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GROUP EXTENSIONS AND TAME PAIRS

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Abstract. Tame pairs of groups were introduced to study the *missing boundary* problem for covers of compact 3-manifolds. In this paper we prove that if $1 \to A \to G \to B \to 1$ is an exact sequence of infinite finitely presented groups or if G is an ascending HNN-extension with base A and H is a certain type of finitely presented subgroup of A, then the pair (G, H) is tame.

Also we develop a technique for showing certain groups cannot be the fundamental group of a compact 3-manifold. In particular, we give an elementary proof of the result of R. Bieri, W. Neumann and R. Strebel:

A strictly ascending HNN-extension cannot be the fundamental group of a compact 3-manifold.

1. Introduction

We introduced the idea of a tame pair $H < G$ of groups in [M1]. The original motivation was to establish a geometric group theoretic approach to attack a well known problem (the missing boundary problem for covers of compact 3-manifolds) in 3-dimensional topology. A 3-manifold M is a *missing boundary manifold* if M is embedded in a compact manifold M_1 such that $M_1 - M$ is a subset of the boundary of M_1. It is conjectured that for any compact P_2-irreducible 3-manifold M and finitely generated subgroup $H < \pi_1(M)$, the cover of M with fundamental group H is a missing boundary manifold. In [M1], we show that if the pair $(\pi_1(M), H)$ is tame, then the cover of M with fundamental group H is a missing boundary manifold. In [M1], we consider very general combings of groups (almost prefix closed combings) and show that subgroups that are rational (quasi-convex) with respect to these combings define tame pairs of groups. Results in [B] and [E] show that the fundamental group of a closed 3-manifold satisfying Thurston's geometrization conjecture has an almost prefix closed combing. A consequence of the main theorem of [M1] is:

Theorem [M1]**.** *If H is a rational subgroup of the automatic group G, then the pair (G, H) is tame.*

Hence if M is a compact P_2-irreducible 3-manifold with automatic fundamental group and H is rational with respect to the automatic structure then the cover of M with fundamental group H is a missing boundary manifold.

As general combings and rational subgroups lead to tame pairs, one wonders what other general classes of pairs of groups are tame. Suppose M is a compact 3-manifold and there is a short exact sequence of infinite finitely generated groups $1 \to A \to \pi_1(M) \to B \to 1$. When $A \neq \mathbb{Z}$, the structure of this exact sequence and

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the structure of M is determined by J. Hempel and W. Jaco in [HJ]. In this case it is straightforward to see that if H is a finitely generated subgroup of A, then the cover of M corresponding to H is a missing boundary manifold. Hence, a natural question to ask is:

" If $1 \to A \to G \to B \to 1$ is an exact sequence of infinite finitely presented groups, which subgroups H of A are such that (G, H) is tame?"

Theorem 1. *Let $1 \to A \to G \to B \to 1$ be a short exact sequence of infinite finitely presented groups, and H a finitely generated subgroup of A of infinite index in A. Then (G, H) is tame.*

If the pair $(G, 1)$ is tame, then G has a *tame combing* in the sense of [MT]. If G has a tame combing, then G is *quasi-simply-filtrated* (see [BM1]) by Theorem 3 of [MT]. We thus have the following generalization of the main theorem of [BM2].

Corollary. *Let $1 \to A \to G \to B \to 1$ be a short exact sequence of infinite finitely presented groups, then G is tame combable.*

In the special case of $B \approx \mathbb{Z}$, Theorem 2 (below) shows that for any finitely generated subgroup H of A, (G, H) is tame.

Theorem 2. *Suppose A is a finitely presented group and $f : A \to A$ is a monomorphism. Let $G = \langle A, t : t^{-1}at = f(a)\rangle$ be the corresponding ascending HNN-extension. If B is any finitely generated subgroup of $N(A)$ ($\equiv$ the normal closure of A in G), then the pair (G, B) is tame.*

An interesting situation arises in the case of Theorem 2; when G is strictly ascending (i.e. when $f : A \to A$ is not an epimorphism), the pair (G, A) is easily shown to be not *semistable at infinity* (see §4), even though (G, A) is tame. But if G were the fundamental group of a compact P_2-irreducible 3-manifold, then we would have that the cover of M with fundamental group A would be a missing boundary manifold. It is straightforward to show that missing boundary manifolds are semistable at infinity. We thus have an elementary proof that a strictly ascending HNN-extension cannot be the fundamental group of a compact 3-manifold, a result first established by R.Bieri, W.Neumann and R.Strebel in [BNS].

This observation opens the possibility of showing a given group G is not a compact 3-manifold group by finding a subgroup H such that (G, H) is tame but not semistable at infinity.

The paper is organized as follows: In §2 we make the relevant definitions and describe the spaces in which we construct certain homotopies. In §3 we prove Theorem 1 and in §4 we prove Theorem 2.

2. Preliminaries

Let $P = \langle g_1, \ldots, g_n : r_1, \ldots, r_m\rangle$ be a presentation for the group G.

Definition. The *standard 2-complex corresponding to P*, denoted X_P, has one vertex $*$, a directed loop at $*$ labeled by g_i for each i and a 2-cell attached to the loop with label r_i for each i.

The universal cover of a space X is denoted $\tilde{X}$. The 1-skeleton of $\tilde{X}_P$ is the Cayley graph of G with respect to the generating set $\{g_1, \ldots, g_n\}$. (Hence the vertices of $\tilde{X}_P$ are the elements of G and the edges of $\tilde{X}_P$ are directed and labeled by the elements of $\{g_1, \ldots, g_n\}$.)

We work in covering spaces of standard 2-complexes. If X is such a space and Y is a subcomplex of X, then $St(Y)$ has as 1-skeleton all edges that intersect Y. A 2-cell is in $St(Y)$ if its boundary is contained in $St(Y)$. Inductively let $St^N(Y) \equiv St^{N-1}(St(Y))$ for $N \geq 1$ ($St^0(Y) \equiv Y$).

Definition. Suppose P is a finite presentation of G, H is a finitely generated subgroup of G and $*$ is a vertex of $\tilde{X}_P$. The pair (G, H) is *tame* if for each integer N there is an integer M such that for any edge path α in $Cl(\tilde{X} - St^N(H*))$ with $\alpha(0)$, $\alpha(1) \in St^N(H*)$, α is homotopic rel$\{0,1\}$ to an edge path β in $St^M(H*)$, by a homotopy in $Cl(\tilde{X} - St^N(H*))$.

In [M1], this definition is shown to be independent of presentation P, for G and Corollary 3 there states:

Theorem [M1]**.** *If M is a compact P^2-irreducible 3-manifold and H is a finitely generated subgroup of $\pi_1(M)$, then $H/\tilde{X}$ is a missing boundary manifold if and only if $(\pi_1(M), H)$ is tame.*

The following definition is used in §4:

Definition. A locally finite CW-complex X is *semistable at infinity* if for any proper ray $r : [0, \infty) \to X$ and compact set $C \subset X$ there exists a compact set D such that for any loop α based on r in $X - D$, and compact set E, α is homotopic rel r to a loop in $X - E$ by a homotopy in $X - C$.

3. The proof of Theorem 1

Let $P \equiv \langle a_1, \ldots a_n,\ h_1, \ldots, h_k,\ b_1, \ldots, b_m : r_1, \ldots r_q, s_1, \ldots, s_t\rangle$ be a presentation for G where $\langle a_1, \ldots, a_n, h_1, \ldots, h_k : r_1, \ldots, r_q\rangle$ is a presentation for the subgroup A of G, $h_1, \ldots, h_k$ are generators of H and for each $b \in \{b_1^{\pm 1}, \ldots, b_m^{\pm 1}\}$ and $a \in \{a_1^{\pm 1}, \ldots, a_n^{\pm 1}, h_1^{\pm 1} \ldots, h_k^{\pm 1}\}$ the conjugation relation $b^{-1}abw(a, b)$, for $w(a, b)$ a word in the letters $\{a_1^{\pm 1}, \ldots, a_n^{\pm 1}, h_1^{\pm 1}, \ldots, h_k^{\pm 1}\}$, is one of the relations s_i.

Let $X \equiv X_P$ and let Y be the subcomplex of X consisting of the loops and 2-cells corresponding to $a_1, \ldots, a_n, h_1, \ldots h_k$ and $r_1, \ldots, r_q$ respectively. Let $\tilde{X} \xrightarrow{q} X$ be the universal cover of X. Observe that $q^{-1}(Y)$ is a disjoint union of copies of the universal cover of Y, one for each element of B.

The edges of X are directed and labeled, one for each generator of P. Take each edge of $\tilde{X}$ to have the label and direction of the edge of X that q maps it to. Let $\tilde{X} \xrightarrow{p} Z$ be the quotient by the action of A on $\tilde{X}$. Observe that Z is an infinite, locally finite 2-complex.

We prove the following result which is equivalent to Theorem 1.

Theorem A. *For any integer N there is an integer S such that if α is an edge path in $Cl(\tilde{X} - St^N(H))$ with $\alpha(0),\ \alpha(1) \in\ St^N(H)$, then α is homotopic rel$(\{0,1\})$ to a path in $St^S(H)$ by a homotopy in $Cl(\tilde{X} - St^N(H))$.*

The proof is an easy consequence of four lemmas.

Lemma 4. *If α is an edge path in $\tilde{X}$ with $p(\alpha(0)) = p(\alpha(1))$, and $im(p\alpha) \cap p(St^N(H)) = \emptyset$, then any edge path β, in A-edges from $\alpha(0)$ to $\alpha(1)$, is homotopic rel$\{0,1\}$ to α in $\tilde{X} - St^N(H)$.*

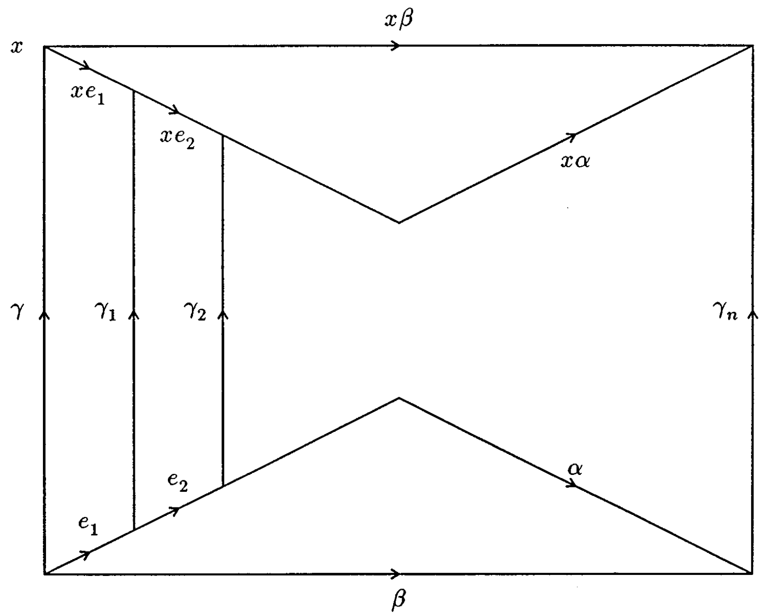

FIGURE 1

Proof. Since $im(p\alpha) \cap p(\mathrm{St}^N(H)) = \emptyset$, the copies of $\tilde{Y}$ in $\tilde{X}$ that intersect α do not intersect $\mathrm{St}^N(H)$. Let $\tilde{Y}_0$ be the copy of $\tilde{Y}$ in $\tilde{X}$ containing $\alpha(0)$ and $\tilde{Y}_*$ be the copy of $\tilde{Y}$ containing H.

Note that the normality of A in G implies:

If $\tilde{Y}_1$ and $\tilde{Y}_2$ are copies of $\tilde{Y}, y$ is a vertex of $\tilde{Y}_1$ and $d(y, \tilde{Y}_2) = n$ (here $d(y, \tilde{Y}_2)$ is the length of a minimal edge path from y to a vertex of $\tilde{Y}_2$), then for every vertex v of $\tilde{Y}_1,\ d(v, \tilde{Y}_2) = n$.

Let β be an edge path in $\tilde{Y}_0$ from $\alpha(0)$ to $\alpha(1)$. Let K be an integer such that the loop $\langle \alpha, \beta^{-1} \rangle$ is homotopically trivial in $\mathrm{St}^K(x)$ for any vertex x of $\langle \alpha, \beta^{-1} \rangle$. As H has infinite index in A, there are vertices of $\tilde{Y}_*$, arbitrarily far from H (and hence from $\mathrm{St}^N(H)$) when measured in $\tilde{Y}_*$. This implies that there are vertices of $\tilde{Y}_*$ arbitrarily far from H when measured in $\tilde{X}$. By the above note there are vertices of $\tilde{Y}_0$ arbitrarily far from $\mathrm{St}^N(H)$.

Let γ be an edge path in $\tilde{Y}_0$ from $\alpha(0)$ to a vertex x such that $\mathrm{St}^K(x) \cap \mathrm{St}^N(H)) = \emptyset$. The translate of $\langle \alpha, \beta^{-1} \rangle$ to x is homotopically trivial by a homotopy missing $\mathrm{St}^N(H)$.

Say $\alpha = \langle e_1, e_2, \ldots, e_n \rangle$. Using the 2-cells corresponding to the conjugation relations we see that $\langle e_1^{-1}, \gamma, xe_1 \rangle$ is homotopic rel$\{0, 1\}$ to an edge path γ_1 (in A-edges), by a homotopy in $\tilde{X} - \mathrm{St}^N(H)$. (In fact the image under p of this homotopy does not intersect $p(St^N(H))$.) (See Figure 1.)

Inductively $\langle e_{i+1}^{-1}, \gamma_i, xe_{i+1} \rangle$ is homotopic rel$\{0, 1\}$ to the edge path γ_{i+1} (in A-edges) by a homotopy in $\tilde{X} - \mathrm{St}^N(H)$. The loop $\langle \beta, \gamma_n, (x\beta)^{-1}, \gamma^{-1} \rangle$ is a loop in $\tilde{Y}_0$ and hence is homotopically trivial in $\tilde{Y}_0$. Patching together these homotopies as in Figure 1 gives the desired homotopy of α to β. □

Remark. The edge path β is homotopic to the A-edge path $\langle \gamma, x\beta, \gamma_n^{-1}\rangle$ by a homotopy in $\tilde{X} - St^N(H)$, and this fact only depends upon A being finitely generated (as opposed to A being finitely presented).

Next we list integers and certain finite subcomplexes of $\tilde{X}$ used extensively in the remainder of the proof.

Choose M so that for any two vertices $v, w \in St(p(\mathrm{St}^N(H)))$, there is an edge path of length $\leq M$ from v to w. Observe that $p(H)$ is a single vertex of Z.

Choose $M' > M$ such that if x, y are vertices of $St(p(\mathrm{St}^N(H))) - p(\mathrm{St}^N(H))$, in the same component of $Z - p(\mathrm{St}^N(H))$, then there is an edge path of length $\leq M'$ from x to y in $Z - p(\mathrm{St}^N(H))$.

Choose L such that if α is an edge path of length $\leq 2M' + 1$ such that $\alpha(0)$ and $\alpha(1)$ are in the same copy of $\tilde{Y}$, then there exists an edge path in A-edges from $\alpha(0)$ to $\alpha(1)$ of length $\leq L$.

Let Q be an integer such that any edge loop γ in $\tilde{X}$ of length $\leq 2M' + L + 1$ is homotopically trivial in $\mathrm{St}^Q(w)$ for any vertex of w of γ.

For each vertex $v \in \mathrm{Bd}(\mathrm{St}^{N+Q}(H))$ such that $p(v) \in Z - p(\mathrm{St}^N(H))$ take α_v to be a shortest edge path from v to a vertex of H. Let β_v be the shortest subpath of α_v beginning at v such that $p\beta_v(1) \in \mathrm{St}(p(\mathrm{St}^N(H)))$. Then β_v is an edge path of length $< Q$ such that $\beta_v(0) = v,\ \mathrm{im}(p\beta_v) \cap p(\mathrm{St}^N(H)) = \emptyset, p\beta_v(1) \in \mathrm{St}(p(\mathrm{St}^N(H)))$ and $\mathrm{im}(\beta_v) \subset \mathrm{St}^{Q+N}(H)$.

Lemma 5. *If α is an edge path in $Cl(\tilde{X} - St^{Q+N}(H))$ with $\alpha(0), \alpha(1) \in St^{Q+N}(H)$, then α is homotopic rel$\{0,1\}$, by a homotopy in $\tilde{X} - St^N(H)$, to an edge path $\langle \beta_1, \tau, \beta_2\rangle$ where for each vertex w of $\tau, p(w) \in St(p(St^N(H)))$, and $im(\beta_i) \subset St^{Q+N}(H)$ for $i \in \{1,2\}$. (I.e. β_i is "close" to H and $p(\tau)$ is "close" to $p(H)$.)*

Proof. Let $x = \alpha(0)$ and $y = \alpha(1)$. If $p(x)(p(y))$ is in $\mathrm{St}(p(St^N(H)))$, then $\beta_1(\beta_2)$ is the constant path. Otherwise let $\beta_1(\beta_2)$ be $\beta_x(\beta_y^{-1})$. We consider the case β_1 and β_2 non-trivial, as the others are completely analogous. Partition the consecutive vertices of $\langle \beta_1^{-1}, \alpha, \beta_2\rangle$ as $v_1, \ldots, v_{n(1)}, w_{n(1)+1}, \ldots, w_{n(2)}, v_{n(2)+1}, \ldots, v_{n(3)}, \ldots, v_{n(k)}$ where $p(v_i) \notin p(\mathrm{St}^N(H))$ and $p(w_i) \in p(St^N(H))$.

Define $n(0)$ to be 0.

Observe that for even i, $p(v_{n(i)+1}), p(v_{n(i+1)}) \in \mathrm{St}(p(\mathrm{St}^N(H))) - p(\mathrm{St}^N(H))$ and they lie in the same component of $Z - p(\mathrm{St}^N(H))$. Hence there is an edge path $\gamma'_{n(i+1)}$ from $p(v_{n(i+1)})$ to $p(v_{n(i)+1})$ of length $\leq M'$ in $Z - p(\mathrm{St}^N(H))$. Lift $\gamma'_{n(i+1)}$ to the vertex $v_{n(i+1)}$ and call the resulting path $\gamma_{n(i+1)}$ (see Figure 2).

For all $i, p(w_i) \in p(\mathrm{St}^N(H))$. So for odd i there is a path $\gamma'_{n(i)+1}$ in $\mathrm{St}(p(\mathrm{St}^N(H)))$ from $p(w_{n(i)+1})$ to $p(v_{n(i-1)+1})$, of length $\leq M$. Lift $\gamma'_{n(i)+1}$ to $w_{n(i)+1)}$ and call the resulting path $\gamma_{n(i)+1}$.

Observe that for odd i, $v_{n(i-1)+1}$ and the end points of $\gamma_{n(i)}$ and $\gamma_{n(i)+1}$ lie in the same copy of $\tilde{Y}$. Furthermore p maps each of these points to $p(v_{n(i-1)+1}) \in St(p(\mathrm{St}^N(H))) - p(\mathrm{St}^N(H))$, so this copy of $\tilde{Y}$ does not intersect $\mathrm{St}^N(H)$. For even i, let $\delta_{n(i)}$ (recall $n(0) = 0$) be an edge path in A-edges from $v_{n(i)+1}$ to the end point of $\gamma_{n(i+1)}$ and $\delta_{n(i+1)}$ an edge path of length $\leq L$, in A-edges, from the end point of $\gamma_{n(i+1)}$ to the end point of $\gamma_{n(i+1)+1}$. (See Figure 2.)

Now $p(\delta_{n(i)}) \subset \mathrm{St}(p(\mathrm{St}^N(H)))$ for all i, and for odd i, $p(\gamma_{n(i)+1}) \subset \mathrm{St}(p(\mathrm{St}^N(H)))$. For odd i, let the subpath of α between $w_{n(i)+1}$ and $v_{n(i+1)+1}$ be $\alpha_{n(i)}$, the subpath

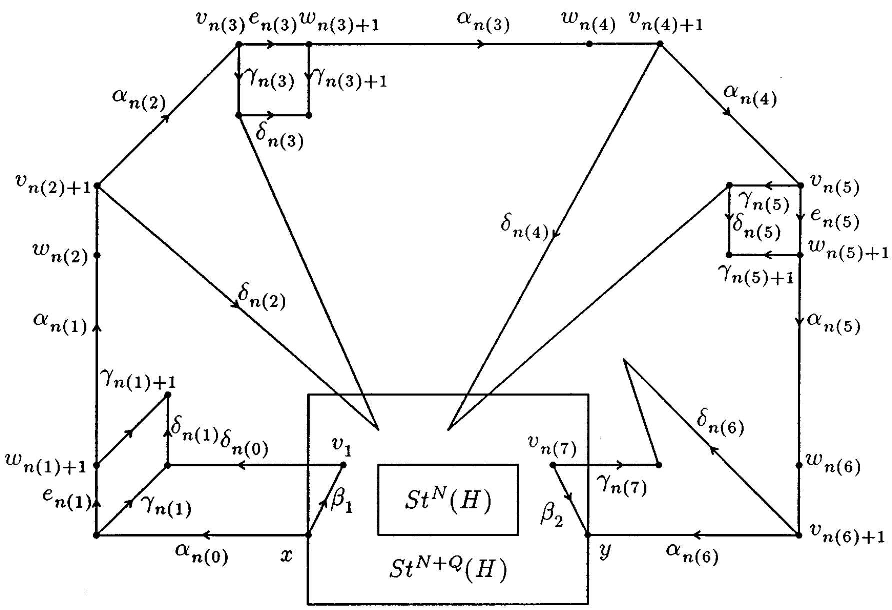

FIGURE 2

of α between $v_{n(i-1)+1}$ and $v_{n(i)}$ be $\alpha_{n(i-1)}$ and the edge between $v_{n(i)}$ and $w_{n(i)+1}$ be $e_{n(i)}$. Now

$$\lambda \equiv \langle \delta_0, \delta_{n(1)}, \gamma^{-1}_{n(1)+1}, \alpha_{n(1)}, \delta_{n(2)}, \delta_{n(3)}, \gamma^{-1}_{n(3)+1}, \alpha_{n(3)}, \dots, \delta_{n(k-1)}, \gamma^{-1}_{n(k)} \rangle$$

is such that $\operatorname{im}(p\lambda) \subset \operatorname{St}(p(\operatorname{St}^N(H)))$. It now suffices to show that λ is homotopic rel$\{0,1\}$ to $\langle \beta_1^{-1}, \alpha, \beta_2 \rangle$ by a homotopy in $\tilde{X} - \operatorname{St}^N(H)$. This follows since:

For even i in $\{1, 2, \dots, k-3\}$ the loops $\langle \delta_{n(i)}, \gamma^{-1}_{n(i+1)}, \alpha^{-1}_{n(i)} \rangle$, and the loops, $\langle \delta_0, \ \gamma^{-1}_{n(1)}, \alpha^{-1}_{n(0)}, \beta_1 \rangle$ and $\langle \delta_{n(k-1)}, \gamma^{-1}_{n(k)}, \beta_2, \alpha^{-1}_{n(k-1)} \rangle$ are homotopically trivial by a homotopy in $\tilde{X} - \operatorname{St}^N(H)$ (see Lemma 4), and for odd i, $\langle \gamma_{n(i)}, \delta_{n(i)}, \gamma_{n(i)+1}, e^{-1}_{n(i)} \rangle$ is a loop of length $\leq 2M' + L + 1$ and so by the definition of Q, is homotopically trivial by a homotopy in $\tilde{X} - \operatorname{St}^N(H)$.

Let $*$ be a vertex of $\operatorname{St}(p(\operatorname{St}^N(H))) - p(\operatorname{St}^N(H))$, and $\tilde{Y}_*$ the copy of $\tilde{Y}$ each of whose vertices is mapped by p to $*$. Note that $\tilde{Y}_* \cap \operatorname{St}^N(H) = \emptyset$. □

Lemma 6. *There is an integer S such that for any two vertices of $St^{M+Q+N}(H) \cap \tilde{Y}_*$ there is a path in A edges between them with image in $St^S(H)$.*

Proof. If v_1, v_2 are vertices of $\operatorname{St}^{M+N+Q}(H) \cap \tilde{Y}_*$ let α_i be an edge path from v_i to $x_i \in H$ of length $\leq M + N + Q$. Let $\langle e_1, \dots, e_n \rangle$ be an edge path in H-edges from x_1 to x_2. (See Figure 3.)

Recall the conjugation relations $b^{-1}abw(a,b)$ for $a \in \{a_1^{\pm 1}, \dots, a_n^{\pm 1}, h_1^{\pm 1}, \dots, h_k^{\pm 1}\}$, $b \in \{b_1^{\pm 1}, \dots b_m^{\pm 1}\}$ and $w(a,b)$ a word in the letters $\{a_1^{\pm 1}, \dots, a_n^{\pm 1}, h_1^{\pm 1}, \dots, h_k^{\pm 1}\}$. If R is an integer such that the length of $w(a,b)$ is less than R for all a, b, then there is an A-edge path between the end points of the path $\langle \alpha_1, e_i, \alpha_1^{-1} \rangle$ of length $\leq R^{|\alpha_1|} \leq R^{M+N+Q}$ for each $i \in \{1, \dots, n\}$. As the end points of each e_i are in H, there is an edge path in A-edges from v_1 to v_3 ($\equiv$ the end point of

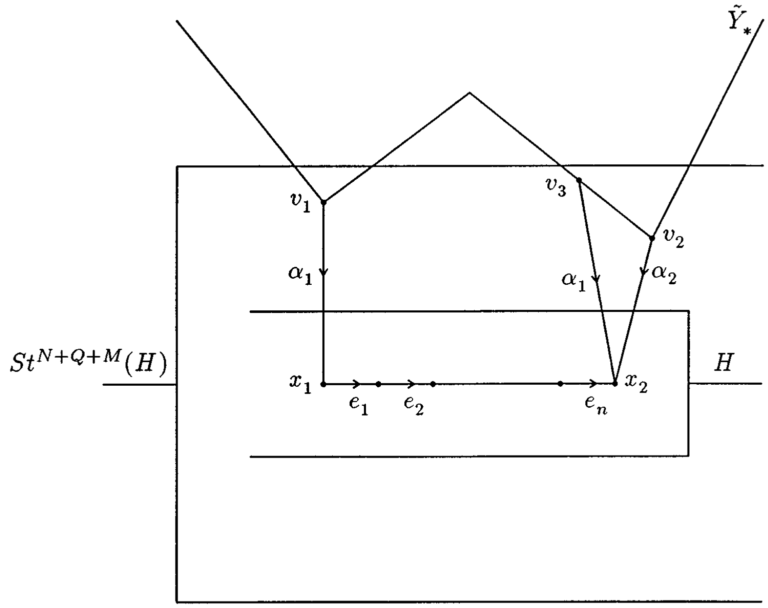

FIGURE 3

$\langle \alpha_1, e_1, \ldots, e_n, \alpha_1^{-1} \rangle$) in $\mathrm{St}^{(M+N+Q)+R^{M+N+Q}}(H)$. This edge path shows that v_3 is in $\tilde{Y}_*$. In $\tilde{X}$ the distance between v_3 and v_2 is $\leq 2(M+N+Q)$, (since both v_2 and v_3 are within $M+N+Q$ of x_2).

Choose T such that if an element of A has length $\leq 2(M+N+Q)$ in $\tilde{X}$, then it has length $\leq T$ in the A-generators. There is an A-edge path between v_3 and v_2 of length $\leq T$. Let $S = \max\{M+N+Q+R^{M+N+Q}, M+N+Q+T\}$. $\square$

Lemma 7. *If λ is an edge path in $\tilde{X} - St^N(H)$ such that $\{\lambda(0), \lambda(1)\} \subset St^{N+Q}(H)$ and for each vertex w of $\lambda, p(w) \in St(p(St^N(H)))$, then λ is homotopic rel$\{0,1\}$ to a path in $St^S(H)$ by a homotopy in $\tilde{X} - St^N(H)$.*

Proof. The path λ can be partitioned as $\langle \tau_1, \xi_1, \tau_2, \xi_2, \ldots, \tau_{n-1}, \xi_{n-1}, \tau_n \rangle$ where τ_i has image in $\mathrm{St}^S(H)$, ξ_i has image in $\mathrm{Cl}(\tilde{X} - \mathrm{St}^{Q+N}(H))$, $\{\xi_i(0), \xi_i(1)\} \subset \mathrm{Bd}(\mathrm{St}^{Q+N}(H))$ and some vertex of ξ_i is in $\tilde{X} - \mathrm{St}^S(H)$. If $\lambda = \tau_1$, we are finished. Otherwise it suffices to show that ξ_i is homotopic rel$\{0,1\}$ to a path in $St^S(H)$ by a homotopy in $\tilde{X} - St^N(H)$. Say the vertices of ξ_i are $w_0, w_1, \ldots, w_n$ and the edge of ξ_i connecting w_j and w_{j+1} is e_{j+1}.

Let γ_j be an edge path of length $\leq M$ from w_j to a vertex of $\tilde{Y}_*$. Let δ_j be an edge path of length $\leq L$ in A-edges from $\gamma_{j-1}(1)$ to $\gamma_j(1)$. As $\langle \gamma_{j-1}, \delta_j, \gamma_j^{-1}, e_j^{-1} \rangle$ is a loop of length $\leq 2M + L + 1$ containing a vertex of $\mathrm{Cl}(\tilde{X} - \mathrm{St}^{Q+N}(H))$, it is homotopically trivial in $\tilde{X} - \mathrm{St}^N(H)$. (See Figure 4.)

Hence ξ_i is homotopic rel$\{0,1\}$ to the path $\langle \gamma_0, \delta_1, \delta_2, \ldots, \delta_n, \gamma_n^{-1} \rangle$ by a homotopy missing $\mathrm{St}^N(H)$. As $\delta_1(0)$ and $\delta_n(1)$ are vertices of $\mathrm{St}^{M+Q+N}(H) \cap \tilde{Y}_*$, Lemma 6 gives an edge path β, in $\mathrm{St}^S(H) \cap \tilde{Y}_*$ from $\delta_1(0)$ to $\delta_n(1)$. Now $\langle \delta_1, \delta_2, \ldots, \delta_n, \beta^{-1} \rangle$ is an edge loop in $\tilde{Y}_*$ and so is homotopically trivial by a homotopy in $\tilde{Y}_*$. In

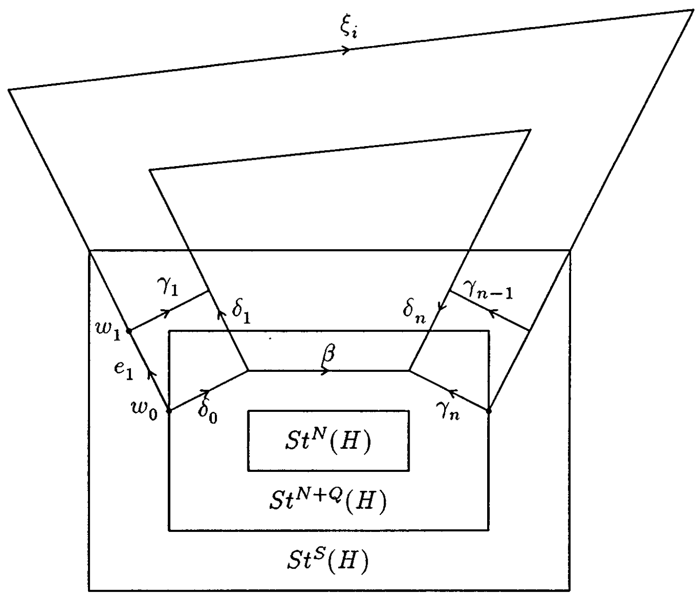

FIGURE 4

particular, this homotopy misses $\mathrm{St}^N(H)$. We have ξ_i homotopic rel$\{0,1\}$ to the path $\langle\gamma_0, \beta, \gamma_n^{-1}\rangle$ (which has image in $\mathrm{St}^S(H)$) by a homotopy in $\tilde{X} - \mathrm{St}^N(H)$. $\square$

To finish the proof of Theorem A (and Theorem 1) let $\langle\delta_0, \alpha_1, \delta_1, \alpha_2, \delta_2, \ldots, \delta_{n+1}\rangle$ be a partition of α, where $\mathrm{im}(\delta_i) \subset \mathrm{St}^S(H)$, $\alpha_i(0), \alpha_i(1) \in \mathrm{Bd}(\mathrm{St}^{N+Q}(H))$, and $\mathrm{im}(\alpha_i) \subset \mathrm{Cl}(\tilde{X} - \mathrm{St}^{N+Q}(H))$. Applying Lemmas 5 and 7 to α_i shows that α_i is homotopic rel$\{0,1\}$ to an edge path in $\mathrm{St}^S(H)$, by a homotopy in $\tilde{X} - St^N(Q)$.

4. The proof of Theorem 2

Before beginning this proof it is convenient to slightly change our definition of St. If P is a finite presentation of a group and $\hat{X}$ is a covering space of X_P then for any subcomplex Y of $\hat{X}$, $St(Y)$ is defined to be the union of Y and all (closed) 2-cells that intersect Y.

As a first step we consider the case when B is a finitely generated subgroup of A.

Proof. Let $Q = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ be a set of generators for A where $\{b_1, \ldots, b_n\}$ generates B and $\langle Q : R\rangle$ is a presentation for A. For each i and j let $w(a_i)$ and $w(b_j)$ be a word in the alphabet Q representing $f(a_i)$ and $f(b_j)$ respectively. Let P be the following presentation of G: $\langle\{t\} \cup Q : R, t^{-1}a_i t = w(a_i), t^{-1}b_j t = w(b_j)$ for each i and $j\rangle$. Let $X = X_P$. The 1-skeleton of $\tilde{X}$ is the Cayley graph of the presentation P of G. So the vertices of $\tilde{X}$ are the elements of G. Let $*$ be the identity of G. Let $h : G \to \mathbb{Z}$ be the homomorphism that kills the normal closure of A. We say that an element g of G (i.e. a vertex of $\tilde{X}$) is in level L if $h(g) = L$. Hence each vertex of the coset xA is in level $h(x)$, and if α is any word in the generators of P, representing x, then $h(x)$ is the exponent sum of t in α. The groups A and B are in level 0. The 2-cells corresponding to the conjugation relations of P can be used to *slide* an A or B edge to an edge path in the next level up. Any A or B edge e can be slid up L levels by a homotopy in $St^L(e)$. I.e. e is homotopic rel$\{0,1\}$ to a path $\langle t^L, \lambda, t^{-L}\rangle$ by a homotopy in $St^L(e)$ where λ is a path in the level, L levels above the level containing e. □

Now we need a lemma.

Lemma 8. *If γ is an edge path in levels $N+1$ and above of $\tilde{X}$ such that the end points of γ are in $St^L(B)$, then γ is homotopic rel$\{0,1\}$ to a path in $St^{2L+N+1}(B)$ by a homotopy in $\tilde{X} - St^N(B)$.*

Proof. Let γ_1, resp. γ_2, be any edge path in $St^L(B)$, from the initial point of γ, resp. from the terminal point of γ, to a point of B. Let γ_3 be an edge path in B-edges from the terminal point of γ_1 to the terminal point of γ_2. As $St^L(B)$ lies between levels $-L$ and L, the edges of the path $\tau = \langle \gamma_1, \gamma_3, \gamma_2^{-1}\rangle$ lie in levels $-L$ and above. Each edge e of τ, that lies below level $N+1$, can be slid up to level $N+1$ by a homotopy with image in $St^{L+N+1}(e) \subset St^{2L+N+1}(B)$. Hence there is a path γ_4, in levels $N+1$ and above, with the same end points as γ, and with image in $St^{2L+N+1}(B)$. As γ_4 and γ have the same end points and both paths lie in levels $N+1$ and above, the loop γ followed by γ_4^{-1} is homotopically trivial in levels $N+1$ and above. (Slide all of the edges of this loop up to a common level. Any loop in a single level lies in a copy of the universal cover corresponding to A.) □

Remark. This is the only place in this proof that we use the fact that A is finitely presented. If A were merely finitely generated and we still knew that any loop in levels K and above were homotopically trivial in levels K and above, then our proof would still work.

Suppose α is an edge path that begins and ends in $St^{3N+2}(B)$ and such that the image of α is a subset of the closure $Cl[\tilde{X} - St^{3N+2}(B)]$. It suffices to show that α is homotopic rel$\{0,1\}$ to a path in $St^{15N+11}(B)$, by a homotopy in $\tilde{X} - St^N(B)$. Clearly we can slide any A or B edge of α that lies below level $-N-1$ to level $-N-1$ by a homotopy that does not intersect $St^N(B)$ (or $St^N(A)$ for that matter). Suppose $\alpha = \langle e_1, \dots, e_k\rangle$. We may assume that each A and B edge of α lies in level $-N-1$ or above, and if e is an edge of α not in level $-N-1$, then e is in $Cl[\tilde{X} - St^{3N+2}(B)]$. We form a new path β, with the same end points as α by:

1) If e is an edge of α in a level from $-N$ to N, then slide e to level $N+1$ by a homotopy with image in $St^{2N+1}(e) \subset \tilde{X} - St^N(B)$. (So e is replaced by a path of the form $\langle t^k, \tau, t^{-k}\rangle$ where τ has image in level $N+1$.)

2) If e is an edge of α in level $-N-1$ and sliding e to level $-N$ does not intersect $St^{3N+2}(B)$, then again slide e to level $N+1$ by a homotopy with image in $\tilde{X} - St^N(B)$.

Canceling any pairs of edges of the form tt^{-1} or $t^{-1}t$ we see that α is homotopic rel$\{0,1\}$ to β, by a homotopy in $\tilde{X} - St^N(B)$, where β can have various forms depending upon where the end points of α lie. In any case, $\beta = \langle u_0, \beta_1, u_1, \beta_2, \dots, u_n, \beta_{n+1}, u_{n+1}\rangle$ such that

1) For each i, $u_i = t^{r(i)}$ and for $i \in \{1, 2, \dots, n\}$, $r(i) = \pm(2N+2)$ where the $r(i)$ alternate in sign.

2) For $i \in \{2, \dots, n\}$, the β_i alternate between edge paths in level $-N-1$ with image in $St^{3N+3}(B)$ (recall edges in level $-N-1$ not in $St^{3N+3}(B)$ were slid to level $N+1$ missing $St^N(B)$) and edge paths that begin and end in level $N+1$ and

lie in levels $N+1$ and above. The β_i of the second type satisfies the hypothesis of Lemma 8 with $L = 5N+5$ since the u_i provide paths of length $\leq 2N+2$ to a point (of a β_i of the first type) in $St^{3N+3}(B)$.

So at this stage we have:

Lemma 9. *The subpath $\langle u_1, \beta_2, \dots, u_n\rangle$ of β is homotopic rel$\{0,1\}$ to a path in $St^{11(N+1)}(B)$ by a homotopy in $\tilde{X} - St^N(B)$.*

Hence we need only deal with the paths $\langle u_0, \beta_1\rangle$ and $\langle \beta_{n+1}, u_{n+1}\rangle$ in various special cases.

If the initial point of α is in a level from $-N$ to N, then $r(0)$ is an integer in $[-2N-1, 2N+1]$, and β_1 is as in 2) above so the argument goes as above for $\langle u_0, \beta_1\rangle$. Similarly for $\langle \beta_{n+1}, u_{n+1}\rangle$ if the terminal point of α is in a level $-N$ to N.

If the initial point of α is in level $N+1$ or above, then $r(0)$ is 0 and β_1 will be an edge path in levels $N+1$ and above, that ends in level $N+1$. (This does include the "awkward" case that β_1 is a power of t.) In this case we have that the initial point of α (and hence the initial point of β_1) is in $St^{3N+2}(B)$ and u_1 is a path from the terminal point of β_1 to a point of $St^{3N+3}(B)$. Hence β_1 satisfies the hypothesis of Lemma 8, again with $L = 5N+5$. Similarly for β_{n+1} if the terminal point of α is in level $N+1$ or above.

Note also that if $n=0$ (i.e. $\beta_1 = \beta_{n+1}$), then Lemma 8 again applies to β_1, with $L \leq 5N+5$.

Finally we consider the case that the initial point of α is in a level below level $-N$. As $St^{3N+2}(B)$ lies between levels $-3N-2$ and $3N+2$, $r(0)$ (the length of u_0) is $\leq 4N+3$. Now either β_1 is in level $-N-1$ (in which case β_1 is in $St^{3N+3}(B)$ and $r(0) \leq 2N+1$) so that $\langle u_0, \beta_1\rangle$ is in $St^{5N+3}(B)$ or u_0 is in $St^{(3N+2)+(4N+3)}(B)$ and β_1 satisfies the hypothesis of Lemma 8 with $L = 7N+5$. In all cases, α is homotopic rel$\{0,1\}$ to a path in $St^{15N+11}(B)$ by a homotopy in $\tilde{X} - St^N(B)$.

This finishes the case of B a finitely generated subgroup of A.

To finish the proof of Theorem 2, suppose $\langle a_1, \dots, a_n : R\rangle$ is a presentation for A. Let $\langle a_1, \dots, a_n, t : R, t^{-1}a_it = w_i\rangle$ be a presentation for G. The Tietze move that adds a generator $h = ta_jt^{-1}$ gives the presentation $Q = \langle a_1, \dots, a_n, h, t : R, t^{-1}a_it = w_i, t^{-1}ht = a_j\rangle$ and we see that G is an ascending HNN-extension with base group, the subgroup H, of G generated by $\{a_1, \dots, a_n, h\}$. The group H need not be finitely presented (see the example following this proof), but if $\tilde{X}$ is the universal cover of the finite 2-complex corresponding to the presentation Q and α is any loop in the levels K and above of $\tilde{X}$, then by sliding all of the edges of α up to a common level we obtain a loop in the edges with labels in $\{a_1^{\pm1}, \dots, a_n^{\pm1}, h\}$. Sliding up one more level gives a loop in the edges with labels in $\{a_1^{\pm1}, \dots, a_n^{\pm1}\}$, which is trivial in that level. Hence (see the above remark), if B is a finitely generated subgroup of H, then (B, G) is tame. Now let B be a finitely generated subgroup of $N(A)$ the normal closure of A in G. Say $b_1, \dots, b_m$ are words in F ($\equiv$ the free group on $\{a_1, \dots, a_n, t\}$) representing a generating set of B. The exponent sum of t in each b_i is zero. Hence there is an integer $N \geq 0$ such that $B \leq \langle a_1, ta_1t^{-1}, \dots, t^Na_1t^{-N}, a_2, ta_2t^{-1}, \dots, t^Na_2t^{-N}, \dots, a_n, ta_nt^{-1}, \dots, t^Na_nt^{-N}\rangle \leq G$.

If we let $a_{ij} = t^ja_it^{-j}$ for all $i \in \{1, \dots, n\}$ and $j \in \{0, 1, \dots, N\}$, then using Tietze moves (as above) we obtain a presentation for G:

$Q = \langle a_{10}, \dots, a_{1N}, a_{20}, \dots a_{2N}, \dots, a_{n0}, \dots a_{nN}, t : R, t^{-1}a_{i0}t = w_i$, for $i \in \{1, \dots, n\}$, $t^{-1}a_{ij}t = a_{i(j-1)}$ for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, N\}\rangle$.

Hence if H is the subgroup of G generated by $\{a_{10}, \dots, a_{1N}, \dots, a_{n0}, \dots, a_{nN}\}$, then $A \leq H$, G is an ascending HNN-extension of H and if $\tilde{X}$ is the universal cover of the finite 2-complex corresponding to Q, then any edge loop α in levels K and above can be slid up to a common level. Sliding up N more levels gives a loop in the edges labeled $a_{10} = a_1, \dots, a_{n0} = a_n$. This loop is homotopically trivial in this level. Hence by the above Remark, we are finished. $\square$

The following example (due to J. Stallings [S] and alluded to in the above proof) is an ascending HNN extension G with base a finitely presented group A so that the subgroup of G generated by A and tat^{-1} (for some $a \in A$) is not finitely presented. (This example shows that Theorem 2 is not a restatement of the first case considered.)

Let $A = (\mathbb{Z}_p * \mathbb{Z}_q) \times (\mathbb{Z}_x * \mathbb{Z}_y)$, (where $\mathbb{Z}_k$ is the infinite cyclic group with generator k). So A has presentation $\langle p, q, x, y : [p, x],\ [p, y],\ [q, x],\ [q, y]\rangle$.

The subgroup K of A with generating set $\{x, p, qy^{-1}\}$ is normal in A and not finitely presented (see [P] or [M2] for instance).

Consider the monomorphism $f : A \to A$ defined by

$$f(p) = p, \quad f(q) = qpq^{-1}, \quad f(x) = x \quad \text{and} \quad f(y) = yxy^{-1}.$$

Let G be the ascending HNN extension of A obtained from f, so that G has presentation:

$$\langle t, p, q, x, u : t^{-1}qt = p, t^{-1}qt = qpq^{-1}, t^{-1}xt = x, t^{-1}yt = yxy^{-1}, \\ [p, x], [p, y], [q, x], [q, y]\rangle.$$

Now $K \leq A \leq G$ and we observe that K is generated by $f(A) \cup \{qy^{-1}\}$. I.e. that $K = \langle p, qpq^{-1}, x, yxy^{-1}, qy^{-1}\rangle$. (This follows since K is generated by $\{x, p, qy^{-1}\}$ and since K is normal in A.)

In G, the subgroup $K = \langle f(A) \cup \{qy^{-1}\}\rangle = \langle t^{-1}At \cup \{qy^{-1}\}\rangle$ is isomorphic to the subgroup $\langle A \cup \{t(qy^{-1})t^{-1}\}\rangle$. Hence $\langle A \cup \{t(qy^{-1})t^{-1}\}\rangle$ is not finitely generated.

Next we devise a technique to show that a finitely presented group is *not* the fundamental group of a compact 3-manifold.

First of all, following the ideas in [M1], one can show that the notion of a pair of groups being semistable is well defined. More specifically:

Proposition 1. *If X_1 and X_2 are finite simplicial complexes and there is an isomorphism of pairs $(\pi_1(X_1), A)$ to $(\pi_1(X_2), B)$, then $A/\tilde{X}_1$ is semistable at infinity iff $B/\tilde{X}_2$ is semistable at infinity.*

The next proposition is shape theoretic in nature and we refer the reader to [MS] as a basic reference.

Proposition 2. *Any missing boundary 3-manifold is semistable at infinity.*

Proof. If M is a missing boundary 3-manifold, then say M is a subset of a compact 3-manifold M_1 such that $M_1 - M$ is a subset of the boundary of M_1. The boundary components of M_1 are surfaces and if S is one such surface, then suppose C is a component of the intersection of S with the closure of M in M_1 (so that C corresponds to an end of M). Now, C is pointed 1-movable. This can be seen by altering K. Borsuk's proof that every pointed continuum in $\mathbb{R}^2$ is 1-movable (see Theorem 5 Ch. II § 8.1 [MS]) or by appealing directly to [K] or [Mc]. Hence by a theorem of J. Krasinkiewicz (see Theorem 4 Ch. II § 8.1 [MS]), C has the shape of a locally connected continuum. Using regular neighborhoods of S, we see that C

is a Z-set in M_1. Hence the end of M corresponding to C is semistable at infinity (see [G]), and so M is semistable at infinity. □

Proposition 3. *Suppose G is a finitely presented group and A is a finitely generated subgroup of G such that the pair (G, A) is tame, but not semistable at infinity. Then G is not the fundamental group of a compact 3-manifold.*

Proof. Suppose M were such a 3-manifold. Then the tameness of $(\pi_1(M), A)$ implies that $A/\tilde{M}$ is a missing boundary manifold and by Proposition 2 is semistable at infinity. But this implies that (G, H) is semistable at infinity, the desired contradiction. □

Proposition 4. *Suppose A has a presentation $\langle a_1, \dots, a_n : r_1, \dots r_m\rangle$, $f : A \to A$ is a monomorphism but not an epimorphism and G is the strictly ascending HNN-extension with presentation $P \equiv \langle t, a_1, \dots, a_n : r_1, \dots, r_m, t^{-1}a_i t = f(a_i)\rangle$. Then $\hat{X}_P \equiv A/\tilde{X}_P$ is not semistable at infinity (and so G is not the fundamental group of a compact 3-manifold).*

The motivating example is $P \equiv \langle t, x : t^{-1}xt = x^2\rangle$.

Proof. Let Y be the subcomplex of $\tilde{X}_P$ consisting of the loops labeled by the a_i union with the 2-cells given by the r_i. If $\tilde{X}_P \xrightarrow{f} X_P$ is the universal covering of X_P and $\tilde{X}_P \xrightarrow{p} \hat{X}_P$ is the quotient map, then $f^{-1}(Y)$ is a disjoint union of copies of $\tilde{Y}$.

Let $\tilde{Y}_i$ be the copy of $\tilde{Y}$ containing the vertex t^i, for $i \in \{0, -1, -2, \dots\}$. We have that $p(\tilde{Y})$ is a copy of Y in $\hat{X}_P$. Furthermore, the copies of $\tilde{Y}_i$ union the 2-cells corresponding to the conjugation relations $t^{-1}a_i t = f(a_i)$ where a_i is an edge in one of the $\tilde{Y}_i$ for $i < 0$ are mapped by p to a sort of mapping telescope T in $\hat{X}_P$. Observe that $T - p(\tilde{Y}_0)$ is a component of $\hat{X}_P$ minus the compact set $p(\tilde{Y}_0)$.

Pick an edge loop α in $p(\tilde{Y}_i)$ labeled by an element of $A - f(A)$. Then α is not homotopic to an edge loop in $p(\tilde{Y}_j)$ for any $j < i$. Hence T is not semistable at infinity and so $\hat{X}_P$ is not semistable at infinity. □

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