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RIESZ TRANSFORMS FOR $1 \leq p \leq 2$

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Abstract. It has been asked (see R. Strichartz, *Analysis of the Laplacian...*, J. Funct. Anal. **52** (1983), 48–79) whether one could extend to a reasonable class of non-compact Riemannian manifolds the L^p boundedness of the Riesz transforms that holds in $\mathbb{R}^n$. Several partial answers have been given since. In the present paper, we give positive results for $1 \leq p \leq 2$ under very weak assumptions, namely the doubling volume property and an optimal on-diagonal heat kernel estimate. In particular, we do not make any hypothesis on the space derivatives of the heat kernel. We also prove that the result cannot hold for $p > 2$ under the same assumptions. Finally, we prove a similar result for the Riesz transforms on arbitrary domains of $\mathbb{R}^n$.

1. Introduction

Let M be a complete Riemannian manifold, d be the geodesic distance on M, and $d\mu$ be the Riemannian measure. Denote by $B(x,r)$ the geodesic ball of center $x \in M$ and radius $r > 0$ and by $V(x,r)$ its Riemannian volume $\mu(B(x,r))$.

One says that M satisfies the doubling volume property if there exists C such that

$$V(x,2r) \leq C\, V(x,r),\ \forall\, x \in M,\, r > 0.$$

Let Δ be the Laplace-Beltrami operator on M, $e^{-t\Delta}$ be the associated heat semigroup, and $p_t(x,y)$ be the heat kernel on M, i.e. the kernel of $e^{-t\Delta}$. Let ∇ be the Riemannian gradient.

For $f \in \mathcal{C}_0^\infty(M)$, denote by $\| f \|_p$ the L^p norm of f with respect to $d\mu$, and by $\| f \|_{1,\infty}$ the quantity $\sup_{\lambda>0} \lambda\mu(\{x; |f(x)| > \lambda\})$.

Our main result is

Theorem 1.1. *Let M be a complete Riemannian manifold satisfying the doubling volume property and such that*

$$p_t(x,x) \leq \frac{C}{V(x,\sqrt{t})},$$

for all $x \in M$, $t > 0$ and some $C > 0$. Then the Riesz transform $T = \nabla\Delta^{-1/2}$ is weak $(1,1)$ and bounded on L^p, $1 < p \leq 2$. That is, there exists C_p, $1 \leq p \leq 2$, such that, $\forall\, f \in \mathcal{C}_0^\infty(M)$,

$$\| \,|\nabla f|\, \|_p \leq C_p \left\| \Delta^{1/2} f \right\|_p,\ 1 < p \leq 2,$$

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and

$$\| \, |\nabla f| \, \|_{1,\infty} \leq C_1 \left\| \Delta^{1/2} f \right\|_1 .$$

Remarks. -Integration by parts shows that

$$\| \, |\nabla f| \, \|_2 = \left\| \Delta^{1/2} f \right\|_2 , \ \forall \, f \in \mathcal{C}_0^\infty(M),$$

therefore T is obviously bounded from L^2 into itself, and the real issue is to prove that T is weak $(1,1)$; the L^p boundedness, $1 < p < 2$, then follows by interpolation.

-It is a standard fact (see [3], p. 172 or [5], p. 36) that for a fixed $p \in]1, +\infty[$ the inequality

$$\| \, |\nabla f| \, \|_p \leq C \left\| \Delta^{1/2} f \right\|_p , \ \forall \, f \in \mathcal{C}_0^\infty(M),$$

implies by duality

$$\left\| \Delta^{1/2} f \right\|_q \leq C \, \| \, |\nabla f| \, \|_q , \ \forall \, f \in \mathcal{C}_0^\infty(M),$$

where q is the conjugate exponent of p. The converse is not clear.

Let us compare Theorem 1.1 with the existing results about Riesz transforms. If M has non-negative Ricci curvature (in which case the assumptions of Theorem 1.1 are satisfied), Bakry proves in [4], using the Littlewood-Paley theory, that the Riesz transforms on M are L^p bounded, $1 < p < +\infty$. Note that he does not go through weak $(1,1)$ type.

Another approach consists in showing that the Riesz transforms are Calderón-Zygmund operators, using estimates of the heat kernel (see [31] for the classical Euclidean setting, and [7], [22] for the Riemannian setting). At first sight, this requires a pointwise estimate of two spatial derivatives of the heat kernel that is highly nontrivial. Already the estimate of the first derivative uses the gradient estimates of Li and Yau ([23]).

This approach has also been followed in the Lie group setting (which is also covered by our method, see the remark below): for $1 \leq p < 2$, Saloff-Coste, in [28], uses a trick that is specific to the group structure to get the estimate on the first derivative and then the estimate on the second one for free. Alexopoulos gets the complete result for $1 \leq p < +\infty$ by more intricate methods ([1]).

Finally, in a general situation, [14] shows that again for $1 \leq p < 2$, it is enough to get an pointwise estimate on the first derivative. Such an estimate follows from some strong form of the parabolic Harnack principle, but there are many natural situations where it can be false, for example on manifolds that are quasi-isometric to a manifold with non-negative Ricci curvature, or where it is not clear, for example on covering manifolds.

However, [19] shows that at least a weighted L^2 estimate of the first spatial derivative of the heat kernel can be derived from the upper estimate of the heat kernel itself without any further assumption. Our contribution here is to point out that this L^2 estimate is enough to apply the method of [14].

Examples. - The assumptions of Theorem 1.1 are satisfied on manifolds where a parabolic Harnack principle holds (see [29]), for instance manifolds that are quasi-isometric to a manifold with non-negative Ricci curvature or cocompact covering manifolds whose deck transformation group has polynomial growth. The L^p boundedness of the Riesz transforms is new in these two situations. One can expect it to hold also for $p > 2$.

- It is easy to construct manifolds that satisfy the assumptions of Theorem 1.1 but where the parabolic Harnack principle is false. A typical example is the following, considered in [30], see also [9], §4: take two copies of $\mathbb{R}^2 \setminus B(0,1)$, and glue them smoothly along the unit circles. For more information, see §4.

- A natural question is whether Theorem 1.1 also holds for $p > 2$. We will use the above example to answer negatively in §4. This answer could be expected from an unpublished counterexample of Kenig [21] (using an idea of Meyers) of a self-adjoint second order elliptic operator (with discontinuous coefficients) on $\mathbb{R}^2$ such that the associated Riesz transforms are unbounded for $p > 2 + \varepsilon$. For a precise statement, see [2], §5, Lemma 4. For another negative result, see [11].

Remark. The Riemannian structure is simply a convenient setting for this paper, but our method covers more general situations, e.g. manifolds endowed with a second-order subelliptic operator. Even more generally, we could consider abstract diffusion semigroups in the sense of Bakry ([3]). It is proved in [27] that the Bakry-Emery assumption $\Gamma_2 \ge 0$ is enough to get Li-Yau's gradient estimate, therefore the heat kernel hypothesis of Theorem 1.1. Note that strangely enough, the inequality given by Theorem 1.1 for $1 < p < 2$ is the one out of the four cases (p smaller or bigger than 2, domination of the "carré du champ" by the square root of the generator or the converse) that Bakry does not get in [3]. Note also that it follows from [31] that the conclusion of Theorem 1.1 cannot hold for Markov semigroups without the diffusion assumption. This was observed by Silverstein (see [26]).

The above theorem admits a local version:

Theorem 1.2. *Let M be a complete Riemannian manifold satisfying the local doubling volume property*

$$\forall\, r_0 > 0,\ \exists\, C_{r_0} \text{ such that } V(x,2r) \le C_{r_0}\, V(x,r),\ \forall\, x \in M,\ r \in]0, r_0[,$$

and whose volume growth at infinity is at most exponential in the sense that

$$V(x,\theta r) \le C e^{c\theta} V(x,r),\ \forall\, x \in M,\ \theta > 1,\ r \le 1.$$

Suppose that

$$p_t(x,x) \le \frac{C'}{V(x,\sqrt{t})},$$

for all $x \in M$ and $t \in]0,1]$. Then there exists C_p, $1 \le p \le 2$, such that, $\forall\, f \in \mathcal{C}_0^\infty(M)$,

$$\|\, |\nabla f|\, \|_p \le C_p \left(\left\| \Delta^{1/2} f \right\|_p + \|\, f\, \|_p \right),\ 1 < p \le 2,$$

and

$$\|\, |\nabla f|\, \|_{1,\infty} \le C_1 \left(\left\| \Delta^{1/2} f \right\|_1 + \|\, f\, \|_1 \right).$$

Again, Theorem 1.2 is a generalisation of the result of Bakry ([4], thm. 4.1) that treats the case of Ricci curvature bounded below. As a corollary, one gets the following generalisation of the result of Lohoué ([24]) about Cartan-Hadamard manifolds. Let us recall that in their specific situations, [4] and [24] are able to treat also the case $p > 2$.

Theorem 1.3. *Let M be a complete Riemannian manifold such that*

$$V(x,2r) \leq C\,V(x,r),\ \forall\, x \in M,\ r \in]0,1[,$$

$$V(x,\theta r) \leq Ce^{c\theta} V(x,r),\ \forall\, x \in M,\ \theta > 1,\ r \leq 1,$$

and

$$p_t(x,x) \leq \frac{C'}{V(x,\sqrt{t})},\ \forall\, x \in M,\ t \in]0,1].$$

Assume further that M has a spectral gap $\lambda > 0$:

$$\lambda \, \|\, f\,\|_2 \leq \|\, \Delta f\,\|_2\,,\ \forall\, f \in \mathcal{C}_0^\infty(M).$$

Then there exists C_p, $1 < p \leq 2$, such that

$$\|\, |\nabla f|\, \|_p \leq C_p \left\| \Delta^{1/2} f \right\|_p,\ \forall\, f \in \mathcal{C}_0^\infty(M).$$

Let us explain here why Theorem 1.3 follows from Theorem 1.2. The spectral gap assumption means that

$$\left\| e^{-t\Delta} \right\|_{2\to 2} \leq e^{-\lambda t}.$$

For $1 \leq p \leq 2$, by interpolation

$$\left\| e^{-t\Delta} \right\|_{p\to p} \leq e^{-2(1-\frac{1}{p})\lambda t},$$

hence

$$\Delta^{-1/2} = \frac{\sqrt{\pi}}{2} \int_0^{+\infty} e^{-t\Delta} \frac{dt}{\sqrt{t}}$$

is bounded on L^p if $1 < p \leq 2$; therefore

$$\left\| \Delta^{1/2} f \right\|_p + \|\, f\,\|_p \leq C \left\| \Delta^{1/2} f \right\|_p.$$

Note that our method to prove Theorem 1.3 also works for non-amenable Lie groups endowed with a family of Hörmander vector fields and therefore gives another proof of the corresponding result in [25].

2. The tools

2.1. **Calderón-Zygmund decomposition.** Let (M,d,μ) be a metric measured space, $B(x,r)$ be the ball of center $x \in M$ and radius r, and $V(x,r)$ be its volume. Suppose that M satisfies the doubling volume property, i.e.

$$V(x,2r) \leq C\,V(x,r),\ \forall\, x \in M,\ r > 0.$$

Then there exists $c = c(M)$ such that, given $f \in L^1(M) \cap L^2(M)$ and $\lambda > 0$, one can decompose f as

$$f \;=\; g + b \;=\; g + \sum_i b_i\,,$$

so that

(a) $|g(x)| \leq c\lambda$ for almost all $x \in M$;

(b) there exists a sequence of balls $B_i = B(x_i, r_i)$ so that the support of each b_i is contained in B_i:

$$\int |b_i(x)| d\mu(x) \le c\lambda\mu(B_i) \text{ and } \int b_i(x) d\mu(x) = 0;$$

(c) $\sum_i \mu(B_i) \le \frac{c}{\lambda} \int |f(x)| d\mu(x)$;
(d) there exists $k \in \mathbb{N}^*$ such that each point of M is contained in at most k balls B_i.

Note that conditions (b) and (c) imply that $\| b \|_1 \le \sum_i \| b_i \|_1 \le c \| f \|_1$. Hence $\| g \|_1 \le (1+c) \| f \|_1$.

For a proof of the existence of the Calderón-Zygmund decomposition in this setting, see for example [8].

2.2. **Upper estimates of the heat kernel and its time derivative.** Our main assumption on M, apart from the doubling volume property, is the heat kernel estimate

$$p_t(x,x) \le \frac{C}{V(x, \sqrt{t})}.$$

A necessary and sufficient geometric condition, in isoperimetric terms, for this estimate to hold is given in [16], Prop. 5.2.

From this on-diagonal estimate, the corresponding off-diagonal estimate automatically follows ([19], thm. 1.1):

$$p_t(x,y) \le \frac{C_\alpha}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \exp\left(-\alpha \frac{d^2(x,y)}{t}\right), \ \forall\, x, y \in M,\, t > 0$$

for any $\alpha \in]0, 1/4[$.

With the doubling volume property, this implies

$$p_t(x,y) \le \frac{C'_\alpha}{V(y,\sqrt{t})} \exp\left(-\alpha \frac{d^2(x,y)}{t}\right), \ \forall\, x, y \in M,\, t > 0$$

for any $\alpha \in]0, 1/4[$. Indeed $B(y, \sqrt{t}) \subset B(x, \sqrt{t} + d(x,y))$. Now an obvious consequence of the doubling volume property is that there exists $D > 0$ such that $V(x, \theta r) \le C\theta^D V(x,r)$, if $\theta > 1$. Therefore

$$V(y,\sqrt{t}) \le V(x, \sqrt{t} + d(x,y)) \le C(1 + \frac{d(x,y)}{\sqrt{t}})^D V(x,\sqrt{t}),$$

and the estimate follows.

A similar estimate for the time derivative of the heat kernel also follows from [13], thm.4 (see also [19], cor. 3.3):

$$|\frac{\partial p_t}{\partial t}(x,y)| \le \frac{C''_\alpha}{tV(y,\sqrt{t})} \exp\left(-\alpha \frac{d^2(x,y)}{t}\right), \ \forall\, x, y \in M,\, t > 0.$$

From now on $\alpha \in]0, 1/4[$ is fixed. Note that since $p_t(x,y) = p_t(y,x)$ one can exchange x and y in the right hand sides of the above estimates.

2.3. **Weighted estimates of the space derivative of the heat kernel.**

Lemma 2.1. *For all* $\gamma > 0$,

$$\int_{d(x,y)\geq t^{1/2}} e^{-2\gamma\frac{d^2(x,y)}{s}}\, d\mu(x) \leq C_\gamma V(y,\sqrt{s})e^{-\gamma t/s},\ \forall\, y\in M,\ s,t>0.$$

Proof. Note first that

$$\int_{d(x,y)\geq t^{1/2}} e^{-2\gamma\frac{d^2(x,y)}{s}}\, d\mu(x) \leq e^{-\gamma\frac{t}{s}}\int_M e^{-\gamma\frac{d^2(x,y)}{s}}\, d\mu(x) = e^{-\gamma\frac{t}{s}}I.$$

Now split the region of integration into annular regions $i\, s^{1/2}\leq d(x,y) < (i+1)\, s^{1/2}$, $i\in\mathbb{N}$, to estimate I. One gets

$$I \leq \sum_{i=0}^{\infty} V(y,(i+1)s^{1/2})e^{-\gamma i^2} \leq V(y,s^{1/2})\sum_{i=0}^{\infty}(i+1)^D e^{-\gamma i^2},$$

which proves the lemma.

Applying the upper bound of $p_t(x,y)$ and Lemma 2.1 with $t=0$, one gets

Lemma 2.2. *For all* $\gamma\in]0,2\alpha[$,

$$\int_M |p_s(x,y)|^2 e^{\gamma\frac{d^2(x,y)}{s}}\, d\mu(x) \leq \frac{C_\gamma}{V(y,\sqrt{s})},\ \forall\, y\in M,\ s>0.$$

From Lemma 2.2, one deduces, using [18], an estimate of a weighted L^2 norm of $|\nabla_x p_s(.,y)|$. Here we need a slightly less sophisticated version of the estimate than in [18]; it admits a simpler proof, that goes over to a more general setting.

Lemma 2.3. *For all* $\gamma\in]0,2\alpha[$,

$$\int_M |\nabla_x p_s(x,y)|^2 e^{\gamma\frac{d^2(x,y)}{s}}\, d\mu(x) \leq \frac{C_\gamma}{V(y,\sqrt{s})}s^{-1},\ \forall\, y\in M,\ s>0.$$

Proof. By integration by parts,

$$\begin{aligned} I(s,y) &= \int_M |\nabla_x p_s(x,y)|^2 e^{\gamma\frac{d^2(x,y)}{s}}\, d\mu(x)\\ &= \int_M p_s(x,y)\Delta_x p_s(x,y) e^{\gamma\frac{d^2(x,y)}{s}}\, d\mu(x)\\ &\quad -\int_M p_s(x,y)\nabla_x p_s(x,y)\nabla_x(e^{\gamma\frac{d^2(x,y)}{s}})\, d\mu(x)\\ &= I_1(s,y)+I_2(s,y). \end{aligned}$$

Now

$$I_1(s,y) = -\int_M p_s(x,y)\frac{\partial p_s}{\partial s}(x,y)e^{\gamma\frac{d^2(x,y)}{s}}\, d\mu(x).$$

Using the estimate

$$|\frac{\partial p_s}{\partial s}(x,y)| \leq \frac{C}{sV(y,\sqrt{s})}\exp\left(-\alpha\frac{d^2(x,y)}{s}\right),$$

(see §2), one gets as in Lemma 2.2

$$|I_1(s,y)| \leq \frac{C_\gamma}{sV(y,\sqrt{s})}.$$

Then

$$I_2(s,y) = -\int_M p_s(x,y)\nabla_x p_s(x,y)\frac{2\gamma d(x,y)}{s}\nabla_x d(x,y)e^{\gamma\frac{d^2(x,y)}{s}}\,d\mu(x);$$

hence, since $|\nabla_x d(x,y)| \le 1$,

$$\begin{aligned}|I_2(s,y)| &\le \int_M p_s(x,y)|\nabla_x p_s(x,y)|\frac{2\gamma d(x,y)}{s}e^{\gamma\frac{d^2(x,y)}{s}}\,d\mu(x)\\ &\le \frac{C}{\sqrt{s}}\int_M p_s(x,y)|\nabla_x p_s(x,y)|e^{\gamma'\frac{d^2(x,y)}{s}}\,d\mu(x)\\ &\le \frac{1}{\sqrt{s}}\left(\int_M |p_s(x,y)|^2 e^{\gamma''\frac{d^2(x,y)}{s}}\,d\mu(x)\right)^{1/2}\\ &\quad\times\left(\int_M |\nabla_x p_s(x,y)|^2 e^{\gamma\frac{d^2(x,y)}{s}}\,d\mu(x)\right)^{1/2}.\end{aligned}$$

Note that since $\gamma < 2\alpha$, γ' and γ'' may be chosen smaller than α. Therefore, according to Lemma 2.2,

$$I_2(s,y) \le C\left(\frac{1}{s\,V(y,\sqrt{s})}\right)^{1/2}\sqrt{I(s,y)},$$

so that

$$I(s,y) \le \frac{C_\gamma}{sV(y,\sqrt{s})} + C\sqrt{\frac{1}{s\,V(y,\sqrt{s})}}\sqrt{I(s,y)}.$$

The lemma follows.

Finally one can state

Lemma 2.4. *There exists $\beta > 0$ such that*

$$\int_{d(x,y)\ge t^{1/2}} |\nabla_x p_s(x,y)|\,d\mu(x) \le Ce^{-\beta t/s}s^{-1/2},\ \forall\, y\in M,\ s,t>0.$$

Proof. Choose $\beta < \alpha$, write

$$\begin{aligned}&\int_{d(x,y)\ge t^{1/2}} |\nabla_x p_s(x,y)|\,d\mu(x)\\ &\quad\le \left(\int_M |\nabla_x p_s(x,y)|^2 e^{2\beta\frac{d^2(x,y)}{s}}\,dx\right)^{1/2}\left(\int_{d(x,y)\ge t^{1/2}} e^{-2\beta\frac{d^2(x,y)}{s}}\,dx\right)^{1/2}\end{aligned}$$

and apply Lemmas 2.1 and 2.3.

For more information about the difficulties that arise when one tries to get pointwise estimates on the gradient of the heat kernel, see [18].

3. The proof of the main result

The first part of the argument is taken from [14]. We reproduce it for the sake of completeness.

Let $T = \nabla\Delta^{-1/2}$. We want to prove that

$$\mu(\{x; |Tf(x)| > \lambda\}) \le C\frac{\|f\|_1}{\lambda},$$

for all $\lambda > 0$, $f \in L^1(M)$.

Fix $f \in L^1(M) \cap L^2(M)$ and $\lambda > 0$, and write the C-Z decomposition of f at the level λ. One has

$$\mu(\{x; |Tf(x)| > \lambda\}) \le \mu(\{x; |Tg(x)| > \lambda/2\}) + \mu(\{x; |Tb(x)| > \lambda/2\}).$$

Using the facts that T is bounded on $L^2(M)$ and that $|g(x)| \le c\lambda$, we obtain

$$\begin{aligned}\mu(\{x; |Tg(x)| > \lambda/2\}) &\le C\lambda^{-2} \|\, Tg \,\|_2^2 \le C'\lambda^{-2} \|\, g \,\|_2^2 \\ &\le C''\lambda^{-1} \|\, g \,\|_1 \le C'''\lambda^{-1} \|\, f \,\|_1 .\end{aligned}$$

Let us estimate now $Tb = \sum_i Tb_i$. Write

$$Tb_i = Te^{-t_i\Delta} b_i + T(I - e^{-t_i\Delta}) b_i,$$

where $t_i = r_i^2$ and r_i is the radius of B_i.

One checks first that

$$\left\| \sum_i e^{-t_i\Delta} b_i \right\|_2^2 \le C\lambda \,\|\, f \,\|_1 .$$

The L^2 boundedness of T will then imply as above

$$\mu(\{x; |T \sum_i e^{-t_i\Delta} b_i| > \lambda/2\}) \le \frac{C'}{\lambda} \,\|\, f \,\|_1 .$$

The heat kernel upper bound, the fact that b_i is supported in $B(x_i, \sqrt{t_i})$ and property (c) of the C-Z decomposition yield

$$\begin{aligned}|e^{-t_i\Delta} b_i(x)| &\le \int_M \frac{e^{-\alpha \frac{d^2(x,y)}{t_i}}}{V(x,\sqrt{t_i})} |b_i(y)| \, d\mu(y) \\ &\le C \frac{e^{-\alpha' \frac{d^2(x,x_i)}{t_i}}}{V(x,\sqrt{t_i})} \int_M |b_i(y)| \, d\mu(y) \\ &\le C' \frac{e^{-\alpha' \frac{d^2(x,x_i)}{t_i}}}{V(x,\sqrt{t_i})} \lambda \mu(B_i) \\ &\le C''\lambda \int_M \frac{e^{-\alpha' \frac{d^2(x,y)}{t_i}}}{V(x,\sqrt{t_i})} 1_{B_i}(y) \, d\mu(y).\end{aligned}$$

It is therefore enough to show

$$\left\| \sum_i \int_M \frac{e^{-\alpha' \frac{d^2(.,y)}{t_i}}}{V(.,\sqrt{t_i})} 1_{B_i}(y) \, d\mu(y) \right\|_2 \le C \left\| \sum_i 1_{B_i} \right\|_2 ,$$

since

$$\lambda^2 \left\| \sum_i 1_{B_i} \right\|_2^2 \le C\lambda^2 \sum_i \mu(B_i) \le C'\lambda \,\|\, f \,\|_1$$

because of properties (c) and (d) of the C-Z decomposition. Now

$$\begin{aligned}&\left\| \sum_i \int_M \frac{e^{-\alpha' \frac{d^2(.,y)}{t_i}}}{V(.,\sqrt{t_i})} 1_{B_i}(y)\, d\mu(y) \right\|_2 \\ &\quad = \sup_{\|u\|_2=1} \left| \int_M \left(\sum_i \int_M \frac{e^{-\alpha' \frac{d^2(x,y)}{t_i}}}{V(x,\sqrt{t_i})} 1_{B_i}(y)\, d\mu(y) \right) u(x) d\mu(x) \right| \\ &\quad \leq \sup_{\|u\|_2=1} \int_M \sum_i \left(\int_M \frac{e^{-\alpha' \frac{d^2(x,y)}{t_i}}}{V(x,\sqrt{t_i})} |u(x)| d\mu(x) \right) 1_{B_i}(y)\, d\mu(y).\end{aligned}$$

Since

$$V(y,\sqrt{t_i}) \leq (1 + \frac{d(x,y)}{\sqrt{t_i}})^D V(x,\sqrt{t_i})$$

(see §2.2), one can estimate

$$\int_M \frac{e^{-\alpha' \frac{d^2(x,y)}{t_i}}}{V(x,\sqrt{t_i})} |u(x)| d\mu(x)$$

by

$$\frac{1}{V(y,\sqrt{t_i})} \int_M e^{-\alpha'' \frac{d^2(x,y)}{t_i}} |u(x)| d\mu(x)$$

for $\alpha'' < \alpha' < \alpha$. Next

$$\begin{aligned} & \frac{1}{V(y,\sqrt{t_i})} \int_M e^{-\alpha'' \frac{d^2(x,y)}{t_i}} |u(x)| d\mu(x) \\ = \quad & \frac{1}{V(y,\sqrt{t_i})} \left(\int_{d(x,y)\leq\sqrt{t_i}} e^{-\alpha'' \frac{d^2(x,y)}{t_i}} |u(x)| d\mu(x) \right. \\ & \left. \qquad + \sum_{k\in\mathbb{N}} \int_{2^k\sqrt{t_i}\leq d(x,y)\leq 2^{k+1}\sqrt{t_i}} e^{-\alpha'' \frac{d^2(x,y)}{t_i}} |u(x)| d\mu(x) \right) \\ \leq \quad & \frac{1}{V(y,\sqrt{t_i})} \left(\int_{B(y,\sqrt{t_i})} |u(x)| d\mu(x) + \sum_{k\in\mathbb{N}} e^{-\alpha'' 2^{2k}} \int_{B(y,2^{k+1}\sqrt{t_i})} |u(x)| d\mu(x) \right) \\ = \quad & \left(1 + \sum_{k\in\mathbb{N}} \frac{V(y,2^{k+1}\sqrt{t_i})}{V(y,\sqrt{t_i})} e^{-\alpha'' 2^{2k}} \right) Mu(y) \\ \leq \quad & \left(1 + \sum_{k\in\mathbb{N}} 2^{(k+1)D} e^{-\alpha'' 2^{2k}} \right) Mu(y), \end{aligned}$$

where $Mu(y) = \sup_{r>0} \frac{1}{V(y,r)} \int_{B(y,r)} |u(x)|\, d\mu(x)$ is the Hardy-Littlewood maximal function. Therefore

$$\left\| \sum_i \int_M \frac{e^{-\alpha' \frac{d^2(.,y)}{t_i}}}{V(.,\sqrt{t_i})} 1_{B_i}(y)\, d\mu(y) \right\|_2 \leq C \sup_{\| u \|_2 = 1} \int_M Mu(y) \sum_i 1_{B_i}(y)\, d\mu(y)$$
$$\leq C' \left\| \sum_i 1_{B_i} \right\|_2 ,$$

since the sublinear operator M is bounded on L^2. This ends the estimate of the term $T \sum_i e^{-t_i \Delta} b_i$.

Consider now the term $T \sum_i (I - e^{-t_i \Delta}) b_i$. Write

$$\mu(\{x; |T \sum_i (I - e^{-t_i \Delta}) b_i| > \lambda\})$$
$$\leq \sum_i \mu(2B_i) + \mu(\{x \in M \setminus \bigcup_i 2B_i; |T \sum_i (I - e^{-t_i \Delta}) b_i| > \lambda\}).$$

The first term is controlled by a constant times $\| f \|_1 / \lambda$ thanks to the doubling volume property and property (c) in the C-Z decomposition. The second term is dominated by

$$\frac{1}{\lambda} \int_{M \setminus \bigcup_i 2B_i} |T \sum_i (I - e^{-t_i \Delta}) b_i(x)|\, d\mu(x) \leq \frac{1}{\lambda} \sum_i \int_{M \setminus 2B_i} |T(I - e^{-t_i \Delta}) b_i(x)|\, d\mu(x).$$

Hence it is enough to show

$$\int_{M \setminus 2B_i} |T(I - e^{-t_i \Delta}) b_i(x)|\, d\mu(x) \leq C \| b_i \|_1 .$$

From now on drop the subscripts i. Let $k_t(x,y)$ be the kernel of the operator $T(I - e^{-t\Delta})$. Since b is supported in B, one has

$$\begin{aligned} \int_{M \setminus 2B} |T(I - e^{-t\Delta}) b(x)|\, d\mu(x) &\leq \int_{M \setminus 2B} \left(\int_B |k_t(x,y)| |b(y)|\, d\mu(y) \right) d\mu(x) \\ &\leq \int_M \left(\int_{d(x,y) \geq t^{1/2}} |k_t(x,y)|\, d\mu(x) \right) |b(y)|\, d\mu(y). \end{aligned}$$

It is therefore enough to prove

$$\int_{d(x,y) \geq t^{1/2}} |k_t(x,y)|\, d\mu(x) \leq C, \ \forall y \in M, \ \forall t > 0.$$

Let us now compute the kernel $k_t(x,y)$. Since

$$\Delta^{-1/2} = \int_0^{+\infty} e^{-s\Delta} \frac{ds}{\sqrt{s}}$$

(we forget a multiplicative constant which plays no rôle), one can write

$$\begin{aligned} \Delta^{-1/2}(I - e^{-t\Delta}) &= \int_0^{+\infty} e^{-s\Delta} \frac{ds}{\sqrt{s}} - \int_0^{+\infty} e^{-(s+t)\Delta} \frac{ds}{\sqrt{s}} \\ &= \int_0^{+\infty} \left(\frac{1}{\sqrt{s}} - \frac{1_{\{s>t\}}}{\sqrt{s-t}} \right) e^{-s\Delta}\, ds, \end{aligned}$$

and

$$T(I-e^{-t\Delta}) = \nabla\Delta^{-1/2}(I-e^{-t\Delta}) = \int_0^{+\infty}\left(\frac{1}{\sqrt{s}} - \frac{1_{\{s>t\}}}{\sqrt{s-t}}\right)\nabla e^{-s\Delta}\,ds.$$

Therefore

$$k_t(x,y) = \int_0^{+\infty}\left(\frac{1}{\sqrt{s}} - \frac{1_{\{s>t\}}}{\sqrt{s-t}}\right)\nabla_x p_s(x,y)\,ds = \int_0^{+\infty} g_t(s)\nabla_x p_s(x,y)\,ds.$$

Now, according to Lemma 2.3,

$$\begin{aligned}\int_{d(x,y)\ge t^{1/2}} |k_t(x,y)|\,d\mu(x) &\le \int_0^{+\infty}|g_t(s)|\left(\int_{d(x,y)\ge t^{1/2}}|\nabla p_s(x,y)|\,d\mu(x)\right)ds\\ &\le C\int_0^{+\infty}|g_t(s)|e^{-\beta t/s}s^{-1/2}\,ds.\end{aligned}$$

Finally

$$\begin{aligned}\int_0^{+\infty}|g_t(s)|e^{-\beta t/s}s^{-1/2}\,ds &= \int_0^t \frac{e^{-\beta t/s}}{s}\,ds + \int_t^{+\infty}\left(\frac{1}{\sqrt{s-t}} - \frac{1}{\sqrt{s}}\right)e^{-\beta t/s}s^{-1/2}\,ds\\ &= I_1 + I_2,\end{aligned}$$

where

$$I_1 = \int_0^1 \frac{e^{-\beta/u}}{u}\,du$$

is finite and independent of t, and

$$\begin{aligned}I_2 \le I_2' &= \int_t^{+\infty}\left(\frac{1}{\sqrt{s-t}} - \frac{1}{\sqrt{s}}\right)s^{-1/2}\,ds\\ &= \int_0^{+\infty}\left(\frac{1}{\sqrt{u}} - \frac{1}{\sqrt{u+t}}\right)\frac{1}{\sqrt{u+t}}\,du.\end{aligned}$$

Again, setting $u = vt$,

$$I_2' = \int_0^{+\infty}\left(\frac{1}{\sqrt{v(v+1)}} - \frac{1}{v+1}\right)dv$$

is finite and independent of t. This ends the proof of Theorem 1.1.

4. Localisation

In this section, we shall explain the modifications that are necessary to prove Theorem 1.2. It is enough to prove that the operator

$$\tilde{T} = \nabla(\Delta + a)^{-1/2} = \int_0^{+\infty} e^{-as}\nabla e^{-s\Delta}\frac{ds}{\sqrt{s}}$$

is weak $(1,1)$, for some $a > 0$. To this end we shall use a version of the localisation technique of [15].

Let $(x_j)_{j\in J}$ be a maximal 1-separated subset of M: the collection of balls $B^j = B(x_j,1)$, $j \in J$, covers M, whereas the balls $B(x_j,1/2)$ are pairwise disjoint. It follows from the local doubling volume property that there exists $N \in \mathbb{N}^*$ such that every $x \in M$ is contained in at most N balls $2B^j = B(x_j,2)$. Consider a C^∞ partition of unity φ_j, $j \in J$ such that φ_j is supported in B^j.

For $f \in \mathcal{C}_0^\infty(M)$, write

$$\tilde{T} f = \sum_j 1_{2B^j} \tilde{T}(f\varphi_j) + \sum_j (1 - 1_{2B^j}) \tilde{T}(f\varphi_j).$$

One has, for $\lambda > 0$,

$$\mu(\{x; |\tilde{T} f(x)| > \lambda\}) \leq \mu(\{x; \sum_j 1_{2B^j} |\tilde{T}(f\varphi_j)(x)| > \lambda/2\}) + \mu(\{x; \sum_j (1 - 1_{2B^j}) |\tilde{T}(f\varphi_j)(x)| > \lambda/2\}).$$

Since at most N terms of the sum $\sum_j 1_{2B^j} |\tilde{T}(f\varphi_j)(x)|$ are non-zero, this yields

$$\mu(\{x; |\tilde{T} f(x)| > \lambda\}) \leq \sum_j \mu(\{x; 1_{2B^j} |\tilde{T}(f\varphi_j)(x)| > \lambda/2N\}) + \frac{2}{\lambda} \sum_j \left\| (1 - 1_{2B^j}) |\tilde{T}(f\varphi_j)| \right\|_1 .$$

The desired estimate of $\mu(\{x; |\tilde{T} f(x)| > \lambda\})$ will follow if we prove that

$$\mu(\{x; 1_{2B^j} |\tilde{T} f(x)| > \lambda\}) \leq \frac{C}{\lambda} \| f \|_1 , \ \forall f \in \mathcal{C}_0^\infty(B^j),$$

and that

$$\int_{M \setminus 2B^j} |\tilde{T} f(x)| \, d\mu(x) \leq C \| f \|_1 , \ \forall f \in \mathcal{C}_0^\infty(B^j),$$

where the constants are independent of $j \in J$. Indeed one can add up these estimates since

$$\sum_j \| f\varphi_j \|_1 \leq C' \| f \|_1 .$$

To prove the first estimate, one performs the C-Z decomposition $f = g + b$ in B^j, which is a space with the doubling volume property. Note that the constants in the C-Z decomposition only depend on the constant in the doubling volume property and therefore are independent of j. One treats the good part as before. As for the bad part, write

$$\tilde{T} b = \sum_i \tilde{T} 1_{3B^j} e^{-t_i \Delta} b_i + \sum_i \tilde{T} (1 - 1_{3B^j}) e^{-t_i \Delta} b_i + \sum_i \tilde{T}(I - e^{-t_i \Delta}) b_i,$$

where $t_i = r_i^2$ and r_i is the radius of B_i. Note that since $B_i \subset B^j$, one has $t_i \leq 1$. Therefore one can use the small time heat kernel and small radii volume estimates to estimate

$$\left\| \sum_i 1_{3B^j} e^{-t_i \Delta} b_i \right\|_2^2 .$$

Indeed, the corresponding argument in Section 3 applies with easy modifications; one is able to use the L^2 boundedness of the Hardy-Littlewood maximal function restricted to $3B^j$ because the doubling volume property holds there.

To control $\sum_i \tilde{T}(I - e^{-t_i \Delta}) b_i$, it is enough to prove that

$$\int_{M \setminus 2B_i} |\tilde{T}(I - e^{-t_i \Delta}) b_i(x)| \, d\mu(x) \leq C \| b_i \|_1 .$$

To control $\sum_i \tilde{T}(1 - 1_{3B^j})e^{-t_i\Delta}b_i$, it is enough to prove that

$$\int_{2B^j} |\tilde{T}f(x)|\, d\mu(x) \le C \,\|\, f\,\|_1\,,\ \forall f \in \mathcal{C}_0^\infty(M \setminus 3B^j).$$

Then it remains to prove

$$\int_{M\setminus 2B^j} |\tilde{T}f(x)|\, d\mu(x) \le C \,\|\, f\,\|_1\,,\ \forall f \in \mathcal{C}_0^\infty(B^j).$$

These three estimates are similar; they follow from kernel estimates. Specifically, the first estimate follows from

$$\int_{d(x,y)\ge t^{1/2}} |\tilde{k}_t(x,y)|\, d\mu(x) \le C,\ \forall y \in M,\, t \le 1, \tag{1}$$

where

$$\tilde{k}_t = \int_0^{+\infty} \left(\frac{e^{-as}}{\sqrt{s}} - \frac{e^{-a(s-t)}1_{\{s>t\}}}{\sqrt{s-t}} \right) \nabla_x p_s(x,y)\, ds$$

is the kernel of the operator $\tilde{T}(I - e^{-t\Delta})$, the second and third estimates follow from

$$\int_{d(x,y)\ge 1} |\tilde{k}(x,y)|\, d\mu(x) \le C,\ \forall y \in M, \tag{2}$$

where

$$\tilde{k}(x,y) = \int_0^{+\infty} e^{-as}\nabla_x p_s(x,y)\frac{ds}{\sqrt{s}}$$

is the kernel of the operator $\tilde{T}$.

Let us gather the estimates on the heat kernel and its derivatives that are still true under our weaker assumptions. Our volume growth assumptions are enough for Lemma 2.1 to hold for $s \le 1$. Lemmas 2.2, 2.3 and 2.4 follow, for all $t > 0$ but only for $s \le 1$. For $s \ge 1$, one uses the integral maximum principle ([17], [18]) according to which the quantity

$$\int_M |p_s(x,y)|^2 e^{\gamma \frac{d^2(x,y)}{s}}\, d\mu(x)$$

is non-increasing in $s > 0$. It follows that

$$\int_M |p_s(x,y)|^2 e^{\gamma \frac{d^2(x,y)}{s}}\, d\mu(x) \le \frac{C_\gamma}{V(y,1)},\ \forall\, y \in M,\, s \ge 1.$$

Since

$$|\frac{\partial p_s}{\partial s}(x,y)| \le \frac{C}{sV(y,1)} \exp\left(-\alpha \frac{d^2(x,y)}{s}\right)$$

([18], [13]), the proof of Lemma 2.3 gives

$$\int_M |\nabla_x p_s(x,y)|^2 e^{\gamma \frac{d^2(x,y)}{s}}\, d\mu(x) \le \frac{C_\gamma}{V(y,1)} s^{-1},\ \forall\, y \in M,\, s \ge 1;$$

therefore by Hölder

$$\int_{d(x,y)\ge t^{1/2}} |\nabla_x p_s(x,y)|\, d\mu(x) \le \left(\frac{C}{sV(y,1)}\right)^{1/2} \left(\int_{d(x,y)\ge t^{1/2}} e^{-2\beta \frac{d^2(x,y)}{s}}\, dx\right)^{1/2}.$$

Proceeding as in Lemma 2.1 and using the fact that the volume growth is at most exponential, one gets

$$\int_{d(x,y)\geq t^{1/2}} |\nabla_x p_s(x,y)|\, d\mu(x) \leq Ce^{-\beta t/s} s^{-1/2} e^{cs},\ \forall\, y \in M,\ s \geq 1,\ t > 0.$$

To prove (2), it is therefore enough to check that

$$\int_0^1 e^{-as} e^{-\beta t/s} \frac{ds}{s} + \int_1^{+\infty} e^{-as} e^{-\beta t/s} e^{cs} \frac{ds}{s}$$

is finite, which is obvious, provided a is chosen large enough.

To prove (1), one checks that

$$\int_0^1 |\frac{e^{-as}}{\sqrt{s}} - \frac{e^{-a(s-t)} 1_{\{s>t\}}}{\sqrt{s-t}}| e^{-\beta t/s} \frac{ds}{\sqrt{s}} + \int_1^{+\infty} |\frac{e^{-as}}{\sqrt{s}} - \frac{e^{-a(s-t)} 1_{\{s>t\}}}{\sqrt{s-t}}| e^{-\beta t/s} e^{cs} \frac{ds}{\sqrt{s}}$$

$$= \int_0^t \frac{e^{-as}}{\sqrt{s}} e^{-\beta t/s} \frac{ds}{\sqrt{s}} + \int_t^1 \left(\frac{e^{-a(s-t)}}{\sqrt{s-t}} - \frac{e^{-as}}{\sqrt{s}} \right) e^{-\beta t/s} \frac{ds}{\sqrt{s}}$$
$$+ \int_1^{+\infty} \left(\frac{e^{-a(s-t)}}{\sqrt{s-t}} - \frac{e^{-as}}{\sqrt{s}} \right) e^{-\beta t/s} e^{cs} \frac{ds}{\sqrt{s}}$$

is uniformly bounded for $t \leq 1$.

Indeed, the treatment of the first and third terms is obvious. As for the second one, write

$$\int_t^1 \left(\frac{e^{-a(s-t)}}{\sqrt{s-t}} - \frac{e^{-as}}{\sqrt{s}} \right) e^{-\beta t/s} \frac{ds}{\sqrt{s}} \leq \int_t^1 \left(\frac{e^{at}}{\sqrt{s-t}} - \frac{1}{\sqrt{s}} \right) \frac{ds}{\sqrt{s}}$$
$$= e^{at} \int_t^1 \left(\frac{1}{\sqrt{s-t}} - \frac{1}{\sqrt{s}} \right) \frac{ds}{\sqrt{s}} + (e^{at} - 1) \int_t^1 \frac{ds}{s}.$$

The first term was treated in Section 3, and the second one is $(e^{at} - 1)(-\log t)$ which is bounded for $t \in]0,1]$.

5. A counterexample for $p > 2$

For $n \geq 2$, let M_n be a manifold consisting in two copies of $\mathbb{R}^n \setminus B(0,1)$, with the Euclidean metric, glued smoothly along the unit circles. One checks easily that on this manifold the volume growth is uniformly Euclidean:

$$C^{-1} r^n \leq V(x,r) \leq C r^n.$$

Moreover, the usual Sobolev inequality (or Nash inequality if $n = 2$) holds on M_n:

$$\|\, f \,\|_{\frac{2n}{n-2}} \leq C \,\|\, |\nabla f| \,\|_2\,,\ \forall\, f \in \mathcal{C}_0^\infty(M_n),$$

and, for $p > n$,

$$|f(x) - f(y)| \leq C d(x,y)^{1-\frac{n}{p}} \,\|\, |\nabla f| \,\|_p\,,\ \forall\, f \in \mathcal{C}_0^\infty(M_n), x, y \in M_n.$$

Fix $p > n$, and suppose that

$$\|\, |\nabla f| \,\|_p \leq C \left\| \Delta^{1/2} f \right\|_p.$$

Fix $z \in M_n$ and choose $f = p_t(., z)$. One gets

$$|p_t(x,z) - p_t(y,z)| \le Cd(x,y)^{1-\frac{n}{p}} \left\| \Delta_x^{1/2} p_t(.,z) \right\|_p .$$

Now

$$\Delta_x^{1/2} p_t(.,z) = \Delta_x^{1/2} e^{-\frac{t}{2}\Delta_x} p_{t/2}(.,z);$$

therefore

$$\left\| \Delta_x^{1/2} p_t(.,z) \right\|_p \le Ct^{-1/2} \left\| p_{t/2}(.,z) \right\|_p$$

since the heat semigroup is analytic on L^p. By the Hölder inequality

$$\| p_t(.,z) \|_p \le \| p_t(.,z) \|_2^{2/p} \| p_t(.,z) \|_\infty^{1-\frac{2}{p}} .$$

Because the Euclidean L^2 Sobolev inequality holds on M_n, one has

$$\| p_t(.,z) \|_2^2 = p_{2t}(z,z) \le Ct^{-n/2}$$

and

$$\| p_t(.,z) \|_\infty \le Ct^{-n/2}$$

(see [33]). Thus

$$\| p_t(.,z) \|_p \le Ct^{-\frac{n}{2}(1-\frac{1}{p})}.$$

This yields

$$|p_t(x,z) - p_t(y,z)| \le Cd(x,y)^{1-\frac{n}{p}} t^{-\frac{1}{2}(1-\frac{n}{p})-\frac{n}{2}}.$$

The uniform lower bound

$$p_t(z,z) \ge ct^{-n/2}$$

holds on M_n (see [10], thm. 7.2); therefore

$$|p_t(x,z) - p_t(y,z)| \le C \left(\frac{d(x,y)}{\sqrt{t}} \right)^{1-\frac{n}{p}} p_t(z,z),$$

in particular

$$|p_t(z,z) - p_t(y,z)| \le C \left(\frac{d(y,z)}{\sqrt{t}} \right)^{1-\frac{n}{p}} p_t(z,z),$$

and for a small enough

$$|p_t(z,z) - p_t(y,z)| \le \frac{1}{2} p_t(z,z)$$

as soon as $d(y,z) \le a\sqrt{t}$. In other words

$$p_t(y,z) \ge \frac{1}{2} p_t(z,z) \ge ct^{-n/2}$$

for $y \in B(z, a\sqrt{t})$. A standard iteration argument shows that this implies the Gaussian lower bound

$$p_t(y,z) \ge ct^{-n/2} \exp\left(-C\frac{d^2(y,z)}{t} \right),\ \forall\, x, y \in M_n.$$

This estimate is false because the probability of going from y in the first copy of $\mathbb{R}^n \setminus B(0,1)$ to z in the other one is at most the probability of reaching the central hole from y times the probability of reaching z from the central hole, therefore substantially smaller than the corresponding probability in $\mathbb{R}^n$. This idea can easily be made precise for $n > 2$ (see [6]), and also, in a more subtle way, for $n = 2$. We owe the latter observation to Laurent Saloff-Coste.

Without computations, one can also say that the upper and lower Gaussian estimates of the heat kernel would imply the parabolic Harnack principle, which is false on M_n because the family of L^2 Poincaré inequalities considered in [29] clearly cannot hold.

Therefore the estimate

$$\| \, |\nabla f| \, \|_p \leq C \left\| \Delta^{1/2} f \right\|_p$$

is false for $p > n$ on M_n, $n \geq 2$.

6. The case of bad domains in $\mathbb{R}^n$

In this section, let Ω be an open subset of the Euclidean space $\mathbb{R}^n$. Define the quadratic form on $\mathcal{C}_c^\infty(\Omega)$ by

$$Q(f) = \int_\Omega \sum_i \frac{\partial f}{\partial x_i} \, \frac{\partial \overline{f}}{\partial x_i} dx.$$

Then Q is closable and the self-adjoint operator Δ associated with the obtained closed form is called the (minus) Laplacian on Ω with Dirichlet boundary conditions. It is well known that Δ generates a holomorphic semigroup and its heat kernel $p_t(x,y)$ satisfies the Gaussian upper bound

$$p_t(x,y) \leq \frac{C}{t^{n/2}} \exp\left(-\alpha \frac{|x-y|^2}{t}\right), \; \forall \, x, y \in \Omega, \, t > 0$$

for some $\alpha > 0$. See for example [12].

A similar estimate is also true for the time derivative of the heat kernel, by the method of [13]:

$$|\frac{\partial p_t}{\partial t}(x,y)| \leq \frac{C_\alpha}{t^{\frac{n}{2}+1}} \exp\left(-\alpha \frac{|x-y|^2}{t}\right), \; \forall \, x, y \in \Omega, \, t > 0.$$

If the domain Ω has Lipschitz boundary, then Ω satisfies the doubling volume property and the method of Theorem 1.1 is applicable, hence the Riesz transform is bounded on $L^p(\Omega)$ for all $1 < p \leq 2$ and is of weak type $(1,1)$. The boundedness on L^p in this case is also one of the results of [20]. However, a general open set need not satisfy the doubling volume property, and the weak $(1,1)$ estimate of the Riesz transform on such a domain is an open question. The difficulty in this case is that the usual Calderón-Zygmund theory is not applicable on a domain which does not satisfy the doubling volume property.

Section 3 of [14] gives us the right tools to prove a positive answer for this question. We first note that the term $t^{n/2}$ in the Gaussian upper bounds of the heat kernel and its time derivative is the volume of the ball of radius $\sqrt{t}$ in $\mathbb{R}^n$, not the volume of the ball in Ω. This observation plays an important rôle in the estimates of our next theorem.

Theorem 6.1. *The Riesz transform $T = \nabla \Delta^{-1/2}$ is weak $(1,1)$ and bounded on $L^p(\Omega)$, $1 < p \leq 2$. That is, there exists C_p, $1 \leq p \leq 2$, such that, $\forall \, f \in \mathcal{C}_0^\infty(\Omega)$,*

$$\| \, |\nabla f| \, \|_p \leq C_p \left\| \Delta^{1/2} f \right\|_p, \; 1 < p \leq 2,$$

and

$$\| \, |\nabla f| \, \|_{1,\infty} \le C_1 \left\| \Delta^{1/2} f \right\|_1 .$$

Proof. The boundedness of $T = \nabla \Delta^{-1/2}$ on $L^2(\Omega)$ is obvious from integration by parts.

Using the argument of Section 3 of [14], we define an associated operator $\mathcal{T}$ on $L^2(\mathbb{R}^n)$ by putting $\mathcal{T}(f)(x) = T(1_\Omega f)(x)$ for $x \in \Omega$ and $\mathcal{T}(f)(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. It is straightforward to check that T is of weak type $(1,1)$ on Ω if and only if $\mathcal{T}$ is of weak type $(1,1)$ on $\mathbb{R}^n$ (and that T is bounded on $L^p(\Omega)$ if and only if $\mathcal{T}$ is bounded on $L^p(\mathbb{R}^n)$). Thus we transform the question of weak type $(1,1)$ estimate of T on the bad domain Ω to that of $\mathcal{T}$ on $\mathbb{R}^n$ which is a space with the doubling volume property. We also observe that if $k(x,y)$ is the associated kernel of T, then $\mathcal{T}$ has an associated kernel $\mathcal{K}(x,y)$ given by $\mathcal{K}(x,y) = k(x,y)$ for $x, y \in \Omega$ and $\mathcal{K}(x,y) = 0$ otherwise. We now extend the semigroup $e^{-t\Delta}$ on $L^2(\Omega)$ to $L^2(\mathbb{R}^n)$ in the same way. Note that all the weighted estimates in subsection 2.3 are still true for $e^{-t\Delta}$ hence are also true for its extension defined as above. We then apply the same method as in Theorem 1.1 to show that $\mathcal{T}$ is of weak type $(1,1)$ on $\mathbb{R}^n$. See [14] for more details.

Remark. We say that a region Ω of $\mathbb{R}^n$ has the extension property if there exists a bounded linear map E from the Sobolev spaces $W^{1,p}(\Omega)$ into the Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ such that Ef is an extension of f from Ω to $\mathbb{R}^n$ for all $f \in W^{1,p}(\Omega)$ and all $1 \le p \le \infty$. Consider the Laplacian Δ with Neumann boundary conditions on a bounded domain Ω with the extension property. Then the Gaussian heat kernel bound is well known, see Chapter 3 of [12]. Using a limiting argument and the definition of the quadratic form, one can see that the estimate using integration by parts in the proof of Lemma 2.3 is still valid. Therefore, Theorem 6.1 is still true for the Laplacian with Neumann boundary conditions.

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References

[1] Alexopoulos G., An application of homogeneisation theory to harmonic analysis: Harnack inequalities and Riesz transforms on Lie groups of polynomial growth, *Can. J. Math.*, 44, 4, 691-727, 1992. MR **93j:**22006

[2] Auscher P., Tchamitchian P., Square root of divergence operators, square functions and singular integrals, Mat. Res. Let., 3, 429-437, 1996. MR **97k:**42026

[3] Bakry D., Transformations de Riesz pour les semi-groupes symétriques, in *Séminaire de Probabilités XIX*, Springer L.N. n^o 1123, 130-175, 1985. MR **89h:**42022

[4] Bakry D., Etude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée, in *Séminaire de Probabilités XXI*, Springer L.N. n^o 1247, 137-172, 1987. MR **89h:**58208
[5] Bakry D., The Riesz transforms associated with second order differential operators, in *Seminar on stochastic processes 88*, Birkhaüser, 1989. MR **90c:**58186
[6] Benjamini I., Chavel I., Feldman E., Heat kernel lower bounds on manifolds using the old ideas of Nash, *Proc. London Math. Soc.*, 72, 215-240, 1992. MR **97c:**58150
[7] Chen Jie-Cheng, Heat kernels on positively curved manifolds and applications, Ph. D. thesis, Hanghzhou University, 1987.
[8] Coifman R., Weiss G., *Analyse harmonique non-commutative sur certains espaces homogènes*, Springer L. N. n^o 242, 1971. MR **58:**17690
[9] Coulhon T., Espaces de Lipschitz et inégalités de Poincaré, *J. Funct. Anal.*, 136, 1, 81-113, 1996. MR **97a:**46040
[10] Coulhon T., Grigor'yan A., On-diagonal lower bounds for heat kernels on non-compact Riemannian manifolds, preprint Mittag-Leffler institute, *Duke Univ. Math. J.*, 89, 1, 133–199, 1997. MR **98e:**58159
[11] Coulhon T., Ledoux M., Isopérimétrie, décroissance du noyau de la chaleur et transformations de Riesz: un contre-exemple, *Arkiv för Mat.*, 32, 63-77, 1994. MR **95e:**58170
[12] Davies E-B., *Heat kernels and spectral theory*, Cambridge University Press, 1989. MR **90e:**35123
[13] Davies E-B., Non-Gaussian aspects of heat kernel behaviour, *J. London Math. Soc.*, 55, 2, 105–125, 1997. MR **97i:**58169
[14] Duong X., McIntosh A., Singular integral operators with non-smooth kernels on irregular domains, preprint, 1995.
[15] Duong X., Robinson D., Semigroup kernels, Poisson bounds and holomorphic functional calculus, *J. Funct. Anal.*, 142, 1, 89-128, 1996. MR **97j:**47056
[16] Grigor'yan A., Heat kernel upper bounds on a complete non-compact manifold, *Rev. Mat. Iberoamericana*, 10, 2, 395-452, 1994. MR **96b:**58107
[17] Grigor'yan A., Integral maximum principle and its applications, *Proc. Edinburgh Roy. Soc.*, 124A, 353-362, 1994. MR **95c:**35045
[18] Grigor'yan A., Upper bounds of derivatives of the heat kernel on an arbitrary complete manifold, *J. Funct. Anal.*, 127, 363-389, 1995. MR **96a:**58183
[19] Grigor'yan A., Gaussian upper bounds for the heat kernel on arbitrary manifolds, *J. Diff. Geom.*, 45, 33–52, 1997. CMP 97:11
[20] Jerison D., Kenig C., The inhomogeneous Dirichlet problem in Lipschitz domains, *J. Funct. Anal.*, 130, 1, 161-219, 1995. MR **96b:**35042
[21] Kenig C., unpublished notes.
[22] Li Jiayu, Gradient estimate for the heat kernel of a complete Riemannian manifold and its applications, *J. Funct. Anal.*, 97, 293-310, 1991. MR **92f:**58174
[23] Li P., Yau S.T., On the parabolic kernel of the Schrödinger operator, *Acta Math.*, 156, 153-201, 1986. MR **87f:**58156
[24] Lohoué N., Comparaison des champs de vecteurs et des puissances du laplacien sur une variété riemannienne à courbure non positive, *J. Funct. Anal.*, 61, 2, 164-201, 1985. MR **86k:**58117
[25] Lohoué N., Transformées de Riesz et fonctions de Littlewod-Paley sur les groupes non moyennables, *C.R.A.S Paris*, 306, I, 327-330, 1988. MR **89b:**43008
[26] Meyer P-A., Correction au volume X du Séminaire: Inégalités de Littlewood-Paley, in *Séminaire de Probabilités XII*, Springer L.N. n^o 649, p. 741, 1978. MR **81m:**60147
[27] Qian Z., Gradient estimates and heat kernel estimates, *Proc. Royal Soc. Edinburgh*, 125A, 975-990, 1995. MR **97c:**58153
[28] Saloff-Coste L., Analyse sur les groupes de Lie à croissance polynômiale, *Arkiv för Mat.*, 28, 2, 315-331, 1990. MR **92d:**22014
[29] Saloff-Coste L., A note on Poincaré, Sobolev and Harnack inequalities, *Duke J. Math.*, 65, I.R.M.N., 27-38, 1992. MR **93d:**58158
[30] Saloff-Coste L., On global Sobolev inequalities, *Forum Mat.*, 6, 271-286, 1994. MR **95f:**58075
[31] Stein E., *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970. MR **44:**7280

[32] Strichartz R., Analysis of the Laplacian on the complete Riemannian manifold, *J. Funct. Anal.*, 52, 48-79, 1983. MR **84m:**58138

[33] Varopoulos N., Une généralisation du théorème de Hardy-Littlewood-Sobolev pour les espaces de Dirichlet, *C.R.A.S Paris*, 299, I, 651-654,1984. MR **86f:**31004

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