in \( \mathbb{C}^{rn} \). If \( \bar{v} \) is an extreme point of \( \Delta_n(\mathcal{K}) \), then, by Lemma 4.5, we may write \( \bar{v} = \xi^* v \xi \).
where \( v \in \partial K_r \) and \( \xi \in \mathbb{M}_{r,n} \) with \( \| \xi \|_2 = 1 \) and \( r \leq n \). By (5), we therefore get that
\[
\Re F(\bar{v}) = \Re F(\xi^* v \xi) = \Re(\Phi_r(v)\xi | \xi) \\
\leq \langle I_n \otimes I_r \xi | \xi \rangle = \| \xi \|_2^2 = 1.
\]
for all extreme points \( \bar{v} \) of \( \Delta_n(K) \). Since \( \Delta_n(K) \) is compact by (4), the Krein-Milman theorem implies that
\[
\Re F(\Delta_n(K)) \leq 1.
\]
This, in turn, implies that for any unit vector \( \xi \in \mathbb{C}^n \) and \( v \in K_r \),
\[
\Re(\Phi_r(v)\xi | \xi) = \Re F(\xi^* v \xi) \leq 1,
\]
i.e., \( \Re \Phi_r(v) \leq I_n \otimes I_r \), contradicting (6). Hence \( K = \overline{\partial K} \). \( \square \)

We remark that an inspection of the above proof reveals that only matrix extreme points in \( K_r \) for \( r \leq n \) are necessary to generate \( K_n \).

The key idea of the above proof is to use the matricial separation theorem and the correspondence between linear functionals on \( \mathbb{M}_n(V) \) and the linear mappings \( V \rightarrow \mathbb{M}_n \) to reduce the matricial problem to a scalar one in \( \mathbb{M}_n(V) \). This naturally leads to introducing the convex set \( \Delta_n(K) \) and establishing a connection between the matrix extreme points of \( K \) and the extreme points of \( \Delta_n(K) \), which allows us to use the classical Krein-Milman theorem.

The converse result, which says that the extreme points are contained in any closed set with closed convex hull equal to the compact convex set in question, is usually considered an integral part of the classical Krein-Milman theorem. Morenz proved a similar condition for his structural elements in the \( C^* \)-convexity case in \( \mathbb{M}_n([17, \text{Theorem 4.5}]) \). We present a similar condition in the matrix convexity case to document that the situation in our Krein-Milman theorem is actually optimal.

**Theorem 4.6.** Let \( K \) be a compact matrix convex set in a locally convex space \( V \), and let \( S = (S_n) \) be a collection of closed subsets \( S_n \subset K_n \) such that \( \nu^* S_m \nu \subset S_n \) for all isometries \( \nu \in \mathbb{M}_{m,n} \). If \( \overline{\partial S} = K \), then
\[
\partial K \subset S.
\]

We note that the condition that \( S \) be closed under isometries is actually necessary. In the case, say, where \( K = CS(A) \) for some \( C^* \)-algebra \( A \), we saw in Example 2.3 that \( \partial K \) consists of all pure matrix states. Using the fact that the minimal Stinespring representation of a pure matrix state is irreducible (cf. [2]), it easily follows that the pure matrix states are closed under isometries. One may therefore remove elements from the isometric orbit of a pure matrix state and still have them generate the whole matrix state space, but there is no canonical way of choosing which pure states to exclude.

The proof of the above theorem follows more or less by reversing the proof of Theorem 4.3. In this respect the lemma below is the converse to Lemma 4.4

**Lemma 4.7.** Let \( \mathcal{R} \) be an operator system. Given a matrix extreme point \( \varphi \in CS_n(\mathcal{R}) \) and an invertible element \( \xi \in \mathbb{M}_n \) satisfying \( \| \xi \|_2 = 1 \), then
\[
\bar{\varphi} = \xi^* \varphi \xi
\]
is an extreme point in \( \Delta_n(CS(\mathcal{R})) \).
Proof. Assume that $\varphi \in CS_n(\mathcal{R})$ is a matrix extreme point, and that $\xi \in \mathbb{M}_n$ satisfying $\|\xi\|_2 = 1$ is invertible. We wish to prove that $\varphi = \xi^*\varphi\xi$ is an extreme point in $\Delta_n(CS(\mathcal{R}))$. Given a proper convex combination

$$\xi^*\varphi\xi = t\xi_1^*\varphi_1\xi_1 + (1-t)\xi_2^*\varphi_2\xi_2$$

with $\varphi_1 \in CS_s(\mathcal{R})$, $\varphi_2 \in CS_s(\mathcal{R})$, right invertible elements $\xi_1 \in \mathbb{M}_{r,n}$, $\xi_2 \in \mathbb{M}_{s,n}$ satisfying $\|\xi_1\|_2$, $\|\xi_2\|_2 = 1$, and $0 < t < 1$, then

$$\varphi = t(\xi_1\xi_1^{-1})^*\varphi_1(\xi_1\xi_1^{-1}) + (1-t)(\xi_2\xi_2^{-1})^*\varphi_2(\xi_2\xi_2^{-1}).$$

This is a proper matrix convex combination since $\varphi$, $\varphi_1$, and $\varphi_2$ are unital, $\xi_1$, $\xi_2$ are right invertible, and $\xi$ is invertible. Hence $n = r = s$, and we have unitaries $u_1$, $u_2 \in \mathbb{M}_n$ such that $\varphi_1 = u_1^*\varphi u_1$ and $\varphi_2 = u_2^*\varphi u_2$, i.e.,

$$\varphi = t(u_1\xi_1\xi_1^{-1})^*\varphi(u_1\xi_1\xi_1^{-1}) + (1-t)(u_2\xi_2\xi_2^{-1})^*\varphi(u_2\xi_2\xi_2^{-1}).$$

If we define the matrix state $\psi : \mathbb{M}_n \to \mathbb{M}_n$ by

$$\psi(\alpha) = t(u_1\xi_1\xi_1^{-1})^*\alpha(u_1\xi_1\xi_1^{-1}) + (1-t)(u_2\xi_2\xi_2^{-1})^*\alpha(u_2\xi_2\xi_2^{-1})$$

for $\alpha \in \mathbb{M}_n$, then the above equation says that $\varphi = \psi \circ \varphi$.

We claim that this implies that $\psi = \text{id}$. The operator system $\varphi(\mathcal{R})$ in $\mathbb{M}_n$ is irreducible, because otherwise $\varphi$ is unitarily equivalent to a diagonal matrix of matrix states, contradicting that $\varphi$ is matrix extreme. The claim now follows from Proposition 1.5.

Since $\psi = \text{id}$, the uniqueness part of Choi’s description of completely positive maps ([4, Remark 4]) implies that

$$\sqrt{t}(u_1\xi_1\xi_1^{-1}) = \sqrt{s}\lambda_11_n, \quad \sqrt{1-t}(u_2\xi_2\xi_2^{-1}) = \sqrt{1-s}\lambda_21_n$$

for $0 \leq s \leq 1$ and $\lambda_1$, $\lambda_2 \in \mathbb{C}$ satisfying $|\lambda_1|$, $|\lambda_2| = 1$. But $t = s$ since

$$t = t\|\xi_1\|_2^2 = \text{Tr}(\sqrt{t}u_1\xi_1) = \text{Tr}(s\lambda_1\xi_1^*\lambda_1) = s\|\xi_1\|_2^2 = s.$$ 

Hence $u_1\xi_1\xi_1^{-1} = \lambda_11_n$, and so

$$\xi_1^*\varphi_1\xi_1 = \xi_1^*u_1^*\varphi u_1\xi_1 = (\lambda_1\xi_1^*)^*\varphi(\lambda_1\xi_1) = \xi^*\varphi\xi.$$ 

Similarly, $\xi_2^*\varphi_2\xi_2 = \xi^*\varphi\xi$, and we are done. $\square$

As extreme points are not necessarily preserved under affine surjections the method of Lemma 4.5 does not lead to an extension of the above lemma to general compact matrix convex sets. Luckily, Lemma 4.7 is all we need.

Proof of Theorem 4.6. By Proposition 3.5 (2) we may assume that $K = CS(\mathcal{R})$ for some operator system $\mathcal{R}$, as the statement of the theorem is preserved under matrix affine homeomorphism.

We begin by proving that $\Delta_n(CS(\mathcal{R}))$ is the closed convex hull $L$ of $\Delta_n(S)$ by reversing the argument in the proof of Theorem 4.3. Clearly $L \subset \Delta_n(CS(\mathcal{R}))$.

For the converse, assume that $\bar{\varphi} \in \Delta_n(CS(\mathcal{R})) \setminus L$. Then there exist a weakly continuous linear functional $F : M_n(\mathbb{R}^n) \to \mathbb{C}$ and $\lambda \in \mathbb{R}$ such that

$$\text{Re} \ F(\bar{\varphi}) \leq \lambda < \text{Re} \ F(\bar{\varphi}_0)$$
for all \( \varphi \in \Delta_n(S) \). If \( F \) corresponds to the weakly continuous linear mapping \( \Phi : \mathcal{R}^* \rightarrow M_n \), as in the proof of Theorem 4.3, and if we write \( \varphi_0 = \xi_0^* \varphi_0 \xi_0 \) with \( \varphi_0 \in K_n \) and \( \xi_0 \in M_{r_0,n} \) satisfying \( \| \xi_0 \|_2 = 1 \), then the above is equivalent to
\[
\text{Re}(\Phi_r(\varphi)|\xi) \leq \lambda < \text{Re}(\Phi_r(\varphi_0)|\xi_0)
\]
for all \( r \in \mathbb{N} \), \( \varphi \in S_r \), and \( \xi \in M_{r,n} \) satisfying \( \| \xi \|_2 = 1 \). Since \( K = CS(\mathcal{R}) = \overline{\partial_r(S)} \), this implies that
\[
\text{Re} \left( \Phi_r(K_r) \right) \leq \lambda_{r,n}
\]
for all \( r \in \mathbb{N} \), contradicting that \( \varphi_0 \in K_n \).

Hence \( \Delta_n(CS(\mathcal{R})) \) is the closed convex hull of \( \Delta_n(S) \). By Krein-Milman, this implies that the extreme points of \( \Delta_n(CS(\mathcal{R})) \) are contained in the closure of \( \Delta_n(S) \). Since \( S \) is closed under isometries, (4) also holds for \( \Delta_n(S) \). Hence \( \Delta_n(S) \) is also closed. From this we claim to be able to show that \( \partial CS(\mathcal{R}) \subset S \).

Let \( \varphi \in \partial CS_n(\mathcal{R}) \). Choosing an arbitrary invertible element \( \xi \in M_n \) satisfying \( \| \xi \|_2 = 1 \), Lemma 4.7 shows that \( \varphi = \xi^* \varphi \xi \) is an extreme point in \( \Delta_n(CS(\mathcal{R})) \). By the above, we may find \( \eta^* \psi \eta \in \Delta_n(S) \) such that \( \varphi = \eta^* \psi \eta \) with \( \psi \in S_r \), i.e.,
\[
\varphi = (\eta \xi^{-1})^* \psi (\eta \xi^{-1})
\]
In particular, \( (\eta \xi^{-1})^* (\eta \xi^{-1}) = I_n \). Since \( S \) is closed under isometries, this shows that \( \varphi \in S_n \).

We wish to conclude with a few remarks about the often mentioned connections with \( \mathcal{C}^* \)-convexity. It is important to observe that the \( \mathcal{C}^* \)-convexity Krein-Milman theorems of Farenick and Morenz do not follow immediately from our work. In both cases additional technical results from their papers are needed. If \( K = (K_n) \) is a compact matrix convex set, then we know that any element in \( K_n \) can be approximated by matrix convex combinations \( \sum \gamma_i^* v_i \gamma_i \) of matrix extreme points \( v_i \in K_{n_i} \), for \( i = 1, \ldots, n \) and \( n_i \leq n \). Even though \( v_1, \ldots, v_n \) are \( \mathcal{C}^* \)-extreme, they need not lie in \( K_n \). We therefore need to alter the matrix convex combination to include \( \mathcal{C}^* \)-extreme points in \( K_n \), i.e., we wish to write
\[
\gamma_i^* v_i \gamma_i = \begin{bmatrix} \gamma_i^* & 0 \\ 0 & * \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & * \end{bmatrix} \begin{bmatrix} \gamma_i \\ 0 \end{bmatrix}
\]
such that
\[
\begin{bmatrix} v & 0 \\ 0 & * \end{bmatrix} \in K_n
\]
is \( \mathcal{C}^* \)-extreme. In the case of a compact matrix convex set in \( \mathbb{C} \), [17, Corollary 5.3] shows how to choose the missing entry, whereas for the matrix state spaces of a \( \mathcal{C}^\ast \)-algebra the choice is given by [9, Theorem 3.3]. With these additional results the corresponding \( \mathcal{C}^\ast \)-convexity versions of Krein-Milman follow.

REFERENCES

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