TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 3, March 1999, Pages 1203–1225
S 0002-9947(99)02424-1

INVARIANT MEASURES FOR SET-VALUED DYNAMICAL SYSTEMS

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Abstract. A continuous map on a compact metric space, regarded as a dynamical system by iteration, admits invariant measures. For a closed relation on such a space, or, equivalently, an upper semicontinuous set-valued map, there are several concepts which extend this idea of invariance for a measure. We prove that four such are equivalent. In particular, such relation invariant measures arise as projections from shift invariant measures on the space of sample paths. There is a similarly close relationship between the ideas of chain recurrence for the set-valued system and for the shift on the sample path space.

1. Introduction

All our spaces will be nonempty, compact, metric spaces. For such a space X let $P(X)$ denote the space of Borel probability measures on X with $\delta : X \to P(X)$ the embedding associating to $x \in X$ the point measure δ_x. The support $|\mu|$ of a measure μ in $P(X)$ is the smallest closed subset of measure 1. If $f : X_1 \to X_2$ is Borel measurable then the induced map $f_* : P(X_1) \to P(X_2)$ associates to μ the measure $f_*(\mu)$ defined by

$$f_*(\mu)(B) = \mu(f^{-1}(B)) \tag{1.1}$$

for all B Borel in X_2.

We regard a continuous map f on X as a dynamical system by iterating. A measure $\mu \in P(X)$ is called an *invariant measure* when it is a fixed point for the map $f_* : P(X) \to P(X)$, i.e. $f_*(\mu) = \mu$.

A *relation* $F : X_1 \to X_2$ is a subset of $X_1 \times X_2$. We associate to $x \in X_1$ the subset of X_2 given by $F(x) = \{y : (x, y) \in F\}$. The relation is a function when each $F(x)$ is a singleton. For $A \subset X_1$, the *image* $F(A) = \bigcup\{F(x) : x \in A\} = \{y : y \in F(x) \text{ for some } x \in A\}$. The *inverse relation* $F^{-1} : X_2 \to X_1$ is $\{(y, x) : (x, y) \in F\}$. So for $B \subset X_2$, $F^{-1}(B) = \{x : F(x) \cap B \neq \emptyset\}$. In particular, the *domain* of F, denoted $\mathrm{Dom}(F)$, is given by $F^{-1}(X_2) = \{x \in X_1 : F(x) \neq \emptyset\}$. F is a *closed* (or *Borel*) *relation* when it is a closed (resp. Borel) subset of $X_1 \times X_2$. A function is a closed relation iff it is a continuous function. If $F : X_1 \to X_2$ and $G : X_2 \to X_3$ are relations then the *composed relation* $G \circ F : X_1 \to X_3$ is $\{(x, z) : z \in G(y) \text{ for some } y \in F(x)\}$, i.e. the projection to $X_1 \times X_3$ of the subset $(F \times X_3) \cap (X_1 \times G) \subset X_1 \times X_2 \times X_3$. The composition of closed relations is closed.

Received by the editors June 14, 1996.
1991 *Mathematics Subject Classification.* Primary 54H20, 58F10, 34C35.
Key words and phrases. Set-valued dynamical system, dynamics of a relation, sample path spaces, invariant measure, basic set, chain recurrence.

For the elementary facts about relations, and the notation we will use, see Akin (1993), Chapter 1.

Both for its own sake and for application to the usual function case, the dynamics of a relation F on X, i.e. $F : X \to X$, have recently received some attention, e.g. Aubin and Frankowska (1990), McGehee (1992) and Akin (1993). As in the function case, the dynamics are obtained by iterating. By convention, F^0 is the identity map 1_X, $F^1 = F$, $F^2 = F \circ F$ and F^n is the n-fold composition for any positive integer n; and we write F^{-n} for $(F^{-1})^n$.

For a relation there are two concepts of invariant measure in the literature which extend the continuous function notion. We introduce two more and will prove in Section 3 that for a closed relation F on X all four definitions are equivalent.

(1) Aubin, Frankowska and Lasota (1991) call a measure $\mu \in P(X)$ invariant for F if

$$\mu(B) \leq \mu(F^{-1}(B)) \tag{1.2}$$

for all Borel subsets B of X. We will then call μ an *invariant*$_1$ *measure*. If $F = f$ is a function then $f^{-1}(A)$ and $f^{-1}(X \backslash A)$ are disjoint, and so, applying (1.2) to $B = A$ and $B = X \backslash A$, we obtain equality. By (1.1) this condition says $\mu = f_*\mu$, and so μ is an invariant measure for the map.

(2) A *Markov kernel* κ from X_1 to X_2 is a function from X_1 to $P(X_2)$, associating to each $x \in X_1$ a measure κ_x on X_2 such that for each Borel set A, $\kappa_x(A)$ is a Borel measurable real valued function on X_1. For example, if $f : X_1 \to X_2$ is a Borel measurable function, then $\delta_f = \delta \circ f : X_1 \to P(X_2)$ is the Markov kernel associated with f, associating to x the point measure $\delta_{f(x)}$. For a Markov kernel $\kappa : X_1 \to P(X_2)$ there is an induced mapping on measures $\kappa_* : P(X_1) \to P(X_2)$ defined by

$$\kappa_*(\mu)(B) = \int_{X_1} \kappa_x(B)\mu(dx) \tag{1.3}$$

for B a Borel subset of X_2. Notice that $\kappa_*(\mu)$ depends only on the measures κ_x for $x \in |\mu|$. Clearly, for $\kappa = \delta_f$, $\kappa_* = f_*$. If $F : X_1 \to X_2$ we say that the Markov kernel $\kappa : X_1 \to P(X_2)$ is *supported* by F when

$$|\kappa_x| \subset F(x) \text{ for all } x \in X_1. \tag{1.4}$$

For example, let $f : X_1 \to X_2$ be a Borel measurable *selection function* for F, a function satisfying $f(x) \in F(x)$ for all $x \in X_1$. Then $\delta_f : X_1 \to P(X_2)$ is a Markov kernel supported by F.

1.1. Lemma. *If $F : X_1 \to X_2$ is a closed relation such that* $\mathrm{Dom}(F) = X_1$*, i.e. $F^{-1}(X_2) = X_1$, then there exists a Borel measurable selection function for F.*

Proof. Let $X_3 \subset [0, 1]$ denote the Cantor set. By Hocking and Young (1961), Theorem 3.28, there exists a continuous map h from X_3 onto the compact metric space X_2. $h^{-1} \circ F$ is a closed relation from X_1 to X_3. Let $\tilde{f}(x) = \sup h^{-1}(F(x))$. Since each $h^{-1}(F(x))$ is nonempty and closed, $\tilde{f}$ is a selection function for $h^{-1} \circ F$. $\tilde{f} : X_1 \to \mathbf{R}$ is upper semicontinuous and so is Borel measurable. $f = h \circ \tilde{f} : X_1 \to X_2$ is therefore a Borel measurable selection map for F. □

For a closed relation F on X, Miller (1995) calls μ invariant for F if there exists a Markov kernel $\kappa : X \to P(X)$ such that

$$|\kappa_x| \subset F(x) \text{ for all } x \in |\mu| \quad \text{and} \quad \mu = \kappa_*(\mu). \tag{1.5}$$

We will then call μ an *invariant*$_2$ *measure*.

If $F = f$ is a continuous function on X, then δ_f is the unique Markov kernel supported by f and $(\delta_f)_*(\mu) = f_*(\mu)$. So condition (1.5) specializes to the usual notion of invariant measure when f is a map.

(3) If μ_{12} is a measure on the product space $X_1 \times X_2$, then the coordinate projections $\pi_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ induce the *marginal measures* $\mu_i = \pi_{i*}(\mu_{12})$ $(i = 1, 2)$.

For a closed relation F on $X \times X$ we will call μ an *invariant*$_3$ *measure* if there exists $\mu_{12} \in P(X \times X)$ such that

$$|\mu_{12}| \subset F, \qquad \pi_{1*}(\mu_{12}) = \mu = \pi_{2*}(\mu_{12}). \tag{1.6}$$

If $F = f$ is a continuous function then the restriction of π_1 to the subset f is a homeomorphism with inverse given by $\mathrm{grph}_f : X \to X \times X$: $\mathrm{grph}_f(x) = (x, f(x))$. The unique measure μ_{12} on $X \times X$ with support in f and satisfying $\pi_{1*}(\mu_{12}) = \mu$ is $(\mathrm{grph}_f)_*(\mu)$. Since $\pi_2 \circ \mathrm{grph}_f = f$, $\pi_{2*}(\mathrm{grph}_f)_*(\mu) = f_*(\mu)$. So again (1.6) specializes to the proper notion.

(4) Let $X^{\mathbf{Z}}$ denote the set of bi-infinite sequences in X regarded as functions of $\mathbf{Z}$ to X. With the product topology it is compact. Let $\pi_0 : X^{\mathbf{Z}} \to X$ be the projection $\pi_0(\xi) = \xi_0$. The *shift homeomorphism* $s : X^{\mathbf{Z}} \to X^{\mathbf{Z}}$ is defined by

$$s(\xi)_i = \xi_{i+1}, \quad i \in \mathbf{Z}. \tag{1.7}$$

For a closed relation F on X we denote by X_F the *sample path space* for F, $\{\xi : \xi_{i+1} \in F(\xi_i) \text{ for all } i \in \mathbf{Z}\}$, a closed, s invariant subset of $X^{\mathbf{Z}}$. So s restricts to a homeomorphism on X_F, which we denote s_F. The measures on X_F are called *sample path measures*. So $\nu \in P(X^{\mathbf{Z}})$ is a sample path measure when $|\nu| \subset X_F$. Such a measure is called an *invariant sample path measure* when it is s invariant or, equivalently, when it is s_F invariant regarded as a measure on X_F.

We will call $\mu \in P(X)$ an *invariant*$_4$ *measure* for F if there exists an invariant sample path measure which projects to μ via $\pi_0 : X^{\mathbf{Z}} \to X$, $\pi_0(\xi) = \xi_0$. That is, there exists $\nu \in P(X^{\mathbf{Z}})$ such that

$$|\nu| \subset X_F, \quad s_*(\nu) = \nu \quad \text{and} \quad \pi_{0*}(\nu) = \mu. \tag{1.8}$$

If $F = f$ is a homeomorphism on X then π_0 restricts to a homeomorphism of X_f to X inducing a conjugacy of s_f with f. So μ is f invariant on X exactly when it is the image of an s_f invariant measure under the conjugacy π_0.

The image of the map $\pi_0 : X_F \to X$ need not be all of X. We pause to describe this subset.

Call a relation F on X *surjective* if $F(X) = X = F^{-1}(X)$ or, equivalently, if $\mathrm{Dom}(F^{-1}) = X = \mathrm{Dom}(F)$. For any subset B of X the restriction of F to B, denoted F_B, is defined by $F_B = F \cap (B \times B)$. A subset B will be called a *surjective subset* when F_B is a surjective relation on B. Thus, B is a surjective subset exactly when

$$B \subset F(B) \cap F^{-1}(B). \tag{1.9}$$

1.2. Lemma. *For a closed relation F on X let $\pi_0 : X_F \to X$ be the projection from the sample path space,*

$$\pi_0(X_F) = \bigcap_{n \in \mathbf{Z}} F^n(X). \tag{1.10}$$

We call this subset the dynamic domain *of F. $\pi_0(X_F)$ is a surjective subset of X, and if B is any surjective subset of X then $B \subset \pi_0(X_F)$. In particular, F is a surjective relation iff $X = \pi_0(X_F)$ and so iff the projection $\pi_0 : X_F \to X$ is surjective.*

Proof. Since s_F is bijective on X_F and the coordinate n projection π_n is $\pi_0(s_F)^n$ for every integer n, we clearly have

$$\pi_n(X_F) = \pi_0(X_F) \quad \text{for all} \quad n \in \mathbf{Z}. \tag{1.11}$$

This implies $\pi_0(X_F)$ satisfies (1.9) and so is a surjective subset. By induction on $n = 1, 2, \dots$ we see that $B \subset F^n(X) \cap F^{-n}(X)$ for any surjective subset B, and so any surjective subset is contained in $\bigcap_{n \in \mathbf{Z}} F^n(X)$. Finally, if x is in this intersection, then for each positive integer m, $\{\xi : \xi_0 = x$ and $\xi_{i+1} \in F(\xi_i)$ for $|i| \leq m\}$ is a nonempty compact subset of $X^{\mathbf{Z}}$. The intersection of this decreasing sequence is $\pi_0^{-1}(x) \cap X_F$, which is therefore nonempty. Thus, equation (1.10) holds and $\pi_0(X_F)$ is the maximum surjective subset. In particular, the entire space X is a surjective subset, i.e. F is surjective, iff $X = \pi_0(X_F)$. □

In Section 2 we will consider the case where X is finite. The general equivalence uses the finite space results, which also have a nice geometrical representation using directed graphs. Furthermore, the finite "pixel" approximations to the space leads us to consider relations even when the original $F = f$ was a homeomorphism. In Section 3 we complete the general proof of the equivalence.

Since map results project from s_F on X_F to F on X, it is useful to compare these as dynamical systems. In Section 4 we consider the chain relations. The set of maximal chain transitive subsets of X_F, the basic sets for s_F, is mapped bijectively by π_0 onto the set of basic sets for F. That is, each s_F basic set maps onto an F basic set via π_0, and each F basic set is the image of a unique s_F basic set.

Finally, in Section 5 we apply these results to show that every invariant measure supported on a basic set can be ϵ approximated by the finite measures associated with periodic ϵ chains, for arbitrarily small ϵ, and can also be expressed as the limit measures of asymptotic chains. The proof uses the embedding of X_F in $X^{\mathbf{Z}}$ even in the case when $F = f$ is a homeomorphism.

We conclude by introducing the particular metrics on $P(X)$ and $X^{\mathbf{Z}}$ which we will use. On $P(X)$ the *Hutchinson metric* is given by

$$d(\mu_1, \mu_2) = \sup\{\int u(x)\mu_1(dx) - \int u(x)\mu_2(dx)\}, \tag{1.12}$$

where the sup is taken over all real-valued functions u with Lipschitz constant ≤ 1. Then δ is an isometric embedding of X into $P(X)$, and if $f : X_1 \to X_2$ is Lipschitz then $f_* : P(X_1) \to P(X_2)$ is Lipschitz with the same Lipschitz constant (see Hutchinson (1981) and Akin (1993), Exercise 8.16).

On the product $X^{\mathbf{Z}}$ we define

$$d(\xi, \eta) = \sup\{\min(d(\xi_i, \eta_i), 1/|i|) : i \in \mathbf{Z}\}, \tag{1.13}$$

where $\min(d, 1/0) = d$ by convention. It is not hard to show that d is a metric on $X^{\mathbf{Z}}$ yielding the product topology, and, for $\epsilon > 0$,

$$d(\xi, \eta) \leq \epsilon \Leftrightarrow d(\xi_i, \eta_i) \leq \epsilon \text{ for all } i \text{ s.t. } |i| < 1/\epsilon. \tag{1.14}$$

In particular, the projection map $\pi_0 : X^{\mathbf{Z}} \to X$ is distance nonincreasing, i.e. it is Lipschitz with Lipschitz constant = 1. Thus, the same is true for $\pi_{0*} : P(X^{\mathbf{Z}}) \to P(X)$.

2. Finite Spaces

When X is a finite set we can represent a relation $F \subset X \times X$ as a directed graph with vertices the points of X and an edge from x to y exactly when $(x, y) \in F$. Then the characteristic function χ_F is a 0 - 1 square matrix.

A measure μ_{12} on F can be regarded as a real valued function on $X \times X$ satisfying

$$0 \leq \mu_{12} \leq \chi_F, \qquad \sum_{(x,y) \in X \times X} \mu_{12}(x, y) = 1, \tag{2.1}$$

i.e. a probability distribution on the edges of the graph. The marginal measures μ_1 and μ_2 induced by μ_{12} are the distributions on the set of vertices X given by

$$\begin{aligned} \mu_1(x) &= \sum_{y \in X} \mu_{12}(x, y) = \sum_{y \in F(x)} \mu_{12}(x, y), \\ \mu_2(y) &= \sum_{x \in X} \mu_{12}(x, y) = \sum_{x \in F^{-1}(y)} \mu_{12}(x, y), \end{aligned} \tag{2.2}$$

where the empty sum is 0 by convention. Thus, $\mu_1(x)$ is the probability that an edge begins at x and $\mu_2(y)$ is the probability that an edge ends at y. The associated Markov kernel is the conditional distribution on the terminal vertices assuming the initial one:

$$\kappa_x(y) = \frac{\mu_{12}(x, y)}{\mu_1(x)}, \tag{2.3}$$

a distribution well defined when $\mu_1(x) > 0$, which is then supported on $F(x)$. For $x \notin |\mu_1|$, i.e. $\mu_1(x) = 0$, we choose κ_x to be an arbitrary distribution on X. Notice that if, for some x, $F(x) = \emptyset$, i.e. no edge begins at x, then κ_x cannot be chosen supported by F. But we do have

$$|\kappa_x| \subset F(x) \quad \text{for} \quad x \in |\mu_1|, \qquad \mu_2(y) = \sum_x \kappa_x(y) \mu_1(x). \tag{2.4}$$

Conversely, if μ_1 and μ_2 are distributions on X and κ is a Markov kernel, then we obtain the associated distribution on $X \times X$:

$$\mu_{12}(x, y) = \kappa_x(y) \mu_1(x). \tag{2.5}$$

If κ satisfies (2.4), then the marginals of μ_{12} are μ_1 and μ_2 and the support of μ_{12} is contained in F.

In particular, a distribution μ on X is an invariant$_3$ measure for F, i.e. there exists μ_{12} satisfying (2.1) with $\mu_1 = \mu = \mu_2$, iff there is a Markov kernel κ satisfying (2.4) with $\mu_1 = \mu = \mu_2$ and so iff μ is invariant$_2$ for F.

From μ_{12} on $X \times X$, supported by F and with common marginals μ, we can define a measure ν on $X^{\mathbf{Z}}$ with support X_F and such that $s_*(\nu) = \nu$. For integers

a, b with $a < b$ and a sequence $\{x_i : a \le i \le b\}$ in X we define ν on the associated cylinder set by

$$\nu(\{\xi \in X^{\mathbf{Z}} : \xi_i = x_i \text{ for } a \le i \le b\}) = \frac{\prod_{i=a}^{b-1} \mu_{12}(x_i, x_{i+1})}{\prod_{i=a+1}^{b-1} \mu(x_i)}, \tag{2.6}$$

where the empty product is 1. Clearly, $(\pi_{n,n+1})_*(\nu) = \mu_{12}$ for any integer n, where $\pi_{n,n+1}$ is the projection from $X^{\mathbf{Z}}$ to $X \times X$ via the n and $n+1$ coordinates. Hence $\pi_{n^*}(\nu) = \mu$, and because $\mu_{12}(x, y) = 0$ unless $(x, y) \in F$ it follows that ν is supported by X_F. Obviously $s_*(\nu) = \nu$, and so ν is an invariant sample path measure.

Conversely, if ν is an invariant sample path measure such that $\pi_{0^*}(\nu) = \mu$ and so $\pi_{n^*}(\nu) = \mu$, then $(\pi_{n,n+1})_*(\nu) = \mu_{12}$ satisfies (2.1) and (2.2) with $\mu_1 = \mu = \mu_2$. Thus, μ is invariant$_4$ for F iff it is invariant$_3$ for F.

Finally, the equivalence of the invariant$_1$ and invariant$_3$ conditions is a special case of the following theorem of Brualdi (1968).

2.1. Theorem. *Let X_1 and X_2 be nonempty finite sets and let F be a subset of the product $X_1 \times X_2$. Let μ_i be a distribution on X_i for $i = 1, 2$. There exists a distribution μ_{12} on $X_1 \times X_2$ supported by F and with marginals μ_1 and μ_2 iff for all subsets B of X_2, $\mu_2(B) \le \mu_1(F^{-1}(B))$, that is, if $A_1 \supset F^{-1}(A_2)$ for subsets A_i of X_i $(i = 1, 2)$ then*

$$\sum_{y \in A_2} \mu_2(y) \le \sum_{x \in A_1} \mu_1(x). \tag{2.7}$$

Proof. Necessity of (2.7) is easy (see the general proof of (3) $\Rightarrow$ (1) in Theorem 3.1 below). We prove sufficiency using Theorem 2.6 of Gale (1960), a separation theorem which says that for an $m \times n$ matrix A and $1 \times n$ row vector b either the equation $cA = b$ has a nonnegative $1 \times m$ row vector solution c, or for some $n \times 1$ column vector d, Ad is nonnegative while $bd < 0$.

Using F to index the rows and disjoint copies of X_1 and X_2 to index the columns we define

$$\begin{aligned} A_{(x,y)z} &= \begin{cases} 1 & \text{if } z = x \text{ or } z = y, \\ 0 & \text{otherwise,} \end{cases} \\ b_z &= \begin{cases} \mu_1(z) & \text{for } z \in X_1, \\ \mu_2(z) & \text{for } z \in X_2. \end{cases} \end{aligned} \tag{2.8}$$

The required distribution μ_{12} is exactly a nonnegative solution c of the equation $cA = b$. We prove it exists by showing that Ad nonnegative implies $bd \ge 0$.

A vector d such that Ad is nonnegative satisfies $d_x + d_y \ge 0$ whenever $(x, y) \in F$. Choose a positive constant k large enough that for all $x \in X_1$, $y \in X_2$:

$$u_1(x) = d_x + k > 0, \qquad u_2(y) = -d_y + k > 0. \tag{2.9}$$

Thus, u_i is a positive real valued function on X_i for $i = 1, 2$, and Ad nonnegative implies

$$u_1(x) \ge u_2(y) \text{ whenever } (x, y) \in F. \tag{2.10}$$

For any $\epsilon \ge 0$, (2.10) implies

$$\{u_1 \ge \epsilon\} \supset F^{-1}(\{u_2 \ge \epsilon\}) \tag{2.11}$$

where $\{u_i \geq \epsilon\} \equiv \{z \in X_i : u_i(z) \geq \epsilon\}$ $(i = 1, 2)$. Thus, condition (2.7) implies

$$\mu_1(\{u_1 \geq \epsilon\}) \geq \mu_2(\{u_2 \geq \epsilon\}). \tag{2.12}$$

Now let $0 \leq \epsilon_0 < \epsilon_1 < \ldots < \epsilon_N$ list all the values of u_1 and u_2 in ascending order. Define $\delta_0 = \epsilon_0$ and $\delta_i = \epsilon_i - \epsilon_{i-1}$ for $i = 1, \ldots, N$. Then

$$\begin{aligned}\sum_x u_1(x)\mu_1(x) &= \sum_{i=0}^{N} \epsilon_i \mu_1(\{u_1 = \epsilon_i\}) = \sum_{i=0}^{N} \sum_{j=0}^{i} \delta_j \mu_1(\{u_1 = \epsilon_i\}) \\ &= \sum_{j=0}^{N} \sum_{i=j}^{N} \delta_j \mu_1(\{u_1 = \epsilon_i\}) = \sum_{j=0}^{N} \delta_j \mu_1(\{u_1 \geq \epsilon_j\}) \\ &\geq \sum_{j=0}^{N} \delta_j \mu_2(\{u_2 \geq \epsilon_j\}) = \sum_y u_2(y)\mu_2(y).\end{aligned} \tag{2.13}$$

Since μ_i is a distribution it follows that $\sum_z k\mu_i(z) = k$ $(i = 1, 2)$, and so (2.13) and (2.9) imply $bd \geq 0$. □

The finite case provides a useful tool for studying dynamical systems on arbitrary spaces. Consider for a compact, metric space X a partition into finitely many sets, or equivalently, let $E \subset X \times X$ be an equivalence relation with a finite number of distinct equivalence classes. We will call such a relation a *finite equivalence relation*. Let $X/E = \{E(x) : x \in X\}$ be the finite set of equivalence classes, with $\pi_E : X \to X/E$ the projection map associating to $x \in X$ its class $E(x)$.

We will call E a *Borel equivalence relation* when E is a Borel subset of $X \times X$. Letting $i_x : X \to X \times X$ be the inclusion given by $i_x(y) = (x, y)$, we see that $E(x) = (i_x)^{-1}(E)$ is Borel when E is Borel. Conversely, a finite equivalence relation $E = \bigcup\{E(x) \times E(x) : x \in X\}$ is Borel when each equivalence class is Borel. Thus, E is Borel exactly when π_E is Borel measurable.

For a finite equivalence relation E the *mesh* of E is the maximum of the diameters of the equivalence classes of E.

2.2. Lemma. *Let E be a finite, Borel equivalence relation on X with mesh $\leq \epsilon$. If $\mu_1, \mu_2 \in P(X)$ satisfy $(\pi_E)_*(\mu_1) = (\pi_E)_*(\mu_2)$ in $P(X/E)$, then the Hutchinson distance $d(\mu_1, \mu_2)$ is at most 2ϵ.*

Proof. Let $\mu_E = (\pi_E)_*(\mu_1) = (\pi_E)_*(\mu_2)$ and let u be a real valued map on X with Lipschitz constant at most 1. Since the diameter of $E(x)$ is at most ϵ, we have for $y \in E(x)$

$$u(x) - \epsilon \leq u(y) \leq u(x) + \epsilon. \tag{2.14}$$

If $\mu_E(E(x))$ $(= \mu_i(E(x))$ for $i = 1, 2)$ is positive, then we can define the conditional expectations (for $i = 1, 2$):

$$E_i(u|E(x)) = \frac{1}{\mu_E(E(x))} \int_{E(x)} u(y)\mu_i(dy) \tag{2.15}$$

and get from (2.14)

$$u(x) - \epsilon \leq E_i(u|E(x)) \leq u(x) + \epsilon, \tag{2.16}$$

and so for every class $E(x)$ of positive measure we have

$$E_1(u|E(x)) - E_2(u|E(x)) \leq 2\epsilon. \tag{2.17}$$

Multiply by $\mu_E(E(x))$ and sum over X/E to get

$$\int u(x)\mu_1(dx) - \int u(x)\mu_2(dx) \leq 2\epsilon. \tag{2.18}$$

By the definition (1.12) of the Hutchinson metric we see that $d(\mu_1, \mu_2) \leq 2\epsilon$. □

A *section* is a choice function for the equivalence classes, i.e. a map $\omega : X/E \to X$ such that $\pi_E \circ \omega = 1_{X/E}$. So if μ_E is any measure on X/E, then

$$(\pi_E)_*\omega_*(\mu_E) = \mu_E, \tag{2.19}$$

where

$$\omega_*(\mu_E) = \sum_{A \in X/E} \mu_E(A)\delta_{\omega(A)}. \tag{2.20}$$

That is, $\omega_*(\mu_E)$ is the weighted average of the point measures $\delta_{\omega(A)}$ with weights $\mu_E(A)$.

Now let F be a relation from X_1 to X_2 and E_i be a finite equivalence relation on X_i $(i = 1, 2)$. Let $E_1 \times E_2$ denote the equivalence relation on $X_1 \times X_2$ such that $(E_1 \times E_2)(x, y) = E_1(x) \times E_2(y)$ and so that the projection $\pi_{E_1 \times E_2}$ is identified with $\pi_{E_1} \times \pi_{E_2}$. Define $F_E = (\pi_{E_1} \times \pi_{E_2})(F)$, a relation from X_1/E_1 to X_2/E_2, so that

$$\begin{aligned} &(E_1(x), E_2(y)) \in F_E \\ &\quad \Leftrightarrow (E_1(x) \times E_2(y)) \cap F \neq \emptyset \\ &\quad \Leftrightarrow F(E_1(x)) \cap E_2(y) \neq \emptyset \\ &\quad \Leftrightarrow E_1(x) \cap F^{-1}(E_2(y)) \neq \emptyset. \end{aligned} \tag{2.21}$$

If F is a closed relation and μ_{12} satisfies $|\mu_{12}| \subset F$ then $\mu_{12E} \equiv (\pi_{E_1} \times \pi_{E_2})_*(\mu_{12})$ is a measure on $(X_1/E_1) \times (X_2/E_2)$ with $|\mu_{12E}| \subset F_E$.

We call a section $\omega : X_1/E_1 \times X_2/E_2 \to X_1 \times X_2$ *adapted to* F if ω maps F_E to F, i.e.

$$(E_1(x), E_2(x)) \in F_E \Rightarrow \omega(E_1(x), E_2(x)) \in F. \tag{2.22}$$

By (2.21) such sections always exist. If μ_{12E} is a measure on $(X_1/E_1) \times (X_2/E_2)$ which is supported by F_E and if ω is a section adapted to F, then $\omega_*(\mu_{12E})$ is a measure on $X_1 \times X_2$ supported by F.

3. Invariant Measures

3.1. Theorem. *For compact metric spaces X_1, X_2 let $F \subset X_1 \times X_2$ be a closed relation. For a pair of measures $(\mu_1, \mu_2) \in P(X_1) \times P(X_2)$ the following conditions are equivalent,*

(1) For every Borel set $A_2 \subset X_2$:

$$\mu_2(A_2) \leq \mu_1(F^{-1}(A_2)). \tag{3.1}$$

(2) There exists a function $\kappa : X_1 \to P(X_2)$ such that for every Borel set $A_2 \subset X_2$ the function $\kappa_x(A_2)$ is a Borel measurable function of $x \in X_1$,

$$\mu_2(A_2) = \int_{X_1} \kappa_x(A_2)\mu_1(dx), \tag{3.2}$$

and for each x in the support of μ_1, the measure κ_x is supported by $F(x)$, i.e.

$$|\kappa_x| \subset F(x) \quad \textit{for all } x \in |\mu_1|. \tag{3.3}$$

(3) There exists a measure $\mu_{12} \in P(X_1 \times X_2)$ which projects to μ_i via $\pi_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$, i.e. for every Borel subset $A_1 \subset X_1$ and $A_2 \subset X_2$,

$$\mu_1(A_1) = \mu_{12}(A_1 \times X_2),$$

$$\mu_2(A_2) = \mu_{12}(X_1 \times A_2), \tag{3.4}$$

and the measure μ_{12} is supported by F, i.e.

$$|\mu_{12}| \subset F. \tag{3.5}$$

Let F_* denote the set of pairs (μ_1, μ_2) which satisfy these conditions. F_* is a closed relation, i.e. a closed subset of $P(X_1) \times P(X_2)$ whose domain satisfies

$$\mathrm{Dom}(F_*) = \{\mu_1 \in P(X_1) : |\mu_1| \subset \mathrm{Dom}(F)\}. \tag{3.6}$$

Furthermore, for the inverse relation $F^{-1} \subset X_2 \times X_1$,

$$(F^{-1})_* = (F_*)^{-1}. \tag{3.7}$$

If $F : X_1 \to X_2$ is a continuous map, then F_* is the induced continuous map from $P(X_1)$ to $P(X_2)$.

Proof. (2) $\Rightarrow$ (3): Given $\mu_1 \in P(X_1)$ and a Markov kernel $\kappa : X_1 \to P(X_2)$, define $\mu_{12} \in P(X_1 \times X_2)$ on product sets by

$$\mu_{12}(A_1 \times A_2) = \int_{A_1} \kappa_x(A_2)\mu_1(dx). \tag{3.8}$$

For u a real, bounded, measurable function on $X_1 \times X_2$

$$\int_{X_1 \times X_2} u(x,y)\mu_{12}(d(x,y)) = \int_{X_1} (\int_{X_2} u(x,y)\kappa_x(dy))\mu_1(dx). \tag{3.9}$$

Clearly, $\mu_{12}(A_1 \times X_2) = \mu_1(A_1)$ in any case, and (3.2) implies $\mu_{12}(X_1 \times A_2) = \mu_2(A_2)$. To prove $|\mu_{12}| \subset F$ it suffices to show that when a continuous function u vanishes on F its integral, given by (3.9), is zero. For x fixed in $|\mu_1|$, $u(x,y) = 0$ for $y \in F(x)$. Because $|\kappa_x| \subset F(x)$ we have, for $x \in |\mu_1|$,

$$\int_{X_2} u(x,y)\kappa_x(dy) = \int_{F(x)} u(x,y)\kappa_x(dy) = 0. \tag{3.10}$$

But the outer integral in (3.9) is unchanged if we restrict the range of integration from X_1 to $|\mu_1|$. So by (3.10) the integral in (3.9) is zero.

(3) $\Rightarrow$ (2): Because μ_{12} is a measure on a separable metric space there exists a regular conditional distribution for the σ algebra of subsets $\mathcal{B}_1 = \{\pi_1^{-1}(A_1) = A_1 \times X_2 : A_1 \subset X_1 \text{ Borel}\}$ (see, e.g., Gihman and Skorohod (1974), Theorem I.3.3). This is a $\mathcal{B}_1$ measurable Markov kernel $\tilde{\kappa} : X_1 \times X_2 \to P(X_1 \times X_2)$ such that for $A_1 \subset X_1$ Borel and $u : X_1 \times X_2 \to \mathbf{R}$ bounded and Borel measurable we have

$$\begin{aligned}&\int_{A_1 \times X_2} u(x,y)\mu_{12}(d(x,y)) \\ &\quad = \int_{A_1 \times X_2} [\int_{X_1 \times X_2} u(\tilde{x},\tilde{y})\tilde{\kappa}_{(x,y)}(d(\tilde{x},\tilde{y}))]\mu_{12}(d(x,y)).\end{aligned} \tag{3.11}$$

We can fix $y = y_0 \in X_2$ to define $\kappa : X_1 \to P(X_2)$ by $\kappa_x = (\pi_2)_*\tilde{\kappa}_{(x,y_0)}$. Because $\tilde{\kappa}$ is $\mathcal{B}_1$ measurable and $\pi_{1*}(\mu_{12}) = \mu_1$, (3.8) follows from (3.11) with u the characteristic function of $X_1 \times A_2$. With $A_1 = X_1$ (3.2) follows because $\pi_{2*}(\mu_{12}) = \mu_2$. If $U_1 \times U_2$ is open in $X_1 \times X_2$ and disjoint from F, then $\mu_{12}(U_1 \times U_2) = 0$, and so from (3.8) $\kappa_x(U_2) = 0$ for μ_1 almost every $x \in U_1$. By letting U_1 and U_2 vary over countable bases for the topologies we can obtain a Borel set N of μ_1 measure zero such that for $x \notin N$, $|\kappa_x| \subset F(x)$. Because $|\mu_{12}| \subset F$ it follows that $|\mu_1| = |\pi_{1*}(\mu_{12})| \subset \pi_1(F) = \mathrm{Dom}(F)$. Use Lemma 1.1 to construct $f : \mathrm{Dom}(F) \to X_2$, a Borel selection function for F, and replace κ_x by $\delta_{f(x)}$ for all $x \in N \cap |\mu_1|$. Condition (3.3) then holds for the adjusted function κ.

(3) $\Rightarrow$ (1): If $A_2 \subset X_2$ is Borel, then $F \cap (X_1 \times A_2)$ is a Borel subset of $X_1 \times X_2$ and its image under π_1 is $F^{-1}(A_2)$. Thus, the latter set, while not necessarily Borel, is analytic and so is measurable with respect to any Borel measure on X_1. So $\mu_1(F^{-1}(A_2))$ is always defined for $A_2 \subset X_2$ Borel.

If $\mu_{12} \in P(X_1 \times X_2)$ satisfies the conditions of (3), then, because $|\mu_{12}| \subset F$,

$$\begin{aligned} \mu_2(A_2) &= \mu_{12}(X_1 \times A_2) = \mu_{12}((X_1 \times A_2) \cap F) \\ &\leq \mu_{12}(F^{-1}(A_2) \times X_2) = \mu_1(F^{-1}(A_2)), \end{aligned} \tag{3.12}$$

proving (3.1).

(1) $\Rightarrow$ (3): Fix $\epsilon > 0$ and for $i = 1, 2$ choose E_i a finite equivalence relation on X_i with mesh at most ϵ. As in Section 2 we let F_E denote $(\pi_{E_1} \times \pi_{E_2})(F) \subset (X_1/E_1) \times (X_2/E_2)$. Let $\nu_i = (\pi_{E_i})_*(\mu_i)$ $(i = 1, 2)$. Choose a section $\omega : (X_1/E_1) \times (X_2/E_2) \to X_1 \times X_2$ which is adapted to F.

For $B_2 \subset X_2/E_2$,

$$(\pi_{E_1})^{-1}(F_E^{-1}(B_2)) = E_1(F^{-1}((\pi_{E_2})^{-1}(B_2))) \supset F^{-1}((\pi_{E_2})^{-1}(B_2)). \tag{3.13}$$

Consequently, (3.1) implies

$$\begin{aligned} \nu_2(B_2) &= \mu_2((\pi_{E_2})^{-1}(B_2)) \leq \mu_1(F^{-1}((\pi_{E_2})^{-1}(B_2))) \\ &\leq \mu_1((\pi_{E_1})^{-1}(F_E^{-1}(B_2))) = \nu_1(F_E^{-1}(B_2)). \end{aligned} \tag{3.14}$$

Thus, condition (2.7) holds for the pair $(\nu_1, \nu_2) \in P(X_1/E_1) \times P(X_2/E_2)$.

Now we apply Brualdi's Theorem, Theorem 2.1, to F_E and obtain a measure ν_{12} on $(X_1/E_1) \times (X_2/E_2)$ which satisfies

$$\begin{aligned} &|\nu_{12}| \subset F_E, \\ &\pi_{i*}(\nu_{12}) = \nu_i \quad (i = 1, 2). \end{aligned} \tag{3.15}$$

Use the section ω to define the finite measure $\mu_E = (\omega)_*(\nu_{12})$ on $X_1 \times X_2$. Because ω is adapted to F and ν_{12} is supported by F_E, it follows that μ_E is supported by F. Also, $\pi_i \circ (\pi_{E_1} \times \pi_{E_2}) = \pi_{E_i} \circ \pi_i : X_1 \times X_2 \to X_i/E_i$ $(i = 1, 2)$, and so we have

$$\begin{aligned} &|\mu_E| \subset F, \\ &(\pi_{E_i})_*(\pi_i)_*(\mu_E) = \nu_i = (\pi_{E_i})_*(\mu_i) \quad (i = 1, 2). \end{aligned} \tag{3.16}$$

Apply Lemma 2.2. Because the mesh of E_i is at most ϵ,

$$d(\mu_i, (\pi_i)_*(\mu_E)) \leq 2\epsilon \quad (i = 1, 2). \tag{3.17}$$

Now vary ϵ and let μ_{12} be a limit point in $P(X_1 \times X_2)$ of the μ_E's as $\epsilon \to 0$. From (3.16) the limit measure is supported by F, and (3.17) implies $(\pi_i)_*(\mu_{12}) = \mu_i$ $(i = 1, 2)$. Thus, (3.4) and (3.5) of (3) hold.

This completes the proof of the equivalence.

The set of measures on $X_1 \times X_2$ supported by the closed set F is closed in $P(X_1 \times X_2)$ (e.g. by Akin (1993), Proposition 8.1, the relation $\mu \to |\mu|$ is lower semicontinuous). So by compactness and (3) the image of this set under $\pi_{1*} \times \pi_{2*}$ is closed in $P(X_1) \times P(X_2)$. Clearly, $|\mu_{12}| \subset F$ implies $|\mu_1| = |\pi_{1*}(\mu_{12})| \subset \mathrm{Dom}(F) = \pi_1(F)$. Conversely, if $|\mu_1| \subset \mathrm{Dom}(F)$, let $f : \mathrm{Dom}(F) \to X_2$ be a measurable selection function for F; $\kappa_x = \delta_{f(x)}$ satisfies (3.3), and so, with μ_2 defined by (3.2), (μ_1, μ_2) is in F_* by (2).

Equation (3.7) is clear from (3), and the case when F is a continuous map was analyzed in Section 1. □

3.2. Theorem. *Let $F \subset X \times X$ be a closed relation on a compact metric space X. A measure $\mu \in P(X)$ is called an* invariant measure *for F when it satisfies the following equivalent conditions.*

(1) For every Borel set $A \subset X$

$$\mu(A) \le \mu(F^{-1}(A)). \tag{3.18}$$

(2) There exists a Markov kernel $\kappa : X \to P(X)$ satisfying

$$x \in |\mu| \Rightarrow |\kappa_x| \subset F(x) \tag{3.19}$$

and

$$\mu = \kappa_*(\mu). \tag{3.20}$$

(3) There exists a measure $\tilde{\mu} \in P(X \times X)$ satisfying

$$|\tilde{\mu}| \subset F \tag{3.21}$$

and

$$\mu = (\pi_1)_*(\tilde{\mu}) = (\pi_2)_*(\tilde{\mu}). \tag{3.22}$$

(4) There exists a measure ν in $P(X^{\mathbf{Z}})$ which is invariant with respect to the shift homeomorphism s on $X^{\mathbf{Z}}$ and satisfying, with X_F the sample path space for F,

$$|\nu| \subset X_F \tag{3.23}$$

and

$$\mu = (\pi_0)_*(\nu). \tag{3.24}$$

The set $P_F(X)$ of F invariant measures is compact and convex in $P(X)$. It is nonempty if and only if X_F is nonempty and so if and only if the dynamic domain $\pi_0(X_F)$ is nonempty. In general, if $\mu \in P_F(X)$ then

$$|\mu| \subset \pi_0(X_F). \tag{3.25}$$

Proof. By Theorem 3.1 each of the first three conditions is equivalent to $(\mu, \mu) \in F_*$.

(4) ⇒ (3): Define $\tilde{\mu} \in P(X \times X)$ by

$$\tilde{\mu} = (\pi_{0,1})_*(\nu). \tag{3.26}$$

Condition (3.22) then follows from (3.24) and shift invariance. Since the set $(\pi_{0,1})^{-1}(X \times X \backslash F)$ is open and disjoint from X_F, (3.23) implies this set has ν measure 0, and so $\tilde{\mu}$ is supported by F.

(2) ⇒ (4): Let $\mu^1 = \mu$ on $P(X^1)$ and inductively use $\mu^n \in P(X^n)$ and $X^n \xrightarrow{\pi_n} X \xrightarrow{\kappa} P(X)$ to define $\mu^{n+1} \in P(X^{n+1})$ via (3.8). Applying the Kolmogorov Product Theorem to this sequence of measures, we obtain from the Markov kernel κ and the stationary measure μ the associated sample path measure ν. The resulting measure is clearly shift invariant and satisfies (3.24). By the proof of (2) ⇒ (3) in

Theorem 3.1, $\mu^2(X \times X \setminus F) = 0$. Since $X^{\mathbf{Z}} \setminus X_F = \bigcup_{i \in \mathbf{Z}} \{(\pi_{i,i+1})^{-1}(X \times X \setminus F)\}$ we see that the complement of X_F has ν measure 0, i.e. (3.23) holds.

The set of s_F invariant measures in $P(X_F)$ is compact and convex and nonempty–by the Krylov-Bogolyubov Theorem–when X_F is nonempty. So its image under π_{0^*} satisfies the same properties. The inclusion (3.25) follows from (3.23) and (3.24). □

For $\epsilon \geq 0$ call $\mu \in P(X)$ an ϵ *invariant measure* for the closed relation F on X if there exists $(\mu_1, \mu_2) \in F_*$ such that $d(\mu_i, \mu) \leq \epsilon$ for $i = 1, 2$. So an invariant measure is exactly a 0 invariant measure. If $(\mu_1, \mu_2) \in F_*$ with $d(\mu_1, \mu_2) \leq \epsilon$, then μ_1 and μ_2 are both ϵ invariant measures. If μ' is an invariant measure and $d(\mu', \mu) \leq \epsilon$, then μ is an ϵ invariant measure. On the other hand,

3.3. Lemma. *Let F be a closed relation on X. For every $\epsilon > 0$ there exists $\epsilon_1 > 0$ so that if μ is an ϵ_1 invariant measure then there exists an invariant measure μ' such that $d(\mu', \mu) < \epsilon$.*

Proof. The set of ϵ_1 invariant measures is a closed and hence compact subset of $P(X)$. As ϵ_1 decreases to 0 the sets decrease toward the set of invariant measures, and so are eventually contained in its ϵ neighborhood. □

From the proof of Theorem 3.1 we extract some useful information about the pixel approximation.

3.4. Theorem. *Let E be a finite Borel equivalence relation on X with mesh $\leq \epsilon/2$. Let $\pi_E : X \to X/E$ be the projection to the set of equivalence classes. Assume $\omega : X/E \to X$ is a section, i.e. $\pi_E \circ \omega = 1_{X/E}$. For a closed relation F on X, let $F_E = (\pi_E \times \pi_E)(F)$ be the induced relation on the finite set X/E.*

(a) For μ a measure on X, let $\mu_E = (\pi_E)_*(\mu)$. If $\mu_\omega = \omega_*(\mu_E)$, then $(\pi_E)_*(\mu_\omega) = \mu_E$ as well, and

$$d(\mu, \mu_\omega) \leq \epsilon. \tag{3.27}$$

If μ is an F invariant measure, then μ_E is an F_E invariant measure.

(b) For μ_E a measure on X/E let $\mu_\omega = \omega_*(\mu_E)$. If μ_E is an F_E invariant measure, then μ_ω is an ϵ invariant measure for F with $(\pi_E)_*(\mu_\omega) = \mu_E$.

Proof. $\pi_E \circ \omega = 1_{X_E}$ implies $\pi_{E^*}(\mu_\omega) = \mu_E$. So Lemma 2.2 implies (3.27).

If $\tilde{\mu} \in P(X \times X)$ is supported by F with $(\pi_i)_*(\tilde{\mu}) = \mu$ for $i = 1, 2$, then $\tilde{\mu}_E \equiv (\pi_E \times \pi_E)_*(\tilde{\mu}) \in P((X/E) \times (X/E))$ is supported by F_E with $(\pi_i)_*(\tilde{\mu}_E) = \mu_E$ for $i = 1, 2$. So μ F invariant implies $\mu_E = (\pi_E)_*(\mu)$ is F_E invariant, proving (a).

Let $\omega' : (X/E) \times (X/E) \to X \times X$ be a section of $\pi_E \times \pi_E$ which is adapted to F. If $\tilde{\mu}_E \in P((X/E) \times (X/E))$ with $(\pi_i)_*(\tilde{\mu}_E) = \mu_E$ for $i = 1, 2$, then $\tilde{\mu}_\omega = (\omega')_*(\tilde{\mu}_E)$ is supported by F. So if $\mu_{i\omega} = (\pi_i)_*(\tilde{\mu}_\omega)$ $(i = 1, 2)$, then $(\mu_{1\omega}, \mu_{2\omega}) \in F_*$. No two of $\{\mu_{1\omega}, \mu_{2\omega}, \mu_\omega\}$ need be equal, but we do have, for $i = 1, 2$,

$$(\pi_E)_*(\mu_{i\omega}) = (\pi_i)_*(\tilde{\mu}_E) = \mu_E = (\pi_E)_*(\mu_\omega). \tag{3.28}$$

By Lemma 2.2 we have $d(\mu_\omega, \mu_{i\omega}) \leq \epsilon$ $(i = 1, 2)$, and so μ_ω is ϵ invariant. □

4. Chain Recurrence

Since F invariant measures on X are the image under π_0 of the s_F invariant sample path measures on X_F, it will be useful to relate the dynamics of s_F on X_F with F on X.

For a closed relation F on X and $\epsilon \geq 0$, an ϵ *chain* is a sequence $\{x_i\}$ indexed by some subinterval I of $\mathbf{Z}$ such that

$$d(x_{i+1}, F(x_i)) \leq \epsilon \quad \text{for } i, i+1 \in I. \tag{4.1}$$

Here I can be finite or infinite, but has length at least one, so that at least one pair $i, i+1$ lies in I. By compactness (4.1) says there exists $z_{i+1} \in F(x_i)$ such that $d(x_{i+1}, z_{i+1}) \leq \epsilon$. In particular, for a chain (= 0 chain) we have $x_{i+1} \in F(x_i)$ for $i, i+1 \in I$. A sequence $\{x_i : i \in I\}$ is called an *asymptotic chain* if I is unbounded on the right and

$$\mathrm{Lim}_{i \to \infty} d(x_{i+1}, F(x_i)) = 0. \tag{4.2}$$

An ϵ *asymptotic chain* is an asymptotic chain which is also an ϵ chain. When $I = \mathbf{Z}$ we obtain points ξ of $X^{\mathbf{Z}}$, and so we speak of the points ξ of $X^{\mathbf{Z}}$ which are ϵ chains, asymptotic chains or ϵ asymptotic chains. In particular, X_F is the set of 0 chains in $X^{\mathbf{Z}}$.

If F is a surjective relation on X, i.e.

$$F(X) = X = F^{-1}(X), \tag{4.3}$$

then for any ϵ chain $\{x_i : i \in I\}$ there exists points $\xi \in X^{\mathbf{Z}}$ such that

$$\begin{aligned} \xi_i &= x_i, \quad i \in I, \\ \xi_{i+1} &\in F(\xi_i) \quad \text{unless} \quad i, i+1 \in I. \end{aligned} \tag{4.4}$$

We will call such a point ξ an *extension* of $\{x_i : i \in I\}$.

The *chain relation* $\mathcal{C}F \subset X \times X$ is defined by $(x, y) \in \mathcal{C}F$ when for every $\epsilon > 0$ there exists an ϵ chain $\{x_0, \ldots, x_n\}$ $(0 < n < \infty)$ such that $x_0 = x$ and $x_n = y$, i.e. there is an ϵ chain from x to y. $\mathcal{C}F$ is a closed, transitive relation containing F. A point x is called *chain recurrent* if $(x, x) \in \mathcal{C}F$. It is easy to see that x is chain recurrent iff for every $\epsilon > 0$ there exists $\xi \in X^{\mathbf{Z}}$, an ϵ chain with $\xi_0 = x$ which is *periodic*, i.e. $\xi_{i+n} = \xi_i$ for some $n > 0$ and all $i \in \mathbf{Z}$. The set of chain recurrent points is denoted $|\mathcal{C}F|$. It is a closed subset of X on which $(\mathcal{C}F) \cap (\mathcal{C}F^{-1})$ is a closed equivalence relation. The equivalence classes are called the *basic sets* for F (see Akin (1993), Chapters 1-3).

F is called *chain transitive* if $\mathcal{C}F = X \times X$, i.e. every point is chain related to every other. A closed subset A of X is called a *chain transitive subset* if the closed relation $F_A = F \cap (A \times A)$ is a chain transitive relation on A. For example, by Akin (1993), Theorem 4.5, any basic A for F is a chain transitive subset.

4.1. Lemma. *For a closed relation F on X*

$$|\mathcal{C}F| \subset \bigcap_{n \in \mathbf{Z}} (\mathcal{C}F)^n(X) = \pi_0(X_F). \tag{4.5}$$

Any basic set for F is a surjective subset. In fact,

$$B = F(B) \cap F^{-1}(B) = (\mathcal{C}F(B)) \cap (\mathcal{C}F^{-1}(B)). \tag{4.6}$$

Proof. It is easy to check that $\mathcal{C}F(X) = F(X)$ (see Akin (1993), Exercise 2.3e) and so, for all $n \in \mathbf{Z}$, $(\mathcal{C}F)^n(X) = F^n(X)$. The equality in (4.5) follows from (1.10). Clearly, $x \in \mathcal{C}F(x)$ implies $x \in (\mathcal{C}F)^n(X)$ for all n.

A basic set B is a chain transitive subset, and so $B = |\mathcal{C}(F_B)|$, where $F_B = F \cap (B \times B)$ is the closed relation obtained by restricting to B. By (4.5) applied to F_B we see that F_B is a surjective relation, and so B is a surjective subset. Since $F \subset \mathcal{C}F$ we have $B \subset F(B) \cap F^{-1}(B) \subset (\mathcal{C}F(B)) \cap (\mathcal{C}F^{-1}(B))$. On the other hand, if $(x_1, x), (x, x_2) \in \mathcal{C}F$ for $x_1, x_2 \in B$, then, because B is a basic set, $(x_2, x_1) \in \mathcal{C}F$, and so the three points x_1, x_2, x are all $\mathcal{C}F$ equivalent. That is, $x \in B$. So (4.6) follows. □

4.2. Lemma. *(a) Assume $\epsilon > 0$ and n is a positive integer. There exists $\epsilon_1 > 0$ so that if $\{y_i : i \in [-k, k]\}$ is an ϵ_1 chain with $k \leq n$, then there exists $\xi \in X_F$ such that $d(y_i, \xi_i) < \epsilon$ for all $i \in [-k, k]$.*

(b) Assume $\{x_i : i \in [-l, l]\}$ is a 0 chain, $\epsilon > 0$ and n is a positive integer. There exists $\epsilon_1 > 0$ so that if $\{y_i : i \in [-k, k]\}$ is an ϵ_1 chain with $l \leq k \leq n$ such that $y_i = x_i$ for all $i \in [-l, l]$, then there exists $\xi \in X_F$ such that $\xi_i = x_i$ for all $i \in [-l, l]$ and $d(y_i, \xi_i) < \epsilon$ for all $i \in [-k, k]$.

Proof. (a) Let $X_{F,\delta}$ denote the set of δ chains in $X^{\mathbf{Z}}$. As δ decreases to 0 this family of compacta decreases to X_F. So there exists $\epsilon_1 > 0$ so that X_{F,ϵ_1} is contained in the open $\tilde{\epsilon} = \min(\epsilon, 1/n)$ neighborhood of X_F. Given an ϵ_1 chain, $\{y_i : i \in [-k, k]\}$ let η be its extension in X_{F,ϵ_1}. There exists $\xi \in X_F$ such that $d(\eta, \xi) < \tilde{\epsilon}$, and as in (1.14) $d(\eta_i, \xi_i) < \epsilon$ for all $i \in [-n, n]$.

For (b) repeat the proof, intersecting $X_{F,\delta}, X_F$ and X_{F,ϵ_1} with the closed set $\{\xi \in X^{\mathbf{Z}} : \xi_i = x_i \text{ for all } i \in [-l, l]\}$. □

4.3. Theorem. *Let F be a closed surjective relation on X, i.e. $F(X) = X = F^{-1}(X)$. Let s_F be the shift homeomorphism on the sample path space X_F and π_0 be the time zero projection of X_F onto X.*

(a) *For a point $\xi \in X_F$ the following conditions are equivalent:*
  (1) *ξ is s_F chain recurrent, i.e. $\xi \in |\mathcal{C}s_F|$.*
  (2) *$\xi_i \in \mathcal{C}F(\xi_{i+1})$ for all $i \in \mathbf{Z}$.*
  (3) *There exists a basic set B for F such that $\xi_i \in B$ for all $i \in \mathbf{Z}$.*

(b) *For points ξ, η in $|\mathcal{C}s_F|$ we have $\xi \in \mathcal{C}s_F(\eta)$ if and only if $\pi_0(\xi) \in \mathcal{C}F(\pi_0(\eta))$.*

In general, $\pi_0 \times \pi_0$ maps $\mathcal{C}s_F$ onto $\mathcal{C}F$, i.e.

$$(\pi_0 \times \pi_0)(\mathcal{C}s_F) = \mathcal{C}F. \tag{4.7}$$

(c) If B is a basic set for F then $X_{F_B} = X_F \cap B^{\mathbf{Z}}$ is a basic set for s_F mapping onto B via π_0. Every basic set for s_F is constructed this way. Thus, π_0 maps distinct s_F basic sets onto distinct F basic sets and

$$\pi_0(|\mathcal{C}s_F|) = |\mathcal{C}F|. \tag{4.8}$$

(d) F is a chain transitive relation on X if and only if s_F is a chain transitive homeomorphism on X_F.

Proof. If $\xi \in X_F$ then $\xi_1 \in F(\xi_0)$ and so $\pi_0(s_F(\xi)) \in F(\pi_0(\xi))$. With the metric on $X^{\mathbf{Z}}$ given by (1.13) the map π_0 is distance nonincreasing (by (1.14)), so the image under π_0 of an ϵ chain for s_F in X_F is an ϵ chain for F in X. In particular, we have

$$(\pi_0 \times \pi_0)(\mathcal{C}s_F) \subset \mathcal{C}F. \tag{4.9}$$

So if two points in X_F are $\mathcal{C}s_F$ equivalent then their images in X are $\mathcal{C}F$ equivalent. It follows that for any s_F basic set $\tilde{B} \subset X_F$ there is a unique F basic set $B \subset X$ such that $\pi_0(\tilde{B}) \subset B$.

Now we describe a construction we will call the *lifting trick*. Given $\epsilon > 0$, choose an integer $n > (1/\epsilon) + 1$. Given fixed 0 chains $\{x_i : i \in [-l_1, l_1]\}$ and $\{y_j : j \in [-l_2, l_2]\}$ with $l_1, l_2 \leq n$, choose $\epsilon_1 > 0$ small enough to satisfy the conditions of Lemma 4.2a for $\epsilon/2$ and n and Lemma 4.2b for $\epsilon/2$, n and each of the two 0 chains.

Now suppose ξ is an ϵ_1 chain in $X^{\mathbf{Z}}$ such that for some integer $N > l_1 + l_2$,

$$\begin{aligned} \xi_i &= x_i, \qquad i \in [-l_1, l_1], \\ \xi_{i+N} &= y_i, \qquad i \in [-l_2, l_2]. \end{aligned} \tag{4.10}$$

By Lemma 4.2 we can choose a sequence $\eta^0, \ldots, \eta^N$ in X_F so that $d(\eta_i^k, s^k(\xi)_i) < \epsilon/2$ for $i \in [-n, n]$, $\eta_i^0 = \xi_i$ for $i \in [-l_1, l_1]$ and $\eta_i^N = \xi_{i+N}$ for $i \in [-l_2, l_2]$.

Observe that for $0 \leq k < N$ and $i \in [-n, n-1]$

$$d(s_F(\eta^k)_i, \eta_i^{k+1}) \leq d(\eta_{i+1}^k, \xi_{i+k+1}) + d(\eta_i^{k+1}, \xi_{i+k+1}). \tag{4.11}$$

So, since $n - 1 > 1/\epsilon$, we have $d(s_F(\eta^k), \eta^{k+1}) \leq \epsilon$ by (1.14). This says that $\{\eta^0, \ldots, \eta^N\}$ is an ϵ chain for s_F connecting η^0 with η^N satisfying

$$\begin{aligned} \eta_i^0 &= x_i, \qquad i \in [-l_1, l_1], \\ \eta_i^N &= y_i, \qquad i \in [-l_2, l_2]. \end{aligned} \tag{4.12}$$

Now we prove the theorem.

(d) If s_F is chain transitive, i.e. $\mathcal{C}s_F = X_F \times X_F$, then (4.9) implies that $\mathcal{C}F = X \times X$ because F is surjective.

Conversely, assume F is chain transitive, $\zeta^1, \zeta^2 \in X_F$ and $\epsilon > 0$. We choose $n > (1/\epsilon) + 1$, and let $l_1 = l_2 = n$, $x_i = \zeta_i^1$ and $y_i = \zeta_i^2$ for $i \in [-n, n]$. Since ζ_n^1 and $\zeta_{-n}^2 \in X$ and F is chain transitive, we can build a sequence ξ so that $\xi_i = \zeta_i^1$ for $i \in [-n, n]$, and $\{\xi_n, \xi_{n+1}, \ldots, \xi_{n+M}\}$ is an ϵ_1 chain for F in X with $\xi_n = \zeta_n^1$ and $\xi_{n+M} = \zeta_{-n}^2$. Let $\xi_{n+M+j} = \zeta_{j-n}^2$ for $j = 0, \ldots, 2n$, and extend to get $\xi \in X^{\mathbf{Z}}$. Then use the lifting trick construction, choosing $\eta^0 = \zeta^1$ and $\eta^N = \zeta^2$, where $N = 2n + M$. We have constructed an ϵ chain for s_F from ζ^1 to ζ^2. As ϵ was arbitrary it follows that $\zeta^2 \in \mathcal{C}s_F(\zeta^1)$, and so s_F is chain transitive.

(a) (1) $\Rightarrow$ (3): If $\tilde{B}$ is an s_F basic set then by (4.9) there is a basic set B for F such that $\pi_0(\tilde{B}) \subset B$. For a map (though not in general for a closed relation) any basic set is invariant (see Akin (1993), Corollary 4.11). So $\xi \in \tilde{B}$ implies $s_F^k(\xi) \in \tilde{B}$ for all $k \in \mathbf{Z}$. Hence $\xi_k = \pi_0(s_F^k(\xi)) \in B$ for all $k \in \mathbf{Z}$.

(3) $\Rightarrow$ (1): On the basic set B the closed relation F_B is chain transitive. By (4.6), part (d) proved above applies to F_B, and so s_F restricted to X_{F_B} is chain transitive. Furthermore, $\pi_0(X_{F_B}) = B$. Since $(\xi_i, \xi_{i+1}) \in F \cap (B \times B) = F_B$ for all $i \in \mathbf{Z}$, $\xi \in X_{F_B}$. Every point of X_{F_B} is chain recurrent because X_{F_B} is a chain transitive subset.

(3) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): $\xi_{i+1} \in F(\xi_i)$ and $F \subset \mathcal{C}F$. So, by assumption of (2), ξ_i and ξ_{i+1} are $\mathcal{C}F$ equivalent for all i. As $\mathcal{C}F$ is transitive, this means that all the ξ_i's are $\mathcal{C}F$ equivalent to one another, and so lie in some common F basic set.

(c) If B is an F basic set then, as above, $X_{F_B} = X_F \cap B^{\mathbf{Z}}$ is a chain transitive subset mapping onto B. So it is entirely contained in some s_F basic set $\tilde{B}$. But π_0 maps $\tilde{B}$ into some F basic set. As distinct basic sets are disjoint, it must be that $\pi_0(\tilde{B}) \subset B$. Then $\tilde{B} \subset X_F \cap B^{\mathbf{Z}}$, and so we have

$$\tilde{B} = X_F \cap B^{\mathbf{Z}} = X_{F_B} \tag{4.13}$$

is a basic set mapping onto B. Any other basic set mapping into B is contained in $X_F \cap B^{\mathbf{Z}}$ by part (a), and so equals $\tilde{B}$ because distinct basic sets are disjoint.

Thus, π_0 maps distinct basic sets of s_F to distinct basic sets of F, and (4.8) holds.

(b) If $\xi \in \mathcal{C}s_F(\eta)$, then $\pi_0(\xi) \in \mathcal{C}F(\pi_0(\eta))$ by (4.9). If ξ, $\eta \in |\mathcal{C}s_F|$, then by (a) all the ξ_i's lie in some common basic set B_1 and all the η_i's lie in some common basic set B_2. If $\xi_0 \in \mathcal{C}F(\eta_0)$ then for all $i, j \in \mathbf{Z}$ we have $\xi_i \in \mathcal{C}F(\eta_j)$ by transitivity of the relation $\mathcal{C}F$. So for any $\epsilon_1 > 0$ and positive integer n we can connect η_n to ξ_{-n} via an ϵ_1 chain. Then, as in the proof of (d), use the lifting trick to connect η to ξ via an ϵ chain for s_F. In general, if $y \in \mathcal{C}F(x)$ then we can connect x to y by an ϵ_1 chain and use the lifting trick again, this time with $l_1 = l_2 = 0$, to get an ϵ chain for $\{\eta^0, \dots, \eta^N\}$ for s_F such that $\eta^0_0 = x$ and $\eta^N_0 = y$. This proves (4.7). □

For any continuous map f on X the omega limit point set of $x \in X$, $\omega f(x)$, is the set of limit points of the orbit sequence $\{f^k(x) : k = 0, 1, \dots\}$.

4.4. Corollary. *Let $\xi \in X^{\mathbf{Z}}$ be an asymptotic chain for a closed surjective relation F on X,*

$$\mathrm{Lim}_{k\to\infty} d(s^k(\xi), X_F) = 0. \tag{4.14}$$

The limit point set $\omega s(\xi)$ is an s_F chain transitive subset of X_F and so is contained in some s_F basic set. The image $\pi_0(\omega s(\xi))$ is the set of limit points of the sequence $\{\xi_k : k = 0, 1, \dots\}$ in X. It is an F chain transitive subset of X and so is contained in some F basic set.

Proof. Given $\epsilon > 0$, choose $n > 1/\epsilon$ and then $\epsilon_1 > 0$ to satisfy the condition of Lemma 4.2a. Because ξ is an asymptotic chain, there exists $N \in \mathbf{Z}$ such that $d(\xi_{i+1}, F(\xi_i)) < \epsilon_1$ for $i \geq N$. Hence, for $k > N + n$, $\{s^k(\xi)_i : i \in [-n, n]\}$ is an ϵ_1 chain in X for F. By Lemma 4.2a there is $\eta^k \in X_F$ such that $d(s^k(\xi)_i, \eta^k_i) < \epsilon$ for $i \in [-n, n]$, and so $d(s^k(\xi), X_F) < \epsilon$ for all $k > N$ by (1.14). This proves (4.14), which implies $\omega s(\xi) \subset X_F$. By Akin (1993), Proposition 4.14, $\omega s(\xi)$ is a chain transitive subset of $X^{\mathbf{Z}}$. Since it is in fact a subset of X_F on which $s = s_F$, it follows that $\omega s(\xi)$ is contained in some basic set for s_F in X_F. By compactness the image under π_0 of the limit point set of the sequence $\{s^k(\xi)\}$ is the limit point set of the sequence $\{\xi_k = \pi_0(s^k(\xi))\}$. By (4.9) $\pi_0(\omega s(\xi))$ is an F chain transitive subset of X and so is contained in an F basic set. □

To apply all these results when the closed relation F is not surjective we restrict to the dynamic domain $\pi_0(X_F)$. By Lemma 1.2, $\pi_0(X_F) = \bigcap_{n\in\mathbf{Z}} F^n(X)$ and the restriction $F_{\pi_0(X_F)}$ is a surjective relation on $\pi_0(X_F)$. Furthermore, by (4.5) and Lemma 1.2, applied to $\mathcal{C}F$, $|\mathcal{C}F| \subset \pi_0(X_F)$ and

$$\pi_0(X_F) = \mathcal{C}F(\pi_0(X_F)) \cap \mathcal{C}F^{-1}(\pi_0(X_F)). \tag{4.15}$$

From Theorem 4.5 of Akin (1993) it follows that

$$\mathcal{C}(F_{\pi_0(X_F)}) = (\mathcal{C}F)_{\pi_0(X_F)} = (\mathcal{C}F) \cap (\pi_0(X_F) \times \pi_0(X_F)). \tag{4.16}$$

We conclude this section by observing that, while chain transitivity for s_F and for F are equivalent, the same is not true for other related ideas.

Let f be a topologically transitive homeomorphism on a space X. Suppose that f has a unique proper minimal subset, a fixed point $e \in X$. For example, the *stopped torus map* described in Chapter 9 of Akin (1993) is such a map. Let $F = f \cup \{e\} \times X$. The only closed subsets A of X such that $F(A) \subset A$ are $\emptyset$ and X. So F is a "minimal" relation. The sample path space X_F consists of X_f together with sequences $\xi \in X^{\mathbf{Z}}$ such that for some $n \in \mathbf{Z}$ we have $\xi_i = e$ for all $i < n$ and $\xi_{i+1} = f(\xi_i)$ for all $i > n$. For any $\xi \in X_F$ the limit point sets $\omega s_F(\xi)$ and $\omega(s_F^{-1})(\xi)$ are contained in X_f, and so the orbit of ξ is not dense in X_F. Thus, s_F is not even topologically transitive.

5. Approximating Invariant Measures

We relate chains to invariant measures, first defining for any finite sequence $\{x_0, \dots, x_{n-1}\}$ in X the finite measure

$$\sigma(x_0, \dots, x_{n-1}) = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{x_i}. \tag{5.1}$$

Recall that $\mu \in P(X)$ is called an ϵ invariant measure for F if there exists $(\mu_1, \mu_2) \in F_*$ such that $d(\mu_1, \mu), d(\mu_2, \mu) \le \epsilon$.

5.1. Lemma. *If $\{x_0, \dots, x_n\}$ is an ϵ chain for F, then $\sigma(x_0, \dots, x_{n-1})$ is an $(\epsilon + \frac{1}{n}d(x_0, x_n))$ invariant measure for F. In particular, if $x_0 = x_n$, then $\sigma(x_0, \dots, x_{n-1})$ is an ϵ invariant measure for F.*

Proof. There exists $\{z_1, \dots, z_n\}$ such that $z_{i+1} \in F(x_i)$ and $d(z_{i+1}, x_{i+1}) \le \epsilon$ for $i = 0, \dots, n-1$. So the pairs $(\delta_{x_i}, \delta_{z_{i+1}}) \in F_*$ and $d(\delta_{z_{i+1}}, \delta_{x_{i+1}}) \le \epsilon$ for $i = 0, \dots, n-1$. By convexity of F_* and convexity properties of the Hutchinson metric we have

$$(\sigma(x_0, \dots, x_{n-1}), \sigma(z_1, \dots, z_n)) \in F_* \tag{5.2}$$

and

$$d(\sigma(z_1, \dots, z_n), \sigma(x_1, \dots, x_n)) \le \epsilon. \tag{5.3}$$

On the other hand, by definition of the Hutchinson metric

$$d(\sigma(x_0, \dots, x_{n-1}), \sigma(x_1, \dots, x_n)) = \frac{1}{n} d(\delta_{x_0}, \delta_{x_n}) = \frac{1}{n} d(x_0, x_n). \tag{5.4}$$

So the result follows from (5.3) and (5.4) by the triangle inequality. □

If f is a continuous map on X, we define

$$\sigma_n(x, f) = \sigma(x, f(x), \dots, f^{n-1}(x)) = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)}. \tag{5.5}$$

By Lemma 5.1, $\sigma_n(x, f)$ is a $\frac{1}{n}d(x, f^n(x))$ invariant measure for f. The set of limit points of the sequence $\{\sigma_n(x, f) : n = 1, 2, \dots\}$ in $P(X)$ is denoted $M(x, f)$. It is a nonempty, compact connected set of invariant measures for f, all of whose supports lie in the omega set $\omega f(x)$ (see, e.g., Akin (1993), Propositions 8.3 and 8.8). We call x a *convergence point* for f when $M(x, f)$ consists of a single measure, then denoted μ_x. So x is a convergence point exactly when the sequence $\{\sigma_n(x, f)\}$ converges in $P(X)$. The limit is then μ_x. For example, if x is a periodic point then it is a convergence point. In fact,

$$f^n(x) = x \Rightarrow \mu_x = \sigma_n(x, f). \tag{5.6}$$

5.2. Lemma. *Let $\{x_0, \ldots, x_{n-1}\}$ and $\{y_0, \ldots, y_{N-1}\}$ be sequences in X and let $\epsilon > 0$. Assume there is an injection $g : \{0, \ldots, n-1\} \to \{0, \ldots, N-1\}$ such that*

$$d(x_i, y_{g(i)}) \leq \frac{\epsilon}{2}, \qquad i = 0, \ldots, n-1, \tag{5.7}$$

and that with D the diameter of the space $P(X)$, with respect to the Hutchinson metric,

$$\frac{N-n}{N} \cdot D < \frac{\epsilon}{2}. \tag{5.8}$$

Then we have

$$d(\sigma(x_0, \ldots, x_{n-1}), \sigma(y_0, \ldots, y_{N-1})) < \epsilon. \tag{5.9}$$

Proof. Let $\tilde{\sigma} = \sigma(y_{g(0)}, \ldots, y_{g(n-1)}) \in P(X)$. From (5.7) we have

$$d(\sigma(x_0, \ldots, x_{n-1}), \tilde{\sigma}) \leq \frac{\epsilon}{2}.$$

Now call the image set of g in $\{0, \ldots, N-1\}$ the set of *good values* and the rest of $\{0, \ldots, N-1\}$ the set of *bad values.* Clearly,

$$\sigma(y_0, \ldots, y_{N-1}) = \frac{1}{N}(\sum_{j \text{ bad}} \delta_{y_j} + n\tilde{\sigma}). \tag{5.10}$$

If we replace each δ_{y_j} (j bad) by $\tilde{\sigma}$ we change the right side to $\tilde{\sigma}$. Hence,

$$d(\tilde{\sigma}, \sigma(y_0, \ldots, y_{N-1})) \leq \frac{1}{N} \sum_{j \text{ bad}} d(\tilde{\sigma}, \delta_{y_j}), \tag{5.11}$$

which is less than $\frac{\epsilon}{2}$ by (5.8). From the triangle inequality, (5.9) follows. □

5.3. Theorem. *Assume that the closed relation F on X is chain transitive, i.e. $\mathcal{C}F = X \times X$. Let $x \in X$. Let X_F be the sample path space with s_F the restriction to X_F of the shift homeomorphism s on $X^{\mathbf{Z}}$.*

(a) For a measure $\nu \in P(X^{\mathbf{Z}})$ the following conditions are equivalent:

(1) ν is an invariant sample path measure, i.e. $|\nu| \subset X_F$ and $s_{F*}(\nu) = \nu$.

(2) There exists $\xi \in X^{\mathbf{Z}}$, an asymptotic chain for F, such that ξ is a convergence point for the shift s and $\mu_\xi = \nu$.

(3) For every $\epsilon > 0$ there exists $\xi \in X^{\mathbf{Z}}$, a periodic ϵ chain for F, such that $\xi_0 = x$ and $d(\mu_\xi, \nu) \leq \epsilon$.

(4) For every $\epsilon > 0$ and positive integer N there exist $\xi \in X^{\mathbf{Z}}$, an ϵ chain for F, and $n \geq N$ such that $d(\sigma_n(\xi, s), \nu) \leq \epsilon$.

(b) For a measure $\mu \in P(X)$ the following conditions are equivalent:

(1) μ is an invariant measure for F, i.e. $\mu \in P_F(X)$.

(2) There exists $\xi \in X^{\mathbf{Z}}$, an asymptotic chain for F, such that

$$\mu = \text{Lim}_{n \to \infty} \sigma(\xi_0, \ldots, \xi_{n-1}).$$

(3) For every $\epsilon > 0$, there exists an ϵ chain $\{x_0, \ldots, x_n\}$ for F with $x_0 = x_n = x$ and such that $d(\sigma(x_0, ..., x_{n-1}), \mu) \leq \epsilon$.

(4) For every $\epsilon > 0$ and positive integer N there exists an ϵ chain $\{x_0, \ldots, x_n\}$ with $n \geq N$ and such that $d(\sigma(x_0, \ldots, x_{n-1}), \mu) \leq \epsilon$.

Proof. Notice first that since all of X consists of a single F basic set, $F(X) = X = F^{-1}(X)$ by Lemma 4.1.

(a) We prove first that conditions (2), (3) and (4) are equivalent, and then that the set of measures satisfying these conditions is exactly the s_F invariant measures on X_F.

(2) $\Rightarrow$ (4): Let ξ be an asymptotic chain which is a convergence point for s with $\mu_\xi = \nu$. Given $\epsilon > 0$, there exists N_0 such that $d(\xi_{i+1}, F(\xi_i)) < \epsilon$ for $i \geq N_0$. Thus, $\{\xi_i : i \geq N_0\}$ is an ϵ chain for F. Let $\eta \in X^{\mathbf{Z}}$ be an extension of this sequence, i.e. $\eta_i = \xi_i$ for $i \geq N_0$ and $\eta_i \in F(\eta_{i-1})$ for $i \leq N_0$. Then η is an ϵ asymptotic chain for F, and, since $\eta_i = \xi_i$ for $i \geq N_0$,

$$\mathrm{Lim}_{k\to\infty} d(s^k(\eta), s^k(\xi)) = 0, \tag{5.12}$$

i.e. the orbit sequences for ξ and η are asymptotic in $X^{\mathbf{Z}}$. It easily follows that

$$\mathrm{Lim}_{k\to\infty} d(\sigma_k(\eta, s), \sigma_k(\xi, s)) = 0, \tag{5.13}$$

and so η is a convergence point for s with $\mu_\eta = \mu_\xi = \nu$. Since the sequence $\{\sigma_n(\eta, s)\}$ converges to ν, condition (4) is clear.

(4) $\Rightarrow$ (3): Given $\epsilon > 0$, we can use compactness and $F(X) = X$ to choose a finite subset K such that K and $F(K)$ are ϵ dense subsets of X, i.e. for every $y \in X$ there exist $z_0, z_1 \in K$ such that $d(z_0, y) < \epsilon$ and $d(\tilde{z}_1, y) < \epsilon$ for some $\tilde{z}_1 \in F(z_1)$. For each point z of K choose an ϵ chain from z to x and an ϵ chain from x to z, and let L be the maximum of the lengths of these chains. Let D be the diameter of $P(X^{\mathbf{Z}})$. Choose positive integers N_0 and N_1 so that

$$\begin{gathered} \frac{1}{N_0} < \frac{\epsilon}{4}, \quad N_1 > N_0, \\ \frac{2(N_0 + L + 1)}{N_1} D < \frac{\epsilon}{4}. \end{gathered} \tag{5.14}$$

By (4) there exist an $\epsilon/2$ chain $\eta \in X^{\mathbf{Z}}$ for F and $n \geq N_1$ such that $d(\sigma_n(\eta, s), \nu) < \epsilon/2$. By definition of K we can choose z^-, $z^+ \in K$ such that $d(F(z^-), \eta_{-N_0}) < \epsilon$ and $d(F(\eta_{n+N_0-1}), z^+) < \epsilon$. Let $\{x_0^-, \ldots, x_{l_-}^-\}$ and $\{x_0^+, \ldots, x_{l_+}^+\}$ be ϵ chains with l_-, $l_+ \leq L$, $x_0^- = x_{l_+}^+ = x$, $x_{l_-}^- = z^-$ and $x_0^+ = z^+$. Now we build a periodic chain $\tilde{\eta}$ as follows: Let $\tilde{\eta}_i = \eta_i$ for $i \in [-N_0, n + N_0 - 1]$. For $i \in [-N_0 - l_- - 1, -N_0 - 1]$ let $\tilde{\eta}_i$ be a shift of the x^- chain from x to z_-. For $i \in [n + N_0, n + N_0 + l_+]$ let $\tilde{\eta}_i$ be a shift of the x^+ chain from z_+ to x. In particular,

$$\tilde{\eta}_{-N_0-l_--1} = x = \tilde{\eta}_{n+N_0+l_+}, \tag{5.15}$$

and so we can extend $\tilde{\eta}$ to define an N_3 periodic ϵ chain for F, where $N_3 = n + 2N_0 + l_- + l_+ + 1$. Because $N_0^{-1} < \epsilon/4$ we have from (1.14)

$$d(s^i(\eta), s^i(\tilde{\eta})) < \epsilon/4 \text{ for } i = 0, \ldots, n-1. \tag{5.16}$$

Now we use Lemma 5.2 to compare the sequences $\{s^i(\eta) : i \in [0, n-1]\}$ and $\{s^i(\tilde{\eta}) : [0, N_3 - 1]\}$ with $g(i) = i$ for $i \in [0, n-1]$. Since $N_3 - n \leq 2(N_0 + L + 1)$, (5.16) and (5.14) imply (5.7) and (5.8) respectively. By periodicity of $\tilde{\eta}$, the lemma implies $d(\sigma_n(\eta, s), \mu_{\tilde{\eta}}) < \epsilon/2$. So $d(\nu, \mu_{\tilde{\eta}}) < \epsilon$. To obtain ξ required by (3) shift $\tilde{\eta}$ to move x to position 0.

(3) $\Rightarrow$ (2): For $k = 1, 2, \ldots$ choose a periodic 2^{-k} chain ξ^k in $X^{\mathbf{Z}}$ with $\xi_0^k = x$ and $d(\mu_k, \nu) \leq 2^{-k}$, where $\mu_k = \mu_{\xi^k}$. So if n_k is a period for ξ^k, i.e. a positive integer such that $\xi_{i+n_k}^k = \xi_i^k$ for all $i \in \mathbf{Z}$, then $\mu_k = \sigma_{n_k}(\xi^k, s)$. For N_k any increasing

sequence of positive integers we can define η to be the extension to an element of $X^{\mathbf{Z}}$ of the sequence $\{\eta_i : i = 0, 1, \dots\}$ defined by first going around the length n_1 initial loop of ξ^1 N_1 times, then the length n_2 initial loop of ξ^2 N_2 times, and so on. Clearly the resulting η is an asymptotic chain. We show that when $\{N_k\}$ is increasing fast enough, η is a convergence point for s with $\mu_\eta = \nu$.

Since any multiple of a period is a period, we can assume that $n_k > 2^{k+1}$ for $k = 1, 2, \dots$. Now let $N_1 \geq 4(2n_1 + 2n_2)$ and inductively choose N_k $(k > 1)$ so that

$$\frac{(\sum_{i=1}^{k-1} n_i N_i) + 2n_k + 2n_{k+1}}{N_k} \leq \frac{1}{2} \cdot 2^{-k}. \tag{5.17}$$

Now suppose N is between $\sum_{i=1}^{k} n_i N_i$ and $\sum_{i=1}^{k+1} n_i N_i$. We regard the "bad values" of $\{0, \dots, N-1\}$ as all those up to $\sum_{i=1}^{k-1} n_i N_i$, the first and last loops of ξ^k, the ξ^{k+1} loop in which N occurs and, if this one is not initial, the initial ξ^{k+1} loop as well. For each remaining, "good", value j, $s^j(\eta)$ on the interval $[-n_k, n_k]$ looks like one of the $N_k - 2$ interior loops of ξ^k or else like a piece of some loops of ξ^{k+1}. So by Lemma 5.2, $\sigma_N(\eta, s)$ is at most 2^{-k} away from some convex combination of μ_k and μ_{k+1}. Each of these is at most 2^{-k} away from ν, and so $\sigma_N(\eta, s)$ is at most $2^{-(k-1)}$ away from ν. Thus the sequence $\{\sigma_N(\eta, s)\}$ converges to ν as $N \to \infty$.

This completes the proof that conditions (2), (3) and (4) on $\nu \in P(X^{\mathbf{Z}})$ are equivalent. If $\nu = \mu_\xi$ with ξ a convergence point for s, then ν is s invariant and $|\nu| \subset \omega s(\xi)$. So if ξ is an asymptotic chain, then $|\nu| \subset X_F$ by Corollary 4.4. Thus, such measures are s_F invariant measures, i.e. (1) holds for such ν's. If ν is an ergodic invariant measure for s_F then by the Birkhoff Ergodic Theorem there exists $\xi \in X_F$, a convergence point for s_F, with $\mu_\xi = \nu$. In fact, the set of such ξ's has ν measure 1. So an ergodic s_F invariant measure certainly satisfies condition (2). Every invariant measure is a weighted average of the ergodic measures. That is, the set of s_F invariant measures is the smallest closed, convex subset of $P(X_F)$ which contains the ergodic measures. Clearly, the set of measures in $P(X_F)$ which satisfy (3) or (4) is a closed set. We complete the proof by showing that the set of measures satisfying (3) is convex. By closure it is enough to show that if ν^1, ν^2 satisfy (3) then the midpoint $\frac{1}{2}\nu^1 + \frac{1}{2}\nu^2$ satisfies (3).

Given $\epsilon > 0$, choose periodic $\epsilon/2$ chains ξ^1, ξ^2 for F in $X^{\mathbf{Z}}$ with $\xi_0^1 = \xi_0^2 = x$ and $d(\mu_{\xi^i}, \nu^i) \leq \frac{\epsilon}{2}$ for $i = 1, 2$. By multiplying a period of ξ^1 times a period of ξ^2 and then multiplying by some large positive integer we can choose a positive integer N_0, a common period for ξ^1 and ξ^2, which satisfies $N_0^{-1} < \frac{\epsilon}{4}$. With D the diameter of $P(X^{\mathbf{Z}})$ choose N_1 a positive integer such that

$$\frac{2}{N_1} D = \frac{4N_0}{2N_1 \cdot N_0} D < \frac{\epsilon}{4}. \tag{5.18}$$

Now let ξ be the sequence in $X^{\mathbf{Z}}$ of period $2N_1 \cdot N_0$ consisting of N_1 copies of the initial loop of ξ^1 of length N_0 followed by N_1 copies of the initial loop of ξ^2 of length N_0. As usual we apply Lemma 5.2; the "bad values" are the first and last loops of ξ^1 and of ξ^2, a total of $4N_0$ out of $2N_1 N_0$. For each of the remaining values j, $s^j(\xi)$ looks on the interval $[-N_0, N_0]$ like one of the ξ^1 loops ($(N_1 - 2)N_0$ values of j) or like one of the ξ^2 loops (same number of values of j). So by Lemma 5.2 we have $d(\mu_\xi, \frac{1}{2}\mu_{\xi^1} + \frac{1}{2}\mu_{\xi^2}) < \frac{\epsilon}{2}$, and by convexity properties of the Hutchinson metric we have $d(\mu_\xi, \frac{1}{2}\nu^1 + \frac{1}{2}\nu^2) < \epsilon$.

(b) (1) ⇒ (2), (3), (4): If μ is an F invariant measure then by Theorem 3.2 there is an s_F invariant measure ν on X_F such that $\mu = \pi_{0^*}(\nu)$. Apply (2), (3) and (4) from (a) to ν and project by π_{0^*} to get (2), (3) and (4) respectively for μ.

(2) ⇒ (4) and (3) ⇒ (4): Obvious.

(4) ⇒ (1): By using (4) and extending to sequences in $X^{\mathbf{Z}}$ we can choose for $k = 1, 2, \dots$ a $1/k$ chain $\xi^k \in X^{\mathbf{Z}}$ and $n_k > k$ such that $d(\sigma(\xi_0^k, \dots, \xi_{n_k-1}^k), \mu) \leq \frac{1}{k}$. By going to a subsequence we can assume that $\{\sigma_{n_k}(\xi^k, s) : k = 1, 2 \dots\}$ converges to some measure $\nu \in P(X^{\mathbf{Z}})$. As ν satisfies (4) of part (a), it is an s_F invariant measure on X_F. Since $\pi_{0^*}(\sigma_{n_k}(\xi^k, s)) = \sigma(\xi_0^k, \dots, \xi_{n_k-1}^k)$, we have $\pi_{0^*}(\nu) = \mu$. Thus, μ satisfies (1). □

5.4. Corollary. *For a closed relation F on X, let X_F be the sample path space for F with s_F the restriction to X_F of the shift homeomorphism s on $X^{\mathbf{Z}}$ and with $\pi_0 : X^{\mathbf{Z}} \to X$ the time zero projection.*

(a) For a measure $\nu \in P(X^{\mathbf{Z}})$ the following conditions are equivalent:

(1) There exists a basic set $\tilde{B} \subset X_F$ for s_F such that ν is an invariant sample path measure supported by $\tilde{B}$, i.e. $|\nu| \subset \tilde{B}$ and $(s_F)_*(\nu) = \nu$.

(2) There exists $\xi \in X^{\mathbf{Z}}$, an asymptotic chain for F, such that ν is among the limit measures associated with ξ, i.e. ν is a limit point for the sequence $\{\sigma_n(\xi, s)\}$ in $P(X^{\mathbf{Z}})$.

(3) For every $\epsilon > 0$ there exists $\xi \in X^{\mathbf{Z}}$, a periodic ϵ chain for F, such that $d(\mu_\xi, \nu) \leq \epsilon$.

Every ergodic measure for $s_F : X_F \to X_F$ satisfies these conditions. Every invariant sample path measure can be decomposed as an average of such measures.

(b) For a measure $\mu \in P(X)$ the following conditions are equivalent:

(1) There exists a basic set $B \subset X$ for F such that μ is an invariant measure for F supported by B, i.e. $|\mu| \subset B$ and $(\mu, \mu) \in F_*$.

(2) There exist a basic set $\tilde{B} \subset X_F$ for s_F and an invariant sample path measure ν supported by $\tilde{B}$ such that $(\pi_0)_*(\nu) = \mu$.

(3) There exists an asymptotic chain $\{x_0, x_1, \dots\}$ for F such that μ is a limit point for the sequence $\{\sigma(x_0, \dots, x_{n-1})\}$ in $P(X)$.

(4) For every $\epsilon > 0$ there exists an ϵ chain $\{x_0, \dots, x_n\}$ for F such that $x_0 = x_n$, and $d(\sigma(x_0, \dots, x_{n-1}), \mu) \leq \epsilon$.

Every extreme point of the compact convex set of invariant measures for F is the image under π_{0^*} of some ergodic measure for s_F, and so satisfies these conditions. Every invariant measure for F can be decomposed as the average of such measures.

Proof. (a) Applying Theorem 4.3 to the restriction of F to $\pi_0(X_F)$, we see that a basic set $\tilde{B}$ for s_F is just $X_F \cap B^{\mathbf{Z}}$, where B is the F basic set $\pi_0(\tilde{B})$. So if ν satisfies (1) then we can apply Theorem 5.3a to F_B because the basic set B is a chain transitive subset (see Akin (1993), Theorem 4.5). We get asymptotic chains in B with limit measure ν and periodic chains in B whose measures approximate ν. Thus, (1) ⇒ (2) and (1) ⇒ (3) are clear.

(2) ⇒ (1): Each limit measure is s invariant, and each is supported by $\omega s(\xi)$, which is contained in an s_F basic set by Corollary 4.4.

(3) ⇒ (1): If $\{\xi^k\}$ is a sequence of $1/k$ chains in $X^{\mathbf{Z}}$ with period n_k such that $\{\mu_{\xi^k}\}$ converges to ν, then, as the limit of s invariant measures, ν is s invariant. By going to a subsequence if necessary we can assume that the sequence of supports of μ_{ξ^k}, the periodic orbits, converges to a set A in the space of closed subsets of $X^{\mathbf{Z}}$

equipped with the Hausdorff metric. By Akin (1993), Proposition 8.1, the support of the limit ν is contained in the limit of the supports, A. By Akin (1993), Exercise 7.37d A, is a chain transitive subset for s. Finally, it follows from Lemma 4.2a that $A \subset X_F$. So A is contained in some basic set for s_F by chain transitivity.

The support of any ergodic measure for s_F is contained in some topologically transitive subset and so is certainly contained in some basic set (see, e.g. Akin (1993), Corollary 8.9). For any invariant measure ν the ergodic decomposition expresses ν as an average (i.e. an integral of a measure on the set) of ergodic measures.

(b) (1) $\Rightarrow$ (2): If $\mu \in F_*$ and $|\mu| \subset B$, then there exists a measure $\tilde{\mu}$ on $X \times X$ with $|\tilde{\mu}| \subset F$ and $\pi_{1*}(\tilde{\mu}) = \mu = \pi_{2*}(\tilde{\mu})$. Hence, $\tilde{\mu}(B \times X) = \tilde{\mu}(X \times B) = 1$, and so $|\tilde{\mu}| \subset F \cap (B \times B)$. Thus, μ is an invariant measure for F_B. By Theorem 3.2 there exists an invariant measure ν on X_{F_B} such that $(\pi_0)_*(\nu) = \mu$. By Theorem 4.3d applied to F_B, X_{F_B} is a chain transitive subset and so is contained in (and in fact equals) a basic set $\tilde{B}$ for s_F.

(2) $\Rightarrow$ (3),(4): ν satisfies (1) of part (a). Applying (2) and (3) of part (a) and projecting by π_{0*}, we get (3) and (4) respectively.

(3) $\Rightarrow$ (1): By Lemma 5.1 any limit measure of the sequence is the limit of measures ϵ invariant for F for any $\epsilon > 0$ and so is an F invariant measure. (To be precise you apply Lemma 5.2 to ignore the bad values, after which the sequence is an $\epsilon/2$ chain.) The support of μ is contained in the set of limit points of the sequence $\{x_0, x_1, \dots\}$, which is, by the proof of Corollary 4.4, a chain transitive subset contained in some F basic set.

(4) $\Rightarrow$ (2): There is a sequence $\{\xi^k\}$ of $1/k$ chains in $X^{\mathbf{Z}}$ with period n_k such that $\pi_{0*}(\mu_{\xi^k})$ converges to μ. By going to a subsequence we can assume that $\{\mu_{\xi^k}\}$ converges to $\nu \in P(X^{\mathbf{Z}})$, which satisfies (3) and hence (1) of (a). As $\mu = \pi_{0*}(\nu)$, μ satisfies (2).

(2) $\Rightarrow$ (1): Obvious, since $\pi_0(\tilde{B})$ is an F basic set.

If μ is an extreme point for the compact convex set $P_F(X)$ then by Theorem 3.2 the set

(5.19)
$$C_\mu = \{\nu : \pi_{0*}(\nu) = \mu \text{ and } \nu \text{ is an } s_F \text{ invariant measure on } X_F\}$$

is nonempty. It is clearly compact and convex, and so contains extrema. Such an extremum ν is an extremum for the entire set of invariant measures. (A segment containing ν maps to a segment containing μ. As μ is extreme, the image segment is constantly μ and so the original segment lies in C_μ. As ν is an extremum of C_μ, the segment is constantly ν.) Thus, ν is ergodic.

Any element of a compact convex set is an average of the extrema. □

If $F = f$ is a continuous map and μ is an extreme point of $P_f(X)$, then μ is an ergodic measure satisfying Birkhoff's Theorem: μ a.e. x of X is a convergence point with $\mu_x = \mu$. The analogue for a closed relation F on X uses the set $C_\mu \subset P_{s_F}(X_F)$ associated to $\mu \in P_F(X)$ by (5.19). If μ is an extreme point of $P_F(X)$, then, for any $\nu \in C_\mu$, ν a.e. $\xi \in X_F$ is a convergence point for s_F for which $(\pi_0)_*\mu_\xi = \mu$.

5.5. Theorem. *For a closed relation F on X let μ be an extreme point of the compact convex set $P_F(X)$ of F invariant measures. Let C_μ be the compact convex subset of $P(X_F)$ consisting of s_F invariant measures ν such that $(\pi_0)_*\nu = \mu$. The* *set of convergence points ξ for s_F such that $\mu_\xi \in C_\mu$ has ν measure 1 for every $\nu \in C_\mu$.*

For every $\nu \in C_\mu$ and $g \in L^1(X,\mu)$,

$$\mathrm{Lim}_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(\xi_k) = \int_X g d\mu, \tag{5.20}$$

for ν a.e. $\xi \in X_F$. In particular, the asymptotic fraction of times a $\nu \in C_\mu$ typical sample path 'visits' a Borel set $A \subset X$ is given by $\mu(A)$.

Proof. Integrating over the set of s_F convergence points, we see from the Ergodic Decomposition Theorem that for any $\nu \in C_\mu$

$$\nu = \int \mu_\xi \nu(d\xi). \tag{5.21}$$

Applying the continuous linear map $(\pi_0)_*$, we get

$$\mu = (\pi_0)_*(\nu) = \int (\pi_0)_*(\mu_\xi)\nu(d\xi). \tag{5.22}$$

Because μ is an extreme point it follows that the measures $(\pi_0)_*(\mu_\xi)$ in $P_F(X)$ are equal to μ for ν a.e. ξ. That is, $\mu_\xi \in C_\mu$ for ν a.e. ξ. The L^1 result follows similarly. □

Acknowledgement

The authors are grateful to F. Y. Hunt for bringing the paper Aubin, Frankowska and Lasota (1991) to our attention.

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