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ON THE DIOPHANTINE EQUATION $(x^3-1)/(x-1)=(y^n-1)/(y-1)$

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Abstract. In this paper we prove that the equation $(x^3-1)/(x-1) = (y^n-1)/(y-1)$, $x, y, n \in \mathbb{N}$, $x > 1$, $y > 1$, $n > 3$, has only the solutions $(x, y, n) = (5, 2, 5)$ and (90,2,13) with y is a prime power. The proof depends on some new results concerning the upper bounds for the number of solutions of the generalized Ramanujan-Nagell equations.

1. Introduction

Let $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. For any positive integer N with $N > 2$, let $s(N)$ denote the number of solutions (x, m) of the equation

$$N = \frac{x^m-1}{x-1}, \quad x, m \in \mathbb{N}, \quad x > 1, \quad m > 2. \tag{1}$$

Eighty years ago, Ratat [13] and Goormaghtigh [4] observed that $s(31) = 2$ and $s(8191) = 2$, respectively. Simultaneously, they conjectured that if $N \notin \{31, 8191\}$, then $s(N) \leq 1$. The problem can be written in the following form

Conjecture A. The equation

$$\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}, \quad x, y, m, n \in \mathbb{N}, \quad x > 1, \quad y > 1, \quad n > m > 2, \tag{2}$$

has only the solutions $(x, y, m, n) = (5, 2, 3, 5)$ and $(90, 2, 3, 13)$.

In addition, Bateman (see [5, Problem B25]) asked the following problem.

Problem B. Is $(x, y, m, n) = (5, 2, 3, 5)$ the only solution of (2) for which x, y and $(x^m-1)/(x-1)$ are all prime powers?

These are two rather difficult questions. By the results of Baker, Davenport, Lewis, Schinzel, Shorey and Tijdeman (see the references of [14]), we see that (2) has only finitely many solutions (x, y, m, n) if any two out of the four variables x, y, m and n are fixed. However, to this day we do not know whether (2) has only finitely many solutions (x, y, m, n) if any one out of x, y, m and n is fixed.

In this paper we discuss the finiteness of solutions of (2) by the means of the generalized Ramanujan-Nagell equations. Let D_1, D_2 be coprime positive integers,

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and let p be a prime with $p \nmid D_1 D_2$. Further let $N(D_1, D_2, p)$ and $N'(D_1, D_2, p)$ denote the numbers of the solutions (r, s) and (r', s') of equations

$$D_1 r^2 + D_2 = p^s, \qquad r, s \in \mathbb{N}, \tag{3}$$

and

$$D_1 r'^2 + D_2 = 2p^{s'}, \qquad r', s' \in \mathbb{N}, \tag{4}$$

respectively. In this paper we prove the following two general results.

Theorem 1. *$N(D_1, D_2, p) \le 2$ except for $N(1, 7, 2) = 5$ and $N(3, 5, 2) = 3$.*

Theorem 2. *If $2 \nmid D_1 D_2$ and p is an odd prime, then $N'(D_1, D_2, p) \le 2$.*

On applying the above-mentioned theorems to the equation (2), we shall deduce the following result.

Corollary. *The equation (2) has only the solutions $(x, y, m, n) = (5, 2, 3, 5)$ and $(90, 2, 3, 13)$ such that $m = 3$ and y is a prime power.*

2. Proof of Theorem 1

First we may assume that D_1 is squarefree. In [1] and [2], Apéry proved that $N(1, D_2, p) \le 2$ except for $N(1, 7, 2) = 5$. In [7] and [8], the author proved that $N(D_1, 1, p) \le 2$ and $N(D_1, D_2, 2) \le 2$ except for $N(1, 7, 2) = 5$ and $N(3, 5, 2) = 3$, respectively. Therefore, we may assume that $D_1 > 1$, $D_2 > 1$ and p is an odd prime.

Lemma 1 ([10, Formula 3.76]). *For any positive integer t and any complex numbers α and β, we have*

$$\alpha^t + \beta^t = \sum_{i=0}^{[t/2]} (-1)^i \begin{bmatrix} t \\ i \end{bmatrix} (\alpha + \beta)^{t-2i} (\alpha\beta)^i,$$

where $[t/2]$ is the largest integer which does not exceed $t/2$,

$$\begin{bmatrix} t \\ i \end{bmatrix} = \frac{(t-i-1)!t}{(t-2i)!i!}, \qquad i = 0, 1, \ldots, \left[\frac{t}{2}\right],$$

are positive integers.

Lemma 2 ([9, Lemma 4]). *If the equation*

$$D_1 X^2 + D_2 Y^2 = p^z, \quad X, Y, Z \in \mathbb{Z}, \quad \gcd(X, Y) = 1, \quad Z > 0, \tag{5}$$

has solutions (X, Y, Z), then it has a unique solution (X_1, Y_1, Z_1) such that $X_1 > 0$, $Y_1 > 0$ and $Z_1 \le Z$, where Z runs through all solutions (X, Y, Z) of (5). (X_1, Y_1, Z_1) is called the least solution of (5). Further, every solution (X, Y, Z) of (5) can be expressed as

$$Z = Z_1 t,\ X\sqrt{D_1} + Y\sqrt{-D_2} = \lambda_1 (X_1 \sqrt{D_1} + \lambda_2 Y_1 \sqrt{-D_2})^t,$$
$$t \in \mathbb{N}, 2 \nmid t, \lambda_1, \lambda_2 \in \{-1, 1\}.$$

Lemma 3. *If $N(D_1, D_2, p) > 2$, then (3) has two solutions (r_1, s_1) and (r_2, s_2) such that*

$$s_1 = Z_1 t_1, \quad s_2 = Z_1 t_2, \quad 1 < t_1 < t_2, \tag{6}$$

where (X_1, Y_1, Z_1) *is the least solution of* (5), t_1 *and* t_2 *are odd primes satisfying*

$$\sum_{i=0}^{(t_j-1)/2} \begin{bmatrix} t_j \\ i \end{bmatrix} (-4D_2)^{(t_j-1)/2-i} p^{Z_1 i} = (-1)^{(t_j-1)(p^{Z_1})/4}, \qquad j=1,2, \tag{7}$$

$$\frac{\pi}{2\arcsin(D_2/p^{s_1})^{1/2}} < t_2 \tag{8}$$

and

$$\left| \arcsin\left(\frac{D_2}{p^{Z_1}}\right)^{1/2} - \frac{k\pi}{t_j} \right| < \frac{\pi}{2t_j p^{Z_1(t_j-1)/2}}, \quad j=1,2, \tag{9}$$

where k *is a positive integer satisfying* $k \le (t_j-1)/2$.

Proof. We see that if (3) has a solution (r,s), then (5) has a solution $(X,Y,Z)=(r,1,s)$. It follows from Lemma 2 that the least solution (X_1,Y_1,Z_1) of (5) satisfies $Y_1=1$. Then $(r,s)=(X_1,Z_1)$ is a solution of (3). Further, let

$$\varepsilon = X_1\sqrt{D_1}+\sqrt{-D_2}, \quad \bar{\varepsilon} = X_1\sqrt{D_1}-\sqrt{-D_2}. \tag{10}$$

By the proof of [9, Theorem 1], if (r,s) is a solution of (3) with $(r,s)\neq(X_1,Z_1)$, then there exists a positive integer t satisfying $t>1$, $2\nmid t$,

$$s = Z_1 t \tag{11}$$

and

$$\left|\frac{\varepsilon^t-\bar{\varepsilon}^t}{\varepsilon-\bar{\varepsilon}}\right| = 1. \tag{12}$$

Therefore, by (11), if $N(D_1,D_2,p)>2$, then (3) has two solutions (r_1,s_1) and (r_2,s_2) satisfying (6). Moreover, we may assume that (3) has no solutions (r,s) satisfying

$$Z_1 < s < s_1, \quad s_1 < s < s_2. \tag{13}$$

Since $\varepsilon-\bar{\varepsilon}=2\sqrt{-D_2}$ and $\varepsilon\bar{\varepsilon}=p^{Z_1}$, by Lemma 1, we get from (12) that

(14)

$$\frac{\varepsilon^t-\bar{\varepsilon}^t}{\varepsilon-\bar{\varepsilon}} = \sum_{i=0}^{(t-1)/2} (-1)^i \begin{bmatrix} t \\ i \end{bmatrix} (\varepsilon-\bar{\varepsilon})^{t-2i-1}(-\varepsilon\bar{\varepsilon})^i = \sum_{i=0}^{(t-1)/2} \begin{bmatrix} t \\ i \end{bmatrix} (-4D_2)^{(t-1)/2-i} p^{Z_1} = \pm 1.$$

Since $t=(-1)^{(t-1)/2} \pmod 4$ and $p^{Z_1}\equiv(-1)^{(p^{Z_1}-1)/2} \pmod 4$, we obtain (7) from (14).

For any positive integer t with $2\nmid t$, let

$$X_t = \left|\frac{\varepsilon^t+\bar{\varepsilon}^t}{2\sqrt{D_1}}\right|, \quad Y_t = \left|\frac{\varepsilon^t-\bar{\varepsilon}^t}{\varepsilon-\bar{\varepsilon}}\right|. \tag{15}$$

By Lemma 1, X_t and Y_t are positive integers satisfying

$$D_1X_t^2 + D_2Y_t^2 = p^{Z_1 t}. \tag{16}$$

Further, by Lemma 2, we see from (16) that (r,s) is a solution of (3) satisfying (11) if and only if $Y_t=1$. From (7), we get

$$Y_{t_1} = Y_{t_2} = 1. \tag{17}$$

If t_1 is not an odd prime, then $t_1 = k_1k_2$, where k_1 and k_2 are positive integers satisfying $k_1 > 1$, $k_2 > 1$ and $2 \nmid k_1k_2$. On applying Lemma 1, we find from (17) that
(18)

$$1 = Y_{t_1} = Y_{k_1}\left|\frac{(\varepsilon^{k_1})^{k_2} - (\overline{\varepsilon}^{k_1})^{k_2}}{\varepsilon^{k_1} - \overline{\varepsilon}^{k_1}}\right| = Y_{k_1}\left|\sum_{i=0}^{(k_2-1)/2}\begin{bmatrix}k_2\\ i\end{bmatrix}(-4D_2Y_{k_1}^2)^{(k_2-1)/2-i}p^{Z_1k_1i}\right|.$$

Hence, we get from (18) that $Y_{k_1} = Y_{t_1}/Y_{k_1} = 1$. It implies that (3) has a solution $(r, s) = (X_{k_1}, Z_1k_1)$ satisfying $Z_1 < s < s_1$, which contradicts (13). Thus t_1 must be an odd prime.

By the same argument, if t_2 is not an odd prime, then from (13) we get $t_2 = t_1^2$ and $Y_{t_2}/Y_{t_1} = 1$. In this case, by Lemma 1, we have

$$\frac{Y_{t_2}}{Y_{t_1}} = \left|\frac{\varepsilon^{t_1^2} - \overline{\varepsilon}^{t_1^2}}{\varepsilon^{t_1} - \overline{\varepsilon}^{t_1}}\right| = \left|\sum_{i=0}^{(t_1-1)/2}\begin{bmatrix}t_1\\ i\end{bmatrix}(-4D_2)^{(t_1-1)/2-i}p^{Z_1t_1i}\right| = 1. \tag{19}$$

Since $(-1)^{(t_1-1)(p^{z_1}+1)/4} = (-1)^{(t_1-1)(p^{z_1t_1}+1)/4}$, we get from (19) that

$$\sum_{i=0}^{(t_1-1)/2}\begin{bmatrix}t_1\\ i\end{bmatrix}(-4D_2)^{(t_1-1)/2-i}p^{z_1t_1i} = (-1)^{(t_1-1)(p^{Z_1}+1)/4}. \tag{20}$$

Further, since $t_1 \geq 3$, we see from (20) that

$$(-4D_2)^{(t_1-1)/2} - (-1)^{(t_1-1)(p^{Z_1}+1)/4} \equiv 0 (\bmod p^{3Z_1}). \tag{21}$$

On the other hand, by (7), we have

$$\begin{aligned}\Big((-4D_2)^{(t_1-1)/2} - (-1)^{(t_1-1)(p^{Z_1}+1)/4}\Big) + \begin{bmatrix}t_1\\ 1\end{bmatrix}(-4D_2)^{(t_1-3)/2}p^{Z_1}\\ + \begin{bmatrix}t_1\\ 2\end{bmatrix}(-4D_2)^{(t_1-5)/2}p^{2Z_1} \equiv 0 \pmod{p^{3Z_1}}.\end{aligned} \tag{22}$$

The combination of (21) and (22) yields

$$t_1\left(4D_2 - \left(\frac{t_1-3}{2}\right)p^{Z_1}\right) \equiv 0 \pmod{p^{2Z_1}}. \tag{23}$$

Since $p \nmid D_2$ and t_1 is an odd prime, (23) is impossible. Thus t_2 is an odd prime too.

By (10), we have

$$\varepsilon = p^{Z_1/2}e^{\theta\sqrt{-1}}, \quad \overline{\varepsilon} = p^{Z_1/2}e^{-\theta/\sqrt{-1}}, \tag{24}$$

where θ is a real number satisfying

$$\sin\theta = \frac{\varepsilon - \overline{\varepsilon}}{2p^{Z_1/2}\sqrt{-1}} = \left(\frac{D_2}{p^{Z_1}}\right)^{1/2}. \tag{25}$$

Since $0 < D_2/p^{Z_1} < 1$, we may assume that θ satisfies

$$0 < \theta < \frac{\pi}{2}. \tag{26}$$

Further, sine $Y_{t_1} = Y_{t_2} = 1$ by (17), we get from (24) and (25) that

$$\sin t_j\theta = (-1)^{(t_j-1)(p^{Z_1}+1)/4}\frac{\sin\theta}{p^{Z_1(t_j-1)/2}}, \quad j = 1, 2. \tag{27}$$

Hence

$$t_j\theta = k\pi + (-1)^{k+(t_j-1)(p^{Z_1}+1)/4}\phi_j, \qquad j=1,2, \tag{28}$$

where k is an integer, $\phi_j (j=1,2)$ are positive numbers satisfying

$$\sin\phi_j = \frac{\sin\theta}{p^{Z_1(t_j-1)/2}}, \quad 0<\phi_j<\frac{\pi}{2}, \qquad j=1,2. \tag{29}$$

Notice that $0<\theta<\pi/2$ by (26). Since $0<\phi_j<\theta<\pi/2 \quad (j=1,2)$ by (26) and (29), we see from (28) that k satisfies $1\le k\le (t_j-1)/2$. Thus, by (25), (28) and (29), we get

$$\begin{aligned}\left|\arcsin\left(\frac{D_2}{p^{Z_1}}\right)^{1/2} - \frac{k\pi}{t_j}\right| &= \left|\theta - \frac{k\pi}{t_j}\right| = \frac{\phi_j}{t_j} = \frac{\arcsin((\sin\theta)/p^{Z_1(t_j-1)/2})}{t_j}\\ &< \frac{\arcsin(p^{-Z_1(t_j-1)/2})}{t_j} < \frac{\pi}{2t_jp^{Z_1(t_j-1)/2}}, \quad j=1,2.\end{aligned}$$

The inequality (9) is proved.

From (25) and (27), we get

$$|\sin t_1\theta| = \left(\frac{D_2}{p^{s_1}}\right)^{1/2}, \quad |\sin t_2\theta| = \left(\frac{D_2}{p^{s_2}}\right)^{1/2}. \tag{30}$$

By (30), there exist suitable nonnegative integers k_1 and k_2 such that

$$|k_1\pi - t_1\theta| = \arcsin\left(\frac{D_2}{p^{s_1}}\right)^{1/2}, \quad |k_2\pi - t_2\theta| = \arcsin\left(\frac{D_2}{p^{s_2}}\right)^{1/2}. \tag{31}$$

Since $s_1<s_2$ and $\arcsin(D_2/p^{s_1})^{1/2} > \arcsin(D_2/p^{s_2})^{1/2}$, we get from (31) that

(32)

$$0<\left|\frac{k_1}{t_1}-\frac{k_2}{t_2}\right|\pi \le \frac{1}{t_1}\arcsin\left(\frac{D_2}{p^{s_1}}\right)^{1/2} + \frac{1}{t_2}\arcsin\left(\frac{D_2}{p^{s_2}}\right)^{1/2} < \frac{2}{t_1}\arcsin\left(\frac{D_2}{p^{s_1}}\right)^{1/2}.$$

Notice that $|k_1/t_1 - k_2/t_2| \ge 1/t_1t_2$ if $k_1/t_1 \ne k_2/t_2$. We obtain (8) from (32). The lemma is proved.

Lemma 4. *If* (3) *has two solutions* (r_1,s_1) *and* (r_2,s_2) *satisfying* (6) *with* $t_1=3$, *then we have*

$$4D_2 = 3p^{Z_1} + (-1)^{(p^{Z_1-1})/2} \tag{33}$$

and

$$t_2 \ge 2p^{2Z_1}+3. \tag{34}$$

Proof. Since $t_1=3$, we get (33) from (7). Let $\delta=(-1)^{(p^{Z_1}+1)/2}$. We get from (17) and (33) that

$$(-3p^{Z_1}+\delta)^{(t_2-1)/2} + t_2(-3p^{Z_1}+\delta)^{(t_2-3)/2}(3p^{Z_1}) \equiv \delta^{(t_2-1)/2} \pmod{p^{2Z_1}},$$

whence we obtain $(t_2-3)/2\equiv 0 \pmod{p^{Z_1}}$. By (7) again, we get

(35)

$$-\binom{\frac{t_2-3}{2}}{}+\left(9\binom{(t_2-1)/2}{2} - 3\begin{bmatrix}t_2\\1\end{bmatrix}\binom{(t_2-3)/2}{1} + \begin{bmatrix}t_2\\2\end{bmatrix}\right)\delta p^{Z_1} \equiv 0 \pmod{p^{2Z_1}}.$$

Notice that

$$\binom{(t_2-1)/2}{2} \equiv \binom{(t_2-3)/2}{1} \equiv \begin{bmatrix} t_2 \\ 2 \end{bmatrix} \equiv 0 \quad \left(\bmod \frac{t_2-3}{2\lambda}\right),$$

where $\lambda = 1$ or 2 according to whether $t_2 \equiv 1 \pmod 4$ or not. We find from (35) that $(t_2-3)/2 \equiv 0 \pmod{p^{2Z_1}}$. Since $t_2 > 3$, it implies (34). The lemma is proved.

Let α be an algebraic number of degree d with conjugates $\sigma_1\alpha, \sigma_2\alpha, \ldots, \alpha_d\alpha$ and the minimal polynomial

$$a_0 z^d + a_1 z^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (z - \sigma_i\alpha) \in \mathbb{Z}[z], \quad a_0 > 0.$$

Then

$$h(\alpha) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max(1, |\sigma_i\alpha|)\right)$$

is called the logarithmic absolute height of α.

Lemma 5 ([6, Théorème 3])**.** *Let α be an algebraic number with $|\alpha| = 1$. Let b_1, b_2 be positive integers, and let $\Lambda = b_1 \log\alpha - b_2\pi\sqrt{-1}$. If α is not a root of unity, then we have*

$$\log|\Lambda| \geq -8.87AB^2,$$

where $A = \max(20, 10.98|\log\alpha| + rh(\alpha))$, $B = \max(17, r^{1/2}/10, 5.03 + 2.35r + r\log(b_1/68.9 + b_2/2A))$, $r = [\mathbb{Q}(\alpha)\colon\ \mathbb{Q}]/2$.

Lemma 6. *If* (3) *has two solutions* (r_1, s_1) *and* (r_2, s_2) *satisfying* (6)*, then we have*

(36) $$t_2 < 2 + 2563.43\left(1 + \frac{10.98\pi}{\log p^{Z_1/2}}\right).$$

Proof. Under the assumption, we get from (15) and (17) that

(37) $$2p^{Z_1/2} > 2D_2^{1/2}|\varepsilon - \overline{\varepsilon}| = |\varepsilon^{t_2} - \overline{\varepsilon}^{t_2}|.$$

Let $\alpha = \varepsilon/\overline{\varepsilon}$. Then from (37) we get

(38) $$\log 2p^{Z_1/2} > \log|\overline{\varepsilon}^{t_2}| + \log|\alpha^{t_2} - 1| = t_2 \log p^{Z_1/2} + \log|\alpha^{t_2} - 1|.$$

Since $t_2 \geq 5$, if $|\alpha^{t_2} - 1| \geq 1$, then (38) is impossible. Therefore, we have

(39) $$|\alpha^{t_2} - 1| \geq |t_2 \log\alpha - k\pi\sqrt{-1}|$$

for some positive integers k with $k \leq t_2$. Let $\Lambda = t_2\log\alpha - k\pi\sqrt{-1}$. From (38) and (39), we get

(40) $$\log 2p^{Z_1/2} - \log|\Lambda| > t_2 \log p^{Z_1/2}.$$

We see from (10) that α satisfies

$$p^{Z_1}\alpha^2 - 2(D_1X_1^2 - D_2)\alpha + p^{Z_1} = 0.$$

It implies that $[\mathbb{Q}(\alpha):\mathbb{Q}]=2$, $h(\alpha)=\log p^{Z_1/2}$ and α is not a root of unity. Further, by (24) and (26), we have $|\alpha|=1$ and $|\log\alpha|=2\theta<\pi$. Since $k\le t_2$, by Lemma 5, we get

$$\log|\Lambda|\ge -8.87(10.98\pi+\log p^{Z_1/2}) \times\left(\max\left(17, 7.38+\log\left(\frac{t_2}{68.9}+\frac{t_2}{2(10.98\pi+\log p^{Z_1/2})}\right)\right)\right)^2. \tag{41}$$

If

$$7.38+\log\left(\frac{t_2}{68.9}+\frac{t_2}{2(10.98\pi+\log p^{Z_1/2})}\right)\ge 17, \tag{42}$$

then we have

$$\log 0.0288015t_2\ge 9.62,$$

whence we conclude that

$$t_2>523063. \tag{43}$$

On the other hand, by (40), (41) and (42), we get

$$3+565.8842(7.38+\log 0.0288015t_2)^2>1+\frac{\log 2}{\log p^{Z_1/2}}+8.87\left(1+\frac{10.98\pi}{\log p^{Z_1/2}}\right) \left(7.38+\log\left(\frac{t_2}{68.9}+\frac{t_2}{2(10.98\pi+\log p^{Z_1/2})}\right)\right)^2>t_2. \tag{44}$$

We calculate from (44) that $t_2<150000$. It contradicts with (43). So we have

$$7.38+\log\left(\frac{t_2}{68.9}+\frac{t_2}{2(10.98\pi+\log p^{Z_1/2})}\right)<17. \tag{45}$$

Thus, by (40), (41) and (45), we obtain (36). The lemma is proved.

Lemma 7 ([3]). *The equation*

$$X^3+X^2Y-2XY^2-Y^3=1,\quad X,Y\in\mathbb{Z}$$

has only the solutions $(X,Y)=(1,0)$, $(0,-1)$, $(-1,1)$, $(2,-1)$, $(-1,2)$, $(5,4)$, $(4,-9)$ *and* $(-9,5)$.

Proof of Theorem 1. We now suppose that $N(D_1,D_2,p)>2$. By Lemma 3, then (3) has two solutions (r_1,s_1) and (r_2,s_2) satisfying (6).

By Lemmas 4 and 6, if $t_1=3$, then we have

$$2p^{2Z_1}+3\le t_2<2+2563.43\left(1+\frac{10.98\pi}{\log p^{Z_1/2}}\right), \tag{46}$$

whence we conclude that $p^{Z_1}\le 137$. Recall that $D_1>1$, $D_2>1$ and D_1 is square free. By (33), we only need to consider the cases $(D_1,D_2,p^{Z_1})=(2,5,7)$, (2,7,9), (3,8,11), (3,10,13), (5,14,19), (6,17,23), (6,19,25), (7,20,27), (7,22,29),(2,23,31), (10,31,41), (11,32,43), (3,35,47), (3,37,49), (13,40,53), (15,44,59), (15,46,61), (17,50,67), (2,53,71), (2,55,73), (5,59,79), (5,61,81), (21,62,83), (22,67,89), (6,73,97), (26,77,103), (3,80,107), (3,82,109), (7,85,113), (30,91,121), (31,94,125), (2,95,127), (33,98,131) and (34,103,137). On the other hand, by (9) and (33), we get

$$\left|\arcsin\left(\frac{3}{4}+\frac{(-1)^{(p^{Z_1}-1)/2}}{4p^{Z_1}}\right)^{1/2}-\frac{k\pi}{t_2}\right|<\frac{\pi}{2t_2p^{Z_1(t_2-1)/2}} \tag{47}$$

for some positive integers k with $k \le (t_2-1)/2$. However, by (46), (47) is false for the above-mentioned cases.

If $t_1 = 5$, then from (7) we get

$$16D_2^2 - 20D_2p^{Z_1} + 5p^{2Z_1} = 1. \tag{48}$$

Since p^{Z_1} is an odd prime power, we see from (48) that $4D_2 \equiv \lambda \pmod{p^{Z_1}}$, where $\lambda \in \{-1,1\}$. So we have $4D_2 = kp^{Z_1} + \lambda$, where $k = 1$ or 3. Hence, by (48), we get

$$p^{Z_1}(k^2-5k+5) = (5-2k)\lambda.$$

This implies that $p^{Z_1} = 3$. Since D_2 is an integer with $D_2 > 1$, this is impossible.

If $t_1 = 7$, then we have

$$(2p^{Z_1}-4D_2)^3 + (2p^{Z_1}-4D_2)^2p^{Z_1} - 2(2p^{Z_1}-4D_2)p^{2Z_1} - p^{3Z_1} = \pm 1 \tag{49}$$

by (7). However, by Lemma 7, (49) is impossible.

From (8) and (36), if $t_1 > 7$, then we have $t_1 \ge 11$ and

$$\begin{aligned} p^{5Z_1} \le p^{Z_1(t_1-1)/2} &< \left(\frac{p^{Z_1t_1}}{D_2}\right)^{1/2} < \frac{\pi}{2\arcsin(D_2/p^{Z_1t_1})^{1/2}} < t_2 \\ &< 2 + 2563.42\left(1+\frac{10.98\pi}{\log p^{Z_1/2}}\right), \end{aligned}$$

whence we calculate that $p^{Z_1} \le 9$. Since, if $(D_1, D_2, p^{Z_1}) = (2,5,7)$ or (2,7,9), then $t_1 = 3$. So we only need to consider the cases $(D_1, D_2, p^{Z_1}) = (2,3,5)$, (3,2,5), (3,4,7), (5,2,7), (5,4,9) and (7,2,9). For the above-mentioned cases, (7) is false if $t_1 = 11$ or 13. So we have $t_1 \ge 17$. Then, by (8) and (36) again, we get

$$390625 \le p^{8Z_1} \le p^{Z_1(t_1-1)/2} < t_2 < 2 + 2563.42\left(1+\frac{10.98\pi}{\log 5^{1/2}}\right) < 112451,$$

a contradiction. All cases have been considered, the proof is complete.

3. Proof of Theorem 2

By the same method as in the proofs of Lemmas 2, 3, 4 and 6, we can prove the corresponding lemmas about the equation (4) without any difficulty.

Lemma 8. *If the equation*

$$D_1X'^2 + D_2Y'^2 = 2p^{Z'}, \quad X', Y', Z' \in \mathbb{Z}, \quad \gcd(X', Y') = 1, \quad Z' > 0, \tag{50}$$

has solutions (X', Y', Z'), *then it has a unique solution* (X_1', Y_1', Z_1') *such that* $X_1' > 0$, $Y_1' > 0$ *and* $Z_1' \le Z'$, *where* Z' *runs through all solutions* (X', Y', Z') *of* (50). (X_1', Y_1', Z_1') *is called the least solution of* (50). *Moreover, every solution* (X', Y', Z') *of* (50) *can be expressed as*

$$Z' = Z_1't', \frac{X'\sqrt{D_1}+Y'\sqrt{-D_2}}{\sqrt{2}} = \lambda_1\left(\frac{X_1'\sqrt{D_1}+\lambda_2Y_1'\sqrt{-D_2}}{\sqrt{2}}\right)^{t^0},$$
$$t' \in \mathbb{N},\ 2 \nmid t', \lambda_1, \lambda_2 \in \{-1,1\}.$$

Lemma 9. *The equation* (4) *has solutions* (r', s') *if and only if* (50) *has solutions* (X', Y', Z') *and its least solution* (X'_1, Y'_1, Z'_1) *satisfies* $Y'_1 = 1$. *Moreover, if* $N'(D_1, D_2, p) > 2$, *then* (4) *has two solutions* (r'_1, s'_1) *and* (r'_2, s'_2) *such that*

$$s'_1 = Z'_1 t'_1, \quad s'_2 = Z'_1 t'_2, \quad 1 < t'_1 < t'_2, \tag{51}$$

where t'_1 *and* t'_2 *are odd primes satisfying*

$$\sum_{i=0}^{(t'_j-1)/2} \begin{bmatrix} t'_j \\ i \end{bmatrix} (-2D_2)^{(t'_j-1)/2-i} p^{Z'_1 i} = -(-1)^{(t'_j-1)(p^{Z'_1}+1)/4}, \quad j = 1, 2, \tag{52}$$

$$\frac{\pi}{2 \arcsin(D_2/2p^{s'_1})^{1/2}} < t'_2 \tag{53}$$

and

$$\left| \arcsin \left(\frac{D_2}{2p^{Z'_1}} \right)^{1/2} - \frac{k\pi}{t'_j} \right| < \frac{\pi}{2t'_j p^{Z'_1(t'_j-1)/2}}, \quad j = 1, 2, \tag{54}$$

where k *is a positive integer satisfying* $k \le (t'_j - 1)/2$.

Lemma 10. *If* (4) *has two solutions* (r'_1, s'_1) *and* (r'_2, s'_2) *satisfying* (51) *with* $t'_1 = 3$, *then we have*

$$2D_2 = 3p^{Z'_1} + (-1)^{(p^{Z'_1}+1)/2} \tag{55}$$

and

$$t'_2 \ge 2p^{2Z'_1} + 3. \tag{56}$$

Lemma 11. *If* (4) *has two solutions* (r'_1, s'_1) *and* (r'_2, s'_2) *satisfying* (51), *then we have*

$$t'_2 < 3 + 2563.43 \left(1 + \frac{10.98\pi}{\log p^{Z'_1/2}} \right). \tag{57}$$

Proof of Theorem 2. We now suppose that $N'(D_1, D_2, p) > 2$. By Lemma 9, then (4) has two solutions (r'_1, s'_1) and (r'_2, s'_2) satisfying (51).

By Lemmas 10 and 11, if $t'_1 = 3$, then we have

$$2p^{2Z'_1} + 3 \le t'_2 < 3 + 2563.43 \left(1 + \frac{10.98\pi}{\log p^{Z'_1/2}} \right), \tag{58}$$

whence we calculate that $p^{Z'_1} \le 137$. Hence, by (55), we only need to consider the cases $(D_1, D_2, p^{Z'_1})$ such that $p^{Z'_1} \le 137$ and $2D_2 = 3p^{Z'_1} + (-1)^{(p^{Z'_1}+1)/2}$. On the other hand, by (54) and (55), we get

$$\left| \arcsin \left(\frac{3}{4} + \frac{(-1)^{(p^{Z'_1}+1)/2}}{4p^{Z'_1}} \right)^{1/2} - \frac{k\pi}{t'_2} \right| < \frac{\pi}{2t'_2 p^{Z'_1(t'_2-1)/2}} \tag{59}$$

for some positive integers k with $k \le (t'_2 - 1)/2$. However, for the above-mentioned cases, if t'_2 satisfies (58), then (59) is impossible.

If $t'_1 = 5$, then from (52) we get

$$4D_2^2 - 10D_2 p^{Z'_1} + 5p^{2Z'_1} = -1.$$

It implies that

$$(4D_2 - 5p^{Z_1'})^2 - 5p^{2Z_1'} = -4. \tag{60}$$

For any nonnegative integer m, let F_m and L_m denote the mth Fibonacci number and the mth Lucas number, respectively. Notice that all solutions (X, Y) of the equation

$$X^2 - 5Y^2 = -4, \quad X, Y \in \mathbb{N}, \quad \gcd(X, Y) = 1$$

are given by $(X, Y) = (L_{6l}, F_{6l+1})$ and (L_{6l+5}, F_{6l+5}), where l runs through all nonnegative integers. We see from (60) that

$$(D_2, p^{Z_1'}) = (\tfrac{1}{2}L_{6l}, F_{6l+1}) \quad \text{or} \quad (\tfrac{1}{2}L_{6l+6}, F_{6l+5}). \tag{61}$$

On the other hand, by (53) and (57), we have

(62)

$$p^{2Z_1'} < \left(\frac{2p^{5Z_1'}}{D_2}\right)^{1/2} < \frac{\pi}{2\arcsin(D_2/2p^{5Z_1'})^{1/2}} < t_2' < 3 + 2563.43\left(1 + \frac{10.98\pi}{\log p^{Z_1'/2}}\right),$$

whence we calculate that $p^{Z_1'} < 200$. Therefore, we find from (61) that $(D_1, D_2, p^{Z_1'}) = (1, 9, 5)$, $(17, 9, 13)$ and $(17, 161, 89)$. Further, by (54), we get

$$\left|\arcsin\left(\frac{D_2}{2p^{Z_1'}}\right)^{1/2} - \frac{k\pi}{t_2'}\right| < \frac{\pi}{2t_2'p^{Z_1'(t_2'-1)/2}} \tag{63}$$

for some positive integers k with $k \le (t_2' - 1)/2$. However, if t_2' satisfies (62), then (63) is impossible.

If $t_2' = 7$, then from (52) we get

(64)

$$(2p^{Z_1'} - 2D_2)^3 + (2p^{Z_1'} - 2D_2)^2 p^{Z_1'} - 2(2p^{Z_1'} - 2D_2)p^{2Z_1'} - p^{3Z_1'} = (-1)^{(p^{Z_1'}+1)2}.$$

By Lemma 7, we obtain from (64) that $(D_1, D_2, p^{Z_1'}) = (7, 11, 9)$. For this case, by (53), (54) and (57), we get

$$729 < t_2' < 83099 \tag{65}$$

and

$$\left|\arcsin\left(\frac{11}{18}\right)^{1/2} - \frac{k\pi}{t_2'}\right| < \frac{\pi}{2\cdot 3^{t_2'-1}t_2'} \tag{66}$$

for some positive integers k with $k \le (t_2' - 1)/2$. However, (66) is impossible if t_2' satisfies (65).

By (53) and (57), if $t_1' > 7$, then $t_1' \ge 11$ and $p^{Z_1'} \le 9$. Except for the already considered cases, we have $(D_1, D_2, p^{Z_1'}) = (5, 1, 3)$, (1,9,5), (9,1,5), (1,13,7) (5.9,7), (9,5,7), (11,3,7), (13,1,7), (1,17,9), (11,7,9), (13,5,9) or (17,1,9). For these cases, (52) is false when $t_1' = 11, 13, 17$ and 19. So we have $t_1' \ge 23$. Then, by (53) and (57) again, we get

$$\begin{aligned}177147 \le p^{11Z_1'} \le p^{Z_1'(t_1'-1)/2} < \frac{\pi}{2\arcsin(D_2/2p^{s_1'})^{1/2}} < t_2' < 3 + 2563.43 \\ \times\left(1 + \frac{10.98\pi}{\log 3^{1/2}}\right) < 153544,\end{aligned}$$

a contradiction. The theorem is proved.

4. Proof of the Corollary

Lemma 12 ([11]). *The equation*

$$z^2+4=y^n, \quad x,y,n\in\mathbb{N}, \quad 2\nmid y, \quad n>3,$$

has no solution (z,y,n).

Proof of Corollary. Let (x,y,m,n) be a solution of (2) such that $m=3$ and y is a prime power. Then we have

$$(y-1)(2x+1)^2+(3y+1)=4y^n, \quad n>3. \tag{67}$$

If $y=2$, then from (67) we get

$$(2x+1)^2+7=2^{n+2}, \quad n>3. \tag{68}$$

By [12], we find from (68) that $(x,y,m,n)=(5,2,3,5)$ and (90,2,3,13).

If $y=2^k$, where k is a positive integer with $k>1$, then

$$(2^k-1)(2x+1)^2+(3\cdot 2^k+1)=2^{kn+2}, \quad n>3. \tag{69}$$

Let $D_1=2^k-1$, $D_2=3\cdot 2^k+1$ and $p=2$. We see from (69) that $(r,s)=(2x+1,kn+2)$ is a solution of (3) with $s>3k+2$. However, by [8], then (3) has exactly two solutions $(r,s)=(1,k+2)$ and $(2^{k+1}+1,3k+2)$. Therefore, (69) is impossible.

If $2\nmid y$, then $y=p^k$, where p is an odd prime and k is a positive integer.

If $y=5$, then from (67) we get

$$(2x+1)^2+4=5^n, \quad n>3. \tag{70}$$

By Lemma 12, (70) is impossible.

If $y\equiv 1 \pmod 4$ and $y>5$, then we have

$$\left(\frac{p^k-1}{4}\right)(2x+1)^2+\left(\frac{3p^k+1}{4}\right)=p^{kn}, \quad n>3. \tag{71}$$

Let $D_1=(p^k-1)/4$ and $D_2=(3p^k+1)/4$. We see from (71) that $(r,s)=(2x+1,kn)$ is a solution of (3) with $s>3k$. Notice that (3) has two solutions $(r,s)=(1,k)$ and $(2p^k+1,3k)$ in this case. Therefore, by Theorem 1, (71) is impossible.

Similarly, if $y\equiv 3 \pmod 4$, then we have

$$\left(\frac{p^k-1}{2}\right)(2x+1)^2+\left(\frac{3p^k+1}{2}\right)=2p^{kn}, \quad n>3. \tag{72}$$

Let $D_1=(p^k-1)/2$ and $D_2=(3p^k+1)/2$. Since (4) has two solutions $(r',s')=(1,k)$ and $(2p^k+1,3k)$ in this case, by Theorem 2, (72) is impossible. Now, the corollary is proved.

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