GROUP EXTENSIONS AND TAME PAIRS

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Abstract. Tame pairs of groups were introduced to study the missing boundary problem for covers of compact 3-manifolds. In this paper we prove that if $1 \to A \to G \to B \to 1$ is an exact sequence of infinite finitely presented groups or if $G$ is an ascending HNN-extension with base $A$ and $H$ is a certain type of finitely presented subgroup of $A$, then the pair $(G, H)$ is tame.

Also we develop a technique for showing certain groups cannot be the fundamental group of a compact 3-manifold. In particular, we give an elementary proof of the result of R. Bieri, W. Neumann and R. Strebel:

A strictly ascending HNN-extension cannot be the fundamental group of a compact 3-manifold.

1. Introduction

We introduced the idea of a tame pair $H < G$ of groups in [M1]. The original motivation was to establish a geometric group theoretic approach to attack a well known problem (the missing boundary problem for covers of compact 3-manifolds) in 3-dimensional topology. A 3-manifold $M$ is a missing boundary manifold if $M$ is embedded in a compact manifold $M_1$ such that $M_1 - M$ is a subset of the boundary of $M_1$. It is conjectured that for any compact $P_2$-irreducible 3-manifold $M$ and finitely generated subgroup $H < \pi_1(M)$, the cover of $M$ with fundamental group $H$ is a missing boundary manifold. In [M1], we show that if the pair $(\pi_1(M), H)$ is tame, then the cover of $M$ with fundamental group $H$ is a missing boundary manifold. In [M1], we consider very general combings of groups (almost prefix closed combings) and show that subgroups that are rational (quasi-convex) with respect to these combings define tame pairs of groups. Results in [B] and [E] show that the fundamental group of a closed 3-manifold satisfying Thurston’s geometrization conjecture has an almost prefix closed combing. A consequence of the main theorem of [M1] is:

Theorem [M1]. If $H$ is a rational subgroup of the automatic group $G$, then the pair $(G, H)$ is tame.

Hence if $M$ is a compact $P_2$-irreducible 3-manifold with automatic fundamental group and $H$ is rational with respect to the automatic structure then the cover of $M$ with fundamental group $H$ is a missing boundary manifold.

As general combings and rational subgroups lead to tame pairs, one wonders what other general classes of pairs of groups are tame. Suppose $M$ is a compact 3-manifold and there is a short exact sequence of infinite finitely generated groups $1 \to A \to \pi_1(M) \to B \to 1$. When $A \neq \mathbb{Z}$, the structure of this exact sequence and
the structure of $M$ is determined by J. Hempel and W. Jaco in [HJ]. In this case it is straightforward to see that if $H$ is a finitely generated subgroup of $A$, then the cover of $M$ corresponding to $H$ is a missing boundary manifold. Hence, a natural question to ask is:

“If $1 \to A \to G \to B \to 1$ is an exact sequence of infinite finitely presented groups, which subgroups $H$ of $A$ are such that $(G,H)$ is tame?”

**Theorem 1.** Let $1 \to A \to G \to B \to 1$ be a short exact sequence of infinite finitely presented groups, and $H$ a finitely generated subgroup of $A$ of infinite index in $A$. Then $(G,H)$ is tame.

If the pair $(G,1)$ is tame, then $G$ has a tame combing in the sense of [MT]. If $G$ has a tame combing, then $G$ is quasi-simply-filtrated (see [BM1]) by Theorem 3 of [MT]. We thus have the following generalization of the main theorem of [BM2].

**Corollary.** Let $1 \to A \to G \to B \to 1$ be a short exact sequence of infinite finitely presented groups, then $G$ is tame combable.

In the special case of $B \approx \mathbb{Z}$, Theorem 2 (below) shows that for any finitely generated subgroup $H$ of $A$, $(G,H)$ is tame.

**Theorem 2.** Suppose $A$ is a finitely presented group and $f:A \to A$ is a monomorphism. Let $G = \langle A, t : t^{-1}at = f(a) \rangle$ be the corresponding ascending HNN-extension. If $B$ is any finitely generated subgroup of $N(A)$ (≡ the normal closure of $A$ in $G$), then the pair $(G,B)$ is tame.

An interesting situation arises in the case of Theorem 2: when $G$ is strictly ascending (i.e. when $f:A \to A$ is not an epimorphism), the pair $(G,A)$ is easily shown to be not semistable at infinity (see §4), even though $(G,A)$ is tame. But if $G$ were the fundamental group of a compact $P_2$-irreducible 3-manifold, then we would have that the cover of $M$ with fundamental group $A$ would be a missing boundary manifold. It is straightforward to show that missing boundary manifolds are semistable at infinity. We thus have an elementary proof that a strictly ascending HNN-extension cannot be the fundamental group of a compact 3-manifold, a result first established by R.Bieri, W.Neumann and R.Strebel in [BNS].

This observation opens the possibility of showing a given group $G$ is not a compact 3-manifold group by finding a subgroup $H$ such that $(G,H)$ is tame but not semistable at infinity.

The paper is organized as follows: In §2 we make the relevant definitions and describe the spaces in which we construct certain homotopies. In §3 we prove Theorem 1 and in §4 we prove Theorem 2.

### 2. Preliminaries

Let $P = \langle g_1, \ldots, g_n : r_1, \ldots, r_m \rangle$ be a presentation for the group $G$.

**Definition.** The *standard 2-complex corresponding to $P$, denoted $X_P$, has one vertex $*$, a directed loop at $*$ labeled by $g_i$ for each $i$ and a 2-cell attached to the loop with label $r_i$ for each $i$.

The universal cover of a space $X$ is denoted $\tilde{X}$. The 1-skeleton of $\tilde{X}_P$ is the Cayley graph of $G$ with respect to the generating set $\{g_1, \ldots, g_n\}$. (Hence the vertices of $\tilde{X}_P$ are the elements of $G$ and the edges of $\tilde{X}_P$ are directed and labeled by the elements of $\{g_1, \ldots, g_n\}$.)
We work in covering spaces of standard 2-complexes. If $X$ is such a space and $Y$ is a subcomplex of $X$, then $St(Y)$ has as 1-skeleton all edges that intersect $Y$. A 2-cell is in $St(Y)$ if its boundary is contained in $St(Y)$. Inductively let $St^N(Y) \equiv St^{N-1}(St(Y))$ for $N \geq 1$ ($St^0(Y) \equiv Y$).

**Definition.** Suppose $P$ is a finite presentation of $G$, $H$ is a finitely generated subgroup of $G$ and $*$ is a vertex of $X_P$. The pair $(G, H)$ is tame if for each integer $N$ there is an integer $M$ such that for any edge path $\alpha$ in $Cl(\tilde{X} - St^N(H*))$ with $\alpha(0), \alpha(1) \in St^N(H*)$, $\alpha$ is homotopic rel $\{0, 1\}$ to an edge path $\beta$ in $St^M(H*)$, by a homotopy in $Cl(\tilde{X} - St^N(H*))$.

In [M1], this definition is shown to be independent of presentation $P$, for $G$ and Corollary 3 there states:

**Theorem [M1].** If $M$ is a compact $P^2$-irreducible 3-manifold and $H$ is a finitely generated subgroup of $\pi_1(M)$, then $H/\tilde{X}$ is a missing boundary manifold if and only if $(\pi_1(M), H)$ is tame.

The following definition is used in §4:

**Definition.** A locally finite CW-complex $X$ is semistable at infinity if for any proper ray $r : [0, \infty) \to X$ and compact set $C \subset X$ there exists a compact set $D$ such that for any loop $\alpha$ based on $r$ in $X - D$, and compact set $E$, $\alpha$ is homotopic rel $r$ to a loop in $X - E$ by a homotopy in $X - C$.

### 3. The proof of Theorem 1

Let $P \equiv \langle a_1, \ldots, a_n, b_1, \ldots, b_m : r_1, \ldots, r_q, s_1, \ldots, s_t \rangle$ be a presentation for $G$ where $\langle a_1, \ldots, a_n, b_1, \ldots, b_m : r_1, \ldots, r_q \rangle$ is a presentation for the subgroup $A$, $h_1, \ldots, h_k$ are generators of $H$ and for each $b \in \{b^k_1, \ldots, b^k_m\}$ and $a \in \{a^k_1, \ldots, a^k_n, b^k_1, \ldots, b^k_m\}$ the conjugation relation $b^{-1}abw(a, b)$, for $w(a, b)$ a word in the letters $\{a^k_1, \ldots, a^k_n, b^k_1, \ldots, b^k_m\}$, is one of the relations $s_i$.

Let $X \equiv X_P$ and let $Y$ be the subcomplex of $X$ consisting of the loops and 2-cells corresponding to $a_1, \ldots, a_n, h_1, \ldots, h_k$ and $r_1, \ldots, r_q$ respectively. Let $\tilde{X}/q \to X$ be the universal cover of $X$. Observe that $q^{-1}(Y)$ is a disjoint union of copies of the universal cover of $Y$, one for each element of $B$.

The edges of $X$ are directed and labeled, one for each generator of $P$. Take each edge of $\tilde{X}$ to have the label and direction of the edge of $X$ that $q$ maps it to. Let $\tilde{X}/q \to X$ be the quotient by the action of $A$ on $\tilde{X}$. Observe that $Z$ is an infinite, locally finite 2-complex.

We prove the following result which is equivalent to Theorem 1.

**Theorem A.** For any integer $N$ there is an integer $S$ such that if $\alpha$ is an edge path in $Cl(\tilde{X} - St^P(H))$ with $\alpha(0), \alpha(1) \in St^N(H)$, then $\alpha$ is homotopic rel $\{0, 1\}$ to a path in $St^S(H)$ by a homotopy in $Cl(\tilde{X} - St^N(H))$.

The proof is an easy consequence of four lemmas.

**Lemma 4.** If $\alpha$ is an edge path in $\tilde{X}$ with $p(\alpha(0)) = p(\alpha(1))$, and $\text{im}(p(\alpha)) \cap p(St^N(H)) = \emptyset$, then any edge path $\beta$, in $A$-edges from $\alpha(0)$ to $\alpha(1)$, is homotopic rel $\{0, 1\}$ to $\alpha$ in $\tilde{X} - St^N(H)$. 

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Proof. Since $\text{im}(p_\alpha) \cap p(\text{St}^N(H)) = \emptyset$, the copies of $\tilde{Y}$ in $\tilde{X}$ that intersect $\alpha$ do not intersect $\text{St}^N(H)$. Let $\tilde{Y}_0$ be the copy of $\tilde{Y}$ in $\tilde{X}$ containing $\alpha(0)$ and $\tilde{Y}_*$ be the copy of $\tilde{Y}$ containing $H$.

Note that the normality of $A$ in $G$ implies:

If $\tilde{Y}_1$ and $\tilde{Y}_2$ are copies of $\tilde{Y}$, $y$ is a vertex of $\tilde{Y}_1$ and $d(y, \tilde{Y}_2) = n$ (here $d(y, \tilde{Y}_2)$ is the length of a minimal edge path from $y$ to a vertex of $\tilde{Y}_2$), then for every vertex $v$ of $\tilde{Y}_1$, $d(v, \tilde{Y}_2) = n$.

Let $\beta$ be an edge path in $\tilde{Y}_0$ from $\alpha(0)$ to $\alpha(1)$. Let $K$ be an integer such that the loop $\langle \alpha, \beta^{-1} \rangle$ is homotopically trivial in $\text{St}^K(x)$ for any vertex $x$ of $\langle \alpha, \beta^{-1} \rangle$. As $H$ has infinite index in $A$, there are vertices of $\tilde{Y}_*$ arbitrarily far from $H$ (and hence from $\text{St}^N(H)$) when measured in $\tilde{Y}_*$. This implies that there are vertices of $\tilde{Y}_*$ arbitrarily far from $H$ when measured in $\tilde{X}$. By the above note there are vertices of $\tilde{Y}_0$ arbitrarily far from $\text{St}^N(H)$.

Let $\gamma$ be an edge path in $\tilde{Y}_0$ from $\alpha(0)$ to a vertex $x$ such that $\text{St}^K(x) \cap \text{St}^N(H) = \emptyset$. The translate of $\langle \alpha, \beta^{-1} \rangle$ to $x$ is homotopically trivial by a homotopy missing $\text{St}^N(H)$.

Say $\alpha = (e_1, e_2, \ldots, e_n)$. Using the 2-cells corresponding to the conjugation relations we see that $\langle e_1^{-1}, \gamma, xe_1 \rangle$ is homotopic rel$\{0, 1\}$ to an edge path $\gamma_1$ (in $A$-edges), by a homotopy in $\tilde{X} - \text{St}^N(H)$. (In fact the image under $p$ of this homotopy does not intersect $p(\text{St}^N(H))$.) (See Figure 1.)

Inductively $\langle e_{i+1}^{-1}, \gamma_i, xe_{i+1} \rangle$ is homotopic rel$\{0, 1\}$ to the edge path $\gamma_{i+1}$ (in $A$-edges) by a homotopy in $\tilde{X} - \text{St}^N(H)$. The loop $\langle \beta, \gamma_n, (x\beta)^{-1}, \gamma^{-1} \rangle$ is a loop in $\tilde{Y}_0$ and hence is homotopically trivial in $\tilde{Y}_0$. Patching together these homotopies as in Figure 1 gives the desired homotopy of $\alpha$ to $\beta$. 

\[\square\]
Remark. The edge path $\beta$ is homotopic to the $A$-edge path $\langle \gamma, x\beta, \gamma_n^{-1} \rangle$ by a homotopy in $\tilde{X} - St^N(H)$, and this fact only depends upon $A$ being finitely generated (as opposed to $A$ being finitely presented).

Next we list integers and certain finite subcomplexes of $\tilde{X}$ used extensively in the remainder of the proof.

Choose $M$ so that for any two vertices $v, w \in St(p(St^N(H)))$, there is an edge path of length $\leq M$ from $v$ to $w$. Observe that $p(H)$ is a single vertex of $Z$.

Choose $M' > M$ such that if $x, y$ are vertices of $St(p(St^N(H))) \setminus p(St^N(H))$, in the same component of $Z - p(St^N(H))$, then there is an edge path of length $\leq M'$ from $x$ to $y$ in $Z - p(St^N(H))$.

Choose $L$ such that if $\alpha$ is an edge path of length $\leq 2M' + 1$ such that $\alpha(0)$ and $\alpha(1)$ are in the same component of $\gamma$ where $\gamma$ is an edge path in $A$-edges from $\alpha(0)$ to $\alpha(1)$ of length $\leq L$.

Let $Q$ be an integer such that any edge loop $\gamma$ in $\tilde{X}$ of length $\leq 2M' + L + 1$ is homotopically trivial in $St^Q(w)$ for any vertex of $w$ of $\gamma$.

For each vertex $v \in Bd(\tilde{X} - St^{Q+N}(H))$ such that $p(v) \in Z - p(St^N(H))$ take $\alpha_v$ to be a shortest edge path from $v$ to a vertex of $H$. Let $\beta_v$ be the shortest subpath of $\alpha_v$ beginning at $v$ such that $p(\beta_v(1)) \in St(p(St^N(H)))$. Then $\beta_v$ is an edge path of length $\leq Q$ such that $\beta_v(0) = v$, $\text{im}(p(\beta_v) \cap p(St^N(H))) = \emptyset$, $\beta_v(1) \in St(p(St^N(H)))$ and $\text{im}(\beta_v) \subset St^{Q+N}(H)$.

**Lemma 5.** If $\alpha$ is an edge path in $C(\tilde{X} - St^{Q+N}(H))$ with $\alpha(0), \alpha(1) \in St^{Q+N}(H)$, then $\alpha$ is homotopic rel$\{0, 1\}$, by a homotopy in $\tilde{X} - St^N(H)$, to an edge path $\langle \beta_1, \tau, \beta_2 \rangle$ where for each vertex $w$ of $\tau, p(w) \in St(p(St^N(H)))$, and $\text{im}(\beta_i) \subset St^{Q+N}(H)$ for $i \in \{1, 2\}$. (I.e. $\beta_1$ is “close” to $H$ and $p(\tau)$ is “close” to $p(H)$.)

**Proof.** Let $x = \alpha(0)$ and $y = \alpha(1)$. If $p(x)(p(y))$ is in $St(p(St^N(H)))$, then $\beta_1(\beta_2)$ is the constant path. Otherwise let $\beta_1(\beta_2)$ be the shortest subpath of $\alpha$ non-trivial, as the others are completely analogous. Partition the consecutive vertices of $\beta_1^{-1}, \alpha, \beta_2$ as $v_1, \ldots, v_n(1), v_n(1)+1, \ldots, v_n(2), v_n(2)+1, \ldots, v_n(k)$ where $p(v_i) \notin p(St^N(H))$ and $p(w_i) \in p(St^N(H))$. Define $n(0)$ be 0.

Observe that for even $i$, $p(v_n(i)+1)$, $p(v_{n(i)+1}) \in St(p(St^N(H))) \setminus p(St^N(H))$ and they lie in the same component of $Z - p(St^N(H))$. Hence there is an edge path $\gamma_{n(i)+1}$ from $p(v_{n(i)+1})$ to $p(v_{n(i)+1})$ of length $\leq M'$ in $Z - p(St^N(H))$. Lift $\gamma_{n(i)+1}$ to the vertex $v_{n(i)+1}$ and call the resulting path $\gamma_{n(i)+1}$ (see Figure 2).

For all $i$, $p(w_i) \in p(St^N(H))$. So for odd $i$ there is a path $\gamma_{n(i)+1}$ in $St(p(St^N(H)))$ from $p(w_n(i)+1)$ to $p(v_{n(i)-1}+1)$, of length $\leq M$. Lift $\gamma_{n(i)+1}$ to $w_{n(i)+1}$ and call the resulting path $\gamma_{n(i)+1}$. Observe that for odd $i$, $w_{n(i)-1}+1$ and the end points of $\gamma_{n(i)}$ and $\gamma_{n(i)+1}$ lie in the same copy of $\tilde{Y}$. Furthermore, $p$ maps each of these points to $p(v_{n(i)-1}+1) \in St(p(St^N(H))) \setminus p(St^N(H))$, so this copy of $\tilde{Y}$ does not intersect $St^N(H)$. For even $i$, let $\Delta_n(i)$ be an edge path in $A$-edges from $v_{n(i)+1}$ to the end point of $\gamma_{n(i)+1}$ and $\delta_{n(i)+1}$ an edge path of length $\leq L$, in $A$-edges, from the end point of $\gamma_{n(i)+1}$ to the end point of $\gamma_{n(i)+1}$. (See Figure 2.)

Now $p(\delta_{n(i)}) \subset St(p(St^N(H)))$ for all $i$, and for odd $i$, $p(\gamma_{n(i)+1}) \subset St(p(St^N(H)))$. For odd $i$, let the subpath of $\alpha$ between $w_{n(i)+1}$ and $v_{n(i)+1}$ be $\alpha_n(i)$, the subpath
Proof. If \( p \) whose vertices is mapped by \( \leq \) is a loop of length \( \langle x \rangle \) of \( v \) of length \( \alpha \) trivial by a homotopy in \( \tilde{X} \). There is an integer \( e \) be an edge path in \( \ast \). Let \( (1) \) be a vertex of \( St(\tilde{X}) \) such that for any two vertices of \( St(\tilde{X}) \) \( \ast \), the loops, \( \langle \delta, \gamma_{n(1)}, \gamma_{n(0)} \rangle \) and the loops, \( \langle \delta, \gamma_{n(1)}, \gamma_{n(0)}, \beta_1 \rangle \) and \( \langle \delta, \gamma_{n(1)}, \gamma_{n(0)}, \beta_2 \rangle \) are homotopically trivial by a homotopy in \( X - \text{St}^N(H) \) (see Lemma 4), and for odd \( i \), \( \langle \gamma_{n(i)} \rangle \) is a loop of length \( \leq 2M + L + 1 \) and so by the definition of \( Q \), is homotopically trivial by a homotopy in \( X - \text{St}^N(H) \).

Let \( \ast \) be a vertex of \( St(p(\text{St}^N(H))) - p(\text{St}^N(H)) \), and \( \tilde{Y} \), the copy of \( Y \) each of whose vertices is mapped by \( p \) to \( \ast \). Note that \( \tilde{Y} \cap \text{St}^N(H) = \emptyset \) \( \Box \)

**Lemma 6.** There is an integer \( S \) such that for any two vertices of \( \text{St}^{M+Q+N}(H) \cap \tilde{Y} \), there is a path in \( A \) edges between them with image in \( \text{St}^S(H) \).

**Proof.** If \( v_1, v_2 \) are vertices of \( \text{St}^{M+Q+N}(H) \cap \tilde{Y} \), let \( \alpha_i \) be an edge path from \( v_i \) to \( x_i \in H \) of length \( \leq M + N + Q \). Let \( \langle e_1, \ldots, e_n \rangle \) be an edge path in \( H \)-edges from \( x_1 \) to \( x_2 \). (See Figure 3.)

Recall the conjugation relations \( b^{-1}ab(a, b) \) for \( a \in \{a_1^{\pm 1}, \ldots, a_n^{\pm 1}, h_1^{\pm 1}, \ldots, h_k^{\pm 1}\} \), \( b \in \{b_1^{\pm 1}, \ldots, b_{cn}^{\pm 1}\} \) and \( w(a, b) \) a word in the letters \( \{a_1^{\pm 1}, \ldots, a_n^{\pm 1}, h_1^{\pm 1}, \ldots, h_k^{\pm 1}\} \). If \( R \) is an integer such that the length of \( w(a, b) \) is less than \( R \) for all \( a, b \), then there is an \( A \)-edge path between the end points of the path \( \langle a_1, e_i, a_1^{-1} \rangle \) of length \( \leq R[a_1] \leq R^{M+N+Q} \) for each \( i \in \{1, \ldots, n\} \). As the end points of each \( e_i \) are in \( H \), there is an edge path in \( A \)-edges from \( v_1 \) to \( v_3 \) (\( \equiv \) the end point of
homotopically trivial in $\tilde{\omega}$ and for each vertex
Proof. The path $\tau B\delta(\tilde{\omega})$ gives an edge path $\xi$.
ished. Otherwise it suffices to show that
in $\tilde{\omega}$ is an edge loop in $\tilde{\omega}$, and the edge of $\xi$ has image in $\tilde{\omega}$.
Let $\gamma_1$ be an edge path of length $\leq M$ from $w_j$ to a vertex of $\tilde{Y}_*$. Let $\delta_j$ be an
each vertex $w$ of $\lambda, p(w) \in St(p(St^N(H)))$, then $\lambda$ is homotopic rel$\{0, 1\}$
to a path in $St^S(H)$ by a homotopy in $\tilde{X} - St^N(H)$.

Lemma 7. If $\lambda$ is an edge path in $\tilde{X} - St^N(H)$ such that $\{\lambda(0), \lambda(1)\} \subset St^{N+Q}(H)$
and for each vertex $w$ of $\lambda, p(w) \in St(p(St^N(H)))$, then $\lambda$ is homotopic rel$\{0, 1\}$
to a path in $St^S(H)$ by a homotopy in $\tilde{X} - St^N(H)$.

Proof. The path $\lambda$ can be partitioned as $\langle \tau_1, \xi_1, \tau_2, \xi_2, \ldots, \tau_{n-1}, \xi_{n-1}, \tau_n \rangle$ where
$\tau_i$ has image in $St^S(H)$, $\xi_i$ has image in $Cl(\tilde{X} - St^{Q+N}(H))$, $\{\xi_i(0), \xi_i(1)\} \subset$
Bd($St^{Q+N}(H)$) and some vertex of $\xi_i$ is in $\tilde{X} - St^S(H)$. If $\lambda = \tau_1$, we are
finished. Otherwise it suffices to show that $\xi_i$ is homotopic rel$\{0, 1\}$ to a path in
$St^S(H)$ by a homotopy in $\tilde{X} - St^N(H)$. Say the vertices of $\xi_i$ are $w_0, w_1, \ldots, w_n$
and the edge of $\xi_i$ connecting $w_j$ and $w_{j+1}$ is $e_{j+1}$.

Let $\gamma_j$ be an edge path of length $\leq M$ from $w_j$ to a vertex of $\tilde{Y}_*$. Let $\delta_j$ be an
edge path of length $\leq L$ in $A$-edges from $\gamma_{j-1}(1)$ to $\gamma_j(1)$. As $\langle \gamma_{j-1}, \delta_j, \gamma_j^{-1}, e_j^{-1} \rangle$
is a loop of length $\leq 2M + L + 1$ containing a vertex of $Cl(\tilde{X} - St^{Q+N}(H))$, it is
homotopically trivial in $\tilde{X} - St^N(H)$. (See Figure 4.)

Hence $\xi_i$ is homotopic rel$\{0, 1\}$ to the path $\langle \gamma_0, \delta_1, \delta_2, \ldots, \delta_n, \gamma_n^{-1} \rangle$ by a homotopy
missing $St^N(H)$. As $\delta_1(0)$ and $\delta_n(1)$ are vertices of $St^{M+Q+N}(H) \cap \tilde{Y}_*$, Lemma 6
gives an edge path $\beta_j$ in $St^S(H) \cap \tilde{Y}_*$ from $\delta_1(0)$ to $\delta_n(1)$. Now $\langle \delta_1, \delta_2, \ldots, \delta_n, \beta^{-1} \rangle$
is an edge loop in $\tilde{Y}_*$ and so is homotopically trivial by a homotopy in $\tilde{Y}_*$. In
particular, this homotopy misses $\text{St}^N(H)$. We have $\xi_i$ homotopic rel\{0,1\} to the path $⟨\gamma_0,\gamma,\gamma^{-1}⟩$ (which has image in $\text{St}^S(H)$) by a homotopy in $\tilde{X} - \text{St}^N(H)$.

To finish the proof of Theorem A (and Theorem 1) let $⟨\delta_0,\alpha_1,\delta_1,\alpha_2,\delta_2,\ldots,\delta_{n+1}⟩$ be a partition of $\alpha$, where $\text{im}(\delta_i) \subset \text{St}^S(H)$, $\alpha_i(0),\alpha_i(1) \in \text{Bd}(\text{St}^N(H))$, and $\text{im}(\alpha_i) \subset \text{Cl}(\tilde{X} - \text{St}^{N+Q}(H))$. Applying Lemmas 5 and 7 to $\alpha_i$ shows that $\alpha_i$ is homotopic rel\{0,1\} to an edge path in $\text{St}^S(H)$, by a homotopy in $\tilde{X} - \text{St}^N(Q)$.

4. The proof of Theorem 2

Before beginning this proof it is convenient to slightly change our definition of $\text{St}$. If $P$ is a finite presentation of a group and $\tilde{X}$ is a covering space of $X_P$ then for any subcomplex $Y$ of $\tilde{X}$, $\text{St}(Y)$ is defined to be the union of $Y$ and all (closed) 2-cells that intersect $Y$.

As a first step we consider the case when $B$ is a finitely generated subgroup of $A$.

Proof. Let $Q = \{a_1,\ldots,a_n,b_1,\ldots,b_m\}$ be a set of generators for $A$ where $\{b_1,\ldots,b_m\}$ generates $B$ and $\langle Q : R \rangle$ is a presentation for $A$. For each $i$ and $j$ let $w(a_i)$ and $w(b_j)$ be a word in the alphabet $Q$ representing $f(a_i)$ and $f(b_j)$ respectively. Let $P$ be the following presentation of $G$: $\langle \{t\} \cup Q : R, t^{-1}a_it = w(a_i), t^{-1}b_jt = w(b_j) \rangle$ for each $i$ and $j$). Let $X = X_P$. The 1-skeleton of $\tilde{X}$ is the Cayley graph of the presentation $P$ of $G$. So the vertices of $\tilde{X}$ are the elements of $G$. Let $*$ be the identity of $G$. Let $h : G \to \mathbb{Z}$ be the homomorphism that kills the normal closure of $A$. We say that an element $g$ of $G$ (i.e. a vertex of $\tilde{X}$) is in level $L$ if $h(g) = L$. Hence each vertex of the coset $xA$ is in level $h(x)$, and if $\alpha$ is any word in the generators of $P$, representing $x$, then $h(x)$ is the exponent sum of $t$ in $\alpha$. The groups $A$ and $B$ are in level 0. The 2-cells corresponding to the conjugation
relations of $P$ can be used to slide an $A$ or $B$ edge to an edge path in the next level up. Any $A$ or $B$ edge $e$ can be slid up $L$ levels by a homotopy in $\widetilde{St}^L(e)$. i.e. $e$ is homotopic rel $\{0,1\}$ to a path $t^L, \lambda, t^{-L}$ by a homotopy in $\widetilde{St}^L(e)$ where $\lambda$ is a path in the level, $L$ levels above the level containing $e$.

Now we need a lemma.

**Lemma 8.** If $\gamma$ is an edge path in levels $N+1$ and above of $\tilde{X}$ such that the end points of $\gamma$ are in $\widetilde{St}^L(B)$, then $\gamma$ is homotopic rel $\{0,1\}$ to a path in $\widetilde{St}^{2L+N+1}(B)$ by a homotopy in $\tilde{X} - \widetilde{St}^N(B)$.

**Proof.** Let $\gamma_1$, resp. $\gamma_2$, be any edge path in $\widetilde{St}^L(B)$, from the initial point of $\gamma$, resp. from the terminal point of $\gamma$, to a point of $B$. Let $\gamma_3$ be an edge path in $B$-edges from the terminal point of $\gamma_1$ to the terminal point of $\gamma_2$. As $\widetilde{St}^L(B)$ lies between levels $-L$ and $L$, the edges of the path $\tau = (\gamma_1, \gamma_3, \gamma_2^{-1})$ lie in levels $-L$ and above. Each edge of $\tau$, that lies below level $N + 1$, can be slid up to level $N+1$ by a homotopy with image in $St^{L+N+1} \subset St^{2L+N+1}(B)$. Hence there is a path $\gamma_4$, in levels $N+1$ and above, with the same end points as $\gamma$, and with image in $St^{2L+N+1}(B)$. As $\gamma_4$ and $\gamma$ have the same end points and both paths lie in levels $N+1$ and above, the loop $\gamma$ followed by $\gamma_4^{-1}$ is homotopically trivial in levels $N+1$ and above. (Slide all of the edges of this loop up to a common level. Any loop in a single level lies in a copy of the universal cover corresponding to $A$.)

**Remark.** This is the only place in this proof that we use the fact that $A$ is finitely presented. If $A$ were merely finitely generated and we still knew that any loop in levels $K$ and above were homotopically trivial in levels $K$ and above, then our proof would still work.

Suppose $\alpha$ is an edge path that begins and ends in $\widetilde{St}^{3N+2}(B)$ and such that the image of $\alpha$ is a subset of the closure $Cl[\tilde{X} - \widetilde{St}^{3N+2}(B)]$. It suffices to show that $\alpha$ is homotopic rel $\{0,1\}$ to a path in $\widetilde{St}^{15N+11}(B)$, by a homotopy in $\tilde{X} - \widetilde{St}^N(B)$. Clearly we can slide any $A$ or $B$ edge of $\alpha$ that lies below level $-N - 1$ to level $-N - 1$ by a homotopy that does not intersect $\widetilde{St}^N(B)$ (or $\widetilde{St}^N(A)$ for that matter). Suppose $\alpha = \langle e_1, \ldots, e_k \rangle$. We may assume that each $A$ and $B$ edge of $\alpha$ lies in level $-N - 1$ or above, and if $e$ is an edge of $\alpha$ not in level $-N - 1$, then $e$ is in $Cl[\tilde{X} - \widetilde{St}^{3N+2}(B)]$. We form a new path $\beta$, with the same end points as $\alpha$ by:

1) If $e$ is an edge of $\alpha$ in a level from $-N$ to $N$, then slide $e$ to level $N + 1$ by a homotopy with image in $\widetilde{St}^{2N+1}(e) \subset \tilde{X} - \widetilde{St}^N(B)$. (So $e$ is replaced by a path of the form $\langle t^k, \tau, t^{-k} \rangle$ where $\tau$ has image in level $N + 1$.)

2) If $e$ is an edge of $\alpha$ in level $-N - 1$ and sliding $e$ to level $-N$ does not intersect $\widetilde{St}^{3N+2}(B)$, then again slide $e$ to level $N + 1$ by a homotopy with image in $\tilde{X} - \widetilde{St}^N(B)$.

Canceling any pairs of edges of the form $tt^{-1}$ or $t^{-1}t$ we see that $\alpha$ is homotopic rel $\{0,1\}$ to $\beta$, by a homotopy in $\tilde{X} - \widetilde{St}^N(B)$, where $\beta$ can have various forms depending upon where the end points of $\alpha$ lie. In any case, $\beta = \langle u_0, \beta_1, u_1, \beta_2, \ldots, u_n, \beta_{n+1}, u_{n+1} \rangle$ such that

1) For each $i$, $u_i = tr(i)$ and for $i \in \{1, 2, \ldots, n\}$, $r(i) = \pm(2N + 2)$ where the $r(i)$ alternate in sign.

2) For $i \in \{2, \ldots, n\}$, the $\beta_i$ alternate between edge paths in level $-N - 1$ with image in $\widetilde{St}^{3N+3}(B)$ (recall edges in level $-N - 1$ not in $\widetilde{St}^{3N+3}(B)$ were slid to level $N + 1$ missing $\widetilde{St}^N(B)$) and edge paths that begin and end in level $N + 1$ and
lie in levels $N + 1$ and above. The $\beta_i$ of the second type satisfies the hypothesis of Lemma 8 with $L = 5N + 5$ since the $u_i$ provide paths of length $\leq 2N + 2$ to a point (of a $\beta_i$ of the first type) in $St^{3N+3}(B)$.

So at this stage we have:

**Lemma 9.** The subpath $(u_1, \beta_2, \ldots, u_n)\) of $\beta$ is homotopic rel$(0,1)$ to a path in $St^{11(N+1)}(B)$ by a homotopy in $X - St^N(B)$.

Hence we need only deal with the paths $(u_0, \beta_1)$ and $(\beta_{n+1}, u_{n+1})$ in various special cases.

If the initial point of $\alpha$ is in a level from $-N$ to $N$, then $r(0)$ is an integer in $[-2N - 1, 2N + 1]$, and $\beta_1$ is as in 2) above so the argument goes as above for $(u_0, \beta_1)$. Similarly for $(\beta_{n+1}, u_{n+1})$ if the terminal point of $\alpha$ is in a level $-N$ to $N$.

If the initial point of $\alpha$ is in level $N + 1$ or above, then $r(0)$ is 0 and $\beta_1$ will be an edge path in levels $N + 1$ and above, that ends in level $N + 1$. (This does include the “awkward” case that $\beta_1$ is a power of $t$.) In this case we have that the initial point of $\alpha$ (and hence the initial point of $\beta_1$) is in $St^{3N+2}(B)$ and $u_1$ is a path from the terminal point of $\beta_1$ to a point of $St^{3N+3}(B)$. Hence $\beta_1$ satisfies the hypothesis of Lemma 8, again with $L = 5N + 5$. Similarly for $\beta_{n+1}$ if the terminal point of $\alpha$ is in level $N + 1$ or above.

Note also that if $n = 0$ (i.e. $\beta_1 = \beta_{n+1}$), then Lemma 8 again applies to $\beta_1$, with $L \leq 5N + 5$.

Finally we consider the case that the initial point of $\alpha$ is in a level below level $-N$. As $St^{3N+2}(B)$ lies between levels $-3N - 2$ and $3N + 2$, $r(0)$ (the length of $u_0$) is $\leq 4N + 3$. Now either $\beta_1$ is in level $-N - 1$ (in which case $\beta_1$ is in $St^{3N+3}(B)$ and $r(0) \leq 2N + 1$) so that $(u_0, \beta_1)$ is in $St^{5N+3}(B)$ or $u_0$ is in $St^{3N+2} + (4N + 3)$ $(B)$ and $\beta_1$ satisfies the hypothesis of Lemma 8 with $L = 7N + 5$. In all cases, $\alpha$ is homotopic rel$(0,1)$ to a path in $St^{15N+11}(B)$ by a homotopy in $X - St^N(B)$.

This finishes the case of $B$ a finitely generated subgroup of $A$.

To finish the proof of Theorem 2, suppose $(a_1, \ldots, a_n : R)$ is a presentation for $A$. Let $(a_1, \ldots, a_n : t : R, t^{-1}a_it = w_i)$ be a presentation for $G$. The Tietze move that adds a generator $h = ta_it^{-1}$ gives the presentation $Q = \langle a_1, \ldots, a_n, h, t : R, t^{-1}a_it = w_i, t^{-1}ht = a_j \rangle$ and we see that $G$ is an ascending HNN-extension with base group, the subgroup $H$, of $G$ generated by $(a_1, \ldots, a_n, h)$. The group $H$ need not be finitely presented (see the example following this proof), but if $X$ is the universal cover of the finite 2-complex corresponding to the presentation $Q$ and $\alpha$ is any loop in the levels $K$ and above of $X$, then by sliding all of the edges of $\alpha$ up to a common level we obtain a loop in the edges with labels in $\{a_1^{\pm 1}, a_2^{\pm 1}, h\}$. Sliding up one more level gives a loop in the edges with labels in $\{a_1^{\pm 1}, a_2^{\pm 1}, a_3^{\pm 1}\}$, which is trivial in that level. Hence (see the above remark), if $B$ is a finitely generated subgroup of $H$, then $(B, G)$ is tame. Now let $B$ be a finitely generated subgroup of $N(A)$ the normal closure of $A$ in $G$. Say $b_1, \ldots, b_m$ are words in $F$ ($\equiv$ the free group on $\{a_1, \ldots, a_n, t\}$) representing a generating set of $B$. The exponent sum of $t$ in each $b_i$ is zero. Hence there is an integer $N \geq 0$ such that $B \leq \langle a_1, a_1t^{-1}, \ldots, t^N a_1t^{-N}, a_2, a_2t^{-1}, \ldots, t^N a_2t^{-N}, \ldots, a_n, a_2t^{-1}, \ldots, t^N a_nt^{-N} \rangle \leq G$.

If we let $a_{ij} = t^i a_{i-1}t^{-j}$ for all $i \in \{1, \ldots, n\}$ and $j \in \{0, 1, \ldots, N\}$, then using Tietze moves (as above) we obtain a presentation for $G$:

$Q = \langle a_{10}, \ldots, a_{1N}, a_{20}, \ldots, a_{2N}, \ldots, a_{n0}, \ldots, a_{nN}, t : R, t^{-1}a_{0t} = w_i, t^{-1}a_{ij}t = a_{i(j-1)} \rangle$ for $i \in \{1, \ldots, n\}$, $t^{-1}a_{ij}t = a_{i(j-1)}$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, N\}$.
Hence if $H$ is the subgroup of $G$ generated by \{\(a_{10}, \ldots, a_{1N}, \ldots, a_{n0}, \ldots, a_{nN}\)\}, then $A \leq H$, $G$ is an ascending HNN-extension of $H$ and if $\tilde{X}$ is the universal cover of the finite 2-complex corresponding to $Q$, then any edge loop $\alpha$ in levels $K$ and above can be slid up to a common level. Sliding up $N$ more levels gives a loop in the edges labeled $a_{10} = a_1, \ldots, a_{n0} = a_n$. This loop is homotopically trivial in this level. Hence by the above Remark, we are finished.

The following example (due to J. Stallings [S] and alluded to in the above proof) is an ascending HNN extension $G$ with base a finitely presented group $A$ so that the subgroup of $G$ generated by $A$ and $tat^{-1}$ (for some $a \in A$) is not finitely presented. (This example shows that Theorem 2 is not a restatement of the first case considered.)

Let $A = (\mathbb{Z}_p * \mathbb{Z}_q) \times (\mathbb{Z}_x * \mathbb{Z}_y)$, (where $\mathbb{Z}_k$ is the infinite cyclic group with generator $k$). So $A$ has presentation $\langle p, q, x, y : [p, x], [p, y], [q, x], [q, y]\rangle$.

The subgroup $K$ of $A$ with generating set $\{x, p, qy^{-1}\}$ is normal in $A$ and not finitely presented (see [P] or [M2] for instance).

Consider the monomorphism $f : A \to A$ defined by

$$f(p) = p, \quad f(q) = qpy^{-1}, \quad f(x) = x \quad \text{and} \quad f(y) = yxy^{-1}.$$ 

Let $G$ be the ascending HNN extension of $A$ obtained from $f$, so that $G$ has presentation:

$$\langle t, p, q, x, u : t^{-1}qt = p, t^{-1}qt = qpy^{-1}, t^{-1}xt = x, t^{-1}yt = yxy^{-1},\rangle$$

$$\langle [p, x], [p, y], [q, x], [q, y]\rangle.$$ 

Now $K \leq A \leq G$ and we observe that $K$ is generated by $f(A) \cup \{qy^{-1}\}$. I.e. that $K = \langle p, qpy^{-1}, x, yxy^{-1}, qy^{-1}\rangle$. (This follows since $K$ is generated by $\{x, p, qy^{-1}\}$ and since $K$ is normal in $A$.)

In $G$, the subgroup $K = \langle f(A) \cup \{qy^{-1}\} \rangle = \langle t^{-1}At \cup \{qy^{-1}\} \rangle$ is isomorphic to the subgroup $\langle A \cup \{t(qy^{-1})t^{-1}\} \rangle$. Hence $\langle A \cup \{t(qy^{-1})t^{-1}\} \rangle$ is not finitely generated.

Next we devise a technique to show that a finitely presented group is not the fundamental group of a compact 3-manifold.

First of all, following the ideas in [M1], one can show that the notion of a pair of groups being semistable is well defined. More specifically:

**Proposition 1.** If $X_1$ and $X_2$ are finite simplicial complexes and there is an isomorphism of pairs $(\pi_1(X_1), A)$ to $(\pi_1(X_2), B)$, then $A/X_1$ is semistable at infinity iff $B/X_2$ is semistable at infinity.

The next proposition is shape theoretic in nature and we refer the reader to [MS] as a basic reference.

**Proposition 2.** Any missing boundary 3-manifold is semistable at infinity.

**Proof.** If $M$ is a missing boundary 3-manifold, then say $M$ is a subset of a compact 3-manifold $M_1$ such that $M_1 - M$ is a subset of the boundary of $M_1$. The boundary components of $M_1$ are surfaces and if $S$ is one such surface, then suppose $C$ is a component of the intersection of $S$ with the closure of $M$ in $M_1$ (so that $C$ corresponds to an end of $M$). Now, $C$ is pointed 1-movable. This can be seen by altering K. Borsuk’s proof that every pointed continuum in $\mathbb{R}^2$ is 1-movable (see Theorem 5 Ch. II § 8.1 [MS]) or by appealing directly to [K] or [Mc]. Hence by a theorem of J. Krasinkiewicz (see Theorem 4 Ch. II § 8.1 [MS]), $C$ has the shape of a locally connected continuum. Using regular neighborhoods of $S$, we see that $C$
is a Z-set in $M_1$. Hence the end of $M$ corresponding to $C$ is semistable at infinity (see [G]), and so $M$ is semistable at infinity.

**Proposition 3.** Suppose $G$ is a finitely presented group and $A$ is a finitely generated subgroup of $G$ such that the pair $(G, A)$ is tame, but not semistable at infinity. Then $G$ is not the fundamental group of a compact 3-manifold.

**Proof.** Suppose $M$ were such a 3-manifold. Then the tameness of $(\pi_1(M), A)$ implies that $A/\tilde{M}$ is a missing boundary manifold and by Proposition 2 is semistable at infinity. But this implies that $(G, H)$ is semistable at infinity, the desired contradiction.

**Proposition 4.** Suppose $A$ has a presentation $\langle a_1, \ldots, a_n : r_1, \ldots, r_m \rangle$, $f : A \to A$ is a monomorphism but not an epimorphism and $G$ is the strictly ascending HNN-extension with presentation $P \equiv \langle t, a_1, \ldots, a_n : r_1, \ldots, r_m, t^{-1}a_it = f(a_i) \rangle$. Then $\hat{X}_P \equiv A/\hat{X}_P$ is not semistable at infinity (and so $G$ is not the fundamental group of a compact 3-manifold).

The motivating example is $P \equiv \langle t, x : t^{-1}xt = x^2 \rangle$.

**Proof.** Let $Y$ be the subcomplex of $\hat{X}_P$ consisting of the loops labeled by the $a_i$ union with the 2-cells given by the $r_i$. If $\hat{X}_P \xrightarrow{f} X_P$ is the universal covering of $X_P$ and $\hat{X}_P \xrightarrow{p} \hat{X}_P$ is the quotient map, then $f^{-1}(Y)$ is a disjoint union of copies of $\hat{Y}$.

Let $\hat{Y}_i$ be the copy of $\hat{Y}$ containing the vertex $t^i$, for $i \in \{0, -1, -2, \ldots \}$. We have that $p(\hat{Y})$ is a copy of $Y$ in $\hat{X}_P$. Furthermore, the copies of $\hat{Y}_i$ union the 2-cells corresponding to the conjugation relations $t^{-1}a_it = f(a_i)$ where $a_i$ is an edge in one of the $\hat{Y}_i$ for $i < 0$ are mapped by $p$ to a sort of mapping telescope $T$ in $\hat{X}_P$. Observe that $T - p(\hat{Y}_0)$ is a component of $\hat{X}_P$ minus the compact set $p(\hat{Y}_0)$.

Pick an edge loop $\alpha$ in $p(\hat{Y}_i)$ labeled by an element of $A - f(A)$. Then $\alpha$ is not homotopic to an edge loop in $p(\hat{Y}_j)$ for any $j < i$. Hence $T$ is not semistable at infinity and so $\hat{X}_P$ is not semistable at infinity.

**References**


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