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ON GRAPHS WITH A METRIC END SPACE

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Abstract. R. Diestel conjectured that an infinite graph contains a topologically end-faithful forest if and only if its end space is metrizable. We prove this conjecture for uniform end spaces.

1. Introduction

An important structural aspect of an infinite graph is the convergence pattern of its 1-way infinite paths. This is formalized by the concept of ends and the associated end space. Since the end structure of a forest is particularly simple, one is interested in classifying the graphs whose end space is reflected faithfully by a subtree or subforest.

In particular the end space of a forest is metrizable, so graphs with an end-faithful subforest have a metrizable end space. This led to the question of Diestel (1990) whether the converse also holds: *Does every graph with metrizable (topological) end space contain a (topologically) end-faithful forest?*

In this paper we prove the analogue of the above for uniform end spaces: Every graph with a metrizable (uniform) end space has a (uniformly) end-faithful subforest.

2. Preliminaries

2.1. **Terminology.** In the following let G be a connected infinite graph. By $V(G)$ we denote the set of vertices and by $E(G)$ the set of edges of G. A *ray* T in G is a 1-way infinite path in G. A *tail* of a ray T is a connected infinite subgraph of T. Two or more paths are *independent* if their interiors are disjoint. For $X, Y \subseteq G$ we call a path $P \subset G$ an X–Y *path* if its endvertices are in X and Y, respectively, and all inner vertices are in $G \backslash (X \cup Y)$. A subgraph $X \subset G$ is *finitely* or *infinitely linked to* a subgraph $Y \subset G$ in G, respectively, if there are finitely or infinitely many pairwise disjoint X–Y paths in G, respectively.

A finite vertex set $S \subset V(G)$ *separates two rays* $T, T' \subseteq G$ in G if any two tails of T and T' belong to different components of $G - S$. We call two rays $T, T' \subseteq G$ *end-equivalent* (or briefly *equivalent*), denoted by $T \sim T'$, if they are infinitely linked in G. In other words, T is equivalent to T' if there is no finite vertex set $S \subset V(G)$ that separates T and T' in G. The relation $\sim$ is an equivalence relation on the set

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of rays of G. The equivalence classes of this relation are called *ends*; the set of all ends is called the *end space* $\Omega(G)$ of G.

A subgraph $H \subseteq G$ *contains* an end $\omega \in \Omega(G)$ if H contains a tail of every ray in ω and if in H every two rays of ω are equivalent. If $S \subseteq V(G)$ is a finite set of vertices, then every end of G is contained in a component of $G - S$. We call two ends $\omega, \omega' \in \Omega(G)$ *equivalent with respect to* S if they are contained in the same component of $G - S$ and denote this by $(\omega \sim_S \omega')$. Otherwise we say that S *separates these ends*.

Let $\eta : \Omega(H) \to \Omega(G)$ be the mapping that maps every end of $\Omega(H)$ to that end of $\Omega(G)$ that is a superset of it. The subgraph H is called *end-faithful* if the mapping η is bijective. The sets $V_S := \{(\omega, \omega') \in \Omega(G) \times \Omega(G) | \omega \sim_S \omega'\}$, where S is taken over all finite subsets of $V(G)$, form a base of a uniform structure on $\Omega(G)$; we denote this base by $\mathcal{V}_G$. This uniform structure induces a topology on $\Omega(G)$. We call the uniform space $\Omega(G)$ *uniform end space* and the associated topological space $\Omega(G)$ *topological end space*. Since for every two ends of G there exists a finite set of vertices that separates these two ends, we have $\bigcap_{V_S \in \mathcal{V}_G} V_S = \{(\omega, \omega) \in \Omega(G) \times \Omega(G)\}$, i.e. the uniform space $\Omega(G)$ is separated. A subgraph H of G is called *topologically end-faithful* if the map η is a homeomorphism between the topological spaces $\Omega(H)$ and $\Omega(G)$, i.e. if η is bijective and η as well as η^{-1} are continuous. The subgraph H is called *uniformly end-faithful* if η is an isomorphism between the uniform spaces $\Omega(H)$ and $\Omega(G)$, i.e. if η is bijective and η as well as η^{-1} are uniformly continuous.

If $B \subseteq G$ is a tree with root v, we define a partial order on the vertex set $V(B)$ of B, the tree order $<_B$: For $x, y \in V(B)$ we set $x <_B y$ if x lies in the unique v–y path in B. The tree B is called normal in G if the initial vertex and the end-vertex of any B–B path in G are comparable in the tree order.

We denote a countably infinite complete graph by $K_{\aleph_0}$ and call its vertices *branch vertices*. Any subdivided $K_{\aleph_0}$ is denoted by $TK_{\aleph_0}$. A fat $TK_{\aleph_0}$ is a subdivided $K_{\aleph_0}$, whose branch vertices are pairwise linked by $\aleph_1$ independent paths. (Two paths are called *independent* if they have no common inner vertices.) The vertices of infinite degree are called *branch vertices*. Let $A \subseteq G$ be a $K_{\aleph_0}$ or a fat $TK_{\aleph_0}$, respectively; we say A is *contained* in a subgraph $H \subseteq G$ if $A \cap H$ is a $K_{\aleph_0}$ or a fat $TK_{\aleph_0}$, respectively. Since all rays of a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$, respectively, are equivalent in G, we say a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$, respectively, *belongs to an end*.

A finite vertex set $S \subseteq V(G)$ *separates a vertex* $x \in V(G)$ *and a ray* $T \subseteq G$ in G if every x–T path in G meets S. The vertex set S *separates a vertex* $x \in V(G)$ *and a* $K_{\aleph_0}$ *or fat* $TK_{\aleph_0}$ in G, respectively, if the vertex x does not lie in that component of $G - S$ that contains the $K_{\aleph_0}$ or fat $TK_{\aleph_0}$. Moreover S *separates an end* $\omega \in \Omega(G)$ and *a vertex* $v \in V(G)$ in G if ω is contained in a component of $G - S$ that does not contain the vertex x. The vertex set S *separates an end* ω *and a ray* $T \subseteq V(G)$ in G if a tail of T and the end ω are contained in different components of $G - S$. We will use the following notation: Let $H \subseteq G$ be a subgraph; then we denote by $N_G(H)$ the set of neighbours of H in $G \backslash H$. By H_G we denote the subgraph of G that consists of $H \cup N_G(H)$ and all H–$N_G(H)$ edges of G.

2.2. **The uniform end structure.** In a tree, clearly, two rays are equivalent if and only if they have a tail in common. Thus an end-faithful tree or forest represents the end-structure of a graph in a very simple way. So it is of interest to study the following question of Halin, [4]: *Does every connected graph contain an end-faithful spanning tree, or at least an end-faithful forest?* Counterexamples of Seymour &

Thomas [11] and of Thomassen [12] show that this is not the case. However, Halin [4] proved that every countable connected graph contains an end-faithful spanning tree.

The end space of a tree B with root v is, as mentioned before, metrizable; consider for example the following metric: Given distinct ends ω, ω', let $x_{\omega,\omega'}$ be the vertex in B that separates ω and ω' and that has maximal distance from v, say $n(\omega, \omega')$; then $d(\omega, \omega') := 1\backslash(n(x_{\omega,\omega'}) + 1)$. It is easy to show that this metric generates the topological end space. This leads to the following conjecture stated by Diestel, [2], as a problem: *A graph contains a topologically endfaithful forest if and only if its end space is metrizable.* An answer to this problem is given by an observation of Diestel [2]: Every normal spanning tree of a connected graph is topologically end-faithful. But which graphs contain a normal spanning tree? Jung proved [8] that this is true for every countable connected graph. Furthermore every connected graph that does not contain $TK_{\aleph_0}$ and every connected graph that contains neither a $K_{\aleph_0}$ nor a fat $TK_{\aleph_0}$ contain a normal spanning tree, as shown by Halin [5]. Hence these graphs contain a topologically end-faithful tree.

For a metrizable uniform space the associated topological space is also metrizable. Since every uniformly end-faithful subgraph is also topologically end-faithful, also the following version of Diestel's question is of interest: *Does every graph with a metrizable uniform end space contain a uniformly end-faithful forest?* This is the case:

Theorem 1. *Every connected graph with metrizable uniform end space contains a uniformly end-faithful tree.*

It should be noted that, unlike in the topological case, the converse of this theorem does not hold. It easy to construct a forest whose uniform end space is not metrizable [13].

3. Construction of a $K_{\aleph_0}$- and fat $TK_{\aleph_0}$-free subgraph

3.1. **Construction of S^*.** For the proof of Theorem 1 we need the following two theorems:

Theorem 2 ([5]). *Every connected graph that does not contain a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ contains a normal spanning tree.*

Theorem 3 ([2]). *Every normal spanning tree is uniformly end-faithful.*

We now sketch the proof of Theorem 1: Throughout the paper let G be a fixed connected graph with a metrizable uniform end-space $\Omega(G)$. Our aim is to construct a connected uniformly end-faithful subgraph of G that contains neither $K_{\aleph_0}$ nor fat $TK_{\aleph_0}$. Then, by Theorems 2 and 3 this subgraph, and hence also G, contain a uniformly end-faithful tree. To do this we first construct a countable vertex set $S^* \subset V(G)$ such that $G - S^*$ decomposes into components that contain a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ only if they are one-ended. In each of these components we select a ray. We then link all the other components of $G - S^*$ containing ends with the selected rays via S^*. In this way we obtain a connected subgraph of G that is uniformly end-faithful to G and does not contain a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$.

The above construction is carried out in two steps: First we construct a graph $G' \subseteq G$ that does not contain a $K_{\aleph_0}$; we then construct a graph $G'' \subseteq G'$ that also does not contain a fat $TK_{\aleph_0}$. Then we show that G'' is uniformly end-faithful to G.

Let us recall a few topological properties that we need for the construction of S^*. By assumption the uniform end space is metrizable. So, by the following theorem, it has a countable base for the uniformity:

Theorem 4 ([10])**.** *A uniform space is metrizable if and only if it is separated and has a countable base for the uniformity.*

If a filter has a countable base, then, as one easily verifies, every base contains a countable base. For this reason the neighbourhood base $\mathcal{V}_G$ introduced in Section 1 contains a countable base $\mathcal{V}'_G$. We set $\tilde{S} := \bigcup_{V_S \in \mathcal{V}'_G} S$ and $\tilde{\mathcal{V}}_G := \{V_S | S \subset \tilde{S}, |S| < \aleph_0\}$. Since $\tilde{S}$ is countable, $\tilde{\mathcal{V}}_G$ is also countable. Furthermore $\tilde{\mathcal{V}}_G$ is a filter base of $\Omega(G)$, since it is a superset of the filter base $\mathcal{V}'_G$ of $\Omega(G)$. Thus we have proved the following lemma:

Lemma 1. *There exists a countable vertex set $\tilde{S} \subset V(G)$ such that the filter base $\tilde{\mathcal{V}}_G := \{V_S | S \subset \tilde{S}, |S| < \aleph_0\}$ generates the same uniform structure on $\Omega(G)$ as the filter base $\mathcal{V}_G = \{V_S | S \subset V(G), |S| < \aleph_0\}$.*

Remark 1. If we say that $\tilde{S}$ generates the uniform structure, we mean this in the sense of Lemma 1. This means in particular that for every two ends of G there exists a finite subset of $\tilde{S}$ that separates these ends. Furthermore we have the following: If an end $\omega \in \Omega(G)$ is separated in G from all other ends by a finite vertex set $S \subset V(G)$, then there exists a finite subset $S' \subseteq \tilde{S}$ that separates ω in G from all other ends. To prove this remember that only one element of the set $V_S \in \mathcal{V}_G$ contains ω, namely (ω, ω). Since $\tilde{\mathcal{V}}_G$ is a filter base, there exists a set $V_{S'} \in \tilde{\mathcal{V}}_G$ with $V_{S'} \subseteq V_S$. In this set the end ω occurs also only in (ω, ω). So the finite vertex set $S' \subseteq \tilde{S}$ separates the end ω from all other ends in G.

If $\tilde{S}$ is finite, every end of G is contained in an one-ended component of $G - \tilde{S}$. Hence, in this case we can select from every end of G a ray that begins in $\tilde{S}$ such that all these rays are pairwise disjoint (except for their initial vertices). We link the vertices in $\tilde{S}$ by a finite tree $B \subset G$. The subgraph of G consisting of B and the chosen rays then contains neither a $K_{\aleph_0}$ nor a fat $TK_{\aleph_0}$, since all vertices of infinite degree lie in B; but $V(B)$ is finite. Since $V(B)$ separates pairwise all ends of this graph, it is clearly uniformly end-faithful to G. For this reason we assume in the following that $\tilde{S}$ is infinite.

Construction of S^*: We now construct inductively the vertex set S^*: Let $\tilde{S} := \{b_1, b_2, ...\}$ and $S_1 := \{b_1\}$. The graph $G - S_1$ decomposes into components. Since S_1 is finite, every $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ is contained in one of these components. Let $\mathcal{C}_1$ be the set of components that contain a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$. By X_1 we denote the set of all vertices in $\tilde{S}$ that lie in one-ended components of $\mathcal{C}_1$. Consider now the set of all vertices in $\tilde{S}$ that belong to a component of $\mathcal{C}_1$ or that are adjacent to such a component. From this set we delete all vertices of X_1 and denote the resulting set by S^1. To formalize this we set $S^1 := \bigcup_{C \in \mathcal{C}_1} (V(C) \cup N_G(C)) \cap (\tilde{S} \backslash X_1)$.

Suppose $X_1, ..., X_{k-1}$ and $S^1, ..., S^{k-1}$ have been defined for some $k > 0$. Let $S_k := \{b_1, ..., b_k\}$. Then $G - (S_k \backslash \bigcup_{i<k} X_i)$ decomposes into components. Since $S_k \backslash \bigcup_{i<k} X_i$ is finite, every $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ is contained in one of these components. Let $\mathcal{C}_k$ be the set of those components that contain a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$. By X_k we denote the set of all vertices in $\tilde{S}$ that lie in one-ended components of $\mathcal{C}_k$. Then let $S^k := \bigcup_{C \in \mathcal{C}_k} (V(C) \cup N_G(C)) \cap (\tilde{S} \backslash \bigcup_{i \leq k} X_i)$. Finally we set $S^* := \bigcap_{i \in \mathbb{N}} S^i$.

Remark 2. By the construction of S^* we have $S_i \setminus \bigcup_{k<i} X_k \subseteq S_j \setminus \bigcup_{k<j} X_k$ for all $i < j$, since $(S_l \setminus \bigcup_{k<l} X_k) \cap X_l = \emptyset$ and $S_{l+1} \supseteq S_l$ for all $l \in \mathbb{N}$. This roughly means that the graph decomposes step by step into finer components. Thus we have $S^i \supseteq S^j$ for $i < j$.

In the construction of G' the graph $G - S^*$ decomposes into components, some of which contain ends. But not every end of G must be contained in one of these components, since S^* may be infinite. So we distinguish different types of ends: On one hand, those ends of G whose rays are infinitely linked to S^* in G and on the other hand, those ends whose rays are only finitely linked to S^*. By definition of an 'end' it is clear that an end belongs either to the first or to the second type. Furthermore the ends of the second type are contained in a component of $G - S^*$.

Our aim is to construct a subgraph of G that contains neither a $K_{\aleph_0}$ nor a fat $TK_{\aleph_0}$. So we distinguish such ends to which a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ belongs and such ends for which this is not the case.

In the construction of G' we take those components of $G - S^*$, that contain ends and the vertices of S^* and link them by edges. By an arbitrary selection of these edges it is quite possible that we generate a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ that is not contained in one of the components of $G - S^*$. So we have to take care which edges we select in the construction of G'. Therefore we distinguish those ends that are contained in a component of $G - S^*$ with finitely many neighbours in S^* and those ends that are contained in a component with infinitely many neighbours in S^*. We now characterize the ends of G by four types:

Type 1: An end $\omega \in \Omega(G)$ is of Type 1 if in G every ray of ω is infinitely linked to S^*.

Type 2: An end $\omega \in \Omega(G)$ is of Type 2 if in G every ray of ω is only finitely linked to S^* and a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ belongs to ω.

Type 3: An end $\omega \in \Omega(G)$ is of Type 3 if in G every ray of ω is only finitely linked to S^*, neither a $K_{\aleph_0}$ nor a fat $TK_{\aleph_0}$ belongs to ω and the set of neighbours $N_G(C)$ of the component C of $G - S^*$ that contains ω is finite.

Type 4: An end $\omega \in \Omega(G)$ is of Type 4, if in G every ray of ω is only finitely linked to S^*, neither a $K_{\aleph_0}$ nor a fat $TK_{\aleph_0}$ belongs to ω and the set of neighbours $N_G(C)$ of the component C of $G - S^*$ that contains ω is infinite.

Remark 3. Clearly each end of G belongs to one of these four types of ends.

We show now that every end of Type 2, 3 and 4 is contained in a component of $G - S^*$ and furthermore we show that such components are one-ended in the case of Type 2.

Lemma 2. *Let $\omega \in \Omega(G)$ be an end such that in G every ray of ω is only finitely linked to S^*. Then ω is contained in a component of $G - S^*$. If furthermore a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ belongs to ω, this component is one-ended and has only finitely many neighbours in S^*.*

Proof. Let $\omega \in \Omega(G)$ be an end such that in G every ray of ω is only finitely linked to S^*. Then every ray $T \in \omega$ contains a tail in $G - S^*$, since T contains at most finitely many vertices of S^*. Furthermore in $G - S^*$ each two rays T, $T' \in \omega$ are equivalent, since from any infinite set of pairwise disjoint T–T' paths at most finitely many of these paths contain vertices of S^*. This means that ω is contained in a component of $G - S^*$.

Now let $\omega \in \Omega(G)$ be such that in G every ray of ω is only finitely linked to S^* and such that a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ belongs to ω. Then there exists a finite vertex set that separates ω in G from all other ends. Suppose this were not the case and let T be an arbitrary ray of ω. Since there are only finitely many pairwise disjoint T–S^* paths in G, there exists a finite vertex set S' that separates ω and S^* in G (that means ω is contained in a component of $G - S'$ that does not contain a vertex of S^*). Furthermore there exists an end $\omega' \in \Omega(G)$ with $\omega' \neq \omega$ that is contained in the same component C_ω of $G - S'$ as ω. Then $V(C_\omega) \cap S^* = \emptyset$, but $V(C_\omega) \cap \tilde{S} \neq \emptyset$, since $\tilde{S}$ generates the uniform structure (see Remark 1).

Since S' is finite, C_ω contains branch vertices of the $K_{\aleph_0}$ and fat $TK_{\aleph_0}$ that belong to ω. We denote by H the set of these branch vertices. Let W be an arbitrary H–$\tilde{S}$ path in C. By v_W we denote the initial vertex of W in H and by e_W the end-vertex of W in $\tilde{S}$. The vertices e_W and v_W are not separable by a finite subset of $\tilde{S}\backslash\{e_W\}$ and hence the vertex e_W is not separable from the associated $K_{\aleph_0}$ or fat $TK_{\aleph_0}$.

Now we show that $e_W \notin \bigcup_{i\in\mathbb{N}} X_i$. Suppose the converse and let $j \in \mathbb{N}$ be minimal with $e_W \in X_j$. Then by definition of X_j the vertex e_W lies in an one-ended component C^j of $G-(S_j\backslash\bigcup_{i<j} X_i)$ that contains a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$. Let ω^* be the end of G that is contained in C^j. Then the $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ belongs to ω^*. We will show that C^j is a component of $G - S^*$. Therefore we consider again the construction of S^*: For every $k > j$ let C^k be the component of $G-(S_k\backslash\bigcup_{i<k} X_i)$ that contains ω^*. As shown in Remark 2 we have $S_k\backslash\bigcup_{i<k} X_i \supseteq S_j\backslash\bigcup_{i<j} X_i$ and hence $C^k \subseteq C^j$ for all $k > j$. On the other hand we have $V(C^j) \cap \tilde{S} \subseteq X_j$ by definition of X_j. So $C^k \supseteq C^j$ for all $k > j$, since C^k is a component of $G-(S_k\backslash\bigcup_{i<k} X_i)$. Hence $C^k = C^j$ for $k > j$. That means C^j is a component of $G-(S_k\backslash\bigcup_{i<k} X_i)$ for all $k \geq j$. Since C^j contains a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ we have $N_G(C^j) \subseteq S^k$ for all $k \geq j$. Furthermore we have $S^k \supseteq S^j$ for all $k < j$ (see Remark 2). So we have $N_G(C^j) \subseteq \bigcap_{i\in\mathbb{N}} S^i = S^*$, which means C^j is a component of $G - S^*$.

By assumption C^j is one-ended. So the vertex e_W lies in the one-ended component C^j of $G - S^*$. But as we assumed above e_W lies in the component C_ω of $G - S'$ that contains more than one end but no vertices of S^*. This means C_ω is a subgraph of the one-ended component C^j. Because of this contradiction we have $e_W \notin \bigcup_{i\in\mathbb{N}} X_i$.

Since there is no finite subset of $\tilde{S}\backslash\{e_W\}$ that separates e_W in G from the branch vertex v_W, we have $e_W \in S^k$ for all $k \in \mathbb{N}$. Thus, $e_W \in S^*$, in contradiction to $S^* \cap V(C_\omega) = \emptyset$. Hence there exists a finite vertex set of G that separates ω in G from all the other ends.

Since $\tilde{S}$ generates the uniform structure on $\Omega(G)$, there exists a finite subset of $\tilde{S}$ that separates ω in G from all the other ends (see Remark 1). Thus, there exists a minimal $j > 0$ such that ω is contained in an one-ended component K of $G - S_j$. We show now that K is also a component of $G-(S_j\backslash\bigcup_{i<j} X_i)$. This is clear for $j = 1$. Hence in the following let $j > 1$. For all $k \leq j$ let K^k be the component of $G-(S_k\backslash\bigcup_{i<k} X_i)$ that contains ω. Furthermore let K^{j-1} be a component of $G-S_{j-1}$. Since $S_j\backslash\bigcup_{i<j} X_i \supseteq S_{j-1}\backslash\bigcup_{i<j-1} X_i$, we have $K^j \supseteq K^{j-1}$. So K^j is also a component of $G-(S_j\backslash X_{j-1})$. But since $j-1 < j$, the component K^{j-1} contains more than one end. Hence $V(K^{j-1}) \cap X_{j-1} = \emptyset$. Thus K^j is also a component of

$G - S_j$, so $K^j = K$. That means K is the component of $G - (S_j \setminus \bigcup_{i<j} X_i)$ that contains ω. We have shown above that a one-ended component of $G - (S_j \setminus \bigcup_{i<j} X_i)$ that contains a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ is also a component of $G - S^*$. Thus K is a component of $G - S^*$ and $N_G(K)$ is finite. □

Remark 4. As we have just seen, the ends of Type 3 and Type 4 are contained in components of $G - S^*$. We denote the set of these components by $\mathcal{C}$. Furthermore the ends of Type 2 are contained in one-ended components of $G - S^*$ that have a finite neighbourhood in S^*. We denote the set of these components by $\mathcal{C}^*$. By the definition of $\mathcal{C}$ and $\mathcal{C}^*$ we have $\mathcal{C} \cap \mathcal{C}^* = \emptyset$. Furthermore we have $\bigcup_{i\in\mathbb{N}} X_i \subseteq \bigcup_{C\in\mathcal{C}^*} C$, by the proof of Lemma 2.

3.2. **Construction of G' by removing every $K_{\aleph_0}$ of G.** We consider the subgraph $\bigcup_{C\in\mathcal{C}} C$ and link the components of this graph by the vertices of $S^* := \{s_0, s_1, s_2, ...\}$: Set $t_0 := s_0$ and $t_1 := s_1$ and choose an arbitrary but fixed t_1–t_0 path W_1 in G; furthermore let $r(x) := 1$ for all $x \in V(W_1)$. Suppose the vertices $t_0, ..., t_{k-1}$ and the paths $W_1, ..., W_{k-1}$ have been chosen for some $k > 1$ and for all $i < k$ let $r(x) = i$ for all $x \in V(B_i \setminus B_{i-1})$; here $B_i := \bigcup_{j\leq i} W_j$ and $B_{i-1} := \bigcup_{j\leq i-1} W_j$. Let t_k be the vertex with minimal index in $S^* \setminus B_{k-1}$. We choose a t_k–B_{k-1} path W_k by the following rules:
R1) The end vertex $v_k \in B_{k-1}$ of W_k should have maximal possible value $r(v_k)$.
R2) When selecting W_k according to rule R1) select v_k such that in B_{k-1} it has maximal distance to the vertex t_0.
Then $B := \bigcup_{i\in\mathbb{N}} W_i$ is a tree with root $v := t_0$. We now consider the ends of Type 2: As we have seen in Lemma 2 each of these ends is contained in a one-ended component of $G - S^*$. As mentioned before we denote the set of these components by $\mathcal{C}^*$. For every end ω of Type 2 we now select a ray T_ω that lies in the component $C_\omega \in \mathcal{C}^*$ that contains ω. Let t_n be the vertex of $N_G(C_\omega)$ with maximal index (by Lemma 2 the set $N_G(C_\omega)$ is finite). We link the ray T_ω and the tree B by a T_ω–t_n path W_ω in ${C_\omega}^G$ (here ${C_\omega}^G$ denotes the graph that consists of $C_\omega \cup N_G(C_\omega)$ and all C_ω–$N_G(C_\omega)$ edges of G). Now we add all C–S^* edges of the components $C \in \mathcal{C}$. We denote the resulting graph by G'.

Remark 5. The graph G' is composed of four different types of subgraphs of G: First it contains the end-containing components of $G - S^*$ that contain neither a $K_{\aleph_0}$ nor a $TK_{\aleph_0}$; second it contains subgraphs of the components of $G - S^*$ that contain a $K_{\aleph_0}$ or $TK_{\aleph_0}$; third it contains edges between S^* and all these components; and fourth it contains a tree B linking the vertices of S^*. By construction this graph is connected. However, one should mention that the tree B is not necessarily disjoint to the components of $G - S^*$.

Definition 1 ([1])**.** Let H be a tree with root b and let $x \in V(H)$. Then the unique x–b path in H is called *down-closure* $[x]$ of x.

Definition 2. Let H be a tree and $x, y \in V(H)$. Then the vertex x lies *above* the vertex y if y lies in the down-closure of x. The vertex x lies below y if x lies in the down-closure of y.

We now prove several lemmata, that we need later on:

Lemma 3. *Let H be a rooted tree and W a path in H and v a vertex in $H \setminus W$. If v lies in the down-closure of a vertex of W, then v lies in the down-closure of every vertex of W.*

Proof. Let t and t' be two vertices of W; then the unique t–t' path $W_{t,t'}$ in H is a subpath of W. This path consists of the vertices of $[t]\backslash[t']$, $[t']\backslash[t]$ and the vertex of $[t] \cap [t']$ with maximal distance to the root. We now assume that there exists a vertex v from $H\backslash W$ and two vertices t and s of W such that v lies in the down-closure of the vertex t but not in the down-closure of the vertex s. Then v lies in $[t]\backslash[s]$ and hence in the path $W_{t,s}$. Thus v lies also in W, in contradiction to the assumption. □

Lemma 4. *The tree B is normal in G.*

Proof. We use the same notation as in the construction of G'. For all $i \in \mathbb{N}$ we denote by t_i the initial vertex and by v_i the end-vertex of the path W_i. Suppose there exists a B–B path W in G with end-vertices $h, h' \in V(B)$ such that neither h nor h' lies in the down-closure of the other. Let $r(h) = i$, $r(h') = k$ and w.l.o.g. $i > k$. Furthermore let $m > k$ be minimal, such that the unique t_m–h path of B contains only vertices x with $r(x) > k$. We denote this path by $\tilde{W}$. Due to the minimality of m we have $r(v_m) \leq k$. Thus v_m does not lie in $\tilde{W}$. But, since v_m is in the down-closure of t_m, by Lemma 3 the vertex v_m is also in the down-closure of h. If $r(v_m) < k = r(h')$, then W_m does not obey rule R1), since the t_m–h' path $W' := \tilde{W} \cup W$ contains only vertices x with $r(x) \geq k$ and such vertices that do not lie in B. Hence $r(v_m) = k$. Then v_m is in the down-closure of h' or conversely. The latter means that h' lies in the down-closure of h, since v_m lies in the down-closure of h. This is a contradiction to our assumption.

Thus we may assume that v_m lies in the down-closure of h'. But then the path W_m does not obey rule R2), since the path $W' := \tilde{W} \cup W$ contains (except for the end vertex h') only vertices x with $r(x) > k$ and vertices that do not lie in B and ends in h', where h' has larger distance to the root as v_m. But this is a contradiction to the construction of B. □

Since B is normal in G, there are two interesting structural properties that we need for the proof of Theorem 1:

Lemma 5 ([1])**.** *For any two vertices $x, y \in V(B)$ we have $[x] \cap [y]$ separates x and y in G.*

Lemma 6 ([1])**.** *If $W = x_1 ... x_n$ is a B–B path in G, then x_1 and x_n are comparable in the tree order $<_B$.*

Lemma 5 and Lemma 6 are equivalent to results of the paper *Normal Tree Orders For Infinite Graphs* by J.-M. Brochet and R. Diestel [1]. The proofs given there may be easily modified to prove the above presented versions.

Lemma 7. *Let $H \supseteq B$ be an arbitray subgraph of G and let $T \subseteq H$ be a ray that is infinitely linked to S^* in H. Then in H the ray T is equivalent to a ray of B.*

Proof. Since T is infinitely linked to S^* in H, we can choose infinitely many pairwise disjoint T–S^* paths W_i in H. For all $i \in \mathbb{N}$ we denote by v_i the initial vertex of the path W_i in T and by e_i the end-vertex in S^*. Furthermore we set $E := \{e_1, e_2, ...\}$ and denote by v the root of B (see construction of G'). For all $i \in \mathbb{N}$ we denote by $\tilde{W}_i$ the unique e_i–v path in B and set $B' := \bigcup_{i \in \mathbb{N}} \tilde{W}_i$. Then B' is a tree with root v. Since the paths W_i are pairwise disjoint, the set E is infinite; hence also B' is infinite. Then by König's Theorem [9] B' contains either a vertex of infinite degree or a ray. Suppose the first is the case; then there exist infinitely many pairwise disjoint (except for the vertex v) v–E paths in B' whose associated end-vertices $e_{i_j}, j \in \mathbb{N}$, lie in B' above v. W.l.o.g. let these be labeled in such a way that the associated end vertices v_{i_j} of the paths W_{i_j} are ordered on T in the order of their labels. For each two vertices $v_{i_{2j}}$, $v_{i_{2j+1}}$ let $W_{i_{2j},i_{2j+1}}$ be the unique $v_{i_{2j}}$–$v_{i_{2j+1}}$ path in T. Due to the ordering of the v_{i_j} on T these paths are pairwise disjoint. Since the paths W_i are pairwise disjoint for all $i \in \mathbb{N}$ and contain no vertices of T (except for the end-vertices), also the $e_{i_{2j}}$–$e_{i_{2j+1}}$ paths $\bar{W}_{i_{2j}} := W_{i_{2j}} \cup W_{i_{2j},i_{2j+1}} \cup W_{i_{2j+1}}$ are pairwise disjoint.

Since B is normal in G, also $B' \subseteq B$ is normal in G. Then, by Lemma 6 every path $\bar{W}_{i_{2j}}$ must contain a vertex of $[e_{i_{2j}}] \cap [e_{i_{2j+1}}] = [v]$. Since $[v]$ is finite (it even contains only the vertex v), this is a contradiction to the pairwise disjointness of the paths $\bar{W}_{i_{2j}}$.

So B' contains a ray R; w.l.o.g we asssume that it starts in the root of B'. Let us show that there are infinitely many pairwise disjoint T–R paths in H. Suppose every set of pairwise disjoint T–R paths in H is finite. Let $\mathcal{W}$ be a set of such paths; then $V(\mathcal{W})$ is finite, say $|V(\mathcal{W})| = n$. Then there exists a vertex $x \in V(R) \backslash V(\mathcal{W})$ such that no vertex of $V(\mathcal{W})$ lies above x. Since R is a ray, all except finitely many vertices of R lie in B above x. We show now that infinitely many vertices $e_{i_j} \in E$, $j \in \mathbb{N}$, lie in B above x. Suppose not; then either there exists a vertex $y \in V(R) \backslash E$ that lies in B above x but no vertex in E lies above y or no vertex of E lies above x (in this case we set $y := x$). But this is a contradiction to the construction of B', since y lies in at least one of the paths $\tilde{W}_i$. Hence infinitely many vertices of E lie in B' above x. We select $n+1$ of these vertices $e_{i_1}, ..., e_{i_{n+1}}$. From assumption the associated T–e_{i_j} paths W_{i_j} are pairwise disjoint. Since $V(\mathcal{W})$ contains only n distinct vertices, at least one of these paths, say W_{i_k}, is disjoint to the paths of $\mathcal{W}$. Since all vertices of R that lie above x (including x) do not lie in $V(\mathcal{W})$, the unique e_{i_k}–x path W in B does not contain a vertex in $V(\mathcal{W})$. Thus also the path $W_{i_k} \cup W$ does not contain a vertex of $V(\mathcal{W})$. This path, however, contains a T–R path as subpath (since it starts in T and ends in R) which is disjoint to $V(\mathcal{W})$. But this is a contradiction to the maximality of $\mathcal{W}$. Thus there exist infinitely many pairwise disjoint T–R paths in H, and hence, R is in H equivalent to T. □

Remark 6. By Lemma 7 the tree B contains a ray of every end of Type 1 and hence also $G' \supseteq B$ does so. Hence by Lemma 2 and Lemma 7 each end of G is represented by a ray in G'. Later on we will show that in G' an end of G does not split into several ends.

We show now that G' does not contain a $K_{\aleph_0}$:

Lemma 8. *The components of $\mathcal{C}$ and $\mathcal{C}^* \cap G'$ do not contain a branch vertex of a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ of G'.*

Proof. By construction of G', we have $G' \backslash B \subseteq \mathcal{C} \cup (\mathcal{C}^* \cap G')$. Furthermore for $C_\omega \in \mathcal{C}^*$ the subgraph $C_\omega \cap G'$ consists of a ray T_ω, a T_ω–S^* path (except for its end-vertex in S^*) and (again except for their end-vertices in S^*) of at most finitely many S^*–S^* paths of B (since $N_G(C_\omega)$ is finite and B is composed of S^*–S^* paths). Since all vertices of $C_\omega \cap G'$ have only finite degree, no branch vertex of a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ of G' lies in $C_\omega \cap G'$.

Let C be a component of $\mathcal{C}$. Suppose a branch vertex of a $K_{\aleph_0}$ or $TK_{\aleph_0}$ of G' lies in C. We denote by ω the end to which the $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ belongs. By the definition of $\mathcal{C}$, the component C contains an end ω' to which neither a $K_{\aleph_0}$ nor a fat $TK_{\aleph_0}$ belongs, so $\omega \neq \omega'$. Since $\tilde{S}$ generates the uniform structure of $\Omega(G)$, there exists a finite subset of $\tilde{S}$ that separates ω and ω' in G (see Remark 2). Since C contains ω' and a branch vertex of a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ belonging to ω, we have $\tilde{S} \cap V(C) \neq \emptyset$.

We denote by H the set of all branch vertices of all $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ that lie in C. Then C contains an H–$\tilde{S}$ path and we denote by v its end-vertex in $\tilde{S}$. Since $C \in \mathcal{C}$ and $\bigcup_{i\in\mathbb{N}} X_i \subseteq \bigcup_{C\in\mathcal{C}^*} C$ and $\mathcal{C} \cap \mathcal{C}^* = \emptyset$, we have $v \notin \bigcup_{k\in\mathbb{N}} X_k$. Then in every step of the construction of S^* the vertex v lies either in the same component as a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ or is adjacent to that component. Thus $v \in S^k$ for all $k \in \mathbb{N}$ and hence $v \in S^*$. But this is a contradiction to $v \in V(C)$. □

Lemma 9. *The graph G' does not contain a $K_{\aleph_0}$.*

Proof. By Lemma 8 the branch vertices of a $K_{\aleph_0}$ of G' lie in $G' \backslash (\mathcal{C} \cup \mathcal{C}^*) = B\backslash(\mathcal{C} \cup \mathcal{C}^*)$. Since this subgraph is a forest, it does not contain a $K_{\aleph_0}$. □

3.3. **Construction of G''.** By Lemma 9 the graph G' does not contain a $K_{\aleph_0}$. But it is quite possible that G' contains some fat $TK_{\aleph_0}$. We will show that their branch vertices lie in S^*. The vertices of S^* lie in B and in G' they are adjacent to the components of $\mathcal{C} \cup \mathcal{C}^*$. But each two branch vertices of a fat $TK_{\aleph_0}$ are linked by uncountably many pairwise independent paths. Since the tree B is countable, in G' uncountably many pairwise independent paths can only run through the components of $\mathcal{C} \cup \mathcal{C}^*$. Thus, if we delete in G' sufficiently many edges between the components of $\mathcal{C} \cup \mathcal{C}^*$ and the vertices in S^* such that there are only countably many pairwise disjoint S^*–S^* paths in the resulting graph G'', then this graph neither contains a $K_{\aleph_0}$ nor a fat $TK_{\aleph_0}$.

For the construction of G'' we need some further lemmata:

Lemma 10. *The branch vertices of a fat $TK_{\aleph_0}$ of G' lie in S^*.*

Proof. By Lemma 8 the branch vertices of a fat $TK_{\aleph_0}$ of G' lie in $G' \backslash (\mathcal{C} \cup \mathcal{C}^*) = B\backslash(\mathcal{C} \cup \mathcal{C}^*)$. By construction the tree B is countable. But a branch vertex of a fat $TK_{\aleph_0}$ has uncountable degree. Thus only such vertices of $B\backslash(\mathcal{C} \cup \mathcal{C}^*)$ that are adjacent in G' to the components of $\mathcal{C} \cup \mathcal{C}^*$ can be branch vertices of a fat $TK_{\aleph_0}$. But only the vertices in S^* have this property. □

Lemma 11. *Each component $C \in \mathcal{C}$ with infinite $N_G(C)$ contains vertices of $\tilde{S}$.*

Proof. Let C be a component of $\mathcal{C}$ with infinite $N_G(C)$. If C contains rays from several ends, we clearly have $V(C) \cap \tilde{S} \neq \emptyset$, since $\tilde{S}$ generates the uniform structure (see Remark 1). Otherwise by definition of $\mathcal{C}$ the component C contains exactly one end ω of Type 4. Each ray $T \in \omega$ has by definition only finitely many pairwise disjoint T–S^* paths in G. Thus, there exists a finite vertex set of C that separates ω from S^*. Suppose this were not the case and let $T \in \omega$ be chosen arbitrary but fixed. Let W be a maximal set of pairwise disjoint T–S^* paths in G. Since $V(W)$ is finite, ω is contained in a component of $G - V(W)$. This component does not contain vertices of S^*, since otherwise there exists a T–S^* path that is disjoint to the paths of W, which is a contradiction to the maximality of W.

Hence there exists a finite vertex set $S \subseteq V(G)$ that separates in G the end ω and the vertex set S^*. Since $N_G(C) \subseteq S^*$, the component of $G - S$ that contains ω is a subgraph of the component C. So S separates the end ω in G from all other ends, since C is one-ended. Then there exists a finite subset of $\tilde{S}$ that separates ω in G

from all other ends, since $\tilde{S}$ generates the uniform structure (see Remark 1). For all $j \in \mathbb{N}$ let S_j be defined as in the construction of S^* and let i be minimal such that S_i separates ω in G from all other ends. Suppose this is not the case for $S_i \setminus \bigcup_{j<i} X_j$. Let ω' be an end that is not separated from ω and let $T \in \omega$ and $T' \in \omega'$ be rays in $G - (S_i \setminus \bigcup_{j<i} X_j)$. Then there exists a T–T' path W in $G - (S_i \setminus \bigcup_{j<i} X_j)$. Since S_i separates the two ends, W contains a vertex $x \in \bigcup_{j<i} X_j$. By definition of the X_j, there is a minimal $k < i$ such that the vertex x lies in a component of $G - (S_k \setminus \bigcup_{j<k} X_j)$ that contains exactly one end and a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$ belong to this end. Since this is not the case for ω (ω is of Type 4), the end ω is not contained in this component. Then $T \cup W$ must contain a vertex of $S_k \setminus \bigcup_{j<k} X_j$. But $S_k \setminus \bigcup_{j<k} X_j \subseteq S_i \setminus \bigcup_{j<i} X_j$ for $k < i$ (see Remark 2), i.e. $T \cup W$ contains a vertex of $S_i \setminus \bigcup_{j<i} X_j$, in contradiction to the choice of W and T. Hence the vertex set $S_i \setminus \bigcup_{j<i} X_j$ separates ω in G from all other ends.

Suppose now that C does not contain a vertex of $\tilde{S}$. Then C does not contain a vertex of $S_i \setminus \bigcup_{j<i} X_j \subseteq \tilde{S}$. But since this set is finite, it separates at most finitely many vertices of $N_G(C)$ from C, and thus from ω. The other vertices are separated from all the other ends. Since neither a $K_{\aleph_0}$ nor a fat $TK_{\aleph_0}$ belongs to ω (ω is of Type 4), C contains neither a $K_{\aleph_0}$ nor fat $TK_{\aleph_0}$. Then all except finitely many vertices of $N_G(C)$ lie in a component of $G - (S_i \setminus \bigcup_{j \leq i} X_j)$ that does not contain a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$. So in the construction of S^* these vertices of $N_G(C)$ lie not in S^i and hence also not in S^*, which is a contradiction to the definition of $N_G(C)$. □

Remark 7. Since the vertex set $\tilde{S}$ is countable and the components of $\mathcal{C}$ are pairwise disjoint, by Lemma 11 there are only countably many components $C \in \mathcal{C}$ with infinite $N_G(C)$.

Definition 3 ([3])**.** Let X be a subgraph of a graph G and let v be a vertex of $G \setminus X$, such that G contains infinitely many pairwise disjoint (except for v) v–X paths. Then the subgraph of G consisting of these paths is called a *v-X fan* in G.

Lemma 12 ([3])**.** *Let U and C be disjoint subgraphs of a graph G such that U is infinite, every vertex of U has a neighbour in C and C is connected. Then G either contains an v–U fan for a v of C, or there exists a ray $R \subset C$ with infinitely many pairwise disjoint R–U paths in G.*

In the following we denote by C^G for every component $C \in \mathcal{C}$ the subgraph of G that consists of $C \cup N_G(C)$ and all C–$N_G(C)$ edges of G.

Lemma 13. *For all $C \in \mathcal{C}$ the graph C^G does not contain a v–S^* fan with $v \in V(C)$.*

Proof. Suppose there exists a component $C \in \mathcal{C}$ such that C^G contains a v–S^* fan with $v \in V(C)$. Then $N_G(C)$ must be infinite and C contains by Lemma 11 vertices of $\tilde{S}$. Thus we can choose a v–$\tilde{S}$ path in C. Let s be the end-vertex of this path in $\tilde{S}$. Then no finite subset of $\tilde{S} \setminus \{s\}$ separates s and $N_G(C)$. Since in any step of the construction of S^* only a finite set of vertices is deleted in G, either the vertex s lies in each of these steps in a component that contains infinitely many vertices of $N_G(C) \subseteq S^*$ or s is adjacent to such a component. Since all but finitely many vertices of S^* lie in any step of the construction of S^* in a component that contains a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$, in any of these steps either the vertex s lies also in such a component or it is adjacent to such a component. But then $s \in S^*$, in contradiction to $s \in V(C)$. □

Corollary 1. *For $C \in \mathcal{C}$ the neighbourhood $N_G(C)$ is infinite if and only if C contains a ray T with infinitely many pairwise disjoint T–S^* paths in G.*

Proof. Let $C \in \mathcal{C}$ be a component with infinite $N_G(C)$. By definition $N_G(C)$ is disjoint to C and every vertex of $N_G(C)$ has a neighbour in C. Furthermore C is connected. So by Lemma 12 and Lemma 13 there exists a ray $T \subseteq C$ with infinitely many pairwise disjoint T–$N_G(C)$ paths in G. Since $N_G(C) \subseteq S^*$, this proves the 'if' part of the proposition. To see the converse remember that $V(C) \cap S^* = \emptyset$ but $N_G(C) \subseteq S^*$. $\square$

Definition 4. Let $\mathcal{R}$ be an infinite set of pairwise disjoint rays that belong to ends of Type 1 and that all start in a vertex $v \in V(G)$; furthermore let all rays of $\mathcal{R}$ be pairwise disjoint (except for the vertex v). Then the subgraph of G that consists of all rays of $\mathcal{R}$ is called a big star with center v.

Lemma 14. *The components $C \in \mathcal{C}$ do not contain a big star.*

Proof. Suppose there exists a component $C \in \mathcal{C}$ that contains a big star with center v. Since every ray T of a big star belongs to an end of Type 1, it has infinitely many pairwise disjoint T–S^* paths in G. Then via an inductive process indexed over the branches of the big star we can select infinitely many v–S^* paths which are pairwise disjoint except for the initial vertex v. But then C^G contains a v–S^* fan with $v \in V(C)$, in contradiction to Lemma 13. $\square$

Now let $C \in \mathcal{C}$ be an arbitrary but fixed component with infinite $N_G(C)$. Then by Corollary 1 C contains a ray of an end of Type 1. Consider C as a subgraph of G' and let $\Omega(C)$ be the end space of C. This end space consists of two sorts of ends: on one hand ends of Type 4 that are contained in C and on the other hand ends that are in G subsets of an end of Type 1. In the following we denote by $\mathcal{E}_C$ the set of ends of the second sort.

Definition 5. Let E be a subset of $\Omega(G)$, and let M be a set of rays in G such that the map $f : M \to E$ that maps every ray of M to that end of $\Omega(G)$ that contains this ray is bijective. Then M is called a set of representatives of E.

Remark 8. Definition 5 means that M contains exactly one ray of every end of E but no further rays.

Definition 6. Let M be a set of rays in G. Then we call a countable vertex set $Y \subset V(G)$ that contains infinitely many vertices of every ray in M an M-cover.

We now select one ray from every end of $\mathcal{E}_C$. Let Γ_C be the set of these rays. Then Γ_C is a set of representatives of $\mathcal{E}_C$. We show now that there exists a Γ_C-cover:

Lemma 15. *For each component $C \in \mathcal{C}$ with infinite $N_G(C)$ there exists a Γ_C-cover.*

Proof. Let C be a component of $\mathcal{C}$ with infinite $N_G(C)$ and let $v_0 \in V(C)$ be an arbitrary but fixed vertex. For every ray $T \in \Gamma_C$ we select a ray $T_{v_0} \subset C$ that begins in v_0 and that contains a tail of T. Then let $\Gamma'_C := \{T_{v_0} | T \in \Gamma_C\}$. Let "$\subseteq$" be an order of Γ'_C and let α be the ordinal number of $(\Gamma'_C, \subseteq)$. We set $T := T_\beta$, if T is in the position β of the order "$\subseteq$" of Γ'_C.

Now for all $\beta < \alpha$ we inductively define a tree B_β: Let $B_0 := T_0$ and suppose that for arbitrary $\beta > 0$ the trees B_γ are defined for all $\gamma < \beta$. Then we set $B_\beta := B'_\beta$

if $|V(B'_\beta \cap T_\beta)| = \infty$ and $B_\beta := B'_\beta \cup xT_\beta$, otherwise. Here $B'_\beta := \bigcup_{\gamma<\beta} B_\beta$ and x is the last vertex of T_β in B'_β and xT_β is the tail of T_β beginning in x. (The vertex x exists, since T_β begins in $v_0 \in B_0 \subset B'_\beta$.)

We set $B := \bigcup_{\beta<\alpha} B_\beta$. Then B is a tree and every $T_\beta \in \Gamma'_C$ has infinitely many vertices in B. Since B is a subgraph of C by Lemma 14, every vertex in B has finite degree, i.e. $V(B)$ is countable. But then the vertex set $V(B)$ is a Γ'_C-cover, and hence also a Γ_C-cover. $\square$

We now construct a subgraph G'' of G' such that G'' does not contain a fat $TK_{\aleph_0}$. As mentioned before this can be achieved by deleting sufficiently many edges between S^* and the components of $\mathcal{C} \cup \mathcal{C}^*$. For every component $C \in \mathcal{C}$ we denote (in analogy to C^G) by $C^{G'}$ the subgraph of G' consisting of $N_{G'}(C) \cup C$ and all C–$N_{G'}(C)$ edges of G'. Since $N_G(C) = N_{G'}(C)$ and since in the construction of G' all C–$N_G(C)$ edges are inserted, we have $C^G = C^{G'}$.

Construction of G'': We delete in G' all edges between S^* and the components $C \in \mathcal{C}$, except those that lie in B. We consider first the components $C \in \mathcal{C}$ with infinite $N_G(C)$: Let $\mathcal{E}_C$ and Γ_C be defined as above. By Lemma 15 there exists a set $Y_{\Gamma_C} \subseteq V(C)$ that is a Γ_C-cover. Let $\mathcal{P} := Y_{\Gamma_C} \times N_G(C)$. Since Y_{Γ_C} and $N_G(C)$ are countable, $\mathcal{P}$ is also countable. For every finite subset P of $\mathcal{P}$ for which this is possible we now choose for every pair $(a, b) \in P$ an a–b path in $C^{G'}$ such that these paths are pairwise independent. Two paths that are chosen to different finite subsets of $\mathcal{P}$, however, must not be independent. We now reinsert all $N_G(C)$–C edges of all chosen paths. For the components $C \in \mathcal{C}$ with finite $N_G(C)$ we proceed as follows: Let $t_{n(C)}$ be the vertex with maximal index in $N_G(C)$. For each of these components we insert a $t_{n(C)}$–C edge. The resulting graph is the graph G''.

Remark 9. Since for every component $C \in \mathcal{C}$ with infinite $N_G(C)$ for every singleton of $\mathcal{P}$ a path is chosen, every vertex of $N_G(C)$ is connected with C. Furthermore for every component $C \in \mathcal{C}$ with finite $N_G(C)$ a C–S^* edges is chosen. Hence G'' is connected.

We now prove that the resulting graph G'' contains neither a $K_{\aleph_0}$ nor a fat $TK_{\aleph_0}$. The proof requires the following lemma:

Lemma 16. *Every set of pairwise independent S^*–S^* paths of G'' is countable.*

Proof. In G'' the components $C \in \mathcal{C}$ with finite $N_G(C)$ are linked with $N_G(C)$ by exactly one S^*–C edge (apart from the edges of B). Since B is countable and the components of $\mathcal{C}$ are pairwise disjoint, at most countably many pairwise independent S^*–S^* paths run through these components. By the same argument, this holds also for the components of $\mathcal{C}^*$. By Lemma 11 there are only countably many components $C \in \mathcal{C}$ with infinite $N_G(C)$. In G'' each of these components is linked with its neighbourhood only by countably many edges that lie not in B. Since B is countable, there exist only countably many pairwise independent S^*–S^* paths in G'' that run through these components. Alltogether it follows that there are at most countably many pairwise independent S^*–S^* paths in G''. $\square$

Corollary 2. *The graph G'' does not contain a $K_{\aleph_0}$ or fat $TK_{\aleph_0}$.*

Proof. By Lemma 9 the graph G'' does not contain a $K_{\aleph_0}$, since it is a subgraph of G'. In Lemma 10 we have shown that the branch vertices of every fat $TK_{\aleph_0}$ of

G' lie in S^*. Since for every pair (v, v') of branch vertices of a fat $TK_{\aleph_0}$ there exist uncountably many pairwise independent v–v' paths, by Lemma 16 the graph G'' does not contain a fat $TK_{\aleph_0}$. □

3.4. **Proof of the uniform end-faithfulness of G'' and G.** In this chapter we prove that the graph G'' is uniformly end-faithful to G. To do this recall the meaning of the term *uniformly end-faithful*. The subgraph $G'' \subseteq G$ is called uniformly end-faithful if the map $\eta : \Omega(G'') \to \Omega(G)$ that maps every end of G'' to the end of G that it contains as subset is an isomorphism between the uniform spaces $\Omega(G)$ and $\Omega(G'')$. We thus have to show that η is a bijection and that η as well as its inverse η^{-1} is uniformly continuous.

The map η is 1–1 if and only if two rays of G'' that are equivalent in G are also equivalent in G''. The map η is onto if and only if G'' contains a ray of every end of $\Omega(G)$.

The map η is uniformly continuous if and only if for every neighbourhood V of the uniform end-space $\Omega(G)$ there exists a neighbourhood V'' of the uniform end-space of $\Omega(G'')$ such that $V'' \subseteq \eta^{-1} \times \eta^{-1}(V)$. Due to the filter-properties of the uniformity it suffices to show this for the elements of the base $\tilde{\mathcal{V}}_G$. But an element of the base $\tilde{\mathcal{V}}_G$ has the form $V_S = \{(\omega, \omega') \in \Omega(G) \times \Omega(G) | \omega \sim_S \omega'\}$. The same holds for the corresponding filter base of the uniformity of $\Omega(G'')$. So it suffices to show that for every finite vertex set $S \subset V(G)$ there exists a finite vertex set $S' \subset V(G'')$ that separates those ends in G'' whose associated ends are separated by S in G. Since G'' is a subgraph of G, the vertex set $S' := S \cap V(G'')$ has this property. Hence the map η is uniformly continuous for every subgraph of G. For the inverse map η^{-1} the situation is different; it is not always uniformly continuous. To summarize, we have to show the following:

1) The subgraph G'' contains a ray from every end of $\Omega(G)$ (that means η is onto);
2) Two rays of G'' that are equivalent in G are also equivalent in G'' (that means η is 1–1);
3) For every finite vertex set $S \subset V(G'')$ there exists a finite vertex set $S' \subset V(G)$, such that S' separates those ends in G whose associated ends are separated by S in G'' (that means η^{-1} is uniformly continuous).

The following two lemmata are needed to show the bijectivity of η. We denote by $C^{G''}$ (in analogy to C^G) the subgraph of G'' that consists of $C \cup N_{G''}(C)$ and all C–$N_{G''}(C)$ edges of G''.

Lemma 17. *Let C be a component of $\mathcal{C}$ and R a ray of C such that there are infinitely many pairwise disjoint R–S^* paths in C^G. Let $\mathcal{W}$ be a set of such paths and let A be the set of their end-vertices in S^*. Then there exist infinitely many pairwise disjoint R–A paths in $C^{G''}$.*

Proof. By assumption $N_G(C)$ is infinite and we define $\mathcal{E}_C, \Gamma_C$ and Y_{Γ_C} as above. Since there exist infinitely many pairwise disjoint R–S^* paths in G, the ray R belongs to an end of Type 1. Then by definition of $\mathcal{E}_C$ the ray R belongs to an end of $\mathcal{E}_C$ and by definition of Γ_C there exists a ray $T_R \in \Gamma_C$ that is equivalent to R in C. Hence there exist infinitely many pairwise disjoint T_R–A paths in C^G. Since $Y_{T_R} := Y_{\Gamma_C} \cap V(T_R)$ is inifinte by the definition of Y_{Γ_C}, there exist also infinitely many pairwise disjoint Y_{T_R}–A paths in C^G.

For every finite subset $P \subset \mathcal{P} = Y_{\Gamma_C} \times N_G(C)$ for which this is possible in G, the graph G'' contains a set of pairwise independent paths, whose initial and end-vertices can be identified with the pairs of P. Since A is a subset of $N_G(C)$ and Y_{T_R} is a subset of Y_{Γ_C}, for every $i \in \mathbb{N}$ there exists a set of pairwise disjoint Y_{T_R}–A paths in $C^{G''}$ with cardinality strictly larger than i.

Suppose that in $C^{G''}$ every set of pairwise disjoint T_R–A paths is finite. Then also every set of pairwise disjoint Y_{T_R}–A paths is finite. So we can select a finite set M of pairwise disjoint Y_{T_R}–A paths, such that there exists no set of such paths in $C^{G''}$ that is a strict superset of M. Then $V(M) := \bigcup_{w \in M} V(w)$ is finite, say $|V(M)| = n$, and it separates the vertex sets Y_{T_R} and A in $C^{G''}$. Thus there exist at most n pairwise disjoint Y_{T_R}–A paths in $C^{G''}$. But this contradicts the construction of G''. Hence there exist infinitely many pairwise disjoint T_R–A paths in $C^{G''} \subseteq G''$ and, since R is equivalent to T_R, there exist also infinitely many pairwise disjoint R–A paths in $C^{G''} \subseteq G''$. $\square$

Lemma 18. *Every two rays $T, T' \subseteq B$ that are equivalent in G have a common tail.*

Proof. W.l.o.g let T and T' be rays starting at the root of B. Since they are equivalent in G there exists an infinite set W of pairwise disjoint T–T' paths $W_i \subseteq G$, $i \in \mathbb{N}$. We denote by t_i the initial vertex of W_i in T and by t'_i the end-vertex of W_i in T'. Furthermore let v be the vertex in B with smallest distance to the root that separates T and T'. Then in $B - \{v\}$ all tails of T lie in a component C whose vertices all lie above v. The same holds for T' and a component $C' \neq C$ of $B - \{v\}$. Otherwise a vertex v' that lies below v would separate the two rays in B, in contradiction to the minimality of the distance between v and the root of B. Thus two vertices $x \in V(C)$ and $y \in V(C')$ are incomparable in the order $<_B$. Futhermore $[x] \cap [y] = [v]$ for all $x \in V(C)$, $y \in V(C')$.

Since $[v]$ is finite, all but finitely many of the vertices t_i and t'_i lie in the components C and C'. But only finitely many of the paths W_i contain vertices of $[v]$, since these paths are pairwise disjoint. Thus there exists a path $W_j = x_1 ... x_n$, whose end-vertices are incomparable with the tree order $<_B$ and that contains no vertex of $[t_j] \cap [t'_j] = [v]$. Let x_k be the vertex of $W_j \cap C$ with maximal index. Then $k < n$, since $y = x_n \notin V(C)$. Let x_l be the vertex with minimal index $l > k$ of $W_j \cap B$. Since x_l does not lie in C and also not in $[v]$, the vertices x_l and x_k are incomparable in the tree order. Then $x_k ... x_l$ is a B–B path of G with (in the order $<_B$) incomparable end-vertices. But this is a contradiction to Lemma 6. $\square$

First part of the proof of Theorem 1, bijectivity of η. To prove that the map $\eta : \Omega(G'') \to \Omega(G)$ is onto, we have to show that G'' contains a ray of every end of $\Omega(G)$. We first consider the ends of Type 3 and 4: By construction the graph G'' contains the components of $\mathcal{C}$. Since every end of Type 3 and 4 is contained in one of these components, G'' contains rays from each of these ends. By Lemma 7 the tree B contains a ray from each end of Type 1. Since B is a subgraph of G'', it follows that G'' contains also rays from each of these ends. By definition every end ω of Type 2 is contained in a component $C_\omega \in \mathcal{C}^*$. By the construction of G' and G'' we have $V(C_\omega) \cap G'' \supseteq T_\omega$, where T_ω is a ray of ω. Thus G'' also contains rays from every end of Type 2. Since every end of G is of one of the four types, the map η is onto.

To prove that the map $\eta : \Omega(G'') \to \Omega(G)$ is 1–1 we have to show that any two rays of G'' that are equivalent in G are so in G''. By the definition of 'contained' it follows immediately that each two rays of an end of Type 3 or 4, respectively, have equivalent tails in a component of $\mathcal{C}$. Since every component of $\mathcal{C}$ is contained in G'', each two rays of such an end are equivalent in G''.

By Lemma 2 each end ω of Type 2 is contained in a one-ended component $C_\omega \in \mathcal{C}^*$ with finite $N_G(C_\omega)$. In the construction of G'' for every end of Type 2 a ray $T_\omega \subseteq C_\omega$ was selected and was linked with B by a finite path. Since the tree B consists of S^*–S^* paths, but the vertex set $N_G(C_\omega)$ is finite, it follows that $V(C_\omega) \cap V(B)$ is finite. Hence the set $V(C_\omega \cap G'') \backslash V(T_\omega)$ is also finite. Since ω is contained in C_ω, in G'' every ray of ω has a common tail with T_ω. Thus in G'' any two rays of ω are equivalent.

Now let ω be an end of Type 1 and let $T \in \omega$ be a ray in G''. Then by definition of Type 1 there exist infinitely many pairwise disjoint T–S^* paths in G. We show now that there are also infinitely many of such paths in G''. If T has a tail in B, then by construction of B there exist infinitely many pairwise disjoint T–S^* paths in B. Suppose this were not the case and let $T := x_1x_2... \subseteq B$ be a ray with only finitely many pairwise disjoint T–S^* paths in B. Let $\mathcal{W}$ be a maximal (by inclusion) set of such paths. Then $V(\mathcal{W})$ is finite and there is a $j \in \mathbb{N}$ such that no vertex of $V(\mathcal{W})$ lies in B above x_j. But, by construction of B, the vertex x_j lies in a path W_k that joins a vertex $t_k \in S^*$ with $\bigcup_{i<k} W_i$. Thus t_k lies in B above x_j, in contradiction to the assumption. So there are infinitely many pairwise disjoint T–S^* paths in B and since $B \subseteq G''$ they are also in G''.

If B contains no tail of T either there are infinitely many vertices of S^* in T and then there are also infinitely many pairwise disjoint T–S^* paths in G'', or a tail of T is contained in a component $C \in \mathcal{C}$. Since there are infinitely many pairwise disjoint T–S^* paths in G, but the component C is linked to the rest of the graph only via the vertices of S^*, all but finitely many of these T–S^* paths lie in C^G. Then by Lemma 17 there are also infintely many of these paths in G''. Hence by Lemma 7 in G'' the ray T is equivalent to a ray of B. So in G'' any two rays T, $T' \in \omega$ are equivalent to two rays of B and those have a common tail by Lemma 18. So the two rays of B are equivalent in B and thus also in G''. Hence the rays T and T' are also equivalent in G''. □

Lemma 19. *Let $S \subseteq V(B)$ be a finite vertex set that separates in B two vertices $x, y \in V(B)$. Then $S' := \bigcup_{t \in S}[t]$ separates the two vertices in G.*

Proof. If S separates the two vertices x and y in B, then S contains a vertex v of the unique x–y path W in B. This path consists of $[x]\backslash[y]$, $[y]\backslash[x]$ and the vertex of $[x] \cap [y]$ with maximal distance to the root. Thus $[v] \supseteq [x] \cap [y]$ and by Lemma 5 the vertex set S' separates the vertices x and y in G. □

Corollary 3. *Let $S \subseteq V(B)$ be a finite vertex set that separates a vertex $x \in V(B)$ and a ray $T \subseteq B$ in B. Then $S' := \bigcup_{t \in S}[t]$ separates these in G.*

Proof. Since S is finite, there exists a component of $B - S$ that contains a tail of T but not the vertex x. By Lemma 19 the vertex set S' separates in G every vertex of this tail from x. Since S' is finite, all (except for finitely many) of these vertices lie in a component of $G - S'$ that does not contain the vertex x and form there a tail of T. □

Corollary 4. *Let $S \subseteq V(B)$ be a finite vertex set that separates two rays $T, T' \subseteq B$ in B. Then $S' := \bigcup_{t \in S}[t]$ separates these two rays in G.*

Proof. Since S is finite, there exist a component C of $B - S$ that contains a tail $\tilde{T}$ of T and a component $C' \neq C$ of $B - S$ that contains a tail $\tilde{T}'$ of T'. By Lemma 19 the vertex set S' separates in G each two vertices of $\tilde{T}$ and $\tilde{T}'$. Since S' is finite, there exist different components of $G - S$ that contain tails of $\tilde{T}$ and $\tilde{T}'$, i.e. the vertex set S' separates T and T' in G. □

Second part of the proof of Theorem 1, the uniform continuity of η^{-1}. In the following we denote by ω, ω' ends of G and by $\alpha := \eta^{-1}(\omega), \alpha' := \eta^{-1}(\omega')$ the associated ends of G''. (Since η is a bijection, this is a unique association.) We now show that the map $\eta^{-1} : \Omega(G) \to \Omega(G'')$ is uniformly continuous. Therefore we have to show that for every finite vertex set $S \subset V(G'')$ there exists a finite vertex set $\bar{S} \subset V(G)$ that separates those ends in G, whose associated ends are separated in $G'' - S$.

For the construction of the vertex set $\bar{S}$ we prove the following: Let C be a component of $\mathcal{C}$ with infinite $N_G(C)$. Moreover, let $S' \subset V(G)$ be a finite vertex set and let $\mathcal{K}_{S'}(C)$ be the set of components of $C - S'$. Then there exists a finite set $A_{S'}(C) \subseteq N_G(C)$ such that for all $K \in \mathcal{K}_{S'}(C)$ we have $S_K := N_{G-S'}(K) \backslash N_{G''-S'}(K) \subseteq A_{S'}(C)$. (Note that by the definition of $\mathcal{K}_{S'}(C)$ we have $N_{G''-S'}(K) \subseteq N_G(C)$ for all $K \in \mathcal{K}_{S'}(C)$.)

First we show that the set $N_{G-S'}(K)$ is infinite only for finitely many components $K \in \mathcal{K}_{S'}(C)$. In order to show this, we prove that every component $K \in \mathcal{K}_{S'}(C)$ with infinite $N_{G-S'}(K)$ contains a ray of an end of Type 1. Then this implies together with Lemma 14 the assertion: If infinitely many components of $\mathcal{K}_{S'}(C)$ contain a ray of an end of Type 1, then there exists a vertex in S' that lies in the neighbourhood of infinitely many of these components, since S' is finite. But then C contains a big star, in contradiction to Lemma 14. Now let $K \in \mathcal{K}_{S'}(C)$ be an arbitrary component with infinite $N_{G-S'}(K)$. We denote by $K^{G-S'}$ the subgraph of G that consists of $K \cup N_{G-S'}(K)$ and all K–$N_{G-S'}(K)$ edges. Since K is connected, by Lemma 12 and Lemma 13 this component contains a ray T with infinitely many pairwise disjoint T–S^* paths in $K^{G-S'}$. By definition this ray belongs to an end of Type 1. Hence $N_{G-S'}(K)$ is infinite for only finitely many components $K \in \mathcal{K}_{S'}(C)$. We now show that the set S_K is finite for every component $K \in \mathcal{K}_{S'}(C)$ with infinite $N_{G-S'}(K)$. Let $K \in \mathcal{K}_{S'}(C)$ be such that S_K is infinite. Since $K^{G-S'}$ is connected and contains by Lemma 13 no v–S^* fan, by Lemma 12 there exists a ray T of K with infinitely many pairwise disjoint T–S_K paths in $K^{G-S'}$. Clearly, all these paths are contained in C^G, since $C^G \supseteq K^{G-S'}$. Hence, by Lemma 17 there are also infinitely many pairwise disjoint T–S_K paths in $C^{G''}$. Since S' is finite, there are also infinitely many such paths in $C^{G''-S'}$. Then also infinitely many pairwise disjoint T–S_k paths are contained in $K^{G''-S'}$, since the ray T is contained in K. But this is a contradiction since in G'' the finite vertex set S' separates the vertex set S_K from K. (Here $K^{G''-S'}$ and $C^{G''-S'}$ are defined in the same way as $C^{G''}$.) Thus we have shown that the set S_K is finite for every component $K \in \mathcal{K}_{S'}(C)$ with infinite $N_{G-S'}(K)$. As we have seen above, the set $N_{G-S'}(K)$ is infinite only for finitely many components $K \in \mathcal{K}_{S'}(C)$. Hence there exists a finite set $A^1_{S'}(C) \subset S^*$ such that for these components we have $S_K \subseteq A^1_{S'}(C)$.

Let $\mathcal{K}'_{S'}(C) := \{K \in \mathcal{K}_{S'}(C) : |N_{G-S'}(K)| < \aleph_0\}$. Next we show that a finite set $A^2_{S'}(C) \subset S^*$ exists, such that $N_{G-S'}(K) \subseteq A^2_{S'}(C)$ for all $K \in \mathcal{K}'_{S'}(C)$. Suppose that every subset of S^* for which this holds is infinite. Then $\mathcal{K}'_{S'}(C)$ is also infinite, because for each $K \in \mathcal{K}'_{S'}(C)$ the set $N_{G-S'}(K)$ is finite. Since the components of $\mathcal{K}_{S'}(C)$ are disjoint and S' is finite and C is connected, there exists a vertex v of C that is adjacent to infinitely many components of $\mathcal{K}'_{S'}(C)$ such that all these components together have infinitely many neighbours in S^*. Then we can inductively construct a v–S^* fan in C^G, since $N_{G-S'}(K)$ is finite for all $K \in \mathcal{K}'_{S'}(C)$. But this is a contradiction to Lemma 13. We now select a finite vertex set $A^2_{S'}(C) \subset S^*$ such that $N_{G-S'}(K) \subseteq A^2_{S'}(C)$ for all $K \in \mathcal{K}'_{S'}(C)$. Furthermore we set $A_{S'}(C) := A^1_{S'}(C) \cup A^2_{S'}(C)$.

Now let S be an arbitrary finite subset of $V(G'')$. For the definition of the vertex set $\bar{S}$ we define three further vertex sets:
$S^1 := \bigcup_{t\in S}[t]$, $S^2 := \{t_1, ..., t_l\}$, where $l := max\{r, s, m\}$ with $t_r := max_{i\in\mathbb{N}}\{t_i \in S^* |$ there exists $C \in \mathcal{C}^*$ with $t_i \in N_G(C)$ and $V(C) \cap (S \cup S^1) \neq \emptyset\}$, $t_s := max_{i\in\mathbb{N}}\{t_i \in S^* |$ there exists $C \in \mathcal{C}$ with $t_i \in N_G(C)$, $|N_G(C)| < \aleph_0$ and $V(C) \cap (S \cup S^1) \neq \emptyset\}$ and $t_m := max\{t_i \in S^* \cap (S \cup S^1)\}$. Furthermore let $\tilde{\mathcal{C}} := \{C \in \mathcal{C} || N_G(C)| = \aleph_0, V(C) \cap S \neq \emptyset\}$. For every $C \in \tilde{\mathcal{C}}$ let the sets $\mathcal{K}_s(C)$ be defined as above. Since S is finite and the components of $\mathcal{C}$ are disjoint, $\tilde{\mathcal{C}}$ is finite. As we have seen before, for each $C \in \tilde{\mathcal{C}}$ there exists a finite set $A_S(C) \subseteq N_G(C)$ such that $N_{G-S}(K) \backslash N_{G''-S}(K) \subseteq A_S(C)$ for all $K \in \mathcal{K}_S(C)$. Then also $S^3 := \bigcup_{C\in\tilde{\mathcal{C}}} A_S(C)$ is finite. We set $\bar{S} := S \cup S^1 \cup S^2 \cup S^3$. Clearly, $\bar{S}$ is finite.

We now consider two ends α and α' of G'' that are separated by S in G''. Let ω and ω' be the corresponding ends of G. Before we show that the set $\bar{S}$ separates ω and ω' in G, we examine again the different types of ends: If ω is an end of Type 1, then ω is reperesented by a ray in B (see Lemma 7). From now on we denote by R_ω such a representing ray of ω. To show that $\bar{S}$ separates in G the end ω and the end ω', it suffices to show that $\bar{S}$ separates in G the ray R_ω and the end ω'.

If ω is an end of Type 2, we denote by C_ω the component of $\mathcal{C}^*$ that contains ω. By T_ω we denote the ray that was selected as representing ray of ω in the construction of G'; by $t_{n(\omega)}$ we denote the vertex of $N_G(C_\omega)$ with maximal index. Note that in the construction of G' the ray T_ω was linked with the vertex $t_{n(\omega)}$ by a path in ${C_\omega}^G$. (Here ${C_\omega}^G$ denotes the subgraph of G that consists of $C_\omega \cup N_G(C_\omega)$ and all C_ω–$N_G(C_\omega)$ edges of G.) Hence $S \cup S^1$ separates the ray T_ω and the vertex $t_{n(\omega)}$ if and only if $t_{n(\omega)} \in S \cup S^1$ or $V(C_\omega) \cap (S \cup S^1) \neq \emptyset$. But then $N_G(C_\omega) \subseteq \{t_1, ..., t_{n(\omega)}\} \subseteq S^2 \subseteq \bar{S}$. Thus, in this case $\bar{S}$ separates in G the complete component C_ω from the rest of the graph. Since C_ω is one-ended, then $\bar{S}$ separates in G the end ω from all other ends. So in $G-\bar{S}$ the end ω is separated from all other ends whose associated ends are separated in $G''-S$ from the end α. Hence, in the following we consider only the case $t_{n(\omega)} \notin S \cup S^1$ and $V(C_\omega) \cap (S \cup S^1) = \emptyset$. In this case in $G - (S \cup S^1)$ the end ω is contained in the same component as the vertex $t_{n(\omega)}$. The same holds for the corresponding end α in $G'' - S$. This means if S separates in G'' the end α and the end α', then S separates the vertex $t_{n(\omega)}$ and the end α'. To show that $\bar{S}$ separates in G the end ω and the end ω' it suffices to show that $S \cup S^1$ separates in G the vertex $t_{n(\omega)}$ and the end ω'.

If ω is an end of Type 3, then we denote by C_ω the component of $\mathcal{C}$ that contains ω and by K_ω the component of $C_\omega - S$ that contains ω. Furthermore let $t_{n(\omega)}$ be the vertex of $N_G(C_\omega)$ with maximal index (by the definition of Type 3 the set

$N_G(C_\omega)$ is finite). If ω is separated in $G'' - (S \cup S^1)$ from the vertex $t_{n(\omega)}$, either $t_{n(\omega)} \in S \cup S^1$ or $V(C) \cap (S \cup S^1) \neq \emptyset$. In both cases it follows from the definition of S^2 that $N_G(C_\omega) \subseteq \{t_1, ..., t_{n(\omega)}\} \subseteq S^2 \subseteq \bar{S}$, which means that in G the vertex set $\bar{S}$ separates the complete component C from the rest of the graph. Since $\bar{S} \supseteq S$, $C - \bar{S}$ decomposes into finer components than $C - S$. So in $G - \bar{S}$ the end ω is separated from all other ends whose associated ends are separated in $G'' - S$ from the end α. Thus in the following we consider only the case $V(C) \cap (S \cup S^1) = \emptyset$ and $t_{n(\omega)} \notin S \cup S^1$, which means that in $G'' - S$ and in $G - (S \cup S^1)$ the vertex $t_{n(\omega)}$ is adjacent to the whole component C. So, if S separates in G'' the end α and the end α', then S separates the vertex $t_{n(\omega)}$ and α'. To show that $\bar{S}$ separates in G the end ω and the end ω' it suffices to show that $S \cup S^1$ separates in G the vertex $t_{n(\omega)}$ and the end ω'.

If ω is an end of Type 4, then let C_ω be the component of $\mathcal{C}$ that contains ω. Let K_ω be the component of $C_\omega - S$ that, considered as subgraph of G, contains ω. Let $\bar{K}_\omega$ be the corresponding component of $C_\omega - \bar{S}$. Since $S \subseteq \bar{S}$, it follows that $K_\omega \supseteq \bar{K}_\omega$.

We now show that $N_{G''-S}(K_\omega) \supseteq N_{G-\bar{S}}(\bar{K}_\omega)$. By the construction of G'' we have $N_{G''}(C) = N_G(C)$ for each component $C \in \mathcal{C}$ with infinite $N_G(C)$. Hence $V(C) \cap S \neq \emptyset$ if $N_{G-S}(C) \backslash N_{G''-S}(C) \neq \emptyset$. As shown above, we have $A_S(C) \supseteq N_{G-S}(K) \backslash N_{G''-S}(K)$ for every component K of $C - S$, thus also for the component K_ω. But this means $N_{G''-S}(K_\omega) \supseteq N_{G-S}(K_\omega) \backslash A_S(C)$. Since $A_S(C) \subseteq S^3 \subseteq \bar{S}$, it follows that $N_{G''-S}(K_\omega) \supseteq N_{G-\bar{S}}(K_\omega)$. So the component $\bar{K}_\omega$ is in $G - \bar{S}$ linked with $G \backslash \bar{K}_\omega$ only by vertices of $N_{G''-S}(K_\omega)$; the same holds for the component K_ω in $G'' - S$. If S separates in G'' the end α' and the end α, then α' is contained in a component of $G'' - S$ that is different from K_ω and S separates the vertex set $N_{G''-S}(K_\omega)$ and the end α'. Since $\bar{K}_\omega \subseteq K_\omega$, also ω' is not contained in $\bar{K}_\omega$ but in another component of $G - \bar{S}$. Thus, for the proof that $\bar{S}$ separates in G the end ω' and the end ω it suffices to show that $\bar{S}$ separates in G the vertex set $N_{G''-S}(K_\omega)$ and the end ω'.

We show now that $\bar{S}$ separates ω and ω' in G. By the above considerations it suffices, depending on the type of ends, to show that $\bar{S}$ separates in G two vertex sets of the type $N_{G''-S}(K_\omega)$ or $\{t_{n(\omega)}\}$, or separates such a vertex set and a ray, or separates two rays. But, since $\bar{S} \supseteq S \cup S^1$, this is proved by Lemma 19 and Corollaries 3 and 4. So the proof of Theorem 1 is complete. □

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