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THE NONARCHIMEDEAN THETA CORRESPONDENCE FOR GSp(2) AND GO(4)

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Abstract. In this paper we consider the theta correspondence between the sets $\mathrm{Irr}(\mathrm{GSp}(2,k))$ and $\mathrm{Irr}(\mathrm{GO}(X))$ when k is a nonarchimedean local field and $\dim_k X = 4$. Our main theorem determines all the elements of $\mathrm{Irr}(\mathrm{GO}(X))$ that occur in the correspondence. The answer involves distinguished representations. As a corollary, we characterize all the elements of $\mathrm{Irr}(\mathrm{O}(X))$ that occur in the theta correspondence between $\mathrm{Irr}(\mathrm{Sp}(2,k))$ and $\mathrm{Irr}(\mathrm{O}(X))$. We also apply our main result to prove a case of a new conjecture of S.S. Kudla concerning the first occurrence of a representation in the theta correspondence.

Suppose k is a nonarchimedean local field of characteristic zero and odd residual characteristic, X is an even dimensional nondegenerate symmetric bilinear space over k and n is a nonnegative integer. Let ω be the Weil representation of $\mathrm{Sp}(n,k) \times \mathrm{O}(X)$ corresponding to a fixed choice of nontrivial additive character of k, and let $\mathcal{R}_X(\mathrm{Sp}(n,k))$ be the set of elements of $\mathrm{Irr}(\mathrm{Sp}(n,k))$ that are nonzero quotients of ω; similarly define $\mathcal{R}_n(\mathrm{O}(X))$. By [W], the condition that $\pi \otimes_{\mathbb{C}} \sigma$ be a nonzero quotient of ω for π in $\mathcal{R}_X(\mathrm{Sp}(n,k))$ and σ in $\mathcal{R}_n(\mathrm{O}(X))$ defines a bijection between $\mathcal{R}_X(\mathrm{Sp}(n,k))$ and $\mathcal{R}_n(\mathrm{O}(X))$. By [R], the extension of ω to the subgroup R of $\mathrm{GSp}(n,k) \times \mathrm{GO}(X)$ consisting of pairs whose entries have the same similitude factor also defines a well behaved correspondence between $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$ and $\mathrm{Irr}(\mathrm{GO}(X))$. Here, $\mathrm{GSp}(n,k)^+$ is the subgroup of elements of $\mathrm{GSp}(n,k)$ whose similitude factors lie in the group of similitude factors of the elements of $\mathrm{GO}(X)$; thus, $\mathrm{GSp}(n,k)^+$ is of index at most two in $\mathrm{GSp}(n,k)$ and contains $\mathrm{Sp}(n,k)$. To be more precise about the correspondence, let $\mathcal{R}_X(\mathrm{GSp}(n,k)^+)$ be the set of elements of $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$ whose restrictions to $\mathrm{Sp}(n,k)$ are multiplicity free and have a constituent in $\mathcal{R}_X(\mathrm{Sp}(n,k))$; similarly define $\mathcal{R}_n(\mathrm{GO}(X))$. Then by [R] the condition

$$\mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \sigma) \neq 0$$

defines a bijection between $\mathcal{R}_X(\mathrm{GSp}(n,k)^+)$ and $\mathcal{R}_n(\mathrm{GO}(X))$. Granted the theta dichotomy conjecture, if $\dim_k X \leq 2n$, then $\mathrm{GSp}(n,k)^+$ may be replaced by $\mathrm{GSp}(n,k)$ in this result.

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Since S.S. Kudla proved the compatibility of the theta correspondence with induction in [K], the characterization of $\mathcal{R}_X(\mathrm{Sp}(n,k))$ and $\mathcal{R}_n(\mathrm{O}(X))$, and analogously, $\mathcal{R}_X(\mathrm{GSp}(n,k)^+)$ and $\mathcal{R}_n(\mathrm{GO}(X))$, remains as a fundamental open problem. One solution might involve epsilon factors. See [HKS] for some progress in this direction in the unitary group case. In light of the many examples relating distinguished representations and functorality, one might also investigate whether distinguished representations could be involved in a solution to this problem. See [GRS] for a global example, again in the case of unitary groups. Recently, S.S. Kudla has made some new conjectures concerning the first occurrence of a representation in the theta correspondence which also shed some light on this problem.

In this paper we consider $\mathcal{R}_2(\mathrm{GO}(X))$ when $\dim_k X = 4$. Our main result is a complete characterization of $\mathcal{R}_2(\mathrm{GO}(X))$ in this case using distinguished representations. As a corollary, we determine $\mathcal{R}_2(\mathrm{O}(X))$. Also, as a consequence of the main theorem and some other results, we prove a case of one of the above mentioned conjectures of S.S. Kudla. Even though we obtain corollaries for the theta correspondence for isometries, we emphasize that the statements of our results and our methods of proof depend strongly on the use of similitudes. It would be interesting to see to what degree our results and methods extend to other situations. The employment of distinguished representations may possibly generalize. See section 4. As far as we know, the case of Kudla's conjecture proven in this paper is the highest dimensional case known. We believe that one of the more valuable aspects of this work is to provide a model for global considerations. It would be very interesting to determine if analogous results hold globally. A summary of previous work on this example appears near the end of this introduction.

To state the main theorem we need some more terminology. Assume $\dim_k X = 4$. Let π be contained in $\mathrm{Irr}(\mathrm{GSO}(X))$. If π induces irreducibly to $\mathrm{GO}(X)$ we say that π is regular; otherwise, we say that π is invariant. If π is invariant, then π has two extensions to $\mathrm{GO}(X)$. If y in X is anisotropic, then the stabilizer in $\mathrm{O}(X)$ of y can be identified with $\mathrm{O}(Y)$, where Y is the orthogonal complement to y. We say that π is distinguished if π is invariant and there is a y such that

$$\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \neq 0,$$

and, if $\mathrm{disc}(X) \neq 1$, then Y is isotropic. We show in Corollary 7.5 that the assumption that Y is isotropic if $\mathrm{disc}(X) \neq 1$ is unnecessary; however, the more restrictive definition is convenient. Suppose that π is distinguished. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) = 1.$$

It follows that for exactly one extension π' of π to $\mathrm{GO}(X)$ we have $\mathrm{Hom}_{\mathrm{O}(Y)}(\pi', \mathbf{1}) \neq 0$. Call this extension π^+; the other extension of π to $\mathrm{GO}(X)$ will be called π^-. For technical reasons, if $\mathrm{disc}(X) \neq 1$, π is one dimensional and invariant, then we also will say that π is distinguished. In this case there are also definitions of π^+ and π^-. The definitions of π^+ and π^- do not depend on the choice of y. See section 4.

Theorem 6.8 (Main Theorem). *Assume* $\dim_k X = 4$. *Let* σ *be in* $\mathrm{Irr}(\mathrm{GO}(X))$. *Then* σ *is in* $\mathcal{R}_2(\mathrm{GO}(X))$ *if and only if* σ *is not of the form* π^- *for some distinguished* π *in* $\mathrm{Irr}(\mathrm{GSO}(X))$.

This result is entirely analogous to the case $\dim_k X = 2n = 2$ considered by Hecke, Weil, Jacquet, Langlands and others. In this case, the role of $\mathrm{SO}(Y)$ is played by $\mathrm{SO}(X)$. For a description of this case, see section 7. Since the elements of $\mathrm{Irr}(\mathrm{GO}(X))$ have multiplicity free restrictions to $\mathrm{O}(X)$ we immediately obtain the following corollary.

Corollary 6.9. *Assume* $\dim_k X = 4$. *Let* σ_1 *be in* $\mathrm{Irr}(\mathrm{O}(X))$. *Then* σ_1 *is in* $\mathcal{R}_2(\mathrm{O}(X))$ *if and only if* σ_1 *is not an irreducible constituent of* $\pi^-|_{\mathrm{O}(X)}$ *for some distinguished* π *in* $\mathrm{Irr}(\mathrm{GSO}(X))$.

To describe the proof and make the theorem concrete, we characterize $\mathrm{GSO}(X)$ in terms of units of quaternion algebras. If $\mathrm{disc}(X) = 1$, there is an isomorphism of $\mathrm{GSO}(X)$ with either $(\mathrm{Gl}(2,k) \times \mathrm{Gl}(2,k))/k^\times$ or $(D^\times \times D^\times)/k^\times$, depending on the Hasse invariant of X. In the first case X is isotropic; in the second case, X is anisotropic. Here, D is the division quaternion algebra over k. If $\mathrm{disc}(X) \neq 1$, then there is an isomorphism of $\mathrm{GSO}(X)$ with $(k^\times \times \mathrm{Gl}(2,K))/K^\times$. Here, $K = k(\sqrt{\mathrm{disc}(X)})$. If $\mathrm{disc}(X) = 1$, we have a bijection between $\mathrm{Irr}(\mathrm{GSO}(X))$ and the subset of $\tau \otimes_{\mathbb{C}} \tau'$ in $\mathrm{Irr}(\mathrm{Gl}(2,k) \times \mathrm{Gl}(2,k))$ or $\mathrm{Irr}(D^\times \times D^\times)$ such that $\omega_\tau = \omega_{\tau'}$. If $\mathrm{disc}(X) \neq 1$, there is a two-to-one map from $\mathrm{Irr}(\mathrm{GSO}(X))$ onto the subset of τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ such that ω_τ factors through N_k^K; the two representations lying over τ correspond to the characters through which ω_τ factors.

Using these identifications, regular, invariant and distinguished have the following meanings for an element π of $\mathrm{Irr}(\mathrm{GSO}(X))$. Suppose $\mathrm{disc}(X) = 1$, and let $\tau \otimes_{\mathbb{C}} \tau'$ correspond to π. Then π is regular if and only if $\tau \ncong \tau'$, and if π is invariant, then π is distinguished. Suppose $\mathrm{disc}(X) \neq 1$, and let π correspond to τ and the quasi-character χ of $k^\times$. In this case, π is regular if and only if τ is not Galois invariant. In contrast to the case $\mathrm{disc}(X) = 1$, not all invariant representations are distinguished. Indeed, if π is invariant and infinite dimensional, so that τ is Galois invariant, then π is distinguished if and only if

$$\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det) \neq 0.$$

Nonvanishing is given by the following theorem. We claim no originality for this result. The proof is a straightforward generalization of arguments from [H] and [F], along with some observations from [HST] or [T]. In the global case, this theorem goes back to [HLR].

Theorem 5.3 (Hakim-Flicker). *Let* τ *in* $\mathrm{Irr}(\mathrm{Gl}(2,K))$ *be infinite dimensional and Galois invariant. Let* $\omega_\tau = \chi \circ \mathrm{N}_k^K$. *Then the following are equivalent:*

1. $\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det) \neq 0$*;*
2. *For every quasi-character* ζ *of* $K^\times$ *extending* χ*,*
$$\epsilon(\tau \otimes_{\mathbb{C}} \zeta^{-1}, 1/2, \psi_K) = \chi(-1);$$
3. τ *is the base change of an element of* $\mathrm{Irr}(\mathrm{Gl}(2,k))$ *with central character* $\chi\omega_{K/k}$.

For explanations of the notation, see section 5.

With these interpretations, we can explain the proof of the main theorem. Under the hypotheses of the theorem we need to show that every element of the form π^- is not in $\mathcal{R}_2(\mathrm{GO}(X))$, and that every element of $\mathrm{Irr}(\mathrm{GO}(X))$ not of the form π^- is in $\mathcal{R}_2(\mathrm{GO}(X))$. The first statement follows by an argument analogous to one in [HK]. This proof depends on a lemma that holds more generally: every distribution on X^2 invariant under $\mathrm{SO}(Y)$ is invariant under $\mathrm{O}(Y)$, for any Y as above.

To prove the second statement, we use the local analogue of the global method of computing a Fourier coefficient. Let σ in $\mathrm{Irr}(\mathrm{GO}(X))$ not be of the form π^-. Let

z be in X^2. If $\det(z_i, z_j) \neq 0$, we will say that z is nondegenerate. As above, if z is nondegenerate, then the components of z generate a nondegenerate subspace, and the stabilizer in $\mathrm{O}(X)$ is isomorphic to $\mathrm{O}(Z)$, where Z is the orthogonal complement of the subspace. By a result analogous to the well known relation between the nonvanishing of global theta lifts and the nonvanishing of period integrals, to show that σ is in $\mathcal{R}_2(\mathrm{GO}(X))$ it suffices to show that

$$\mathrm{Hom}_{\mathrm{O}(Z)}(\sigma^\vee, \mathbf{1}) \neq 0,$$

for some nondegenerate z. See section 6 for a proof and an explanation of the analogy. First consider the case when σ is not induced from a regular element of $\mathrm{Irr}(\mathrm{GSO}(X))$ or is not of the form π^+. Then $\mathrm{disc}(X) \neq 1$, and σ is the extension of an element of $\mathrm{Irr}(\mathrm{GSO}(X))$ corresponding to a τ in $\mathrm{Irr}(\mathrm{Gl}(2, K))$ and a quasi-character χ of $k^\times$ such that

$$\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det) = 0.$$

With a proper choice of z and quasi-character ζ of $K^\times$ extending χ, using the Kirillov model of $\tau^\vee$, we show

$$L(f) = Z(\zeta^{-1}, f, 1/2)$$

is the required linear functional. Here, $Z(\zeta^{-1}, f, s)$ is the zeta function associated to f in $\tau^\vee$ and ζ. In particular, the invariance of L follows from the functional equation for $Z(\zeta^{-1}, f, s)$. When σ is induced from a regular element π of $\mathrm{Irr}(\mathrm{GSO}(X))$ or is of the form π^+ there is a simplification. In this case, by Theorem 4.4, it suffices to show that

$$\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi^\vee, \mathbf{1}) \neq 0$$

for some nondegenerate z. When X is isotropic we accomplish this by some Kirillov model constructions, in part analogous to those of the previous paragraph, and when X is anisotropic, we use Tunnell's work [T].

In combination with some other results, we use the main theorem to prove a case of a conjecture of S.S. Kudla. To state the conjecture, suppose for the moment that $\dim_k X$ is arbitrary. For σ_1 in $\mathrm{Irr}(\mathrm{O}(X))$, let $n(\sigma_1)$ be the smallest integer n such that σ_1 occurs in the theta correspondence with $\mathrm{Sp}(n, k)$.

Conjecture 7.1 (S.S. Kudla). *If σ_1 is in $\mathrm{Irr}(\mathrm{O}(X))$, then*

$$n(\sigma_1) + n(\sigma_1 \otimes_{\mathbb{C}} \mathrm{sign}) = \dim_k X.$$

This conjecture is known to be true when $\dim_k X = 0$ or 2, but is open for all other cases. There is another conjecture of S.S. Kudla for representations of $\mathrm{Sp}(n, k)$. Recently, S.S. Kudla and S. Rallis have announced considerable progress on this complementary conjecture. See section 7. We prove the following theorem.

Theorem 7.8. *Let $\dim_k X = 4$ and σ be in $\mathrm{Irr}(\mathrm{GO}(X))$. Then*

$$n(\sigma) + n(\sigma \otimes_{\mathbb{C}} \mathrm{sign}) = 4.$$

Here, $n(\sigma)$ is defined just as in the case of isometries. To prove the theorem, we characterize $\mathcal{R}_1(\mathrm{GO}(X))$ and $\mathcal{R}_3(\mathrm{GO}(X))$. That is, under the same assumptions as in the theorem, we specify $n(\sigma)$. To do so, we make use of the main theorem and [S] and [Co]. See Theorem 7.4 and Lemma 7.7. For a presentation of the information, see the tables in section 7. The following result is an immediate corollary of the theorem.

Corollary 7.9. *If* $\dim_k X = 4$*, then S.S. Kudla's Conjecture 7.1 is true.*

In this paper we do not consider applications to functorality and the theory of L-packets. For some discussion of these topics see [V] and [HST].

We will now make some remarks about previous work on the Weil representation and theta correspondence for similitudes when $\dim_k X = 4$ and $n = 2$. In [PSS] and [So1], in the case $\mathrm{disc}(X) = 1$ and X isotropic, the induced Weil representation [R] is used to lift elements of $\mathrm{Irr}(\mathrm{GSO}(X))$ to representations of $\mathrm{GSp}(2, k)$. This construction is an analogue of the global definition of theta lifts, and uses elements of Whittaker models in place of automorphic forms. The problem of whether these representations of $\mathrm{GSp}(2, k)$ are irreducible is not resolved in [PSS] or [So1]. The work [HPS] in part investigates the case $\mathrm{disc}(X) = 1$ and X anisotropic. In this case, as a consequence of Theorem 9.1 of [HPS], every element of $\mathrm{Irr}(\mathrm{GSO}(X))$ is an $\mathrm{SO}(X)$ quotient of ω. Using this result, one could prove the main theorem in this case using Theorems 4.3 and 4.4. Using the induced Weil representation, results from the previously mentioned papers, and the strong multiplicity one theorem for GSp(2) of [So2], a global argument in [V] lifts elements of $\mathrm{Irr}(\mathrm{GSO}(X))$ that are the local components of cuspidal, not invariant, automorphic representations of $\mathrm{GSO}(X)$ to $\mathrm{Irr}(\mathrm{GSp}(2, k))$. Included in these representations are the supercuspidal representations. Since it uses Whittaker models, in the case $\mathrm{disc}(X) \neq 1$, this method fails to construct the representations that correspond to one of the extensions to $\mathrm{GO}(X)$ of the invariant but not distinguished elements of $\mathrm{Irr}(\mathrm{GSO}(X))$. Finally, [HST] makes many remarks and observations about the cases when X is isotropic, though it is mainly concerned with a certain global theta lifting, and its application to another problem. In particular, after the computation of the Fourier coefficient of the global theta lift it makes a conjecture essentially equivalent to the main theorem in the case where X is isotropic; see the guess on page 399. However, instead of using the concept of distinguished representations, the guess is phrased in terms of ϵ factors. Even so, we rely heavily on the understanding of these ϵ factors from Lemma 14 of [HST].

In the first section we recall the theory of the theta correspondence for similitudes from [R]. In the second section we characterize $\mathrm{GO}(X)$ in terms of the units of quaternion algebras. Using this account, in the third section we parameterize $\mathrm{Irr}(\mathrm{GO}(X))$. In the fourth section we define the concept of being distinguished and relate it to the theta correspondence. Distinguished representations for $\mathrm{disc}(X) \neq 1$ are investigated in the fifth section. The main theorem is proven in the sixth section. In the remaining section we make the application to S.S. Kudla's conjecture.

I would like to thank S.S. Kudla for many useful comments, especially for telling me about his conjectures and the proof of Lemma 4.2. Thanks are also due to J. Hakim for some helpful conversations concerning his theorem.

We use the following notation. Let J be a group of td-type, as in [C]. Then $\mathrm{Irr}(J)$ is the set of equivalence classes of smooth admissible irreducible representations of J. If π is in $\mathrm{Irr}(J)$, then $\pi^\vee$ in $\mathrm{Irr}(J)$ is the contragredient representation of π, and ω_π is the central character of π. A quasi-character of J is a continuous homomorphism from J to $\mathbb{C}^\times$, and a unitary character of J is a continuous homomorphism from J to the group of complex numbers of absolute value 1. The trivial representation of J on $\mathbb{C}$ will be denoted by $\mathbf{1}$. We will also use the notation of [GK] for restriction theory. Throughout the paper k is a nonarchimedean local field of characteristic zero and odd residual characteristic. Let D be the division quaternion algebra over k, with canonical involution $*$ and reduced norm N defined by $N(x) = xx^* = x^*x$. The canonical involution of the quaternion algebra $\mathrm{M}_2(k)$ will also be denoted by $*$; in this case the reduced norm is det. We let $\mathbb{H}$ denote the hyperbolic plane defined over k. Let $(\ ,\ )_k$ denote the Hilbert symbol of k. If K is a quadratic extension of k, then $\omega_{K/k}$ is the nontrivial character of $k^\times/\,\mathrm{N}_k^K(K^\times)$. For d in $k^\times/k^{\times 2}$ we let $\epsilon(d) = (-1, -d)_k$.

1. The theta correspondence for similitudes

In this section we recall some results and definitions from [R]. Suppose that $(X, (\ ,\ ))$ is a nondegenerate symmetric bilinear space over k of even dimension m, and let n be a nonnegative integer. Let $\mathrm{GO}(X)$ be the group of k linear automorphisms h of X such that there exists λ in $k^\times$ such that $(h(x), h(y)) = \lambda(x, y)$ for x and y in X. If h is in $\mathrm{GO}(X)$, then such a λ is unique, and will be denoted by $\lambda(h)$. Let $\mathrm{O}(X)$ be the subgroup of all h in $\mathrm{GO}(X)$ such that $\lambda(h) = 1$. Let sign : $\mathrm{GO}(X) \to \{\pm 1\}$ be the unitary character defined by $\mathrm{sign}(h) = \det(h)/\lambda(h)^{m/2}$. We let $\mathrm{GSO}(X) = \ker(\mathrm{sign})$. We will often describe $\mathrm{GO}(X)$ in terms of $\mathrm{GSO}(X)$ and an extra element of $\mathrm{GO}(X)$. Let h_0 in $\mathrm{GO}(X)$ be such that $h_0^2 = 1$ and h_0 is not in $\mathrm{GSO}(X)$. There is an action of the group $\{1, h_0\}$ on $\mathrm{GSO}(X)$ given by $h_0 \cdot h = h_0 h h_0$, and an isomorphism $\mathrm{GSO}(X) \rtimes \{1, h_0\} \cong \mathrm{GO}(X)$ that takes (h, δ) to $h\delta$. Next, let $\mathrm{GSp}(n, k)$ be the group of all g in $\mathrm{Gl}(2n, k)$ such that for some λ in $k^\times$,

$$
{}^t g \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g = \lambda \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.
$$

Again, if g is in $\mathrm{GSp}(2n, k)$, then such a λ is unique and will be denoted by $\lambda(g)$. Let $\mathrm{Sp}(n, k)$ be the subgroup of all g in $\mathrm{GSp}(n, k)$ such that $\lambda(g) = 1$. Let $\mathrm{GSp}(n, k)^+$ be the subgroup of all g in $\mathrm{GSp}(n, k)$ such that there exists h in $\mathrm{GO}(X)$ such that $\lambda(g) = \lambda(h)$. The group $\mathrm{GSp}(n, k)^+$ depends on X and is a proper subgroup of $\mathrm{GSp}(n, k)$ if and only if $\mathrm{disc}(X) \neq 1$. Here, $\mathrm{disc}(X)$ is the discriminant of X, as defined in [Sc]. If $\mathrm{disc}(X) \neq 1$, then $[\mathrm{GSp}(n, k) : \mathrm{GSp}(n, k)^+] = 2$. Fix a nontrivial additive character ψ of k.

To ψ, X and n, there is associated the **Weil representation** ω of $\mathrm{Sp}(n, k) \times \mathrm{O}(X)$ on $\mathcal{S}(X^n)$. For the most part, in this paper we only will need to know the action of $\omega(1, h)$ for h in $\mathrm{O}(X)$, which is given by left translation:

$$
\omega(1, h) \cdot \varphi(x) = L(h)\varphi(x) = \varphi(h^{-1}x).
$$

There exists of an extension of ω to a representation of the larger group

$$
R = \{(g, h) \in \mathrm{GSp}(n, k) \times \mathrm{GO}(X) : \lambda(g) = \lambda(h)\}.
$$

This extension, called the **extended Weil representation**, will also be denoted by ω, and is very simply defined by

$$
\omega(g, h)\varphi = |\lambda(h)|^{-\frac{mn}{4}} \omega(g \begin{pmatrix} 1 & 0 \\ 0 & \lambda(g)^{-1} \end{pmatrix}, 1) L(h)\varphi.
$$

Note that R involves only $\mathrm{GSp}(n, k)^+$. Occasionally, to indicate the dependence of ω on X and n, we will write $\omega_{X,n}$, ω_n or ω_X for ω.

The Weil representation defines a correspondence between $\mathrm{Irr}(\mathrm{Sp}(n, k))$ and $\mathrm{Irr}(\mathrm{O}(X))$. Let $\mathcal{R}_X(\mathrm{Sp}(n, k))$ be the set of all elements of $\mathrm{Irr}(\mathrm{Sp}(n, k))$ that are nonzero quotients of ω, and similarly define $\mathcal{R}_n(\mathrm{O}(X))$. As a consequence of a more general theorem of [W], we have

Theorem 1.1 (Waldspurger). *The set*

$$\{(\pi,\sigma) \in \mathcal{R}_X(\mathrm{Sp}(n,k)) \times \mathcal{R}_n(\mathrm{O}(X)) : \mathrm{Hom}_{\mathrm{Sp}(n,k)\times \mathrm{O}(X)}(\omega, \pi \otimes_{\mathbb{C}} \sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}_X(\mathrm{Sp}(n,k))$ *and* $\mathcal{R}_n(\mathrm{O}(X))$.

A correspondence for similitudes is defined by the extended Weil representation. Let $\mathcal{R}_X(\mathrm{GSp}(n,k)^+)$ be the set of σ in $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$ such that $\sigma|_{\mathrm{Sp}(n,k)}$ is multiplicity free and has a constituent in $\mathcal{R}_X(\mathrm{Sp}(n,k))$. Similarly define $\mathcal{R}_n(\mathrm{GO}(X))$. From [R], section 4, we have

Theorem 1.2. *The set*

$$\{(\pi,\sigma) \in \mathcal{R}_X(\mathrm{GSp}(n,k)^+) \times \mathcal{R}_n(\mathrm{GO}(X)) : \mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}_X(\mathrm{GSp}(n,k)^+)$ *and* $\mathcal{R}_n(\mathrm{GO}(X))$.

If π is in $\mathcal{R}_X(\mathrm{GSp}(n,k)^+)$ or σ is in $\mathcal{R}_n(\mathrm{GO}(X))$, then we denote the corresponding elements of $\mathcal{R}_n(\mathrm{GO}(X))$ and $\mathcal{R}_X(\mathrm{GSp}(n,k)^+)$ by $\theta(\pi)$ and $\theta(\sigma)$, respectively.

The problem of whether the extended Weil representation defines a well behaved correspondence between $\mathrm{Irr}(\mathrm{GSp}(n,k))$ and $\mathrm{Irr}(\mathrm{GO}(X))$ when $\mathrm{GSp}(n,k)^+$ is a proper subgroup of $\mathrm{GSp}(n,k)$ is also dealt with in [R]. To describe the results, suppose that $\mathrm{GSp}(n,k)^+$ is a proper subgroup of $\mathrm{GSp}(n,k)$, i.e., that $\mathrm{disc}(X) \neq 1$. Then the multiplicity free assumption is unnecessary since $[\mathrm{GSp}(n,k)^+ : k^\times \cdot \mathrm{Sp}(n,k)] = [\mathrm{GO}(X) : k^\times \cdot \mathrm{O}(X)] = 2$. See, for example, [GK]. One would like to know if the condition

$$\mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \sigma) \neq 0$$

defines a bijection between $\mathcal{R}_X(\mathrm{GSp}(n,k))$, the set of all π in $\mathrm{Irr}(\mathrm{GSp}(n,k))$ such that some constituent of $\pi|_{\mathrm{Sp}(n,k)}$ lies in $\mathcal{R}_X(\mathrm{Sp}(n,k))$, and $\mathcal{R}_n((\mathrm{GO}(X)))$. In [R] it is shown that this condition defines such a bijection if and only if a certain criterion is satisfied.

To state this criterion, we need to introduce the other nondegenerate symmetric bilinear space X' of dimension m and discriminant $\mathrm{disc}(X)$. From the Witt decomposition theorem we see that X' can be taken to have the same vector space as X, but with symmetric bilinear form multiplied by an element of $k^\times$. Assume that X' has this form. Then $\mathrm{GO}(X) = \mathrm{GO}(X')$, and the restrictions of the Weil representations ω and ω' associated to X and X', respectively, to $\mathrm{O}(X) = \mathrm{O}(X')$ are identical. It follows that $\mathcal{R}_n(\mathrm{O}(X)) = \mathcal{R}_n(\mathrm{O}(X'))$ and $\mathcal{R}_n(\mathrm{GO}(X)) = \mathcal{R}_n(\mathrm{GO}(X'))$. However, the correspondences defined by ω and ω' may differ. In [R] it is proven that the above condition defines a bijection if and only if the correspondences defined by ω and ω' are disjoint, i.e., $\mathcal{R}_X(\mathrm{Sp}(n,k)) \cap \mathcal{R}_{X'}(\mathrm{Sp}(n,k)) = \emptyset$.

Suppose $\mathcal{R}_X(\mathrm{Sp}(n,k)) \cap \mathcal{R}_{X'}(\mathrm{Sp}(n,k)) = \emptyset$. From [R] we have the following. Let g be a representative for the nontrivial coset of $\mathrm{GSp}(n,k)/\,\mathrm{GSp}(n,k)^+$. Let σ be in $\mathcal{R}_n(\mathrm{GO}(X)) = \mathcal{R}_n(\mathrm{GO}(X'))$, and let π and π' in $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$ correspond to σ with respect to ω and ω', respectively. Then $g \cdot \pi = \pi'$, and

$$\mathrm{Ind}_{\mathrm{GSp}(n,k)^+}^{\mathrm{GSp}(n,k)} \pi$$

in $\mathcal{R}_X(\mathrm{GSp}(n,k))$ corresponds to σ.

Whether the criterion is expected to hold depends on m and n. If the underlying bilinear spaces lie in the stable range, i.e., if $m \geq 4n+2$, then the criterion does not hold. From [HKS], we have have the following conjecture.

Conjecture 1.3 (Theta dichotomy). *If $m \leq 2n$, then*

$$\mathcal{R}_X(\mathrm{Sp}(n,k)) \cap \mathcal{R}_{X'}(\mathrm{Sp}(n,k)) = \emptyset.$$

For progress on the conjecture, see [KR] and [HKS]. The theta dichotomy conjecture follows from another strong and precise conjecture of S.S. Kudla. Recently, S.S. Kudla and S. Rallis have announced considerable progress on this stronger conjecture. In particular, it follows from their result that no supercuspidal representation is contained in the intersection of Conjecture 1.3. See section 7.

2. Four dimensional symmetric bilinear spaces

In this section we recall the characterization of the group of similitudes of a four dimensional symmetric bilinear space in terms of the units of a quaternion algebra. For the remainder of this paper, d will be an element of $k^\times/k^{\times 2}$. If $d=1$, then let $K = k \times k$; if $d \neq 1$, then let $K = k(\sqrt{d})$. Let $\mathrm{Gal}(K/k) = \{1, -\}$.

Let B be a quaternion algebra over K, with canonical involution $*$. A k linear ring automorphism s of B is a **Galois action** on B if $s^2 = 1$ and $s(ax) = \overline{a}s(x)$ for a in K and x in B. Let s be a Galois action on B. Let $X(s)$ be the set of all x in B such that $s(x) = x^*$. Then $X(s)$ is a four dimensional vector space over k, and equipped with the restriction of the symmetric bilinear form corresponding to the reduced norm of B, $X(s)$ is a nondegenerate symmetric bilinear space. The discriminant and Hasse invariant of $X(s)$ are d and $\epsilon(d)\epsilon(s)$, respectively. Here, to define $\epsilon(s)$, let $B(s)$ be the fixed points of s. Then $B(s)$ is a quaternion algebra over k, and $\epsilon(s) = 1$ if $B(s)$ is split and $\epsilon(s) = -1$ if $B(s)$ is ramified.

The elements of $k^\times \times B^\times$ give elements $\mathrm{GSO}(X)$. Define a left action ρ of $k^\times \times B^\times$ on $X(s)$ by $\rho(t,g)x = t^{-1}gxs(g)^*$. Then $\rho(t,g)$ is in $\mathrm{GSO}(X(s))$ for (t,g) in $k^\times \times B^\times$. There is an inclusion of $K^\times$ in $k^\times \times B^\times$ that sends a to $(\mathrm{N}_k^K(a), a)$.

The following result is well known.

Theorem 2.1. *For every four dimensional nondegenerate symmetric bilinear space X of discriminant d over k there exists a quaternion algebra B over K and a Galois action s on B such that $X \cong X(s)$ as symmetric bilinear spaces. For every quaternion algebra B over K and Galois action s on B the sequence*

$$1 \to K^\times \to k^\times \times B^\times \xrightarrow{\rho} \mathrm{GSO}(X(s)) \to 1$$

is exact.

We now define concrete realizations of the two four dimensional nondegenerate symmetric bilinear spaces $X(d,\epsilon)$ of discriminant d and Hasse invariant ϵ in $\{\pm 1\}$. Suppose first $d = 1$. Let B be $\mathrm{M}_2(k) \times \mathrm{M}_2(k)$ or $D \times D$. Define a Galois action on B by $s(x,y) = (y,x)$. Then $X(s)$ is isomorphic to $\mathrm{M}_2(k)$ or D. We find that $\mathrm{M}_2(k)$ and D have discriminant 1 and Hasse invariant $\epsilon(d)$ and $-\epsilon(d)$, respectively. We let $X(1,\epsilon(d)) = \mathrm{M}(k)$ and $X(1,-\epsilon(d)) = D$. The above exact sequence simplifies to

$$1 \to k^\times \to \mathrm{Gl}(2,k) \times \mathrm{Gl}(2,k) \xrightarrow{\rho} \mathrm{GSO}(X(1,\epsilon(1))) \to 1$$

and

$$1 \to k^\times \to D^\times \times D^\times \xrightarrow{\rho} \mathrm{GSO}(X(1, -\epsilon(1))) \to 1,$$

where ρ is now defined by $\rho(g, g')x = gxg'^*$, and the inclusion of $k^\times$ sends x to (x, x^{-1}).

Suppose that $d \neq 1$. Let $B = \mathrm{M}_2(K)$. Then $B = K \otimes_k \mathrm{M}_2(k)$ or $B = K \otimes_k D$. Here we regard D as a subalgebra of B by letting

$$D = \{ \begin{pmatrix} a & b\delta \\ \overline{b} & \overline{a} \end{pmatrix} : a, b \in K\}$$

where δ is a representative for the nontrivial coset of $k^\times / \mathrm{N}_k^K(K^\times)$. Define Galois actions s and s' on B corresponding to the decompositions $B = K \otimes_k \mathrm{M}_2(k)$ and $B = K \otimes_k D$, respectively, by $s(a \otimes x) = \overline{a} \otimes x$ and $s(a \otimes y) = \overline{a} \otimes y$ for a in K, x in $\mathrm{M}_2(k)$ and y in D. $X(s)$ and $X(s')$ have discriminant d and Hasse invariants $\epsilon(d)$ and $-\epsilon(d)$, respectively. We let $X(d, \epsilon(d)) = X(s)$ and $X(d, -\epsilon(d)) = X(s')$. There are exact sequences

$$1 \to K^\times \to k^\times \times \mathrm{Gl}(2, K) \xrightarrow{\rho} \mathrm{GSO}(X(d, \pm\epsilon(d))) \to 1.$$

Explicitly,

$$X(d, \epsilon(d)) = \{ \begin{pmatrix} a & b\sqrt{d} \\ c\sqrt{d} & \overline{a} \end{pmatrix} : a \in K, b, c \in k\}$$

and

$$X(d, -\epsilon(d)) = \{ \begin{pmatrix} b & -\delta a \\ \overline{a} & c \end{pmatrix} : a \in K, b, c \in k\}.$$

As mentioned in section 1, there is a similitude between $X(d, \epsilon(d))$ and $X(d, -\epsilon(d))$. Let us construct this similitude explicitly. Since $s \circ s'$ is a K algebra isomorphism, by the Skolem-Noether theorem there exists a u in $\mathrm{Gl}(2, K)$ such that $s'(x) = us(x)u^{-1}$ for x in $\mathrm{M}_2(K)$. We may assume that $us(u) = us'(u) = \det(u)$. Then the map $S : X(d, \epsilon(d)) \to X(d, -\epsilon(d))$ defined by $S(x) = xu^{-1}$ is a well defined similitude such that $S(\rho(t, g)x) = \rho(t, g)S(x)$ for (t, g) in $k^\times \times \mathrm{Gl}(2, K)$ and x in $X(d, \epsilon(d))$.

For the remainder of this paper, ϵ will be in $\{\pm 1\}$, and $X = X(d, \epsilon)$. Also, h_0 is the element of $\mathrm{GO}(X)$ that sends x to x^*. Because of the remarks in section 1 concerning the theta correspondence for similitudes when $d \neq 1$, we will disregard the case $d \neq 1$ and $\epsilon = -\epsilon(d)$. Thus, if $d \neq 1$, then $X = X(d, \epsilon(d))$. We will let ω denote the extended Weil representation associated to X and the nonnegative integer n. If necessary, the dependence of ω on n will be indicated by a subscript.

3. Representations

In this section we make some definitions and elementary observations concerning the relationship between representations of $\mathrm{GO}(X)$ and $\mathrm{GSO}(X)$ and the quaternion algebras from the last section. We remind the reader that the case $d \neq 1$ and $\epsilon = -\epsilon(d)$ for our purposes can be and will be ignored. We also point out that by [HPS], Lemma 7.2, the restriction of representations of $\mathrm{GO}(X)$ to $\mathrm{O}(X)$ is multiplicity free.

Suppose first that $d = 1$. Let $\mathrm{Irr}_f(\mathrm{Gl}(2, k) \times \mathrm{Gl}(2, k))$ be the set of pairs of representations in $\mathrm{Irr}(\mathrm{Gl}(2, k))$ with the same central character. We define

$\mathrm{Irr}_f(D^\times \times D^\times)$ similarly. There are bijections

$$\mathrm{Irr}(\mathrm{GSO}(X(1,\epsilon(1)))) \xrightarrow{\sim} \mathrm{Irr}_f(\mathrm{Gl}(2,k)\times \mathrm{Gl}(2,k))$$

and

$$\mathrm{Irr}(\mathrm{GSO}(X(1,-\epsilon(1)))) \xrightarrow{\sim} \mathrm{Irr}_f(D^\times \times D^\times)$$

that take π to the representation that sends (g,g') to $\pi(\rho(g,g'))$. If (τ,τ') is contained in $\mathrm{Irr}_f(\mathrm{Gl}(2,k)\times\mathrm{Gl}(2,k))$ or $\mathrm{Irr}_f(D^\times\times D^\times)$, then the corresponding element $\pi(\tau,\tau')$ of $\mathrm{Irr}(\mathrm{GSO}(X(1,\pm\epsilon(d))))$ has as space the space of $\tau\otimes_{\mathbb{C}}\tau'$ and is defined by $\pi(\tau,\tau')(\rho(g,g')) = \tau(g)\otimes\tau'(g')$. The central character of $\pi(\tau,\tau')$ is $\omega_\tau = \omega_{\tau'}$, and the contragredient of $\pi(\tau,\tau')$ is $\pi(\tau,\tau')^\vee = \pi(\tau^\vee,\tau'^\vee)$.

Suppose that $d \neq 1$. Let $\mathrm{Irr}_f(\mathrm{Gl}(2,K))$ be the set of elements of $\mathrm{Irr}(\mathrm{Gl}(2,K))$ with Galois invariant central character. Recall that if a quasi-character of $K^\times$ is Galois invariant, then it factors through N_k^K via exactly two quasi-characters of $k^\times$. There is a two-to-one surjective map

$$\mathrm{Irr}(\mathrm{GSO}(X(d,\epsilon(d)))) \to \mathrm{Irr}_f(\mathrm{Gl}(2,K))$$

that takes π to the representation that has space the space of π and is defined by $g\mapsto \pi(\rho(1,g))$. If τ is in $\mathrm{Irr}(\mathrm{Gl}(2,K))$, and χ and χ' are the two quasi-characters of $k^\times$ such that $\omega_\tau = \chi\circ\mathrm{N}_k^K$ and $\omega_\tau = \chi'\circ\mathrm{N}_k^K$, then the two elements $\pi(\tau,\chi)$ and $\pi(\tau,\chi')$ of $\mathrm{Irr}(\mathrm{GSO}(X(d,\epsilon(d))))$ lying over τ are defined by $\pi(\tau,\chi)(\rho(t,g)) = \chi(t)^{-1}\tau(g)$ and $\pi(\tau,\chi')(\rho(t,g)) = \chi'(t)^{-1}\tau(g)$. The central character of $\pi(\tau,\chi)$ is χ, and the contragredient of $\pi(\tau,\chi)$ is $\pi(\tau,\chi)^\vee = \pi(\tau^\vee,\chi^{-1})$.

Having described the representations of $\mathrm{GSO}(X)$, we consider their relationship to representations of $\mathrm{GO}(X)$. Let π be in $\mathrm{Irr}(\mathrm{GSO}(X))$. If the induced representation of π to $\mathrm{GO}(X)$ is irreducible, we say that π is **regular**, and if the induced representation of π to $\mathrm{GO}(X)$ is reducible, we say that π is **invariant**. If π is regular, we denote the induced representation of π to $\mathrm{GO}(X)$ by π^+.

We can describe regular and invariant representations in terms of the above characterizations.

Proposition 3.1. *Let π be in $\mathrm{Irr}(\mathrm{GSO}(X))$. If $d=1$, then π is invariant if and only if $\pi=\pi(\tau,\tau)$ for some τ in $\mathrm{Irr}(\mathrm{Gl}(2,k))$ or $\mathrm{Irr}(B^\times)$. If $d\neq 1$, then π is invariant if and only if $\pi=\pi(\tau,\chi)$ for some Galois invariant τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$.*

It also will be useful to have an explicit description of the finite dimensional elements of $\mathrm{Irr}(\mathrm{GSO}(X))$ and $\mathrm{Irr}(\mathrm{GO}(X))$ in the case $d\neq 1$. Assume $d\neq 1$. The finite dimensional elements of $\mathrm{Irr}(\mathrm{GSO}(X))$ are one dimensional, and of the form $\pi(\beta,\chi)$, where β and χ are quasi-characters of $K^\times$ and $k^\times$, respectively, such that $\beta^2=\chi\circ\mathrm{N}_k^K$. Here, to avoid excessively complicated notation, in $\pi(\beta,\chi)$ we regard β as the quasi-character of $\mathrm{Gl}(2,K)$ that sends g in $\mathrm{Gl}(2,K)$ to $\beta(\det(g))$. It follows that the finite dimensional elements of $\mathrm{Irr}(\mathrm{GO}(X))$ are either one or two dimensional. One dimensional elements arise from invariant elements of $\mathrm{Irr}(\mathrm{GSO}(X))$, while two dimensional elements come from regular elements of $\mathrm{Irr}(\mathrm{GSO}(X))$.

Proposition 3.2. *Suppose $d\neq 1$. The subset of invariant one dimensional elements of $\mathrm{Irr}(\mathrm{GSO}(X))$ consists of the $\pi(\alpha\circ\mathrm{N}_k^K,\chi)$, where α and χ are quasi-characters of $k^\times$ and $\chi\circ\mathrm{N}_k^K=\alpha^2\circ\mathrm{N}_k^K$. The set of finite dimensional regular elements of $\mathrm{Irr}(\mathrm{GSO}(X))$ consists of the $\pi(\beta,\chi)$, where β is a Galois noninvariant quasi-character of $K^\times$ and χ is a quasi-character of $k^\times$ such that $\beta^2=\chi\circ\mathrm{N}_k^K$.*

Moreover, if $\pi(\beta,\chi)$ *is such a regular element of* $\mathrm{Irr}(\mathrm{GSO}(X))$*, then* $\chi = \beta|_{k^\times}\mu$ *for some nontrivial quadratic quasi-character* μ *of* $k^\times$ *different from* $\omega_{K/k}$*.*

4. Distinguished representations and the correspondence

In this section we will define what it means for an invariant representation of $\mathrm{GSO}(X)$ to be distinguished, and we will consider what effect being distinguished has on what extensions of the representation to $\mathrm{GO}(X)$ occur in the theta correspondence. The idea that certain extensions of a distinguished representation cannot occur in the theta correspondence is due to [HK]. This appears in Theorem 4.3 below. We go a step further, and show how an extension of a distinguished representation can be proven to occur in the theta correspondence. See Theorem 4.4.

Let π be in $\mathrm{Irr}(\mathrm{GSO}(X))$. To define what it means for π to be distinguished, suppose y in X is anisotropic. Then the stabilizer in $\mathrm{O}(X)$ of y can be identified with $\mathrm{O}(Y)$, where Y is the orthogonal complement to y, and we will write $\mathrm{O}(Y)$ for this stabilizer. We say that π is **generically distinguished** if π is invariant, and there is an anisotropic y in X such that

$$\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \neq 0,$$

and, if $d \neq 1$, then Y is isotropic. We will see in Corollary 7.5 that the assumption that Y be isotropic when $d \neq 1$ is unnecessary, but it is convenient to make it. We will say that π is **distinguished** if π is generically distinguished or $d \neq 1$ and π is invariant and one dimensional.

We regard distinguished, but not generically distinguished, representations as boundary cases. Our reason for including them is, of course, that they behave like generically distinguished representations with respect to the theta correspondence. We also note that they are irreducible subquotients of generically distinguished reducible principal series representations.

In fact, a representation is generically distinguished if and only if it is generically distinguished with respect to a certain anisotropic y_0 in X. Define y_0 in the following way. If $d = 1$, let $y_0 = 1$. If $d \neq 1$, also let $y_0 = 1$. Using the Witt cancellation theorem and the Witt extension theorem, one can show that if y is as in the last paragraph, then there exists h in $\mathrm{GSO}(X)$ such that $h(y) = y_0$. It follows that a representation is generically distinguished if and only if it is generically distinguished with respect to y_0. For the remainder of this paper we let Y be the orthogonal complement to y_0.

The group $\mathrm{SO}(Y)$ can be concretely described. If $d = 1$, then $\mathrm{SO}(Y)$ is the image under ρ of the subgroup $\{(g, g^{*-1}) : g \in \mathrm{Gl}(2,k)\}$ or $\{(g, g^{*-1}) : g \in D^\times\}$. If $d \neq 1$, then by Hilbert's Theorem 90, $\mathrm{SO}(Y)$ is the image under ρ of the subgroup $\{(\det(g), g) : g \in \mathrm{Gl}(2,k)\}$. We also note that h_0 fixes y_0, and thus is contained in $\mathrm{O}(Y)$. Together, $\mathrm{SO}(Y)$ and h_0 generate $\mathrm{O}(Y)$.

In the case $d = 1$, the next proposition completely identifies all the distinguished representations. We will consider the case $d \neq 1$ in greater detail in the next section.

Proposition 4.1 (Hakim). *Let* $\pi \in \mathrm{Irr}(\mathrm{GSO}(X))$*. Assume that* π *is invariant. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \leq 1.$$

If $d = 1$, then π is generically distinguished. If $d \neq 1$ and $\pi = \pi(\tau, \chi)$, then π is generically distinguished if and only if

$$\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det) \neq 0.$$

Proof. Suppose that $d = 1$. Since π is invariant, it follows that $\pi = \pi(\tau, \tau)$ for some τ in $\mathrm{Irr}(\mathrm{Gl}(2, k))$ or τ in $\mathrm{Irr}(D^\times)$. Now $\tau^\vee \cong \omega_\tau^{-1} \otimes_{\mathbb{C}} \tau$. It follows that there is an isomorphism

$$\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \cong \mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau \otimes_{\mathbb{C}} \tau^\vee, \mathbf{1})$$

or an isomorphism

$$\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \cong \mathrm{Hom}_{D^\times}(\tau \otimes_{\mathbb{C}} \tau^\vee, \mathbf{1}).$$

Here $\mathrm{Gl}(2, k)$ or $D^\times$ is embedded on the diagonal. It is well known that the second homomorphism space has dimension one.

Suppose that $d \neq 1$. Then there is an isomorphism

$$\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \cong \mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det).$$

By an argument as in [H], this space has dimension less than or equal to 1. □

If a representation is distinguished, then we will identify its extensions to $\mathrm{GO}(X)$ in the following way. Suppose first that π in $\mathrm{Irr}(\mathrm{GSO}(X))$ is generically distinguished. Since by Proposition 4.1

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) = 1,$$

it follows that for exactly one extension π' of π to $\mathrm{GO}(X)$ we have $\mathrm{Hom}_{\mathrm{O}(Y)}(\pi', \mathbf{1}) \neq 0$. Call this extension π^+; the other extension of π to $\mathrm{GO}(X)$ will be called π^-. From our above remarks, the definitions of π^+ and π^- do not depend on the choice of y. Suppose next that $d \neq 1$ and $\pi = \pi(\alpha \circ \mathrm{N}_k^K, \chi)$ is invariant and one dimensional. See Proposition 3.2. We will denote the extension of π that sends h_0 to $\chi(-1)$ by π^+; the other extension of π will be denoted by π^-. It can be verified that the two definitions of π^+ and π^- in the case where $d \neq 1$ and π is generically distinguished and one dimensional are the same.

The following lemma will be essential in determining which extensions of a generically distinguished representation occur in the theta correspondence. As we mentioned earlier, the proof of this lemma is general. A form of a case of this lemma appears in Proposition 4.5 of [P1].

Lemma 4.2. *If $n < \dim_k Y$, i.e., if $n = 1$ or 2, then any distribution on X^n invariant under $\mathrm{SO}(Y)$ is invariant under $\mathrm{O}(Y)$.*

Proof. Assume $n < \dim_k Y$. It is straightforward to see that any distribution on X^n invariant under $\mathrm{SO}(Y)$ is invariant under $\mathrm{O}(Y)$ if and only if $\mathrm{Hom}_{\mathrm{O}(Y)}(\omega_{X,n}, \mathrm{sign}) = 0$. Assume that $\mathrm{Hom}_{\mathrm{O}(Y)}(\omega_{X,n}, \mathrm{sign}) \neq 0$. Since $X = Y \perp Y^\perp$, we have $\omega_{X,n} \cong \omega_{Y,n} \otimes \omega_{Y^\perp,n}$, as $\mathrm{O}(Y) \times \mathrm{O}(Y^\perp)$ representations. Since $\mathrm{Hom}_{\mathrm{O}(Y)}(\omega_{X,n}, \mathrm{sign}) \neq 0$, it follows that $\mathrm{Hom}_{\mathrm{O}(Y)}(\omega_{Y,n}, \mathrm{sign}) \neq 0$. This contradicts the appendix of [Ra]. □

The next theorem shows that at most one of the extensions of a generically distinguished representation can occur in the theta correspondence when n is 1 or 2. As we pointed out above the idea is due to [HK]. In [HK] the case $n = 1$ was considered. We shall see in Lemma 6.5 that the result holds for all distinguished representations, not just for generically distinguished representations.

Theorem 4.3. *Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X))$. If π is generically distinguished, then π^- is not in $\mathcal{R}_n(\mathrm{GO}(X))$ for $n < \dim_k Y$, i.e., for $n = 1$ and 2.*

Proof. We begin with two comments concerning π. First, $\pi|_{\mathrm{SO}(X)}$ is multiplicity free. For let

$$\pi|_{\mathrm{SO}(X)} = m \cdot \pi_1 \oplus \cdots \oplus m \cdot \pi_M,$$

where the $\pi_i \in \mathrm{Irr}(\mathrm{SO}(X))$ are mutually inequivalent, and m and M are positive integers. Then

$$\sum_{i=1}^{M} m \cdot \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_i, \mathbf{1}) = 1,$$

which implies that $m = 1$, and that $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_i, \mathbf{1}) = 1$ for exactly one i, say $i = 1$, and $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_i, \mathbf{1}) = 0$ for $i > 1$. Second, suppose that V_i is the space of π_i; we assert that $\pi^+(h_0)V_i = V_i$ for all i. Let us prove this first when $i = 1$. Let $\pi^+(h_0)V_1 = V_i$. Let f in $\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_1, \mathbf{1})$ be nonzero. Define a linear functional f' on V_i by $f'(v) = f(\pi^+(h_0)v)$. Then f' is in $\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_i, \mathbf{1})$. Since $f' \neq 0$, $i = 1$. Let i be arbitrary. There exists h in $\mathrm{GSO}(X)$ such that $\pi(h)V_1 = V_i$. We have $\pi^+(h_0)V_i = \pi^+(h_0 h)V_1 = \pi^+(h_0 h h_0)V_1 = \pi(h)\pi(h^{-1}h_0 h h_0)V_1 = \pi(h)V_1 = V_i$, since $h^{-1}h_0 h h_0$ is in $\mathrm{SO}(X)$.

Suppose that π^- is in $\mathcal{R}_n(\mathrm{GO}(X))$ for $n = 1$ or 2. Then there exists a nonzero $\mathrm{O}(X)$ map T from ω_n to π^-. Let $f \in \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1})$ be nonzero. We may assume the composition $f \circ T$ is nonzero. This is a nonzero $\mathrm{SO}(Y)$ invariant distribution on X^n. By Lemma 4.2, $f \circ T$ is invariant under h_0. But since T is an $\mathrm{O}(X)$ map and by the definition of π^-, the composition of h_0 with $f \circ T$ is $-f \circ T$. Since $f \circ T \neq 0$, this is a contradiction. □

The next theorem gives a sufficient condition for one of the extensions of a generically distinguished representation to occur in the theta correspondence. Again, we shall see in Lemma 6.5 that the result holds for all distinguished representations, not just for generically distinguished representations.

Theorem 4.4 . *Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X))$. Suppose π is regular or generically distinguished, and $n < \dim_k Y$, i.e., $n = 1$ or 2. Then*

$$\mathrm{Hom}_{\mathrm{SO}(X)}(\omega_n, \pi) \neq 0 \implies \mathrm{Hom}_{\mathrm{O}(X)}(\omega_n, \pi^+) \neq 0.$$

Proof. Suppose first that π is regular. Let V be the space of π. As a model for π^+ we can take the representation with space $V \oplus V$ and action defined by $\pi^+(h)(v \oplus v') = \pi(h)v \oplus \pi(h_0 h h_0)v'$ for h in $\mathrm{GSO}(X)$ and $\pi^+(h_0)(v \oplus v') = v' \oplus v$. Let L in $\mathrm{Hom}_{\mathrm{SO}(X)}(\omega_n, \pi)$ be nonzero. Define $L' : \omega_n \to \pi^+$ by $L'(\varphi) = L(\varphi) \oplus L(\omega_n(h_0)\varphi)$. Then L' is in $\mathrm{Hom}_{\mathrm{O}(X)}(\omega_n, \pi^+)$ and L' is nonzero.

Suppose that π is generically distinguished. We will use the notation of the proof of Theorem 4.3. Let L in $\mathrm{Hom}_{\mathrm{SO}(X)}(\omega_n, \pi)$ be nonzero. We may assume that the composition L_1 of L with the projection of V onto V_1 is nonzero. To complete the proof it suffices to show that $L_1 \circ \omega_n(h_0) = \pi^+(h_0) \circ L_1$. We first show that $\omega_n(h_0)\ker(L_1) = \ker(L_1)$. Suppose not, i.e., suppose that $L_1(\omega_n(h_0)\ker(L_1)) \neq 0$. Then by the irreducibility of π_1, $L_1(\omega_n(h_0)\ker(L_1)) = V_1$. Let f in $\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_1, \mathbf{1})$ be nonzero. Consider $f \circ L_1$. This distribution is nonzero and $\mathrm{SO}(Y)$ invariant. By Lemma 4.2, $f \circ L_1$ is invariant under h_0, so that $f(V_1) = f(L_1(\omega_n(h_0)\ker(L_1))) = f(L_1(\ker(L_1))) = 0$, contradicting $f \neq 0$. Now since $\ker(L_1)$ is invariant under

$\omega_n(h_0)$, it follows that $\mathcal{S}(X^n)/\ker(L_1)$ is an $\mathrm{O}(X)$ space. Via the $\mathrm{SO}(X)$ isomorphism given by L_1 between $\mathcal{S}(X^n)/\ker(L_1)$ and V_1 we can define an action of h_0 on V_1 so that L_1 is an $\mathrm{O}(X)$ map. By Theorem 4.3, this extension must be π^+. □

A similar argument proves the following statement. Let π be in $\mathrm{Irr}(\mathrm{GSO}(X))$ and Π be in $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$, for $n = 1$ or 2. Assume that π is regular or generically distinguished. Then

$$\mathrm{Hom}_{R'}(\omega_n, \Pi \otimes_{\mathbb{C}} \pi) \neq 0 \implies \mathrm{Hom}_R(\omega_n, \Pi \otimes_{\mathbb{C}} \pi^+) \neq 0.$$

Here R' is the subset of elements of R whose first entries are in $\mathrm{GSO}(X)$.

This result has some interesting consequences. It implies that if a regular or generically distinguished element of $\mathrm{Irr}(\mathrm{GSO}(X))$ corresponds to an element of $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$, in the obvious sense, then that element of $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$ is unique. In particular, since all elements of $\mathrm{Irr}(\mathrm{GSO}(X))$ are either regular or generically distinguished when $\mathrm{disc}(X) = 1$, it follows that in this case if Π is as above, then Π is always uniquely determined. When $\mathrm{disc}(X) = 1$ and $n = 1$ this helps one to understand the Jacquet-Langlands correspondence from the point of view of the theta correspondence. See section 7 and [S]. When $\mathrm{disc}(X) = 1$ and $n = 2$, using the relation to the alternate approach to similitudes via the induced Weil representation [R], this gives a different argument for part of the proof of the strong multiplicity one theorem for $\mathrm{GSp}(2)$ as in [So2]. It would be interesting to see if a complete proof could be obtained along these lines. This would require that the results of this section be extended to the case when X is the split six dimensional space. To do so, it would seem to be necessary to use a Y with $\dim_k Y = 4$. We also mention that the above development is used implicitly in the classical case $\dim_k X = 2n = 2$. In this case one uses a Y with $\dim_k Y = 2$, i.e., $Y = X$. For more remarks about this case, see section 7.

Finally, the results of this section generalize considerably. To generalize the definitions, suppose that X is an even dimensional nondegenerate symmetric bilinear space, and n is a positive integer. In place of the one dimensional subspace $k \cdot y$ above, one could take any nondegenerate subspace, and again define Y as the orthogonal complement to this subspace. The definition of a generically distinguished representation would be as above. Then the results generalize in the following way. Lemma 4.2 and its proof hold for $n < \dim_k Y$. If one assumes the dimension statement of Proposition 4.1, then Theorem 4.3 and Theorem 4.4 also hold for $n < \dim_k Y$. Thus, the remaining key issue for generalization is the validity of the dimension statement of Proposition 4.1. If $\dim_k Y$ is small with respect to $\dim_k X$, then we cannot expect the dimension statement to hold in general.

5. Distinguished $\mathrm{Gl}(2,K)$ representations

In the last section we reduced the problem of determining the distinguished representations of $\mathrm{GSO}(X)$ in the case $d \neq 1$ to a problem concerning the corresponding representations of $\mathrm{Gl}(2,K)$. The problem of determining distinguished $\mathrm{Gl}(2,K)$ representations has essentially been solved by several authors. See [H] and [F]. Ultimately, the consideration of distinguished $\mathrm{Gl}(2,K)$ representations goes back to a global result of [HLR]. However, since a complete account does not appear in the literature we need to give an exposition.

We begin by defining some notation and recalling some facts. Essentially, we will follow [G]. In this section we assume that $d \neq 1$ so that K is a quadratic extension of k. Let π_K be a uniformizer for K, and let ψ_K be a nontrivial Galois invariant additive character of K. If τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ is infinite dimensional, let $K(\tau,\psi_K)$ be the Kirillov model of τ with respect to ψ_K. Let τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ be infinite dimensional. For g in $\mathrm{Gl}(2,K)$, ζ a quasi-character of $K^\times$, f in $K(\tau,\psi_K)$, and s in $\mathbb{C}$, let

$$Z(g,\zeta,f,s) = \int_{K^\times} \tau(g)f(x)\zeta(x)|x|^{s-1/2}\,dx.$$

This integral converges absolutely if $\Re(s)$ is sufficiently large. Moreover, the function defined by the integral for sufficiently large $\Re(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$. There exists a meromorphic function $\gamma(\tau\otimes_{\mathbb{C}}\zeta,s,\psi_K)$ on the complex plane such that

$$\gamma(\tau\otimes_{\mathbb{C}}\zeta,s,\psi_K)Z(g,\zeta,f,s) = Z(\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} g,\zeta^{-1}\omega_\tau^{-1},f,1-s)$$

for g in $\mathrm{Gl}(2,K)$ and f in $K(\tau,\psi_K)$. Let

$$\epsilon(\tau\otimes_{\mathbb{C}}\zeta,s,\psi_K) = \gamma(\tau\otimes_{\mathbb{C}}\zeta,s,\psi_K)\frac{L(\tau\otimes_{\mathbb{C}}\zeta,s)}{L(\tau\otimes_{\mathbb{C}}\omega_\tau^{-1}\zeta,1-s)}.$$

Here the L factors are as in [G]. The function $\epsilon(\tau\otimes_{\mathbb{C}}\zeta,s,\psi_K)$ is entire, and has no zeros. The notation for irreducible principal series and special representations of $\mathrm{Gl}(2,K)$ will be as in [GL]. Let $\pi(\mu_1,\mu_2)$ be a principal series representation of $\mathrm{Gl}(2,K)$. Then $\pi(\mu_1,\mu_2)$ is Galois invariant if and only if $\mu_1\circ - = \mu_1$ and $\mu_2\circ - = \mu_2$, or $\mu_1\circ - = \mu_2$. Let $\sigma(\mu_1,\mu_2)$ be a special representation. Then $\sigma(\mu_1,\mu_2)$ is Galois invariant if and only if $\mu_1\circ - = \mu_1$ and $\mu_2\circ - = \mu_2$.

Lemma 5.1. *Let τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ be infinite dimensional and Galois invariant. Let $\omega_\tau = \chi\circ\mathrm{N}_k^K$ and let ζ be a quasi-character of $K^\times$ whose restriction to $k^\times$ is χ. If τ is not a principal series representation $\pi(\mu_1,\mu_2)$ with μ_1 and μ_2 Galois invariant, then the integral $Z(g,\zeta^{-1},f,1/2)$ is absolutely convergent for all g in $\mathrm{Gl}(2,K)$ and f in $K(\tau,\psi_K)$.*

Proof. The claim follows if τ is supercuspidal. Assume that τ is a principal series representation. Then $\tau=\pi(\mu_1,\mu_2)$ with $\mu_1\circ - = \mu_2$. It suffices to show that for $f\in\mathcal{S}(K)$ the integral

$$\int_{K^\times} |x|^{1/2}\mu_1(x)f(x)\zeta(x)^{-1}\,d^\times x$$

is absolutely convergent. An estimate shows that this integral converges absolutely if

$$|\mu_1(\pi_K)\zeta(\pi_K)^{-1}| < |\pi_K|^{-1/2}.$$

Since $|\mu_1(\pi_K)\zeta(\pi_K)^{-1}|^2 = 1$, our claim follows. Suppose that τ is a special representation. Then $\tau=\sigma(\mu_1,\mu_2)$ with $\mu_1\circ - = \mu_1$, $\mu_2\circ - = \mu_2$ and $\mu_1=\mu_2|\ |$. Again, it suffices to show that the above integral is absolutely convergent. We have $|\mu_1(\pi_K)\zeta(\pi_K)^{-1}| = |\pi_K|^{1/2} < |\pi_K|^{-1/2}$. □

Lemma 5.2. *Let τ, χ and ζ be as in the last lemma. Then $\gamma(\tau\otimes_{\mathbb{C}}\zeta^{-1},s,\psi_K)$ is defined at $1/2$ and*

$$\gamma(\tau\otimes_{\mathbb{C}}\zeta^{-1},1/2,\psi_K) = \epsilon(\tau\otimes_{\mathbb{C}}\zeta^{-1},1/2,\psi_K).$$

Proof. By definition,

$$\epsilon(\tau\otimes_{\mathbb{C}}\zeta^{-1},s,\psi_K)=\gamma(\tau\otimes_{\mathbb{C}}\zeta^{-1},s,\psi_K)\frac{L(\tau\otimes_{\mathbb{C}}\zeta^{-1},s)}{L(\tau\otimes_{\mathbb{C}}\omega_\tau^{-1}\zeta,1-s)}.$$

Since $\epsilon(\tau\otimes_{\mathbb{C}}\zeta^{-1},s,\psi_K)$ is an entire function, and since the L functions are defined at $1/2$ by Lemma 5.1, it suffices to show that

$$\frac{L(\tau\otimes_{\mathbb{C}}\zeta^{-1},1/2)}{L(\tau\otimes_{\mathbb{C}}\omega_\tau^{-1}\zeta,1/2)}=1.$$

If τ is supercuspidal this is clear. Suppose that τ is a principal series representation $\pi(\mu_1,\mu_2)$ with $\mu_1\circ -=\mu_2$. Then

$$\frac{L(\tau\otimes_{\mathbb{C}}\zeta^{-1},1/2)}{L(\tau\otimes_{\mathbb{C}}\omega_\tau^{-1}\zeta,1/2)}=\frac{L(\mu_1\zeta^{-1},1/2)L(\mu_2\zeta^{-1},1/2)}{L(\mu_1^{-1}\zeta,1/2)L(\mu_2^{-1}\zeta,1/2)}.$$

It will suffice to show that $\mu_1(\pi_K)^2=\zeta(\pi_K)^2$ if $\mu_1\zeta^{-1}$ is unramified and $\mu_2(\pi_K)^2=\zeta(\pi_K)^2$ if $\mu_2\zeta^{-1}$ is unramified. By symmetry, it is enough to prove one of these statements. Suppose $\mu_1\zeta^{-1}$ is unramified. If K/k is unramified, then this follows since we can take π_K in $k^\times$, and $\mu_1\mu_2=\zeta\zeta\circ -$ and $\mu_1\circ -=\mu_2$. Suppose that K/k is ramified. Since the residual characteristic of k is odd, we can assume that $\overline{\pi_K}=-\pi_K$ and π_K^2 is a uniformizer of k. Then $\mu_1(\pi_K)^2=\mu_1(-1)\mu_1(\pi_K)\mu_2(\pi_K)=\mu_1(-1)\zeta(\pi_K)\zeta(\overline{\pi_K})=\mu_1(-1)\zeta(-1)\zeta(\pi_K)^2=\zeta(\pi_K)^2$, since $\zeta(-1)=\mu_1(-1)$ because $\mu_1\zeta^{-1}$ is unramified. The case when τ is a special representation is analogous; for details, see the similar case treated in the remark below. □

The last lemma does not hold for all irreducible principal series representations $\pi(\mu_1,\mu_2)$ with μ_1 and μ_2 Galois invariant. Indeed, we claim that if $\tau=\pi(\mu_1,\mu_2)$ is an irreducible principal series representation with μ_1 and μ_2 Galois invariant, then

$$\gamma(\tau\otimes_{\mathbb{C}}\zeta^{-1},1/2,\psi_K)=\epsilon(\tau\otimes_{\mathbb{C}}\zeta^{-1},1/2,\psi_K)$$

unless $\mu_1\zeta^{-1}$ is unramified and $\mu_1(\pi_K)\zeta(\pi_K)^{-1}=|\pi_K|^{-1/2}$ or $\mu_2\zeta^{-1}$ is unramified and $\mu_2(\pi_K)\zeta(\pi_K)^{-1}=|\pi_K|^{-1/2}$; in these last cases,

$$\gamma(\tau\otimes_{\mathbb{C}}\zeta^{-1},1/2,\psi_K)=-\epsilon(\tau\otimes_{\mathbb{C}}\zeta^{-1},1/2,\psi_K).$$

To prove these claims we proceed as in the proof of Lemma 5.2. We need to compute

$$\lim_{s\to 1/2}\frac{L(\tau\otimes_{\mathbb{C}}\zeta^{-1},s)}{L(\tau\otimes_{\mathbb{C}}\omega_\tau^{-1}\zeta,1-s)}=\lim_{s\to 1/2}\frac{L(\mu_1\zeta^{-1},s)L(\mu_2\zeta^{-1},s)}{L(\mu_1^{-1}\zeta,1-s)L(\mu_2^{-1}\zeta,1-s)}.$$

We first show that $\mu_1\zeta^{-1}$ is unramified if and only if $\mu_2\zeta^{-1}$ is. Suppose that $\mu_1\zeta^{-1}$ is unramified. Then $\zeta(u)=\mu_1(u)$ for all $u\in\mathfrak{O}_K^\times$. Since $\mu_1(\ker(\mathrm{N}_k^K))=1$ and $\ker(\mathrm{N}_k^K)$ is contained in $\mathfrak{O}_K^\times$, $\zeta(\ker(\mathrm{N}_k^K))=1$. So, $\zeta\circ -=\zeta$. Now $\mu_1\mu_2=\zeta\zeta\circ -=\zeta^2$. Hence, $\mu_1\zeta^{-1}=(\mu_2\zeta^{-1})^{-1}$, and $\mu_2\zeta^{-1}$ is unramified. The converse follows by symmetry. Note that we also have shown that if $\mu_1\zeta^{-1}$ and $\mu_2\zeta^{-1}$ are unramified, then $\mu_1(\pi_K)\mu_2(\pi_K)=\zeta(\pi_K)^2$, i.e., $\mu_1(\pi_K)\zeta(\pi_K)^{-1}=\mu_2(\pi_K)^{-1}\zeta(\pi_K)$. If now $\mu_1\zeta^{-1}$ and $\mu_2\zeta^{-1}$ are ramified or $\mu_1\zeta^{-1}$ and $\mu_2\zeta^{-1}$ are unramified and $\mu_1(\pi_K)\zeta(\pi_K)^{-1}\neq|\pi_K|^{-1/2}$ and $\mu_2(\pi_K)\zeta(\pi_K)^{-1}\neq|\pi_K|^{-1/2}$, then the limit is 1. Suppose $\mu_1\zeta^{-1}$ and $\mu_2\zeta^{-1}$ are unramified and $\mu_1(\pi_K)\zeta(\pi_K)^{-1}=|\pi_K|^{-1/2}$ or $\mu_2(\pi_K)\zeta(\pi_K)^{-1}=|\pi_K|^{-1/2}$. Then exactly one of $\mu_1(\pi_K)\zeta(\pi_K)^{-1}$ and $\mu_2(\pi_K)\zeta(\pi_K)^{-1}$ is $|\pi_K|^{-1/2}$.

Without loss of generality, we may assume that $\mu_1(\pi_K)\zeta(\pi_K)^{-1} = |\pi_K|^{-1/2}$. Then

$$\lim_{s\to 1/2} \frac{L(\mu_1\zeta^{-1}, s)L(\mu_2\zeta^{-1}, s)}{L(\mu_1^{-1}\zeta, 1-s)L(\mu_2^{-1}\zeta, 1-s)} = \lim_{s\to 1/2} \frac{L(\mu_1\zeta^{-1}, s)}{L(\mu_1\zeta^{-1}, 1-s)} \lim_{s\to 1/2} \frac{L(\mu_2\zeta^{-1}, s)}{L(\mu_2\zeta^{-1}, 1-s)}$$
$$=(-1)\cdot 1 = -1.$$

The proof of Theorem 5.3 that we now give follows essentially from [H] and from [T], as interpreted in [HST]. The previous discussion shows that in Theorem 5.3 it is essential to use ϵ instead of γ factors. Note also that ψ_K differs from the additive character in [H]. There it is assumed that ψ_K is trivial on k.

Proof of Theorem 5.3. Assume first that $\tau \neq \pi(\mu_1, \mu_2)$ with μ_1 and μ_2 Galois invariant.

(1) $\iff$ (2): The equivalence follows from Lemmas 5.1 and 5.2 and an argument essentially as in the proof of Theorem 4.1 of [H].

(2) $\iff$ (3): Since τ is Galois invariant τ and $\tau \neq \pi(\mu_1, \mu_2)$ with μ_1 and μ_2 Galois invariant, τ is the base change of a discrete series representation of $\mathrm{Gl}(2, k)$ that has central character χ or $\chi\omega_{K/k}$. The equivalence of (2) and (3) is 4 of Lemma 14 of [HST].

Now suppose that $\tau = \pi(\mu_1, \mu_2)$ with μ_1 and μ_2 Galois invariant. We will show that (1), (2) and (3) all hold. The statement (2) follows from Lemma 14 of [HST]. To see (3), note that μ_1 and μ_2 factor through N_k^K via, say, μ_1' and μ_2', respectively. By replacing μ_1' by $\omega_{K/k}\mu_1'$, if necessary, we may assume that $\mu_1'\mu_2' = \chi$. Since $\mu_1\mu_2^{-1} \neq |\ |_K^{\pm 1}$ it follows that $\mu_1'\mu_2'^{-1} \neq |\ |_k^{\pm 1}$. It follows that $\pi(\mu_1', \mu_2')$ is defined, and the base change of $\pi(\mu_1', \mu_2')$ is τ. To show (1), we proceed as in Proposition 9 of [F]. Let

$$g_0 = \begin{pmatrix} -\sqrt{d} & \sqrt{d} \\ 1 & 1 \end{pmatrix}$$

and

$$T = \{\begin{pmatrix} a & bd \\ b & a \end{pmatrix} : a, b \in k, a^2 - db^2 \neq 0\}, \quad T' = \{\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} : z \in K^\times\}.$$

Then $g_0 T' g_0^{-1} = T$ and

$$g_0^{-1}\,\mathrm{Gl}(2, k) g_0 = \{\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : a\overline{a} - b\overline{b} \neq 0\}.$$

Define $L : \pi(\mu_1, \mu_2) \to \mathbb{C}$ by

$$L(f) = \int_{T\backslash \mathrm{Gl}(2,k)} f(g_0^{-1}g)\chi(\det(g))^{-1}\, dg.$$

A computation shows that the integrand is well defined. Moreover, one can show that $T\backslash \mathrm{Gl}(2, k)$ has finite measure and that the integrand is bounded, so that the integral converges. Finally, L is nonzero and contained in $\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi\circ\det)$. □

6. The main theorem

In this section we prove the main theorem. Our method for showing that a representation occurs in the theta correspondence is entirely analogous to the global technique of computing a Fourier coefficient of a global theta lift. To explain the analogy, we first give some definitions. Let N_n be the unipotent radical of the Siegel parabolic in $\mathrm{Sp}(n,k)$, i.e., the elements of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

for b in $\mathrm{M}_n(k)$ with ${}^t b = b$. Given β in $\mathrm{M}_n(k)$ with ${}^t\beta = \beta$ we can define a character ψ_β of N_n by

$$\psi_\beta \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \psi(\frac{1}{2}\operatorname{tr}(b\beta)).$$

Here, ψ is our fixed nontrivial character of k. Let u be in X^n. We say that u is **nondegenerate** if the components of u generate a nondegenerate subspace of X, or, equivalently, if $\det(u_i, u_j) \neq 0$. Let u be nondegenerate. The stabilizer of u in $\mathrm{O}(X)$ can be identified with $\mathrm{O}(U)$, where U is the orthogonal complement to the subspace generated by the components of u.

With this notation we can explain our method and its analogy to the global computation of a Fourier coefficient. Suppose for the moment k is a number field, and σ_1 is an irreducible cuspidal automorphic representation of $\mathrm{O}(X(\mathbb{A}))$. Let f be in σ_1, let φ be in $\mathcal{S}(X(\mathbb{A})^n)$, and let F be the theta lift of f to $\mathrm{Sp}(n,\mathbb{A})$ with respect to φ. Let u in $X(k)^n$ be nondegenerate, and set $\beta = (u_i, u_j)$. The Fourier coefficient of F with respect to β is the function

$$F_\beta(g) = \int_{N_n(k)\backslash N_n(\mathbb{A})} F(ng)\overline{\psi_\beta(n)}\, dn.$$

A computation shows that this is

$$\int_{\mathrm{O}(U(\mathbb{A}))\backslash\, \mathrm{O}(X(\mathbb{A}))} \omega_n(g,h)\varphi(u) \int_{\mathrm{O}(U(k))\backslash\, \mathrm{O}(U(\mathbb{A}))} f(h'h)\, dh'\, dh.$$

In conclusion, we find that the theta lift $\Theta(\sigma_1)$ of σ_1 to $\mathrm{Sp}(n,\mathbb{A})$ has a nonzero Fourier coefficient with respect to β if and only if

$$\int_{\mathrm{O}(U(k))\backslash\, \mathrm{O}(U(\mathbb{A}))} f(h)\, dh \neq 0$$

for some f in σ_1. Assume again that k is a local field, as before. Then this statement has a local analogue. The local analogue of the global theta lift of σ_1 to $\mathrm{Sp}(n,\mathbb{A})$ is the unique smooth representation $\Theta(\sigma_1)$ of $\mathrm{Sp}(n,k)$ such that

$$\omega_n(\sigma_1) \cong \Theta(\sigma_1) \otimes \sigma_1$$

as $\mathrm{Sp}(n,k) \times \mathrm{O}(X)$ representations; here, $\omega_n(\sigma_1)$ is the quotient of ω_n by the intersection of all the kernels of the elements of $\mathrm{Hom}_{\mathrm{O}(X)}(\omega_n, \sigma_1)$. See [R] for a discussion. In particular, $\Theta(\sigma_1)$ is nonzero if and only if σ_1 is in $\mathcal{R}_n(\mathrm{O}(X))$, and if σ_1 is in $\mathcal{R}_n(\mathrm{O}(X))$, then $\Theta(\sigma_1)$ has a unique nonzero irreducible quotient, which is $\theta(\sigma_1)$.

Lemma 6.1. *Let σ_1 be in $\mathrm{Irr}(\mathrm{O}(X))$, and let u in X^n be nondegenerate. Let $\beta = (u_i, u_j)$. Then*

$$\mathrm{Hom}_{N_n}(\Theta(\sigma_1), \psi_\beta) \cong \mathrm{Hom}_{\mathrm{O}(U)}(\sigma_1^\vee, \mathbf{1})$$

as $\mathbb{C}$ vector spaces.

Proof. By Frobenius reciprocity, as in 2.29 of [BZ],

$$\begin{aligned}\mathrm{Hom}_{\mathrm{O}(U)}(\sigma_1^\vee, \mathbf{1}) &\cong \mathrm{Hom}_{\mathrm{O}(U)}(\mathbf{1}, (\sigma_1^\vee)|_{\mathrm{O}(U)}^\vee) \\ &\cong \mathrm{Hom}_{\mathrm{O}(X)}(\text{c-Ind}_{\mathrm{O}(U)}^{\mathrm{O}(X)} \mathbf{1}, \sigma_1) \\ &\cong \mathrm{Hom}_{\mathrm{O}(X)}(\mathcal{S}(\mathrm{O}(X) \cdot u), \sigma_1).\end{aligned}$$

Here, the last statement follows by 1.6 of [BZ]. By Lemma 2.3 of [KR], there is an $\mathrm{O}(X)$ isomorphism

$$(\omega_n)_{N_n,\psi_\beta} \cong \mathcal{S}(\mathrm{O}(X) \cdot u),$$

where $(\omega_n)_{N_n,\psi_\beta}$ is the Jacquet module of ω_n with respect to N_n and ψ_β. We thus obtain

$$\begin{aligned}\mathrm{Hom}_{\mathrm{O}(U)}(\sigma_1^\vee, \mathbf{1}) &\cong \mathrm{Hom}_{\mathrm{O}(X)}((\omega_n)_{N_n,\psi_\beta}, \sigma_1) \\ &\cong \mathrm{Hom}_{N_n \times \mathrm{O}(X)}((\omega_n)_{N_n,\psi_\beta}, \psi_\beta \otimes \sigma_1) \\ &\cong \mathrm{Hom}_{N_n \times \mathrm{O}(X)}(\omega_n, \psi_\beta \otimes \sigma_1) \\ &\cong \mathrm{Hom}_{N_n \times \mathrm{O}(X)}(\omega_n(\sigma_1), \psi_\beta \otimes \sigma_1) \\ &\cong \mathrm{Hom}_{N_n \times \mathrm{O}(X)}(\Theta(\sigma_1) \otimes \sigma_1, \psi_\beta \otimes \sigma_1) \\ &\cong \mathrm{Hom}_{N_n}(\Theta(\sigma_1), \psi_\beta).\end{aligned}$$

□

Corollary 6.2. *Suppose that σ is in $\mathrm{Irr}(\mathrm{GO}(X))$ and u in X^n is nondegenerate. If*

$$\mathrm{Hom}_{\mathrm{O}(U)}(\sigma^\vee, \mathbf{1}) \neq 0,$$

then σ is in $\mathcal{R}_n(\mathrm{GO}(X))$.

Proof. Let σ_1 in $\mathrm{Irr}(\mathrm{O}(X))$ be an irreducible constituent of $\sigma|_{\mathrm{O}(X)}$ such that

$$\mathrm{Hom}_{\mathrm{O}(U)}(\sigma_1^\vee, \mathbf{1}) \neq 0.$$

By Lemma 6.1, we have in particular, $\Theta(\sigma_1) \neq 0$. This implies that σ_1 is in $\mathcal{R}_n(\mathrm{O}(X))$, and hence σ is in $\mathcal{R}_n(\mathrm{GO}(X))$. □

Lemma 6.3. *Suppose that π is in $\mathrm{Irr}(\mathrm{GSO}(X))$, and u in X^n is nondegenerate. Assume that $n = 1$, 2 or 3. Then*

$$\mathrm{Hom}_{\mathrm{SO}(U)}(\pi^\vee, \mathbf{1}) \neq 0 \implies \mathrm{Hom}_{\mathrm{SO}(X)}(\omega_n, \pi) \neq 0.$$

Proof. Let $\beta = (u_i, u_j)$. Since $n = 1$, 2 or 3, it follows that $\mathrm{SO}(X) \cdot u = \mathrm{O}(X) \cdot u$. Hence by an argument as in the proof of Lemma 6.1,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SO}(U)}(\pi^\vee, \mathbf{1}) &\cong \mathrm{Hom}_{\mathrm{SO}(X)}(\mathcal{S}(\mathrm{O}(X) \cdot u), \pi) \\ &\cong \mathrm{Hom}_{\mathrm{SO}(X)}((\omega_n)_{N_n,\psi_\beta}, \pi) \\ &\cong \mathrm{Hom}_{N_n \times \mathrm{SO}(X)}(\omega_n, \psi_\beta \otimes \pi).\end{aligned}$$

This implies the lemma. □

Lemma 6.4. *Let π be in $\mathrm{Irr}(\mathrm{GSO}(X))$. If π is generically distinguished, then π^+ is in $\mathcal{R}_1(\mathrm{GO}(X))$.*

Proof. By Proposition 4.1 and Theorem 5.3, $\pi^\vee$ is also generically distinguished. Hence, $\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi^\vee, \mathbf{1}) \neq 0$. By Lemma 6.3 we have $\mathrm{Hom}_{\mathrm{SO}(X)}(\omega_n, \pi) \neq 0$. By Theorem 4.4, π^+ is in $\mathcal{R}_1(\mathrm{GO}(X))$. □

Lemma 6.5. *Suppose that $d \neq 1$. Let π in $\mathrm{Irr}(\mathrm{GSO}(X))$ be one dimensional. If π is regular, then π^+ is in $\mathcal{R}_2(\mathrm{GO}(X))$. If π is invariant, then π^+ is in $\mathcal{R}_1(\mathrm{GO}(X))$ and π^- is not in $\mathcal{R}_1(\mathrm{GO}(X))$ and $\mathcal{R}_2(\mathrm{GO}(X))$.*

Proof. Suppose first that $\pi = \pi(\beta, \chi)$ is regular. By Theorem 4.4, it suffices to show that there is a nonzero $\mathrm{SO}(X)$ map from ω_2 to π. To prove this we show first that there exist generically distinguished elements π_1 and π_2 of $\mathrm{Irr}(\mathrm{GSO}(X))$ such that $\mathrm{Hom}_{\mathrm{GSO}(X)}(\pi_1 \otimes_{\mathbb{C}} \pi_2, \pi) \neq 0$. By Proposition 3.2, there exists a nontrivial quadratic character μ different from $\omega_{K/k}$ such that $\chi = \beta|_{k^\times}\mu$. Since $\beta^2 = \chi \circ \mathrm{N}_k^K$, we have $\beta(\beta \circ -)^{-1} = \mu \circ \mathrm{N}_k^K$. Let $\pi_1 = \pi(\pi(\mathbf{1}, \mu \circ \mathrm{N}_k^K), \mu)$ and $\pi_2 = \pi(\pi(\beta, \beta \circ -), \beta|_{k^\times})$. Then π_1 and π_2 are generically distinguished. We have $\pi_2 \otimes_{\mathbb{C}} \pi^{-1} = \pi_1^\vee$, which proves our claim.

By Lemma 6.4, π_1^+ and π_2^+ are contained in $\mathcal{R}_1(\mathrm{GO}(X))$. It follows that each irreducible constituent of $\pi_1|_{\mathrm{SO}(X)}$ and $\pi_2|_{\mathrm{SO}(X)}$ is a nonzero quotient of $\omega_1|_{\mathrm{SO}(X)}$. Since $\omega_2 \cong \omega_1 \otimes_{\mathbb{C}} \omega_1$ as representations of $\mathrm{O}(X)$, by tensoring and composing we obtain a nonzero $\mathrm{SO}(X)$ map from ω_2 to π.

Now suppose that π is invariant. The proof that $\sigma = \pi^+$ is in $\mathcal{R}_1(\mathrm{GO}(X))$ has several steps. We first claim that $\mathbf{1}$ is contained in $\mathcal{R}_X(\mathrm{Sl}(2,k))$. To see this, note that $\pi(\omega_{K/k}, \mathbf{1})|_{\mathrm{Sl}(2,k)}$ has an irreducible component π' such that π' is in $\mathcal{R}_{\mathbb{H}}(\mathrm{Sl}(2,k))$ and $\pi'^\vee$ is in $\mathcal{R}_{(K,\mathrm{N}_k^K)}(\mathrm{Sl}(2,k))$; here, $\pi(\omega_{K/k}, \mathbf{1})$ is the irreducible principal series representation of $\mathrm{Gl}(2,k)$. See, for example, our summary in section 7 of the theta correspondence for $\mathrm{Sl}(2,k)$ and $\mathrm{O}(V)$, when V is two dimensional. Since $X = \mathbb{H} \perp (K, \mathrm{N}_k^K)$, it follows that $\omega_1 \cong \omega_{\mathbb{H}} \otimes_{\mathbb{C}} \omega_{(K,\mathrm{N}_k^K)}$ as representations of $\mathrm{Sl}(2,k)$. Hence, there is a nonzero $\mathrm{Sl}(2,k)$ map of ω_1 onto $\mathbf{1}$.

Using that $\mathbf{1}$ is in $\mathcal{R}_X(\mathrm{Sl}(2,k))$, we will show that $\pi(\mathbf{1}, \omega_{K/k})^+$ is in $\mathcal{R}_1(\mathrm{GO}(X))$. It is a straightforward exercise using Corollary 2.6 of [K] to verify that $\theta(\mathbf{1})$ in $\mathrm{Irr}(\mathrm{O}(X))$ is an irreducible subquotient of $\pi(\varrho(|\ |_K^{1/2}, |\ |_K^{-1/2}), \omega_{K/k})'|_{\mathrm{O}(X)}$, where $\pi(\varrho(|\ |_K^{1/2}, |\ |_K^{-1/2}), \omega_{K/k})'$ is an extension of $\pi(\varrho(|\ |_K^{1/2}, |\ |_K^{-1/2}), \omega_{K/k})$ to $\mathrm{GO}(X)$; here $\rho(|\ |_K^{1/2}, |\ |_K^{-1/2})$ is the representation of $\mathrm{Gl}(2,K)$ induced from $|\ |_K^{1/2}$ and $|\ |_K^{-1/2}$. The notation is as in [GL]. Hence, $\theta(\mathbf{1})$ is either an irreducible constituent of $\pi(\sigma(|\ |_K^{1/2}, |\ |_K^{-1/2}), \omega_{K/k})^\pm|_{\mathrm{O}(X)}$ or $\theta(\mathbf{1}) = \pi(\mathbf{1}, \omega_{K/k})^\pm|_{\mathrm{O}(X)}$. The first possibility contradicts either our summary in section 7 of M. Cognet's results [Co] or Theorem 4.4. It follows that $\theta(\mathbf{1}) = \pi(\mathbf{1}, \omega_{K/k})^\pm|_{\mathrm{O}(X)}$. Assume $\theta(\mathbf{1}) = \pi(\mathbf{1}, \omega_{K/k})^-|_{\mathrm{O}(X)}$. Using that there is a nonzero $\mathrm{O}(X)$ map from ω_1 to $\pi(\pi(\mathbf{1}, \mathbf{1}), \omega_{K/k})^+|_{\mathrm{O}(X)}$ from Lemma 6.4 we obtain a nonzero $\mathrm{O}(X)$ map from $\omega_2 \cong \omega_1 \otimes_{\mathbb{C}} \omega_1$ to

$$\pi(\pi(\mathbf{1}, \mathbf{1}), \omega_{K/k})^+ \otimes_{\mathbb{C}} \pi(\mathbf{1}, \omega_{K/k})^-.$$

Making use of the explicit construction of the nonzero $\mathrm{SO}(Y)$ invariant functional on $\pi(\pi(\mathbf{1}, \mathbf{1}), \omega_{K/k})$, one can verify that

$$\pi(\pi(\mathbf{1}, \mathbf{1}), \omega_{K/k})^+ \otimes_{\mathbb{C}} \pi(\mathbf{1}, \omega_{K/k})^- = \pi(\pi(\mathbf{1}, \mathbf{1}), \mathbf{1})^-,$$

so that $\pi(\pi(\mathbf{1}, \mathbf{1}), \mathbf{1})^-$ is in $\mathcal{R}_2(\mathrm{GO}(X))$. This contradicts Theorem 4.3. Hence, $\theta(\mathbf{1}) = \pi(\mathbf{1}, \omega_{K/k})^+|_{\mathrm{O}(X)}$, and $\pi(\mathbf{1}, \omega_{K/k})^+$ is in $\mathcal{R}_1(\mathrm{GO}(X))$.

To see now that for an arbitrary one dimensional, invariant, but not generically distinguished $\pi(\alpha \circ \mathrm{N}_k^K, \chi)$, we have $\pi(\alpha \circ \mathrm{N}_k^K\ \chi)^+$ in $\mathcal{R}_1(\mathrm{GO}(X))$, note that

$$\pi(\alpha \circ \mathrm{N}_K^k, \chi)^+ = \pi(\mathbf{1}, \omega_{K/k})^+ \otimes_{\mathbb{C}} (\alpha \circ \lambda).$$

Here, λ is the similitude factor map from Section 1. Since

$$\mathrm{Hom}_R(\omega_1, \theta(\pi(\mathbf{1}, \omega_{K/k})^+) \otimes_{\mathbb{C}} \pi(\mathbf{1}, \omega_{K/k})^+) \neq 0,$$

it follows that

$$\mathrm{Hom}_R(\omega_1, (\theta(\pi(\mathbf{1}, \omega_{K/k})^+) \otimes_{\mathbb{C}} (\alpha^{-1} \circ \lambda)) \otimes_{\mathbb{C}} \pi(\alpha \circ \mathrm{N}_k^K, \chi)^+) \neq 0.$$

Hence, $\pi(\alpha \circ \mathrm{N}_k^K, \chi)^+$ is in $\mathcal{R}_1(\mathrm{GO}(X))$.

Finally, we show that π^- is not in $\mathcal{R}_n(\mathrm{GO}(X))$ for $n = 1$ and 2. Suppose that π^- is in $\mathcal{R}_n(\mathrm{GO}(X))$ for $n = 1$ or 2. From above, $\pi(\alpha^{-1} \circ \mathrm{N}_k^K, \chi^{-1})^+$ is in $\mathcal{R}_1(\mathrm{GO}(X))$. Since $\omega_{n+1} = \omega_n \otimes_{\mathbb{C}} \omega_1$ as representations of $\mathrm{O}(X)$,

$$\pi(\alpha \circ \mathrm{N}_k^K, \chi)^- \otimes_{\mathbb{C}} \pi(\alpha^{-1}, \circ \mathrm{N}_k^K, \chi^{-1})^+ = \mathrm{sign}$$

is in $\mathcal{R}_{n+1}(\mathrm{GO}(X))$. This contradicts the appendix of [Ra], since $n + 1 < 4$. □

Lemma 6.6. *Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X))$. In the case $d \neq 1$ assume that π is infinite dimensional. Then there exists a nondegenerate z in X^2 with stabilizer $\mathrm{SO}(Z)$ in $\mathrm{SO}(X)$ such that*

$$\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi, \mathbf{1}) \neq 0.$$

Proof. Suppose first $d = 1$ and $\epsilon = \epsilon(1)$. Let $\pi = \pi(\tau, \tau')$. Suppose that τ and τ' are infinite dimensional. Let

$$z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then z is nondegenerate, and the stabilizer of z is

$$\mathrm{SO}(Z) = \{\rho(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}) : a \in k^\times\}.$$

We will use the Kirillov models $K(\tau, \psi)$ and $K(\tau', \psi)$ of τ and τ' with respect to our additive character ψ, respectively. Let n be so large that

$$\int_{k^\times} f(x)|x|^n \, dx$$

converges absolutely for f in $K(\tau, \psi)$ and f in $K(\tau', \psi)$. Define $L : \pi \to \mathbf{1}$ by

$$L(f \otimes f') = \int_{k^\times} f(x)|x|^n \, dx \cdot \int_{k^\times} f'(x)|x|^n \, dx.$$

Then L is a well defined nonzero $\mathbb{C}$ linear map, and L is $\mathrm{SO}(Z)$ invariant.

Suppose next that exactly one of τ and τ', say τ, is infinite dimensional. Since τ' is finite dimensional, τ' is one dimensional, and there exists a quasi-character β' of $k^\times$ such that $\tau' = \beta' \circ \det$. By hypothesis, $\beta'^2 = \omega_{\tau'} = \omega_\tau$. Suppose that τ is a supercuspidal or special representation. We claim that

$$\int_{k^\times} \beta'(x)^{-1} f(x) \, dx$$

converges absolutely for f in $K(\tau, \psi)$. This is clear if τ is supercuspidal. If τ is the special representation $\sigma(\mu_1, \mu_2)$ with $\mu_1 = \mu_2| \ |$, then this follows from the estimate $|\pi_k|^{1/2} |\beta'(\pi_k)|^{-1} |\mu_1(\pi_k)| = |\pi_k| < 1$. Now define $L : \pi \to \mathbf{1}$ by

$$L(f \otimes z) = z \int_{k^\times} \beta'(x)^{-1} f(x) \, dx.$$

Then L is a nonzero element of $\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi, \mathbf{1})$. Suppose that τ is a principal series representation. In this case, we require another nondegenerate element of X^2. Every quadratic extension E of k is contained in $\mathrm{M}_2(k)$ as a k algebra, and for every quadratic extension E of k contained in $X = \mathrm{M}_2(k)$, $\mathrm{Gal}(E/k) = \{1, *\}$, and there exists a nondegenerate z in X^2 such that $\mathrm{SO}(Z) = \{\rho(x, x^{*-1}) : x \in E^\times\}$. Fix a quadratic extension E of k in X and such a z in X^2. Let α be the quasi-character of $E^\times$ defined by $\alpha(x) = \beta'(\det(x))$. Then α extends ω_τ. By [T], we have $\mathrm{Hom}_{E^\times}(\tau, \alpha) \neq 0$ if and only if $\epsilon(\mathrm{BC}_{E/k}(\tau) \otimes_{\mathbb{C}} \alpha^{-1}, 1/2, \psi_E) = \omega_\tau(-1)$. By Lemma 14 of [HST], $\epsilon(\mathrm{BC}_{E/k}(\tau) \otimes_{\mathbb{C}} \alpha^{-1}, 1/2, \psi_E) = \omega_\tau(-1)$, so that $\mathrm{Hom}_{E^\times}(\tau, \beta' \circ \det) \neq 0$. Let f in $\mathrm{Hom}_{E^\times}(\tau, \beta' \circ \det)$ be nonzero. Define $L : \pi \to \mathbf{1}$ by $L(v \otimes z) = z f(v)$. Then L is a nonzero element of $\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi, \mathbf{1})$.

Suppose that τ and τ' are both finite dimensional, i.e., one dimensional. Let β and β' be quasi-characters of $k^\times$ such that $\tau = \beta \circ \det$ and $\tau' = \beta' \circ \det$. Since $\omega_\tau = \omega_{\tau'}$, we have $\beta^2 = \beta'^2$. This implies that $\beta = \beta'$ or $\beta = \omega_{E/k}\beta'$ for some quadratic extension E of k, since the residual characteristic of k is odd. Let E be contained in X and let z in X^2 be as above. Since $\det(x) = \mathrm{N}_k^E(x)$ for x in $E^\times$, it follows that $\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi, \mathbf{1}) \neq 0$.

Now suppose $d = 1$ and $\epsilon = -\epsilon(1)$. Since $\mathrm{SO}(X)$ is compact, it will suffice to show that there exists nonzero v in π and nondegenerate z in X^2 such that $\pi(h)v = v$ for h in $\mathrm{SO}(Z)$. Since for every quadratic extension E of k we have again that E is contained in D as a k algebra, $\mathrm{Gal}(E/k) = \{1, *\}$, and there exists a nondegenerate z in X^2 such that $\mathrm{SO}(Z) = \{\rho(x, x^{*-1}) : x \in E^\times\}$, to prove the existence of the required v and z it will suffice to show that there exists a quadratic extension E of k contained in D, a quasi-character ϕ of $E^\times$, and nonzero vectors w in τ and w' in τ' such that $\tau(x)w = \phi(x)w$ and $\tau'(x)w' = \phi(x^*)w'$ for x in $E^\times$.

If τ and τ' are one dimensional, then an argument as in the case $\epsilon = \epsilon(1)$ works.

Suppose $\dim \tau > 1$ and $\dim \tau' > 1$. We will use terminology and results from [T]. We first assert that we can assume that τ and τ' are minimal. To see this, let $\alpha = \omega_\tau = \omega_{\tau'}$. Consider $\alpha|_{1+\pi_k \mathfrak{O}_k}$. For some large n, we can regard α as a character of $1 + \pi_k \mathfrak{O}_k / 1 + \pi_k^n \mathfrak{O}_k$. This is a finite group of odd order. It follows that squaring is an automorphism of the group of characters of this group. Hence, there exists a quasi-character η of $k^\times$ such that $\eta^2 = \alpha$ on $1 + \pi_k \mathfrak{O}_k$. Consider $\tau \otimes_{\mathbb{C}} \eta^{-1}$ and $\tau' \otimes_{\mathbb{C}} \eta^{-1}$. The common central character of these representations has conductor less than or equal to 1. Since any element of $\mathrm{Irr}(D^\times)$ of dimension larger than 1 with central character of conductor less than or equal to 1 is minimal, $\tau \otimes_{\mathbb{C}} \eta^{-1}$ and $\tau' \otimes_{\mathbb{C}} \eta^{-1}$ are minimal. Since our claim holds for $\tau \otimes_{\mathbb{C}} \eta^{-1}$ and $\tau' \otimes_{\mathbb{C}} \eta^{-1}$ if and only if it holds for τ and τ', we may assume that τ and τ' are minimal.

Let $\mathrm{JL}(\tau)$ and $\mathrm{JL}(\tau')$ be the representations corresponding to τ and τ' under the Jacquet-Langlands correspondence, respectively. Since $\dim \tau > 1$ and $\dim \tau' > 1$, these representations are supercuspidal. Let $a(\mathrm{JL}(\tau))$ and $a(\mathrm{JL}(\tau'))$ be the conductors of $\mathrm{JL}(\tau)$ and $\mathrm{JL}(\tau')$, respectively. Without loss of generality, we may assume that $\dim(\tau) \geq \dim(\tau')$. Using the formulas for $\dim \tau$ and $\dim \tau'$ in terms of $a(\mathrm{JL}(\tau))$ and $a(\mathrm{JL}(\tau'))$, respectively, one can show that $a(\mathrm{JL}(\tau)) \geq a(\mathrm{JL}(\tau'))$. Note that the formula in [T] for $\dim \tau$ when $a(\mathrm{JL}(\tau))$ is odd appears incorrectly: it should be $(q+1)q^{(c-3)/2}$ instead of $(q+1)^{(c-3)/2}$. Let E be a quadratic extension of k whose ramification index e has the same parity as $a(\mathrm{JL}(\tau))$. Let S be the set of all quasi-characters of $E^\times$ whose conductors are less than or equal to $e(a(\mathrm{JL}(\tau)) - 1)/2$ and which extend α, and let S' be the set of all quasi-characters of $E^\times$ whose conductors are less than or equal to $[e(a(\mathrm{JL}(\tau')) - 1)/2 + 1/2]$ and which extend α. Since $a(\mathrm{JL}(\tau)) \geq a(\mathrm{JL}(\tau'))$ we have $S' \subset S$. By the proof of Lemma 3.2 of [T], $\tau|_{E^\times}$ is the direct sum of the elements of S. By the proof of Lemma 3.1 of [T] every quasi-character of $E^\times$ that occurs in $\tau'|_{E^\times}$ is contained in S'. It follows that there exists a quasi-character ϕ of $E^\times$ that occurs in $\tau|_{E^\times}$ and $\tau'|_{E^\times}$. Since the conductor of $\phi \circ *$ is the same as the conductor of ϕ, it follows that $\phi \circ *$ also occurs in $\tau|_{E^\times}$, which proves our claim.

The case when, say, $\dim(\tau) > 1$ and $\dim(\tau') = 1$ remains. Let $\tau' = \beta' \circ \mathrm{N}$. Then $\beta'^2 = \alpha$. It follows that the common central character of $\tau \otimes_{\mathbb{C}} \beta'^{-1}$ and $\tau' \otimes_{\mathbb{C}} \beta'^{-1} = \mathbf{1}$ is trivial. Thus, we may assume that τ is minimal and $\tau' = \mathbf{1}$. Let S be as in the last paragraph. Since α is trivial, it follows that the trivial character of $E^\times$ lies in S, and so we can take ϕ to be the trivial character of $E^\times$.

Suppose now $d \neq 1$. Let $\pi = \pi(\tau, \chi)$. By assumption, τ is infinite dimensional. Let

$$z = \begin{pmatrix} 0 & \sqrt{d} \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ \sqrt{d} & 0 \end{pmatrix}.$$

A computation shows that

$$\mathrm{SO}(Z) = \{\rho(1, \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}) : u \in \ker(\mathrm{N}_k^K)\}.$$

Since $\mathrm{SO}(Z)$ is compact it will suffice to show that there exists a nonzero vector v in τ such that

$$\tau \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} v = v$$

for u in $\ker(\mathrm{N}_k^K)$. We will use the Kirillov model $K(\tau, \psi_K)$ of τ. Let f be the characteristic function of $\mathfrak{O}_K^\times$. Then f is in $K(\tau, \psi_K)$, and since $\ker(\mathrm{N}_k^K)$ is contained in $\mathfrak{O}_K^\times$, we have $f(ux) = f(x)$ for x in $K^\times$ and u in $\ker(\mathrm{N}_k^K)$. Thus, f is the desired vector. □

Lemma 6.7. *Suppose that $d \neq 1$. Let π be in* $\mathrm{Irr}(\mathrm{GSO}(X))$. *Assume that π is infinite dimensional, invariant, but not distinguished. Let π_1 and π_2 be the two extensions of π to* $\mathrm{GO}(X)$. *Then there exists a nondegenerate z in X^2 with stabilizer* $\mathrm{O}(Z)$ *in* $\mathrm{O}(X)$ *such that*

$$\mathrm{Hom}_{\mathrm{O}(Z)}(\pi_1, \mathbf{1}) \neq 0, \qquad \mathrm{Hom}_{\mathrm{O}(Z)}(\pi_2, \mathbf{1}) \neq 0.$$

Proof. Let $\pi = \pi(\tau, \chi)$. Then τ is infinite dimensional. Let the notation be as in section 5. Let

$$z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix}.$$

Then

$$\mathrm{SO}(Z) = \{\rho(a, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) : a \in k^\times\},$$

and $\mathrm{O}(Z)$ is generated by $\mathrm{SO}(Z)$ and

$$\rho(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) h_0.$$

Since π is invariant, by Proposition 3.1, τ is Galois invariant. From the explicit form of $K(\tau, \psi_K)$ it follows that $K(\tau, \psi_K)$ is invariant under composition by $-$, and a computation shows that $(\tau(g)f)\circ - = \tau(\overline{g})(f\circ -)$. We may assume that $\pi_1(h_0)$ is given by $\pi_1(h_0)f = f\circ -$ and $\pi_2(h_0)$ is given by $\pi_2(h_0)f = -f\circ -$. Since π is not distinguished, by Proposition 4.1 we have that $\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau,\chi\circ\det)=0$. By Theorem 5.3, it follows that τ is not the base change of an element of $\mathrm{Irr}(\mathrm{Gl}(2,k))$ with central character $\chi\omega_{K/k}$. In particular, τ is not $\pi(\mu_1,\mu_2)$ for some Galois invariant quasi-characters μ_1 and μ_2 of $K^\times$. Let ζ be a quasi-character of $K^\times$ that extends χ. By Lemma 5.1,

$$Z(g,\zeta^{-1},f,1/2)=\int_{K^\times}\tau(g)f(x)\zeta(x)^{-1}\,dx$$

converges absolutely for all g in $\mathrm{Gl}(2,K)$ and f in $K(\tau,\psi_K)$. Define $L_\zeta:\pi\to\mathbf{1}$ by

$$L_\zeta(f)=Z(1,\zeta^{-1},f,1/2).$$

Then L_ζ is nonzero, and a computation shows that L_ζ is in $\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi,\mathbf{1})$. Moreover,

$$L_\zeta(\pi_1(\rho(1,\begin{pmatrix}0&1\\-1&0\end{pmatrix})h_0)f)=Z(\begin{pmatrix}0&1\\-1&0\end{pmatrix},\zeta\omega_\tau^{-1},f,1/2),$$

and

$$L_\zeta(\pi_2(\rho(1,\begin{pmatrix}0&1\\-1&0\end{pmatrix})h_0)f)=-Z(\begin{pmatrix}0&1\\-1&0\end{pmatrix},\zeta\omega_\tau^{-1},f,1/2),$$

for f in π. By the local functional equation for τ and Lemma 5.2, we thus have

$$L_\zeta(\pi_1(\rho(1,\begin{pmatrix}0&1\\-1&0\end{pmatrix})h_0)f)=\epsilon(\tau\otimes_{\mathbb{C}}\zeta^{-1},1/2,\psi)L_\zeta(f)$$

and

$$L_\zeta(\pi_2(\rho(1,\begin{pmatrix}0&1\\-1&0\end{pmatrix})h_0)f)=-\epsilon(\tau\otimes_{\mathbb{C}}\zeta^{-1},1/2,\psi)L_\zeta(f)$$

for f in $K(\tau,\chi)$. Since τ is not the base change of an element of $\mathrm{Irr}(\mathrm{Gl}(2,k))$ with central character $\chi\omega_{K/k}$, by Lemma 14 of [HST], there exist quasi-characters ζ and ζ' of $K^\times$ extending χ such that

$$\epsilon(\tau\otimes_{\mathbb{C}}\zeta^{-1},1/2,\psi)=\chi(-1),\qquad \epsilon(\tau\otimes_{\mathbb{C}}\zeta'^{-1},1/2,\psi)=-\chi(-1).$$

This completes the proof. □

Proof of Theorem 6.8. The only if part of the theorem follows from Theorem 4.3 and Lemma 6.5.

To prove the if part of the theorem, let π be in $\mathrm{Irr}(\mathrm{GSO}(X))$. We need to show that if π is regular or π is invariant and distinguished, then π^+ is in $\mathcal{R}_2(\mathrm{GO}(X))$, and if π is invariant but not distinguished, then both extensions of π to $\mathrm{GO}(X)$ lie in $\mathcal{R}_2(\mathrm{GO}(X))$.

Suppose $d=1$. Then this follows from Lemma 6.6, Lemma 6.3 and Theorem 4.4.

Suppose now $d\neq 1$. Suppose first π is regular or invariant and distinguished. If π is one dimensional, then we use Lemma 6.5. If π is infinite dimensional, then an argument as in the case $d=1$ is sufficient. Finally, suppose π is invariant but not distinguished. By definition, π is infinite dimensional. By Lemma 6.7 and Corollary 6.2, the extensions π_1 and π_2 of π to $\mathrm{GO}(X)$ are in $\mathcal{R}_2(\mathrm{GO}(X))$. □

Proof of Corollary 6.9. Let σ_1 be in $\mathrm{Irr}(\mathrm{O}(X))$. Suppose that σ_1 is in $\mathcal{R}_2(\mathrm{O}(X))$. Suppose that σ_1 is an irreducible constituent of $\pi^-|_{\mathrm{O}(X)}$ for some distinguished π in $\mathrm{Irr}(\mathrm{GSO}(X))$. Then by definition, π^- is in $\mathcal{R}_2(\mathrm{GO}(X))$, contradicting Theorem 6.8. Suppose that σ_1 is not an irreducible constituent of $\pi^-|_{\mathrm{O}(X)}$ for all distinguished π in $\mathrm{Irr}(\mathrm{GSO}(X))$. By [GK], there exists a σ in $\mathrm{Irr}(\mathrm{GO}(X))$ such that $\sigma|_{\mathrm{O}(X)}$ has σ_1 as an irreducible constituent. By Theorem 6.8, σ is in $\mathcal{R}_2(\mathrm{GO}(X))$. By Lemma 4.2 of [R], σ_1 is in $\mathcal{R}_2(\mathrm{O}(X))$. □

7. A case of a conjecture of Kudla

S.S. Kudla has made some important conjectures about the first occurrence of a representation in the theta correspondence. In the introduction to the paper we described one of S.S. Kudla's conjectures. This conjecture is known to be true when $\dim_k X = 0$ and 2. We will prove this conjecture when $\dim_k X = 4$. We will also show that the isotropy assumption in the definition of a generically distinguished representation in the case $d \neq 1$ is unnecessary. This will be a consequence of a new proof of a result of J. Hakim and D. Prasad on distinguished representations. Tables summarizing the main results of this paper appear at the end of this section.

There is also a conjecture of S.S. Kudla for elements of $\mathrm{Irr}(\mathrm{Sp}(n,k))$ which we will give for completeness; we will partially prove and use a case of the conjecture. To state this conjecture we need some more notation. Fix d in $k^\times/k^{\times 2}$. Then there are, up to equivalence, exactly two anisotropic even dimensional symmetric bilinear spaces X_+ and X_- of discriminant d. From X_+ and X_- we can create two series of even dimensional symmetric bilinear spaces by adding hyperbolic planes to X_+ and X_-. For π in $\mathrm{Irr}(\mathrm{Sp}(n,k))$, let $m_+(\pi)$ be the smallest nonnegative even integer m such that π occurs in the theta correspondence with the m dimensional space with anisotropic component X_+; define $m_-(\pi)$ similarly.

Conjecture 7.2 (S.S. Kudla). *If π is in $\mathrm{Irr}(\mathrm{Sp}(n,k))$, then*

$$m_+(\pi) + m_-(\pi) = 4n + 4.$$

Recently, S.S. Kudla and S. Rallis have announced a proof of the equality of this conjecture for a set of representations π in $\mathrm{Irr}(\mathrm{Sp}(n,k))$ that includes all the supercuspidal representations. One can make completely analogous definitions and conjectures for the theta correspondence for similitudes.

The following lemma will used in the proof of Theorem 7.4.

Lemma 7.3. *Let π be in $\mathrm{Irr}(\mathrm{Sp}(1,k)) = \mathrm{Irr}(\mathrm{Sl}(2,k))$. Then*

$$m_+(\pi) + m_-(\pi) \geq 8.$$

Proof. Suppose that $m_+(\pi)+m_-(\pi) < 8$ for some π in $\mathrm{Irr}(\mathrm{Sl}(2,k))$. We will obtain a contradiction. Let X_1 and X_2 be the symmetric bilinear spaces of dimension $m_+(\pi)$ and $m_-(\pi)$ with anisotropic components X_+ and X_-, respectively. By hypothesis, we have surjective $\mathrm{Sl}(2,k)$ maps of ω_{X_1} and ω_{X_2} onto π. Let g_0 in $\mathrm{Gl}(2,k)$ be such that $\det(g_0) = -1$. By the theorem on page 91 of [MVW], $g_0 \cdot \pi \cong \pi^\vee$. It follows that there is a surjective $\mathrm{Sl}(2,k)$ map of $g_0 \cdot \omega_{X_2}$ onto $\pi^\vee$. Now $g_0 \cdot \omega_{X_2} \cong \omega_{-X_2}$ as representations of $\mathrm{Sl}(2,k)$, where $-X_2$ is the symmetric bilinear space with the same underlying vector space as X_2, but with bilinear form multiplied by -1. Since $\omega_{X_1} \otimes_{\mathbb{C}} \omega_{-X_2} \cong \omega_{X_1 \perp -X_2}$ as representations of $\mathrm{Sl}(2,k)$, and since there is a nonzero $\mathrm{Sl}(2,k)$ map from $\pi \otimes_{\mathbb{C}} \pi^\vee$ to $\mathbf{1}$, it follows that

$$\mathrm{Hom}_{\mathrm{Sl}(2,k)}(\omega_{X_1 \perp -X_2}, \mathbf{1}) \neq 0.$$

Now the anisotropic component of $X_1 \perp -X_2$ is the four dimensional anisotropic symmetric bilinear space $X_a = X(1, -\epsilon(1))$. Since the dimension of $X_1 \perp -X_2$ is less than 8, $X_1 \perp -X_2$ is X_a or $X_a \perp \mathbb{H}$. Suppose $X_1 \perp -X_2 \cong X_a$. Then $\mathbf{1}$ is in $\mathcal{R}_{X_a}(\mathrm{Sl}(2,k))$. This is a contradiction, since the restriction of ω_{X_a} to $\mathrm{Sl}(2,k)$ is the direct sum of square-integrable representations of $\mathrm{Sl}(2,k)$, and hence $\mathcal{R}_{X_a}(\mathrm{Sl}(2,k))$ contains only square-integrable representations. See the discussion in [G], for example. Suppose $X_1 \perp -X_2 \cong X_a \perp \mathbb{H}$. Then $\omega_{X_1 \perp -X_2} \cong \omega_{X_a} \otimes_{\mathbb{C}} \omega_{\mathbb{H}}$ as representations of $\mathrm{Sl}(2,k)$. By the just mentioned fact concerning the decomposition of the restriction of ω_{X_a} to $\mathrm{Sl}(2,k)$, there exists a square-integrable representation τ in $\mathrm{Irr}(\mathrm{Sl}(2,k))$ such that

$$\mathrm{Hom}_{\mathrm{Sl}(2,k)}(\tau \otimes_{\mathbb{C}} \omega_{\mathbb{H}}, \mathbf{1}) \neq 0.$$

This implies that there is a nonzero $\mathrm{Sl}(2,k)$ map from $\omega_{\mathbb{H}}$ to $\tau^\vee$, so that $\tau^\vee$ is in $\mathcal{R}_{\mathbb{H}}(\mathrm{Sl}(2,k))$. This is a contradiction, since $\mathcal{R}_{\mathbb{H}}(\mathrm{Sl}(2,k))$ contains no square-integrable representations. See, for example, the summary following the lemma. □

Suppose X is again as in defined in Section 2. To prove Conjecture 7.1 in this case, we need to understand $\mathcal{R}_1(\mathrm{GO}(X))$ and $\mathcal{R}_3(\mathrm{GO}(X))$. To characterize $\mathcal{R}_1(\mathrm{GO}(X))$ we need to recall some facts about the theta correspondence when the dimension of the underlying bilinear spaces is two and about the theta correspondence between $\mathrm{Irr}(\mathrm{GO}(X))$ and $\mathrm{Irr}(\mathrm{Gl}(2,k)^+)$ in the case $d \neq 1$.

Let V be a nondegenerate two dimensional symmetric bilinear space of discriminant d. Then $\mathrm{GSO}(V)$ is abelian, and all the elements of $\mathrm{Irr}(\mathrm{GSO}(V))$ are one dimensional. We define regular and invariant representations exactly as in section 3. If α in $\mathrm{Irr}(\mathrm{GSO}(V))$ is regular, α^+ will again denote the induced representation of α to $\mathrm{GO}(V)$. Moreover, we say that α in $\mathrm{Irr}(\mathrm{GSO}(V))$ and is distinguished, if and only if

$$\mathrm{Hom}_{\mathrm{SO}(V)}(\alpha, \mathbf{1}) \neq 0.$$

Thus, $\mathrm{SO}(V)$ plays the role that $\mathrm{SO}(Y)$ did in section 4, and if α is in $\mathrm{Irr}(\mathrm{GSO}(V))$ and is distinguished, then we define α^+ and α^- just as in section 4. A result entirely analogous to the main theorem holds: If β is in $\mathrm{Irr}(\mathrm{GO}(V))$, then β is in $\mathcal{R}_1(\mathrm{GO}(V))$ if and only if β is not of the form α^- for some distinguished α in $\mathrm{Irr}(\mathrm{GSO}(V))$. Moreover, by Theorem 1.9 of [Ca], Conjecture 1.3 (theta dichotomy) holds for $X = V$ and $2n = 2$, and the remarks preceding Conjecture 1.3 apply. If one makes the identification of V with $K = k(\sqrt{\mathrm{disc}(V)})$, then elements of $\mathrm{GSO}(V)$ can be identified with quasi-characters of $K^\times$. The map that takes a quasi-character α of $K^\times$ to $\theta(\alpha^+)^\vee$ is just the usual map that associates to a quasi-character an element of $\mathrm{Irr}(\mathrm{Gl}(2,k))$. In particular, if V is anisotropic, and $\alpha = \chi \circ \mathrm{N}_k^K$ is invariant, then $\theta(\alpha^+)^\vee = \pi(\chi, \chi\omega_{K/k})$; and if V is isotropic, so that $K^\times = k^\times \times k^\times$ and $\alpha = (\alpha_1, \alpha_2)$, then $\theta(\alpha^+)^\vee = \pi(\alpha_1, \alpha_2)$. See, for example, [G].

The case when V is anisotropic contains information about the restriction of representations of $\mathrm{Gl}(2,k)$ which we will use in the proof of the next theorem. Let π be in $\mathrm{Irr}(\mathrm{Gl}(2,k))$. It is well known that the restriction of π to $\mathrm{Sl}(2,k)$ is multiplicity free and that $\pi|_{\mathrm{Sl}(2,k)}$ is reducible if and only if π is a theta lift of an element of $\mathrm{Irr}(\mathrm{GO}(V))$ for some anisotropic V. See [Sh]. Let π be a theta lift of σ in $\mathrm{Irr}(\mathrm{GO}(V))$ with V anisotropic. Then from Lemma 4.2 of [R] and the remarks in section 1 it follows that the restriction of π to $\mathrm{Sl}(2,k)$ has two irreducible components if and only if $\sigma \ncong \alpha^+$ with α such that $\alpha|_{\mathrm{SO}(V)} \neq 1$, and $\alpha|^2_{\mathrm{SO}(V)} = 1$.

Let α in $\mathrm{Irr}(\mathrm{GSO}(V))$ be such that $\alpha|_{\mathrm{SO}(V)} \neq 1$ and $\alpha|^2_{\mathrm{SO}(V)} = 1$, and assume $\pi = \theta(\alpha^+)$. Then again from Lemma 4.2 of [R] the restriction of π to $\mathrm{Sl}(2,k)$ has four components. Finally, from Theorem 1.9 (d) of [Ca] it follows that every such π, that is, every π in $\mathrm{Irr}(\mathrm{Gl}(2,k))$ whose restriction to $\mathrm{Sl}(2,k)$ has four components, is a theta lift from every anisotropic V.

In the case $d \neq 1$ we also need to make some remarks about the theta correspondence between $\mathrm{Irr}(\mathrm{GO}(X))$ and $\mathrm{Irr}(\mathrm{Gl}(2,k)^+)$. This was considered in [Co] using the induced Weil representation Ω of $\mathrm{Gl}(2,k) \times \mathrm{GO}(X)$; see [R] for the definition. By an argument as in the proof of Proposition 3.5 of [R], as representations of $\mathrm{Gl}(2,k) \times \mathrm{GSO}(X)$,

$$\Omega \cong \text{c-Ind}_{R'}^{\mathrm{Gl}(2,k)\times\mathrm{GSO}(X)} \omega,$$

where R' is as in the remark after Theorem 4.4. Using Frobenius reciprocity, the main result of [Co] now states that for every infinite dimensional ϱ in $\mathrm{Irr}(\mathrm{Gl}(2,k))$, if $\mathrm{BC}(\varrho^\vee)$ is the base change of $\varrho^\vee$ to $\mathrm{Gl}(2,K)$ and $\pi = \pi(\mathrm{BC}(\varrho^\vee), \omega_{K/k}\omega_{\varrho^\vee})$, then

$$\mathrm{Hom}_{R'}(\omega, \varrho \otimes_{\mathbb{C}} \pi) \neq 0.$$

If $\mathrm{BC}(\varrho^\vee)$ is infinite dimensional, then by Proposition 4.1 and Theorem 5.3, π is generically distinguished. If $\mathrm{BC}(\varrho^\vee)$ is one dimensional, then one can check directly that π is generically distinguished. By the remark following Theorem 4.4, it follows that

$$\mathrm{Hom}_{R}(\omega, \varrho \otimes_{\mathbb{C}} \pi^+) \neq 0.$$

In the case $d = 1$, the next theorem essentially follows from Lemma 6.1, as one would expect. In the case $d \neq 1$, our definition of distinguished representations is more restrictive, and we need to use explicit knowledge of the relevant theta correspondence. The extra effort yields Corollaries 7.5 and 7.6.

Theorem 7.4. *Let σ be in $\mathrm{Irr}(\mathrm{GO}(X))$. Then σ is in $\mathcal{R}_1(\mathrm{GO}(X))$ if and only if σ is of the form π^+ for some distinguished π in $\mathrm{Irr}(\mathrm{GSO}(X))$.*

Proof. Suppose that σ is in $\mathcal{R}_1(\mathrm{GO}(X))$. Suppose first $d = 1$. Consider $\theta(\sigma)$ in $\mathrm{Irr}(\mathrm{Gl}(2,k))$. Assume that $\theta(\sigma)$ is infinite dimensional. It is well known that $\theta(\sigma)$ admits a Whittaker functional, i.e., $\mathrm{Hom}_{N_1}(\theta(\sigma), \psi_1) \neq 0$. By Lemma 4.2 of [R], it follows that $\mathrm{Hom}_{N_1}(\theta(\sigma_1), \psi_1) \neq 0$ for some σ_1 in $\mathrm{Irr}(\mathrm{O}(X))$ which is a constituent of $\sigma|_{\mathrm{O}(X)}$. Since $\theta(\sigma_1)$ is a quotient of $\Theta(\sigma_1)$, it follows that $\mathrm{Hom}_{N_1}(\Theta(\sigma_1), \psi_1) \neq 0$. By Lemma 6.1, this implies that $\mathrm{Hom}_{\mathrm{O}(Y)}(\sigma_1^\vee, \mathbf{1}) \neq 0$ and so $\mathrm{Hom}_{\mathrm{O}(Y)}(\sigma^\vee, \mathbf{1}) \neq 0$. It is not hard to see that this implies that $\sigma|_{\mathrm{GSO}(X)}$ is irreducible and generically distinguished. By Theorem 4.3, $\sigma = \pi^+$. Assume that $\theta(\sigma)$ is one dimensional. By [S], $\epsilon = -\epsilon(1)$. By (v) of Theorem 2.2 of [KR] , $\Theta(\mathbf{1}_{\mathrm{O}(X)})$ has $\mathbf{1}_{\mathrm{Sl}(2,k)}$ as a quotient, i.e., $\theta(\mathbf{1}_{\mathrm{O}(X)}) = \mathbf{1}_{\mathrm{Sl}(2,k)} = \theta(\sigma)|_{\mathrm{Sl}(2,k)}$, so that by Lemma 4.2 of [R] $\sigma|_{\mathrm{O}(X)}$ has $\mathbf{1}$ as constituent. This implies that $\sigma = \pi^+$ for some distinguished π.

Now suppose $d \neq 1$ and $\epsilon = \epsilon(d)$. In this case, we cannot proceed as in the last paragraph and use Lemma 6.1. This is because $\theta(\sigma)$ is now in $\mathrm{Irr}(\mathrm{Gl}(2,k)^+)$, and we cannot conclude that $\mathrm{Hom}_{N_1}(\theta(\sigma), \psi_1) \neq 0$. Instead, we must use the explicit knowledge developed in the paragraphs before the theorem. Suppose first $\theta(\sigma)$ extends to a representation ϱ of $\mathrm{Gl}(2,k)$. Assume that $\theta(\sigma)$ is one dimensional. Then $\theta(\sigma) = \eta \circ \det$ for some quasi-character η of $\mathrm{N}_k^K(K^\times)$. Hence, $\theta(\sigma)|_{\mathrm{Sl}(2,k)} = \mathbf{1}$. We saw in the proof of Lemma 6.2 that $\theta(\mathbf{1}) = \pi(\mathbf{1}, \omega_{K/k})^+|_{\mathrm{O}(X)}$. By Lemma 4.2 of [R], $\theta(\theta(\sigma))|_{\mathrm{O}(X)} = \sigma|_{\mathrm{O}(X)}$ has $\pi(\mathbf{1}, \omega_{K/k})^+|_{\mathrm{O}(X)}$ as an irreducible component.

This implies that $\sigma = \pi(\alpha \circ \mathrm{N}_k^K, \chi)^+$ for some one dimensional $\pi(\alpha \circ \mathrm{N}_k^K, \chi)$ in $\mathrm{Irr}(\mathrm{GO}(X))$. Next, assume that $\theta(\sigma)$ is infinite dimensional. Then if the notation is as in the discussion preceding the lemma, we find that $\sigma = \theta(\varrho|_{\mathrm{Gl}(2,k)^+}) = \pi^+$.

Suppose that $\theta(\sigma)$ induces irreducibly to $\mathrm{Gl}(2,k)$, and assume that σ is not of the form π^+ for some distinguished π. Let ϱ be the induction of $\theta(\sigma)$ to $\mathrm{Gl}(2,k)$. Then ϱ is infinite dimensional. Again, there is a nonzero R' map from ω to $\varrho \otimes_{\mathbb{C}} \pi$. Let g in $\mathrm{Gl}(2,k)$ be a representative for the nontrivial coset of $\mathrm{Gl}(2,k)/\mathrm{Gl}(2,k)^+$. It follows that at least one of

$$\mathrm{Hom}_{R'}(\omega, \theta(\sigma) \otimes_{\mathbb{C}} \pi), \qquad \mathrm{Hom}_{R'}(\omega, g\theta(\sigma) \otimes_{\mathbb{C}} \pi)$$

is nonzero. If the first space is nonzero, then we find as in the last paragraph that $\sigma = \pi^+$, a contradiction. It follows that the first space is zero and the second is nonzero. This implies that

$$\mathrm{Hom}_{\mathrm{Sl}(2,k)}(\omega', \theta(\sigma)) \neq 0,$$

where $\omega' = g^{-1}\omega$ is the extended Weil representation corresponding to the other four dimensional symmetric bilinear space of discriminant d. Hence, $m_+(\theta(\sigma)) \leq 4$ and $m_-(\theta(\sigma)) \leq 4$. By Lemma 7.3, this implies that $m_+(\theta(\sigma)) = m_-(\theta(\sigma)) = 4$. It follows that ϱ is not a lift from a two dimensional symmetric bilinear space with discriminant d. However, the restriction of ϱ to $\mathrm{Gl}(2,k)^+$ is reducible, and so by our above discussion ϱ is a lift from an anisotropic two dimensional symmetric bilinear space of discriminant different from d. This, along with the fact that ϱ has a reducible restriction to $\mathrm{Gl}(2,k)^+$, implies that the restriction to $\mathrm{Sl}(2,k)$ of ϱ has four distinct irreducible components. But then by our above remarks, ϱ is a lift from a two dimensional symmetric bilinear space of the same discriminant as X. This is a contradiction. □

Corollary 7.5. *Assume $d \neq 1$. Let π in $\mathrm{Irr}(\mathrm{GSO}(X))$ be infinite dimensional and invariant. Let y' be anisotropic, and assume that the orthogonal complement Y' of y' is anisotropic. If*

$$\mathrm{Hom}_{\mathrm{SO}(Y')}(\pi, \mathbf{1}) \neq 0,$$

then π is generically distinguished.

Proof. First, it is not difficult to see that we may assume that $y' = S^{-1}(1)$, where S is the intertwining similitude from section 2. It follows that $\mathrm{SO}(Y')$ is the group of $\rho(\det(g), g)$ for g in $D^\times$, with $D^\times$ embedded in $\mathrm{Gl}(2,K)$ as in section 2. Moreover, $\mathrm{O}(Y')$ is generated by $\mathrm{SO}(Y')$, and $h_0' = S^{-1} \circ * \circ S$. Second, we note that Proposition 4.1 and Lemma 4.2 hold for Y' in place of Y. Since Proposition 4.1 holds, we can define $\pi^{+\prime}$ as we defined π^+ in section 4, with Y' in place of Y. An analysis of the proofs of Theorem 4.4 and Lemma 6.1 shows that these results still hold, so that $\pi^{+\prime}$ is in $\mathcal{R}_1(\mathrm{GO}(X))$. By Theorem 7.4, $\pi^{+\prime} = \pi''^{+}$ for some distinguished π'' in $\mathrm{Irr}(\mathrm{GSO}(X))$. It follows that $\pi = \pi''$, and π is generically distinguished. □

When $\omega_\tau = 1$ and $\chi = 1$, the following result is also a consequence of some of the results of [H] and [P2]. There, different means are employed.

Corollary 7.6. *Let τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ be infinite dimensional and Galois invariant. Let $\omega_\tau = \chi \circ \mathrm{N}_k^K$. Then*

$$\mathrm{Hom}_{D^\times}(\tau, \chi \circ \det) \neq 0 \implies \mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det) \neq 0.$$

Proof. Assume the first homomorphism space is nonzero. Then

$$\mathrm{Hom}_{\mathrm{SO}(Y')}(\pi(\tau,\chi),\mathbf{1}) \neq 0,$$

where Y' is as in the proof of Corollary 7.5. By Corollary 7.5, $\pi(\tau,\chi)$ is generically distinguished. The result now follows by Proposition 4.1. □

Lemma 7.7. *Let* σ *be in* $\mathrm{Irr}(\mathrm{GO}(X))$. *If* $\sigma|_{\mathrm{O}(X)} \neq \mathrm{sign}$, *then* σ *is in* $\mathcal{R}_3(\mathrm{GO}(X))$.

Proof. By Theorem 6.8 and the principle of persistence [MVW], p. 67, it suffices to show that if π in $\mathrm{Irr}(\mathrm{GSO}(X))$ is distinguished and $\pi|_{\mathrm{SO}(X)} \neq \mathbf{1}$, then π^- is in $\mathcal{R}_3(\mathrm{GO}(X))$. We will use the technique of section 6. Let u_1 in X^3 be such that the components of u_1 form a basis for the orthogonal complement of $k \cdot 1$. If $d \neq 1$, also let

$$u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix} \oplus \begin{pmatrix} 0 & \sqrt{d} \\ \sqrt{d} & 0 \end{pmatrix}$$

and

$$u_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix} \oplus \begin{pmatrix} 0 & \sqrt{d} \\ \delta^{-1}\sqrt{d} & 0 \end{pmatrix}.$$

Here, δ is a representative for the nontrivial coset of $k^\times / \mathrm{N}_k^K(K^\times)$. Then $\mathrm{O}(U_i) = \{1, h_i\}$ is the stabilizer of u_i in $\mathrm{O}(X)$ where $h_1 = -h_0$ and

$$h_2 = \rho(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})h_0, \qquad h_3 = \rho(\delta, \begin{pmatrix} 0 & \delta \\ -1 & 0 \end{pmatrix})h_0.$$

By Corollary 6.2, it suffices to show that for $i = 1$, 2 or 3, there exists a nonzero vector v in the space of π such that $\pi^-(h_i)v = v$; to prove this it suffices to show that for either $i = 1$, 2 or 3, $\pi^-(h_i) \neq -1$. To this end, suppose that $\pi^-(h_i) = -1$ for $i = 1$, 2 and 3.

Assume first $d = 1$. Let $\pi = \pi(\tau,\tau)$. Since $\pi^-(h_1) = -1$, we have $\pi(h) = \pi(h_0 h h_0^{-1})$ for h in $\mathrm{GSO}(X)$, so that $\tau(g) \otimes \tau(g') = \tau(g') \otimes \tau(g)$ for g in $\mathrm{Gl}(2,k)$ or $D^\times$. This is a contradiction, since by the assumption that $\pi|_{\mathrm{SO}(X)} \neq \mathbf{1}$ the dimension of σ is larger than one.

Assume next $d \neq 1$. If π is infinite dimensional, then there is a contradiction as in the case $d = 1$. Suppose finally that $\pi = \pi(\alpha \circ \mathrm{N}_k^K, \chi)$ is one dimensional. Using that $\pi|_{\mathrm{SO}(X)} \neq \mathbf{1}$ a computation shows that $\pi^-(h_2) = -\pi^-(h_3)$, a contradiction. □

Proof of Theorem 7.8. Let π in $\mathrm{Irr}(\mathrm{GSO}(X))$ be a constituent of the restriction of σ to $\mathrm{GSO}(X)$. Suppose first π is regular so that $\sigma = \pi^+$. By Theorem 7.4, we have $n(\sigma) = n(\sigma \otimes_{\mathbb{C}} \mathrm{sign}) \geq 2$. By Theorem 6.8, it follows that $n(\pi^+) = n(\pi^+ \otimes_{\mathbb{C}} \mathrm{sign}) = 2$.

Suppose next that π is distinguished. Without loss of generality, we may assume that $\sigma = \pi^+$. Suppose $\pi|_{\mathrm{SO}(X)} \neq \mathbf{1}$. Then by Theorem 7.4, $n(\sigma) = 1$ and by Theorem 4.3, Lemma 6.5 and Lemma 7.7, $n(\sigma \otimes_{\mathbb{C}} \mathrm{sign}) = n(\pi^-) = 3$. Suppose that $\pi|_{\mathrm{SO}(X)} = \mathbf{1}$. Then $\sigma|_{\mathrm{O}(X)} = \mathbf{1}$, and by the appendix of [Ra], $n(\sigma) = 0$ and $n(\sigma \otimes_{\mathbb{C}} \mathrm{sign}) = n(\pi^-) = 4$.

Finally, suppose that $d \neq 1$ and π is invariant but not distinguished. Then by Theorem 7.4 and Theorem 6.8, $n(\sigma) = n(\sigma \otimes_{\mathbb{C}} \mathrm{sign}) = 2$. □

Proof of Corollary 7.9. Let σ_1 be in $\mathrm{Irr}(\mathrm{O}(X))$. As we pointed out in the proof of Theorem 7.8, there exists a σ in $\mathrm{Irr}(\mathrm{GO}(X))$ such that σ_1 is a constituent of $\sigma|_{\mathrm{O}(X)}$.

By definition and Lemma 4.2 of [R], $n(\sigma_1) = n(\sigma)$. The corollary follows now from Theorem 7.8. □

The following table summarizes the results when $d = 1$.

$d = 1$, $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$		
σ	$n(\sigma)$	$n(\sigma \otimes_{\mathbb{C}} \mathrm{sign})$
$\sigma\|_{\mathrm{O}(X)} = \mathbf{1}$	0	4
$\sigma\|_{\mathrm{O}(X)} \neq \mathbf{1}$, $\sigma = \pi^+$, π invariant	1	3
$\sigma = \pi^+$, π regular	2	2
$\sigma\|_{\mathrm{O}(X)} \neq \mathrm{sign}$, $\sigma = \pi^-$, π invariant	3	1
$\sigma\|_{\mathrm{O}(X)} = \mathrm{sign}$	4	0

The next table summarizes the information when $d \neq 1$.

$d \neq 1$, $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$		
σ	$n(\sigma)$	$n(\sigma \otimes_{\mathbb{C}} \mathrm{sign})$
$\sigma\|_{\mathrm{O}(X)} = \mathbf{1}$	0	4
$\sigma\|_{\mathrm{O}(X)} \neq \mathbf{1}$, $\sigma = \pi^+$, π distinguished	1	3
$\sigma\|_{\mathrm{GSO}(X)}$ invariant, not distinguished	2	2
$\sigma = \pi^+$, π regular	2	2
$\sigma = \pi^-$, π distinguished	3	1
$\sigma\|_{\mathrm{O}(X)} = \mathrm{sign}$	4	0

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