

## THE NILPOTENCE HEIGHT OF $P_t^s$ FOR ODD PRIMES

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ABSTRACT. K. G. Monks has recently shown that the element  $P_t^s$  has nilpotence height  $2\left[\frac{s}{t}\right] + 2$  in the mod 2 Steenrod algebra. Here the method and result are generalized to show that for an odd prime  $p$  the element  $P_t^s$  has nilpotence height  $p\left[\frac{s}{t}\right] + p$  in the mod  $p$  Steenrod algebra.

### 1. INTRODUCTION AND MAIN RESULTS

Using Monks' method [14] and some new results, we prove the following theorem. Recall that  $P_t^s$  is the Milnor basis element  $P(r_1, r_2, \dots, r_t)$  with  $r_t = p^s$  and  $r_i = 0$  for  $i < t$ .

**Theorem 1.** *For all  $s \geq 0$ ,  $t \geq 1$ ,*

$$(i) \quad (P_t^s)^{p\left[\frac{s}{t}\right]+p} = 0,$$

$$(ii) \quad (P_t^s)^{p\left[\frac{s}{t}\right]+p-1} \neq 0.$$

Thus the nilpotence height of  $P_t^s$  is exactly  $p\left[\frac{s}{t}\right] + p$ . The upper bound (i) on the nilpotence order was proved in [14] for the case  $p = 2$ . The lower bound (ii) was proved in [13] for the case  $p = 2$ . Here we generalize these arguments.

Let  $P_t(r_1, r_2, \dots, r_m)$  be the Milnor basis element  $P(s_1, s_2, \dots, s_{tm})$ , where  $s_{ti} = r_i$  and  $s_j = 0$  if  $t$  does not divide  $j$ . In particular,  $P_t(p^s) = P_t^s$  and  $P_1(n) = P(n)$ .

If  $R = (r_1, r_2, \dots, r_m)$  is a sequence of nonnegative integers, we will write  $P_t(R)$  for the corresponding Milnor basis element. The degree of  $P_t(R)$  is  $2|R|_t$ , where  $|R|_t = \sum_{i=1}^{\infty} (p^{it} - 1)r_i$ , and the excess of  $P_t(R)$  is  $2e(R)$ , where  $e(R) = \sum_{i=1}^{\infty} r_i$ . For a fixed  $t$  let  $B_t$  be the vector subspace of  $A$  with basis the set of all  $P_t(R)$ . For  $P_t^s \in B_t$  write  $\widehat{P}_t^s$  for  $(-1)^s \chi(P_t^s)$ , where  $\chi$  denotes the canonical antiautomorphism of  $B_t$ .

We introduce some notation. Each natural number  $a$  has a unique expansion

$$(1) \quad a = \sum_{i=0}^{\infty} \alpha_i(a) p^i$$

with  $0 \leq \alpha_i(a) < p$ . Let  $0 \leq b < p^t$ , and  $i < t$ . Then the following are obvious:

$$(2) \quad \alpha_i(a) = \alpha_{i+t}(p^t a + b),$$

$$(3) \quad \alpha_i(b) = \alpha_i(p^t a + b).$$

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**Definition.** We say that  $m$  and  $n$  have no carries if  $\alpha_i(m) + \alpha_i(n) \leq p - 1$  for all  $i$ . This is equivalent to the condition that the binomial coefficient  $\binom{m+n}{m}$  is nonzero mod  $p$ . If  $m$  and  $n$  have no carries, we will write  $m \asymp n$ .

Define  $\gamma_t(m) = \sum_{i=0}^{m-1} p^{it}$  for any integer  $m \geq 0$  and  $t \geq 1$ , where  $\gamma_t(0) = 0$ . The following fact is immediate.

**Fact 2.**  $\gamma_t(m + 1) = p^t \gamma_t(m) + 1$ .

**Proposition 3.** For any  $t \geq 1$ ,  $m \geq 0$ , and  $i < t$ ,

$$(4) \quad P_t((p - 1)p^i \gamma_t(m + 1)) \cdot P_t^{m+1} = 0.$$

*Proof.* By Milnor’s product formula,  $P_t(a) \cdot P_t(b) = \sum u_j \cdot P_t(a + b - (p^t + 1)j, j)$ , where the sum is taken over all  $j$  such that  $(a - p^t j) \asymp (b - j)$ , and  $u_j$  is a unit mod  $p$ . So it is sufficient to show that  $(p - 1)p^i \gamma_t(m + 1) - p^t j$  and  $p^{m+1} - j$  have at least one carry for any  $0 \leq j \leq p^i(p - 1)\gamma_t(m)$ . This is the content of the following lemma.  $\square$

**Lemma 4.** For all  $t \geq 1$ ,  $m \geq 0$ ,  $i < t$  and  $0 \leq j \leq p^i(p - 1)\gamma_t(m)$ , there exists a nonnegative integer  $k$  such that  $\alpha_{kt+i}(p^i(p - 1)\gamma_t(m + 1) - p^t j) + \alpha_{kt+i}(p^{m+1} - j) \geq p$ .

*Proof.* We will proceed by induction on  $m$ . Let

$$(5) \quad X = (p - 1)p^i \gamma_t(m + 1) - p^t j$$

$$(6) \quad Y = p^{m+1} - j.$$

If  $m = 0$ , then  $j = 0$ ,  $X = (p - 1)p^i$ , and  $Y = p^i$ . So we have  $\alpha_i(X) + \alpha_i(Y) = p$ .

Assume that for all  $t \geq 1$ ,  $0 \leq i < t$  and  $0 \leq j \leq p^i(p - 1)\gamma_t(m - 1)$ , there exists a nonnegative integer  $k$  such that  $\alpha_{kt+i}(p^i(p - 1)\gamma_t(m) - p^t j) + \alpha_{kt+i}(p^{(m-1)t+i} - j) \geq p$ .

Choose  $t \geq 1$ ,  $i < t$  and  $0 \leq j \leq p^i(p - 1)\gamma_t(m)$ . If  $j = 0$ , then

$$\begin{aligned} X &= (p - 1)p^i \gamma_t(m + 1) \\ &= (p - 1) \left( p^{m+1} + p^{(m-1)t+i} + p^{(m-2)t+i} + \dots + p^{t+i} + p^i \right) \\ Y &= p^{m+1} \end{aligned}$$

Hence  $\alpha_{m+1}(X) + \alpha_{m+1}(Y) = p$ .

Suppose  $j \neq 0$ .

**Case 1 :**  $\alpha_i(j - 1) \neq p - 1$ .

$$X = (p - 1)p^i(p^t \gamma_t(m) + 1) - p^t j = p^t [(p - 1)p^i \gamma_t(m) - j] + (p - 1)p^i.$$

Since  $\alpha_i(j - 1) \neq p - 1$ ,  $j - 1 = p^{i+1}q + p^i h + r$ , where  $0 \leq r < p^i$  and  $0 \leq h < p - 1$ . Then

$$\begin{aligned} Y &= p^{m+1} - j = p^{m+1} - (p^{i+1}q + p^i h + r + 1) \\ &= p^{i+1}[p^{m-t} - (q + 1)] + (p - 1 - h)p^i + (p^i - (r + 1)). \end{aligned}$$

Since  $r < p^i$ , the term  $(p^i - (r + 1))$  is nonnegative and less than  $p^i$ . So  $\alpha_i(X) + \alpha_i(Y) = 2p - h - 2 \geq p$ .

**Case 2 :**  $\alpha_i(j - 1) = p - 1$ .

Let  $j - 1 = p^t q + r$ , where  $0 \leq r < p^t$ . Since  $\alpha_i(j - 1) = p - 1$ ,  $p^i(p - 1) \leq r < p^t$  by (3). In the assumption we have  $j \leq p^i(p - 1)\gamma_t(m)$ . Then  $p^t q + r + 1 \leq p^i(p - 1)(p^t \gamma_t(m - 1) + 1)$ . Solving for  $q + 1$  shows that

$$q + 1 \leq p^i(p - 1)\gamma_t(m - 1) + \frac{p^i(p - 1) - r - 1 + p^t}{p^t}.$$

Since  $p^i(p - 1) \leq r$ ,  $\frac{p^i(p - 1) - r - 1 + p^t}{p^t} < 1$ . Thus  $q + 1 \leq p^i(p - 1)\gamma_t(m - 1)$ , and so by the inductive hypothesis there exists a nonnegative integer  $k$  such that

$$\alpha_{kt+i}((p - 1)p^i\gamma_t(m) - p^t(q + 1)) + \alpha_{kt+i}(p^{(m-1)t+i} - (q + 1)) \geq p.$$

Let  $k$  be any such value. We will show that  $\alpha_{(k+1)t+i}(X) + \alpha_{(k+1)t+i}(Y) \geq p$ , which will complete the induction and hence the proof.

Now since  $\alpha_i(j - 1) = p - 1$ ,  $\alpha_i(r) = p - 1$ . Hence  $\alpha_i(p^t - 1 - r) = 0$ . Therefore  $p^t - 1 - r + (p - 1)p^i < p^t$ . Then we have

$$\begin{aligned} &\alpha_{kt+i}((p - 1)p^i\gamma_t(m) - p^t(q + 1)) \\ &= \alpha_{kt+i}((p - 1)p^i\gamma_t(m) - p^t(q + 1) + p^t - 1 - r) \\ &= \alpha_{(k+1)t+i}(p^t [(p - 1)p^i\gamma_t(m) - p^t(q + 1) + p^t - 1 - r] + (p - 1)p^i) \\ &= \alpha_{(k+1)t+i}(X) \end{aligned}$$

by (2) and (3). Also,

$$Y = p^{mt+i} - (p^t q + r + 1) = p^t(p^{(m-1)t+i} - (q + 1)) + (p^t - (1 + r)).$$

Thus

$$\begin{aligned} \alpha_{kt+i}(p^{(m-1)t+i} - (q + 1)) &= \alpha_{(k+1)t+i}(p^t(p^{(m-1)t+i} - (q + 1)) + (p^t - (1 + r))) \\ &= \alpha_{(k+1)t+i}(Y) \end{aligned}$$

by (2). By the induction hypothesis,  $\alpha_{(k+1)t+i}(X) + \alpha_{(k+1)t+i}(Y) \geq p$ . □

Using Davis' method [6], we can derive

**Proposition 5.**

$$(8) \quad P_t(u) \cdot \widehat{P}_t(v) = \sum_R \binom{|R|_t + e(R)}{p^t u} P_t(R),$$

$$(9) \quad \widehat{P}_t(u) \cdot P_t(v) = \sum_R \binom{e(R)}{v} P_t(R),$$

where the sum is taken over all  $R$  for which  $|R|_t = (p^t - 1)(u + v)$ .

*Proof.* The formula (8) is exactly Corollary 1a in [8]. So we will prove formula (9). Using Gallant's Proposition 1 that  $\chi(P_t^s)$  is  $(-1)^s$  times the sum of all  $P_t(R)$  in the appropriate dimension, we see that the product  $\widehat{P}_t(u) \cdot P_t(v)$  contains a term

$$\prod_i \binom{r_i}{s_i} P_t(r_1, r_2, \dots)$$

for each Milnor matrix,

$$\begin{pmatrix} * & 0 & \dots & 0 & s_1 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ r_1 - s_1 & 0 & \dots & 0 & s_2 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ r_2 - s_2 & 0 & \dots & 0 & s_3 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \end{pmatrix}$$

such that  $v = \sum_i s_i$ . Thus

$$\widehat{P}_t(u) \cdot P_t(v) = \sum_R \sum_S \prod_i \binom{r_i}{s_i} P_t(R)$$

where  $S$  ranges over sequences  $(s_1, s_2, \dots)$  having  $\sum_i s_i = v$ . The equation

$$\sum_S \prod_i \binom{r_i}{s_i} = \binom{\sum_i r_i}{v}$$

follows immediately from considering the coefficient of  $x^v$  in the expansion of

$$(1 + x)^{r_1} \cdot (1 + x)^{r_2} \dots (1 + x)^{r_n}.$$

Hence we have

$$\widehat{P}_t(u) \cdot P_t(v) = \sum_R \binom{\sum_i r_i}{v} P_t(R) = \sum_R \binom{e(R)}{v} P_t(R).$$

□

Using these formulae, we can prove the following proposition.

**Proposition 6.** *If  $a, b$  and  $t$  are positive integers with  $a \geq t$ , then*

$$\widehat{P}_t(p^a(p-1)) \cdot P_t(p^{a+b} - p^a(p-1)) = P_t(p^{a+b} - p^{a-t}(p-1)) \cdot \widehat{P}_t(p^{a-t}(p-1)).$$

*Proof.* Using (8) and (9), we have

$$P_t(p^{a+b} - p^{a-t}(p-1)) \cdot \widehat{P}_t(p^{a-t}(p-1)) = \sum_R \binom{|R|_t + e(R)}{p^{a+b+t} - p^a(p-1)} P_t(R)$$

and

$$\widehat{P}_t(p^a(p-1)) \cdot P_t(p^{a+b} - p^a(p-1)) = \sum_R \binom{e(R)}{p^{a+b} - p^a(p-1)} P_t(R)$$

where  $1 \leq e(R) \leq p^{a+b}$  and  $|R|_t = (p^t - 1)p^{a+b}$ .

In order to prove these sums are equal, we need to show that their binomial coefficients are equivalent mod  $p$ . Note that

$$\binom{e(R)}{p^{a+b} - p^a(p-1)} = \binom{e(R)}{e(R) - p^{a+b} + p^a(p-1)},$$

$$\binom{|R|_t + e(R)}{p^{a+b+t} - p^a(p-1)} = \binom{|R|_t + e(R)}{e(R) - p^{a+b} + p^a(p-1)}.$$

**Case 1 :** If  $1 \leq e(R) < p^{a+b} - p^a(p-1)$ , then both binomial coefficients are zero because  $e(R) - p^{a+b} + p^a(p-1)$  is a negative integer.

**Case 2 :** If  $p^{a+b} - p^a(p-1) \leq e(R) < p^{a+b}$ , then

$$\begin{aligned} \binom{|R|_t + e(R)}{e(R) - p^{a+b} + p^a(p-1)} &= \binom{\sum_{i=a+b}^{a+b+t-1} \alpha_i(|R|_t)p^i + \sum_{i=0}^{a+b-1} \alpha_i(e(R))p^i}{\sum_{i=0}^{a+b-1} \alpha_i(e(R) - p^{a+b} + p^a(p-1))p^i} \\ &\equiv \prod_{i=a+b}^{a+b+t-1} \binom{\alpha_i(|R|_t)}{0} \prod_{i=0}^{a+b-1} \binom{\alpha_i(e(R))}{\alpha_i(e(R) - p^{a+b} + p^a(p-1))} \pmod p \\ &\equiv \binom{e(R)}{e(R) - p^{a+b} + p^a(p-1)} \pmod p. \end{aligned}$$

Hence

$$\binom{|R|_t + e(R)}{p^{a+b+t} - p^a(p-1)} \equiv \binom{e(R)}{p^{a+b} - p^a(p-1)} \pmod p.$$

**Case 3 :** If  $e(R) = p^{a+b}$ , then

$$\binom{e(R)}{p^{a+b} - p^a(p-1)} = \binom{p^{a+b}}{p^a(p-1)} \equiv 0 \pmod p$$

and

$$\binom{|R|_t + e(R)}{p^{a+b+t} - p^a(p-1)} = \binom{p^{a+b+t}}{p^a(p-1)} \equiv 0 \pmod p.$$

So the result holds. □

If  $n, k$  and  $t$  are positive integers with  $n = k + mt$  for some  $m$ , we define

$$(X_t)_k^n = P_t(p^n(p-1)) \cdot P_t(p^{n-t}(p-1)) \cdots P_t(p^k(p-1)).$$

The following corollary is immediate from Proposition 5.

**Corollary 7.** *Let  $m \geq 1, c \geq 0$ , and  $a = c + mt$ . Then*

$$\chi((X_t)_{c+at}^a) \cdot P_t(p^{a+b} - p^a(p-1)) = P_t(p^{a+b} - p^c(p-1)) \cdot \chi((X_t)_c^{a-t}).$$

For any Milnor basis element  $\theta$ , define  $\kappa_\theta : A \rightarrow A$  by

$$(10) \quad \phi(x) = \sum \kappa_\theta(x) \otimes \theta$$

where  $\phi$  is the diagonal map in  $A$ . These operations were first defined by Kristensen [10]. We will call it “stripping by  $\theta$ ”. It follows from (10) that stripping a Milnor basis element by  $\theta = P(t_1, t_2, \dots)$  is given by

$$\kappa_\theta(P(r_1, r_2, \dots)) = P(r_1 - t_1, r_2 - t_2, \dots)$$

where the right hand side is taken to be 0 if  $r_i < t_i$  for any  $i$  (see [18] and [19] for more details).

For any  $x, y \in A$ ,

$$\begin{aligned} \sum \kappa_{P(R)}(x \cdot y) \otimes P(R) &= \left(\sum \kappa_{P(I)}(x) \otimes P(I)\right) \left(\sum \kappa_{P(J)}(y) \otimes P(J)\right) \\ &= \sum \kappa_{P(I)}(x) \kappa_{P(J)}(y) \otimes P(I)P(J) = \sum \lambda_R^{I,J} \kappa_{P(I)}(x) \kappa_{P(J)}(y) \otimes P(R) \end{aligned}$$

where  $\lambda_R^{I,J} = \langle P(I)P(J), \xi^R \rangle$  and the sums are taken over all  $I, J, R$  such that  $R = I + J$ . So

$$\kappa_{P(R)}(x \cdot y) = \sum_{I, J, R} \lambda_R^{I,J} \kappa_{P(I)}(x) \kappa_{P(J)}(y).$$

In particular, if  $\kappa_t^s = \kappa_{P_t^s}$ , then  $\kappa_t^s(x \cdot y) = \sum_{j=0}^t \kappa_{t-j}^{s+j}(x) \kappa_j^s(y)$  follows from the formula  $\phi(\xi_k) = \sum_{i=0}^k \xi_{k-i}^{p^i} \otimes \xi_i$ .

**Lemma 8.** *If  $x$  and  $y$  belong to  $B_t$  then stripping by  $P_t^s$  is a derivation, i.e.*

$$\kappa_t^s(x \cdot y) = \kappa_t^s(x) \cdot y + x \cdot \kappa_t^s(y).$$

*Proof.* The proof follows from the formula above, since all of the other terms in the sum correspond to stripping by Milnor basis elements which are not in  $B_t$  and therefore give zero. □

A sequence  $R = (r_1, r_2, \dots)$  is called a  $t$ -representation of a positive integer  $m$  if  $m = \sum_i \gamma_t(i)r_i$ . We define  $\mu_t(m) = \min\{\sum_i r_i : (r_1, r_2, \dots) \text{ is a } t\text{-representation of } m\}$ .

**Lemma 9.** *For  $P_t(R) \in B_t$  and  $m \geq 0$ ,*

$$\mu_t(m) = \min\{e(P_t(R)) : |R|_t = (p^t - 1)m\}.$$

*Proof.* There is a 1 – 1 correspondence between Milnor basis elements  $P_t(R)$  satisfying  $|R|_t = (p^t - 1)m$  and  $t$ -representations of  $m$  given by

$$P_t(R) \longleftrightarrow m = \sum_i r_i \gamma_t(i).$$

Under this correspondence,  $e(P_t(R))$  corresponds to the number  $\sum_i r_i$  which is used in determining  $\mu_t(m)$ . The lemma follows immediately from this observation. □

Thus we have following lemma which is analogous to Davis’ formula [6, Theorem 1].

**Lemma 10.** *For all  $m \geq 0$  and  $t \geq 1$ ,*

$$\widehat{P}_t((p - 1)p^{t-1}\gamma_t(m)) = (X_t)_{t-1}^{mt-1}.$$

*Proof.* We will prove it by induction on  $m$ . If  $m = 1$  then the result holds since  $P_t(p^{t-1}(p - 1))$  is the only  $P_t(R)$  in this dimension. Assume that the result holds for  $m - 1$ . By (8),

$$\begin{aligned} (X_t)_{t-1}^{mt-1} &= P_t((p - 1)p^{mt-1}) \cdot (X_t)_{t-1}^{(m-1)t-1} \\ &= P_t((p - 1)p^{mt-1}) \cdot \widehat{P}_t((p - 1)p^{t-1}\gamma_t(m - 1)) \\ &= \sum_R \binom{|R|_t + e(R)}{(p - 1)p^{(m+1)t-1}} P_t(R) \end{aligned}$$

where the sum is over all  $R$  such that  $|R|_t = \sum_i (p^{it} - 1)r_i = (p - 1)p^{t-1}(p^{mt} - 1)$ . By Lemma 9,

$$\mu_t((p - 1)p^{t-1}\gamma_t(m)) \leq \sum_i r_i \leq (p - 1)p^{t-1}\gamma_t(m).$$

By [8, Proposition 2], the  $t$ -representation of  $(p - 1)p^{t-1}\gamma_t(m)$  with  $r_m = (p - 1)p^{t-1}$  and all other  $r_i = 0$  has minimal  $\sum_i r_i$ . Hence  $\mu_t((p - 1)p^{t-1}\gamma_t(m)) = (p - 1)p^{t-1}$ . So we have

$$(p - 1)p^{t-1}(p^{mt} - 1) + (p - 1)p^{t-1} \leq \sum_i p^{it}r_i \leq (p - 1)p^{t-1}\gamma_t(m + 1);$$

that is,

$$(p - 1)p^{(m+1)t-1} \leq \sum_i p^{it}r_i \leq (p - 1)p^{t-1}\gamma_t(m + 1).$$

Hence  $\binom{\sum_i p^{it}r_i}{(p-1)p^{(m+1)t-1}} \equiv 1 \pmod p$ . By [8, Proposition 1],

$$(11) \quad \widehat{P}_t((p - 1)p^{t-1}\gamma_t(m)) = \sum_R P_t(R)$$

where the sum is taken over all  $R$  such that  $|R|_t = (p - 1)p^{t-1}(p^{mt} - 1)$ . Therefore we have  $(X_t)_{t-1}^{mt-1} = \widehat{P}_t((p - 1)p^{t-1}\gamma_t(m))$ .  $\square$

The following lemma is analogous to [19, Lemma 1.4].

**Lemma 11.** (i) For  $s \geq 0$  and  $t \geq 1$  let  $\kappa_t^s$  denote the operation of stripping by  $P_t^s$ . Then

$$\kappa_t^s(\widehat{P}_t(k)) = \widehat{P}_t(k - p^s)$$

where  $\widehat{P}_t(k - p^s) = 0$  for  $k < p^s$ .

(ii) Let  $\Theta$  be an element in  $B_t$  which gives zero when stripped by  $P_t^s$ , and assume  $\Theta \cdot \widehat{P}_t(ip^s) \cdot (P_t^s)^m = 0$ . Then  $\Theta \cdot \widehat{P}_t((i - 1)p^s) \cdot (P_t^s)^{m+1} = 0$ .

*Proof.* (i) We will prove it by induction on  $k$ . Note that  $\sum_{n \in \mathbb{Z}} P_t(n) \cdot \chi(P_t(k - n)) = 0$ . Hence we have

$$(12) \quad \sum_{n \in \mathbb{Z}} (-1)^{k-n} P_t(n) \cdot \widehat{P}_t(k - n) = 0$$

with the convention that  $P_t(i) = 0$  and  $\widehat{P}_t(i) = 0$  if  $i < 0$ . Applying the derivation  $\kappa_t^s$  to (12), we have

$$\kappa_t^s(\widehat{P}_t(k)) + \sum_{n > 0} (-1)^n \{ \kappa_t^s(P_t(n)) \cdot \widehat{P}_t(k - n) + P_t(n) \cdot \kappa_t^s(\widehat{P}_t(k - n)) \} = 0.$$

By the induction hypothesis, we obtain

$$\begin{aligned} \kappa_t^s(\widehat{P}_t(k)) + \sum_{n \neq 0} (-1)^n P_t(n - p^s) \cdot \widehat{P}_t(k - n) \\ + \sum_{m \neq p^s} (-1)^{m-p^s} P_t(m - p^s) \cdot \widehat{P}_t(k - m) = 0. \end{aligned}$$

All terms cancel except the term with  $n = p^s$  in the first sum. Hence

$$\kappa_t^s(\widehat{P}_t(k)) = \widehat{P}_t(k - p^s).$$

This completes the induction.

(ii) If  $\Theta \cdot \widehat{P}_t(ip^s) \cdot (P_t^s)^m = 0$  then

$$\kappa_t^s(\Theta) \cdot \widehat{P}_t(ip^s) \cdot (P_t^s)^m + \Theta \cdot \kappa_t^s(\widehat{P}_t(ip^s)) \cdot (P_t^s)^m + \Theta \cdot \widehat{P}_t(ip^s) \cdot \kappa_t^s(P_t^s)^m = 0.$$

By part (i),

$$\Theta \cdot \widehat{P}_t((i-1)p^s) \cdot (P_t^s)^m + m\Theta \cdot \widehat{P}_t(ip^s) \cdot (P_t^s)^{m-1} = 0.$$

So

$$\Theta \cdot \widehat{P}_t((i-1)p^s) \cdot (P_t^s)^{m+1} = -m\Theta \cdot \widehat{P}_t(ip^s) \cdot (P_t^s)^m = 0.$$

□

Let  $E$  be the exterior subalgebra of  $A$  generated by  $\{P_t^0 | t \geq 1\}$ . There is an algebra isomorphism between  $A$  and  $A//E$ . This isomorphism  $F : A \rightarrow A//E$  is given by

$$F(P(t_1, t_2, t_3 \dots)) = [P(pt_1, pt_2, pt_3 \dots)]$$

where  $[x]$  is equivalence class in  $A//E$  of  $x \in A$  (see [11, Chapter 15]). Let  $(P_t^s)^n = 0$ . Then  $[0] = [P_t^s]^n = F((P_t^{s-1})^n)$ . Since  $F$  is an algebra isomorphism,  $(P_t^{s-1})^n = 0$ . So by iterating we see that if  $(P_t^s)^n = 0$  then  $(P_t^i)^n = 0$  for all  $i \leq s$ . This proves the following lemma.

**Lemma 12.** *Let  $t > 1$ .*

(i) *If Theorem 1(i) holds for all  $s \equiv -1 \pmod t$ , then it holds for all  $s$ .*

(ii) *If Theorem 1(ii) holds for all  $s \equiv 0 \pmod t$ , then it holds for all  $s$ .*

## 2. THE PROOF OF THE MAIN RESULT

The proof of Theorem 1(i) now follows by imitating the proof of [14] for the case  $p = 2$ . We proceed by proving the following two equations by induction on  $k$  for  $1 \leq k \leq m$ :

$$(13) \quad \chi((X_t)_{kt-1}^{mt-1}) \cdot (P_t^{mt-1})^{p^{(k-1)+1}} = 0,$$

$$(14) \quad \chi((X_t)_{kt-1}^{(m-1)t-1}) \cdot (P_t^{mt-1})^{p^k} = 0.$$

If  $k = 1$  then

$$\chi((X_t)_{t-1}^{mt-1}) \cdot P_t^{mt-1} = P_t((p-1)p^{t-1}\gamma_t(m)) \cdot P_t^{mt-1} = 0$$

by Lemma 10 and Proposition 3.

Next we show that equation (13) for  $k$  implies equation (14) for  $k$ . The assumption says that

$$\chi((X_t)_{kt-1}^{(m-1)t-1})\chi(P_t(p^{mt-1}(p-1))) (P_t^{mt-1})^{p^{(k-1)+1}} = 0.$$

On applying Lemma 11  $(p-1)$  times with  $\Theta = \chi((X_t)_{kt-1}^{(m-1)t-1})$ , we obtain

$$\chi((X_t)_{kt-1}^{(m-1)t-1}) \cdot (P_t^{mt-1})^{p^k} = 0,$$

as desired.

Now we show that equation (14) for  $k$  implies equation (13) for  $k+1$ . Indeed we have the following equations with  $z = p^{mt} - p^{kt-1}(p-1)$ . At the second step we



use Corollary 7 with  $a = mt - 1$ ,  $b = 1$ , and  $c = kt - 1$ . The induction hypothesis is used at the third step. This gives us

$$\begin{aligned} \chi((X_t)_{(k+1)t-1}^{mt-1}) \cdot (P_t^{mt-1})^{pk+1} &= \chi((X_t)_{kt-1+t}^{mt-1}) \cdot (P_t^{mt-1}) \cdot (P_t^{mt-1})^{pk} \\ &= P_t(z) \cdot \chi((X_t)_{kt-1}^{mt-1-t}) \cdot (P_t^{mt-1})^{pk} \\ &= P_t(z) \cdot 0 = 0. \end{aligned}$$

Thus by induction on  $k$  we see that both equations (13) and (14) hold for  $1 \leq k \leq m$ . Equation (14) for  $k = m$  proves the theorem for all  $s \equiv -1 \pmod t$ . By Lemma 12(i) this proves Theorem 1(i).  $\square$

*Proof of Theorem 1 (ii).* We follow the method of Davis, using properties of the coproduct  $\phi$  in  $A$  to find elements  $\xi^{R_{m,t}(i)}$  in the dual algebra  $A^*$  for  $i < pm$  such that

$$(15) \quad \langle (P_t^{(m-1)t})^i, \xi^{R_{m,t}(i)} \rangle \neq 0.$$

If  $i = kp + w$  with  $0 \leq k \leq m - 1$  and  $0 \leq w \leq p - 1$ , then let

$$(16) \quad \xi^{R_{m,t}(i)} = \xi_t^{r_1(i)} \xi_{2t}^{r_2(i)} \dots \xi_{(k+1)t}^{r_{k+1}(i)}$$

where  $r_l(i) = a_l(i) + b_l(i)$ , with

$$a_l(i) = \begin{cases} (w - p + 1)p^{(m-1)t} & \text{if } l = 1, \\ 0 & \text{if } l > 1 \end{cases}$$

and

$$b_l(i) = \begin{cases} (p - 1)p^{(m-k-1)t} & \text{if } l = k + 1, \\ (p^{t+1} - p + 1)p^{(m-k-1)t} & \text{if } l = k, \\ (p^t - 1)p^{(m-l-1)t+1} & \text{if } 1 \leq l < k. \end{cases}$$

For example,

1. for  $k = 0$ ,  $R_{m,t}(i) = (wp^{(m-1)t})$ ,
2. for  $k = 1$ ,  $R_{m,t}(i) = ((w + 1)p^{(m-1)t} - (p - 1)p^{(m-2)t}, (p - 1)p^{(m-2)t})$ ,
3. for  $k = 2$ ,  $R_{m,t}(i) = ((w + 1)p^{(m-1)t} - p^{(m-2)t+1}, p^{(m-2)t+1} - (p - 1)p^{(m-3)t}, (p - 1)p^{(m-3)t})$ ,
4. for  $k = 3$ ,  $R_{m,t}(i) = ((w + 1)p^{(m-1)t} - p^{(m-2)t+1}, (p^t - 1)p^{(m-3)t+1}, p^{(m-3)t+1} - (p - 1)p^{(m-4)t}, (p - 1)p^{(m-4)t})$ .

The proof of (15) depends on the following calculation.

**Lemma 13.**

$$\phi(\xi^{R_{m,t}(i)}) = (z\xi^{R_{m,t}(i-1)} + \xi') \otimes \xi_t^{p^{(m-1)t}} + \sum_j a_j \otimes b_j$$

where  $b_j \neq \xi_t^{p^{(m-1)t}}$  for all  $j$ ,  $\xi'$  is divisible by  $\xi_t^{p^{(m-1)t+1}}$ , and  $z = w$  if  $w > 0$ ,  $z = 1$  if  $w = 0$ .

This implies (15), since it follows by induction on  $i$  that

$$\langle (P_t^{(m-1)t})^i, \xi^{R_{m,t}(i)} \rangle = \langle (P_t^{(m-1)t})^{i-1}, w\xi^{R_{m,t}(i-1)} \rangle \langle P_t^{(m-1)t}, \xi_t^{p^{(m-1)t}} \rangle \neq 0,$$

since the evaluation of  $\xi'$  on all elements of the subalgebra  $A(mt - 1)$  is zero. By Lemma 12(ii), this implies Theorem 1(ii).

*Proof of Lemma 13.* Let  $R_{m,t}(i)$  be as in (16) above. Since we are only interested in terms of the form  $A \otimes \xi_t^{p^{(m-1)t}}$ , we will work mod terms involving  $\xi_{jt}$  for  $j \leq 1$  in the second factor. So we have

$$(17) \quad \phi(\xi^{R_{m,t}(i)}) = \phi\left(\prod_{l=1}^{k+1} \xi_{lt}^{r_l(i)}\right) = \prod_{l=1}^{k+1} \phi(\xi_{lt}^{r_l(i)}) \equiv \prod_{l=1}^{k+1} (\xi_{lt} \otimes 1 + \xi_{(l-1)t}^{p^t} \otimes \xi_t)^{r_l(i)}.$$

Note that

$$\sum_{l=1}^{k+1} r_l(i) = (w + 1)p^{(m-1)t}.$$

We only want terms of the form  $A \otimes \xi_t^{p^{(m-1)t}}$ . When  $w = 0$ , the only such term is obtained from the product  $\prod_{l=1}^{k+1} (\xi_{(l-1)t}^{p^t} \otimes \xi_t)^{r_l(i)}$ , and hence

$$A = \prod_{l=1}^{k+1} \xi_{(l-1)t}^{p^t r_l(i)} = \prod_{l=1}^k \xi_{lt}^{p^t b_{l+1}(i)} = \prod_{l=1}^k \xi_{lt}^{b_l(i-1)} = \xi^{R_{m,t}(i-1)}$$

since  $i - 1 = (k - 1)p + p - 1$ . Suppose  $w > 0$ . The  $l$ -th factor of (17) can be written as

$$\begin{cases} (\xi_{(k+1)t}^{p^{(m-k-1)t}} \otimes 1 + \xi_{kt}^{p^{(m-k)t}} \otimes \xi_t^{p^{(m-k-1)t}})^{p-1} & \text{if } l = k + 1, \\ (\xi_{kt}^{p^{(m-k-1)t}} \otimes 1 + \xi_{(k-1)t}^{p^{(m-k)t}} \otimes \xi_t^{p^{(m-k-1)t}})^{p^{t+1}-p+1} & \text{if } l = k, \\ (\xi_{lt}^{p^{(m-l-1)t+1}} \otimes 1 + \xi_{(l-1)t}^{p^{(m-l)t+1}} \otimes \xi_t^{p^{(m-l-1)t+1}})^{p^t-1} & \text{if } 1 < l < k, \\ (\xi_t^{p^{(m-2)t+1}} \otimes 1 + 1 \otimes \xi_t^{p^{(m-2)t+1}})^{(w+1)p^{t-1}-1} & \text{if } l = 1, 1 < k, \\ (\xi_t^{p^{(m-2)t}} \otimes 1 + 1 \otimes \xi_t^{p^{(m-2)t}})^{(w+1)p^t-p+1} & \text{if } l = 1, k = 1. \end{cases}$$

We expand each of these factors, and if we choose for each  $l$  the term with the  $j_l$ -th power of the second term, then we must have ( $k \geq 2$ )

$$(18) \quad j_1 p^{(m-2)t+1} + \sum_{l=2}^{k-1} j_l p^{(m-l-1)t+1} + j_k p^{(m-k-1)t} + j_{k+1} p^{(m-k-1)t} = p^{(m-1)t}$$

and  $\binom{(w+1)p^{t-1}-1}{j_1}$  is a unit,  $0 \leq j_l \leq p^t - 1$  for  $2 \leq l \leq k - 1$ ,  $\binom{p^{t+1}-p+1}{j_k}$  is a unit, and  $0 \leq j_{k+1} \leq p - 1$ .

To satisfy (18) we must have  $j_{k+1} + j_k \equiv 0 \pmod{p^{t+1}}$ . Since  $\binom{p^{t+1}-p+1}{j_k}$  is a unit, either  $j_k \equiv 0 \pmod{p}$  or  $j_k \equiv 1 \pmod{p}$ . Therefore either  $j_{k+1} = 0$  or  $p - 1$  since  $j_{k+1} \leq p - 1$ .

If  $j_{k+1} = 0$ , then  $j_k = 0$ . So we have the following equation:

$$(19) \quad j_1 p^{(m-2)t+1} + \sum_{l=2}^{k-1} j_l p^{(m-l-1)t+1} = p^{(m-1)t}.$$

$j_{k-1}$  is multiple of  $p^t$  because  $j_{k-1} p^{(m-k)t+1} \equiv 0 \pmod{p^{(m-k+1)t+1}}$ . Therefore  $j_{k-1} = 0$  since  $j_{k-1} \leq p^t - 1$ . By a similar argument,  $j_{k-2} = j_{k-3} = \dots = j_2 = 0$ . So we have  $j_1 p^{(m-2)t+1} = p^{(m-1)t}$ , which implies  $j_1 = p^{t-1}$ . Therefore we obtain a

term of the form  $A \otimes \xi_t^{p^{(m-1)t}}$  with  $A = A_1 A_2 \cdots A_k A_{k+1}$ , where

$$\begin{aligned} A_1 &= w \xi_t^{wp^{(m-1)t} - p^{(m-2)t+1}} \\ A_2 &= \xi_{2t}^{(p^t-1)p^{(m-3)t+1}} \\ &\dots \\ A_{k-1} &= \xi_{(k-1)t}^{(p^t-1)p^{(m-k)t+1}} \\ A_k &= \xi_{kt}^{(p^{t+1}-p+1)p^{(m-k-1)t}} \\ A_{k+1} &= \xi_{(k+1)t}^{(p-1)p^{(m-k-1)t}}. \end{aligned}$$

Hence  $A = \xi^{R_{m,t}(i-1)}$ .

If  $j_{k+1} = p - 1$ , then we have  $j_k = p^{t+1} - p + 1$ . From (18),

$$j_1 p^{(m-2)t+1} + \sum_{l=2}^{k-2} j_l p^{(m-l-1)t+1} + j_{k-1} p^{(m-k)t+1} + p^{(m-k)t+1} = p^{(m-1)t}.$$

So  $j_{k-1} \equiv -1 \pmod{p^t}$ . Hence  $j_{k-1} = p^t - 1$ , since  $j_{k-1} \leq p^t - 1$ . By a similar argument,  $j_{k-2} = j_{k-3} = \cdots = j_2 = p^t - 1$ . If we put these  $j_l$  in the equation (18) for  $2 \leq l \leq k - 2$ , we will have  $(j_1 + 1)p^{(m-2)t+1} = p^{(m-1)t}$ . This implies that  $j_1 = p^{t-1} - 1$ . Therefore we obtain the other term of the form  $B \otimes \xi_t^{p^{(m-1)t}}$  with  $B = B_1 \cdot B_2 \cdots B_{k+1}$ , where

$$\begin{aligned} B_1 &= \xi_t^{wp^{(m-1)t}} \\ B_2 &= \xi_t^{(p^t-1)p^{(m-2)t+1}} \\ &\dots \\ B_{k-1} &= \xi_{(k-2)t}^{(p^t-1)p^{(m-k+1)t+1}} \\ B_k &= \xi_{(k-1)t}^{(p^{t+1}-p+1)p^{(m-k)t}} \\ B_{k+1} &= \xi_{kt}^{(p-1)p^{(m-k)t}}. \end{aligned}$$

So we have

$$\phi(\xi^{R_{m,t}(i)}) = (A + B) \otimes \xi_t^{p^{(m-1)t}}$$

where  $A = w \xi^{R_{m,t}(i-1)}$  and  $B = \xi'$  is divisible by  $\xi_t^{p^{(m-1)t+1}}$ , as claimed. □

REFERENCES

- [1] J.F. Adams and H.R. Margolis, *Modules Over the Steenrod Algebra*, *Topology* **10** (1971), 271–282. MR **45**:3520
- [2] J.F. Adams and H.R. Margolis, *Sub-Hopf-Algebra of the Steenrod Algebra*, *Proc. Camb. Phil. Soc.* **76** (1974), 45–52. MR **49**:6238
- [3] D.W. Anderson and D.M. Davis, *A Vanishing Theorem in Homological Algebra*, *Comment. Math. Helvet.* **48** (1973), 318–327. MR **48**:12526
- [4] D. Arnon, *Monomial Bases in the Steenrod Algebra*, *J. Pure and Applied Algebra* **96** (1994), 215–223. MR **95i**:55015
- [5] D.P. Carlisle and R.M.W. Wood, *On an Ideal Conjecture in the Steenrod Algebra*, preprint (1994).
- [6] D.M. Davis, *The Antiautomorphism of the Steenrod Algebra*, *Proc. Amer. Math. Soc.* **44** (1974), 235–236. MR **48**:7276

- [7] D.M. Davis, *On the height of  $Sq^{2^n}$* , preprint (1985).
- [8] A.M. Gallant, *Excess and Conjugation in the Steenrod Algebra*, Proc. Amer. Math. Soc. **76** (1979), 161–166. MR **81a**:55029
- [9] B. Gray, *Homotopy Theory*, Academic Press (1975). MR **53**:6528
- [10] L. Kristensen, *On a Cartan Formula for Secondary Cohomology Operations*, Math. Scand. **16** (1965), 97–115. MR **33**:4926
- [11] H.R. Margolis, *Spectra and the Steenrod Algebra*, vol 29 North Holland Math.Library (1983). MR **86j**:55001
- [12] J. Milnor, *The Steenrod Algebra and Its Dual*, Ann. of Math. **67** (1958), 150–171. MR **20**:6092
- [13] K.G. Monks, *Nilpotence in the Steenrod Algebra*, Bol.Soc.Mat.Mexicana **37** (1992), 401–416. MR **95j**:55032
- [14] K.G. Monks, *The Nilpotence Height of  $P_t^s$* , Proc. Amer. Math. Soc. **124** (1966), 1297–1303. MR **96g**:55023
- [15] J.H. Silverman, *Conjugation and Excess in the Steenrod Algebra*, Proc. Amer. Math. Soc. **119** (1993), 657–661. MR **93k**:55020
- [16] J.H. Silverman, *Hit Monomials and the Canonical Antiautomorphism of the Steenrod Algebra*, Proc. Amer. Math. Soc. **123** (1995), 627–637. MR **95c**:55023
- [17] P. D. Straffin, Jr., *Identities for Conjugation in the Steenrod Algebra*, Proc. Amer. Math. Soc. **49** (1975), 253–255. MR **52**:1693
- [18] G.Walker and R.M.W. Wood, *The Nilpotence Height of  $Sq^{2^n}$* , Proc. Amer. Math. Soc. **124** (1966), 1291–1295. MR **96g**:55024
- [19] G.Walker and R.M.W. Wood, *The Nilpotence Height of  $P^{p^n}$* , preprint (1995).
- [20] R.M.W. Wood, *A Note on Bases and Relations in the Steenrod Algebra*, Bull. London Math. Soc. **27** (1995), 380–386. MR **96e**:55013
- [21] R.M.W. Wood, *Differential Operators and the Steenrod Algebra*, preprint (1995).

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