TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 4, April 1999, Pages 1531–1550
S 0002-9947(99)02087-5

SUPPORTS OF DERIVATIONS, FREE FACTORIZATIONS, AND RANKS OF FIXED SUBGROUPS IN FREE GROUPS

GEORGE M. BERGMAN

ABSTRACT. For F a free group of finite rank, it is shown that the fixed subgroup of any set B of endomorphisms of F has rank $\leq \mathrm{rank}(F)$, generalizing a recent result of Dicks and Ventura. The proof involves the combinatorics of derivations of groups. Some related questions are examined.

INTRODUCTION

Suppose G and L are groups, $G * L$ their coproduct (often called their free product), and

$$f:\ G * L \to G \tag{1}$$

the homomorphism which acts as the identity on G and annihilates L. We shall show that if G is finitely generated and torsion-free, then for any two *sections* (right inverses) of f,

$$\sigma_1,\, \sigma_2:\ G \to G * L,$$

the equalizer

$$\{g \in G \mid \sigma_1(g) = \sigma_2(g)\}$$

is a free factor in G (a member of a coproduct decomposition).

Now every surjective homomorphism of *free* groups can be written in the form (1). Using the above free factorization result in this case, and a recent result of Dicks and Ventura [7] which states that the fixed subgroup of any family of *injective* endomorphisms of a free group F has rank $\leq \mathrm{rank}(F)$, we shall show in §5 that the latter result also holds without the injectivity hypothesis.

How is the assertion that an equalizer of sections of (1) is a free factor proved? The idea is roughly as follows. Letting N denote the kernel of (1), any two sections σ_1, σ_2 are determined by *derivations* (definition recalled in §1) d_1, d_2 from G into N as a group with G-action. Now N is a coproduct of conjugates ${}^{g}L$ of L, which are permuted freely under the action of G. Let us define the "support" of an element $x \in N$ to be the set of $g \in G$ such that members of ${}^{g}L$ occur in the expression for x. In §§2-4 below, we shall examine the combinatorics of the function δ that associates to each element $g \in G$ the support of $d(g)$, for d a derivation $G \to N$; or more precisely, we shall study a class of functions δ from groups G to subsets of G-sets S, to which the "support functions" associated with

Received by the editors April 5, 1996 and, in revised form, April 8, 1997.
1991 *Mathematics Subject Classification.* Primary 20E05, 20E06, 20J05; Secondary 05E20, 20C07, 20E08.
This work was done while the author was partly supported by NSF contract DMS 93-03379.

such derivations belong. This class also includes the support functions obtained by looking at derivations from G to the additive group of the group algebra $k\,G$, and in fact we shall see that every function δ in our class arises from a derivation of the latter sort. Hence, known results on the kernels of such additive derivations (or equivalently, the combinatoric facts from which those results are proved) become applicable to kernels and equalizers of derivations $G \to N$, allowing us to obtain the asserted free factorization.

§§6-10 contain related results, examples, and open questions.

I am greatly indebted to Warren Dicks for his part in an extensive e-mail discussion which culminated in these results – in particular, for twice nipping incorrect arguments in the bud, and for acquainting me with relevant results in the literature. I am also grateful to the referee for several useful suggestions.

1. Background on derivations

Throughout this note, group actions will mean *left* actions, and when a group G acts on a set X, we shall write the image of $x \in X$ under $g \in G$ as ${}^g x$.

Suppose G is a group, and N a group on which G acts by automorphisms. We recall

Definition 1. *A left derivation* (*henceforth simply a* derivation) *from G to N means a set-map $d\colon G \to N$ such that for all $g, h \in G$,*

$$d(gh) \ = \ d(g)\ {}^g d(h). \tag{2}$$

Immediate consequences of (2) are

$$d(1) \ = \ 1, \tag{3}$$

$$d(g^{-1}) \ = \ {}^{g^{-1}} d(g)^{-1}. \tag{4}$$

By a *section* of a (surjective) set-map f we shall mean a right inverse to f. When we speak of a section of a group homomorphism, this will be understood to be a homomorphism as well, unless we indicate the contrary by calling it a "set-theoretic section". Occasionally we will emphasize that a homomorphism is meant by writing "group-theoretic section".

Now suppose N is a group on which G acts by automorphisms, let $N \rtimes G$ be the semidirect product determined by this action, and let $f : N \rtimes G \to G$ be the canonical projection $(x, g) \mapsto g$. Thus, a set-theoretic section σ of this map has the form $\sigma(g) = (d(g), g)$ for some set-map $d\colon G \to N$. An easy calculation shows that σ is a group homomorphism if and only if d is a derivation; thus, derivations correspond to sections of these canonical projections.

For any $x \in N$, the map $d_x\colon G \to N$ given by

$$d_x(g) \ = \ x\,({}^g x)^{-1} \tag{5}$$

is a derivation, called the *inner* derivation determined by x. This corresponds to the homomorphism $G \to N \rtimes G$ obtained by conjugating the *trivial* section, $g \mapsto (1, g)$, by the element $(x, 1)$.

Recall also that if E, G are groups and $f\colon E \to G$ is a homomorphism admitting a group-theoretic section $\sigma\colon G \to E$, then E can be written as a semidirect product of $N = \ker(f)$ and G. Indeed, if we let G act on N by

$${}^g x \ = \ \sigma(g)\,x\,\sigma(g)^{-1} \qquad (g \in G,\ x \in N), \tag{6}$$

and form the semidirect product $N \rtimes G$ based on this action, then the map $N \rtimes G \to E$ given by $(x, g) \mapsto x\,\sigma(g)$ is an isomorphism. Moreover, when we use this isomorphism to identify E with $N \rtimes G$, the given map f becomes the canonical projection $(x, g) \mapsto g$, and σ becomes the trivial section $g \mapsto (1, g)$.

Hence if σ_1 and σ_2 are two sections of the above homomorphism f, and we decompose E as a semidirect product using σ_1, and let G act on N via conjugation by elements of $\sigma_1(G)$, then σ_2 is determined by a derivation of G into N, given by $d(g) = \sigma_2(g)\,\sigma_1(g)^{-1} \in N$. In this sense, whenever σ_1 and σ_2 are two sections of a group homomorphism, the map $g \mapsto \sigma_2(g)\,\sigma_1(g)^{-1}$ is a derivation.

However, we may have some other distinguished section σ_0 of f, and have reasons for considering G to act on N via conjugation by elements of $\sigma_0(G)$. Then the map defined by $d(g) = \sigma_2(g)\,\sigma_1(g)^{-1}$ will *not* in general be a derivation with respect to this preferred action. This subtle point will prove important in §§3-4 below.

2. Derivation support functions

As indicated in the Introduction, there are several constructions involving derivations which lead to the concept which we shall call a "derivation support function". We will motivate the definition below using the simplest of these. Throughout this section, G will be a group, and S a (left) G-set.

If L is any other group, and we form the group $N = L^S$ of L-valued functions on S with componentwise operations, then G acts on N by translation automorphisms, described by

$$({}^g x)(s) \;=\; x\,({}^{g^{-1}}s) \qquad (x \in N = L^S,\ g \in G,\ s \in S). \tag{7}$$

Now if d is any derivation $G \to N$ with respect to this action, and for each $g \in G$ we let $\delta(g)$ denote the *support* of $d(g)$ in S, i.e., the set of $s \in S$ such that the sth coordinate of $d(g)$ is $\neq 1$, then from (3), (4) and (2) respectively it is easy to deduce that

$$\delta(1) \;=\; \varnothing, \tag{8}$$

$$\delta(g^{-1}) \;=\; {}^{g^{-1}}\delta(g), \tag{9}$$

$$\delta(gh) \;\subseteq\; \delta(g) \cup {}^g\delta(h). \tag{10}$$

Let us note some consequences of (8)-(10). If in (10) we put gh in place of g and h^{-1} in place of h, and use (9) to evaluate $\delta(h^{-1})$, we get $\delta(g) \subseteq \delta(gh) \cup {}^g\delta(h)$, which we can rewrite as $\delta(g) - {}^g\delta(h) \subseteq \delta(gh)$ (where "$-$" denotes relative complement). If, instead, we put gh for h and g^{-1} for g in (10), manipulate the resulting relation in the same way, and apply ${}^g(\ )$ to the result, we get ${}^g\delta(h) - \delta(g) \subseteq \delta(gh)$. Writing ∇ for symmetric difference of sets, the conjunction of these two inclusions says that

$$\delta(g)\ \nabla\ {}^g\delta(h) \;\subseteq\; \delta(gh). \tag{11}$$

Note that (10) and (11) sandwich $\delta(gh)$ between two Boolean expressions in $\delta(g)$ and ${}^g\delta(h)$. This double inclusion can be expressed as saying that every element of $\delta(gh) \cup \delta(g) \cup {}^g\delta(h)$ belongs to at least two terms of this union; we may put this in a slightly more elegant form by writing $gh = f^{-1}$, applying (9) to $\delta(f^{-1})$ in the resulting statement, and then applying ${}^{f}(\ )$ to the whole formula; getting

$$\text{If } f\,g\,h = 1 \text{ in } G, \text{ then every element of} \\ \delta(f) \cup {}^{f}\delta(g) \cup {}^{fg}\delta(h) \\ \text{belongs to at least two terms of this union.} \tag{12}$$

The above arguments apply not only to a function δ obtained as above from a derivation, but to any function $G \to \mathbf{P}(S)$ (where $\mathbf{P}(S)$ denotes the power set of S) satisfying (8), (9) and (10). Examining our arguments more closely, we get

Lemma 2. *For any function* $\delta\colon G \to \mathbf{P}(S)$ *satisfying* (8)*, we have*

$$(9) \wedge (10) \ \Leftrightarrow\ (11) \ \Leftrightarrow\ (12).$$

Proof. Let us first show that in the presence of (8) *and* (9), we have (10) $\Leftrightarrow$ (11) $\Leftrightarrow$ (12). We have proved the rightward half of the first implication, and our proof thereof really showed that (11) was equivalent to the conjunction of two inclusions, each of which was equivalent to (10); hence we also have the reverse implication. We also showed that the conjunction of (10) and (11) was equivalent to (12); this now gives the second double implication.

It remains only to show that in the presence of (8), each of (11) and (12) implies (9). We get the first of these implications by taking $h = g^{-1}$ in (11), and the second by taking $f = 1$ in (12), in each case using (8) to evaluate $\delta(1)$. $\square$

Let us make

Definition 3. *A map* $\delta\colon G \to \mathbf{P}(S)$ *satisfying* (8)-(10) (*and hence also* (11) *and* (12)) *will be called a* derivation support function *from* G *to subsets of* S*.*

Returning to the case where δ is obtained from a derivation $d : G \to L^S$, let us note that if d is an inner derivation d_x, then $\delta(g)$ can be described as $\{s \in S \mid x({}^{g^{-1}}s) \neq x(s)\}$ (cf. (5) and (7)). Thus, if we define an equivalence relation on S by letting $s \sim t$ if and only if $x(s) = x(t)$, we have

$$s \in \delta(g) \ \Leftrightarrow\ {}^{g^{-1}}s \not\sim s. \tag{13}$$

Conversely, given any equivalence relation $\sim$ on S, it is immediate that (13) describes a derivation support function. (To see this without further calculation, take any group L whose cardinality is at least that of $S/{\sim}$, let x be an element of L^S such that $x(s) = x(t)$ if and only if $s \sim t$, and note that the derivation support function determined by d_x is precisely the function δ given by (13).)

It is natural to ask whether every derivation support function on a G-set S arises as in (13) from an equivalence relation on S. It turns out that this is not true in general (though we shall soon show that it holds when S is a *free* G-set). To see how derivation support functions can arise which are not of that form, let us consider a modification of the way we have made G act on the group $N = L^S$. If we assume some action of G on L by automorphisms is given initially, we may incorporate this into its action on L^S, and in place of (7) define

$$({}^{g}x)(s) \ =\ {}^{g}(x({}^{g^{-1}}s)) \qquad (g \in G,\ x \in L^S,\ s \in S). \tag{14}$$

A derivation $d\colon G \to N$ with respect to this action is easily seen to induce, exactly as before, a derivation support function $\delta : G \to \mathbf{P}(S)$, and when d is an inner derivation d_x this function still takes $g \in G$ to $\{s \in S \mid ({}^{g}x)(s) \neq x(s)\}$ (cf. (5)); but in view of (14), to tell whether, for a given s, one has $({}^{g}x)(s) = x(s)$ we must now not only compare the values of x at the two points s and ${}^{g^{-1}}\!s$, but also know the element g by which we go from one point to the other. This is too much information to capture in an equivalence relation on S. But we can capture it with a slightly more sophisticated construction.

Recall that a *groupoid* is a category in which every morphism is invertible. If S is a G-set, let us define the groupoid $S_{\mathbf{cat}}$ to have for objects the elements of S, and for morphisms $s \to t$ all 3-tuples (t,g,s) with $g \in G$ such that $t = {}^g s$. (This notation is redundant, but suggestive: (t,g,s) is "g used as a way of getting to t from s".) Composition, identity morphisms, and inverses are defined by

$$(t,g,s)(s,h,u) \;=\; (t,gh,u),$$

$$1_s \;=\; (s,1,s),$$

$$(t,g,s)^{-1} \;=\; (s,\, g^{-1},t).$$

In characterizing derivation support functions on an arbitrary G-set S, our idea is to replace the set of questions "Are things the same at s and t?" $(s,t \in S)$, which would be addressed by an equivalence relation, with the set of questions "Do things look the same if we go from s to t by the element $g \in G$?" $(s \in S,\ g \in G,\ t = {}^g s)$, which is addressed by a *subgroupoid* of $S_{\mathbf{cat}}$. In fact, we have

Lemma 4. *Suppose we establish a bijective correspondence between maps $\delta \colon G \to \mathbf{P}(S)$ and subsets C of the morphism-set of $S_{\mathbf{cat}}$ by the condition*

$$s \in \delta(g) \;\Leftrightarrow\; (s,\, g,\, {}^{g^{-1}}\!s) \notin C. \tag{15}$$

Then δ is a derivation support function if and only if C is the morphism-set of a subgroupoid of $S_{\mathbf{cat}}$ with object-set S.

Proof. To be the morphism-set of a subgroupoid with object-set S, the set C must contain all the identity morphisms of S, and be closed under taking inverses and composites. We claim that these conditions translate, respectively, to conditions (8)-(10) on δ.

The first two equivalences are clear; let us show the last one. Since every member of $S_{\mathbf{cat}}$ is of the form $(s,\, g,\, {}^{g^{-1}}\!s)$, to say that C is closed under composition is to say that if both of $(s,\, g,\, {}^{g^{-1}}\!s)$ and $({}^{g^{-1}}\!s,\, h,\, {}^{h^{-1}g^{-1}}\!s)$ are in C, then so is $(s,\, gh,\, {}^{h^{-1}g^{-1}}\!s)$. The contrapositive statement is that if the latter element does not lie in C, then one of the first two fails to lie in C; equivalently, if $s \in \delta(gh)$, then either $s \in \delta(g)$ or ${}^{g^{-1}}\!s \in \delta(h)$. Rewriting the last inclusion as $s \in {}^g\delta(h)$, we get (10). $\square$

Examples of derivation support functions that do not arise as in (13) from equivalence relations will be noted in §6. At this point, let us apply the above lemma to show that for certain S, all derivation support functions do arise in this way.

Theorem 5. *Let S be a* free *G-set. Then every derivation support function $\delta :$ $G \to \mathbf{P}(S)$ is induced as in* (13) *by an equivalence relation $\sim$ on S.*

In this situation, the restriction of $\sim$ to each G-orbit of S is uniquely determined by δ.

Proof. Given δ, let C be the subgroupoid of $S_{\mathbf{cat}}$ described in the preceding lemma, and let $s \sim t$ mean that C contains a morphism $s \to t$. From the definition of a groupoid, it is clear that this is an equivalence relation. Since S is free, we see that given $g \in G$ and $s \in S$, the element g will be the only element of G carrying

${}^{g^{-1}}s$ to s; this makes (15) equivalent to (13), so $\sim$ does indeed induce δ under (13). The final assertion is clearly true of any derivation support function induced by an equivalence relation $\sim$ via (13), without restriction on S. $\square$

Theorem 5 can be thought of as a combinatorial analog of Hochschild's Theorem, which says that for L an abelian group, every derivation $G \to L^G$ is inner.

For future reference, let us make

Definition 6. *If $\delta : G \to \mathbf{P}(S)$ is a derivation support function, the* kernel *of δ will mean the set*

$$\ker(\delta) \;=\; \{g \in G \mid \delta(g) = \varnothing\}.$$

This is easily seen to be a subgroup of G. When δ arises from a derivation $d\colon G \to L^S$, we see that $\ker(\delta) = \ker(d)$, while in the situation of (13), $\ker(d)$ is the group of elements of G that stabilize all the equivalence classes of $\sim$.

3. General constructions of derivation support functions from derivations

We will now give two more general constructions of derivation support functions from derivations, which we will use together with Theorem 5 in the next section. The reader will note that the first of these covers all the examples we have described and sketched so far. For instance, if $N = L^S$, with G acting by (7), or more generally, as in (14), then for each $s \in S$ we may take U_s in the proposition below to be the subgroup of N consisting of all elements whose support does not contain s. Likewise, if N is the coproduct (free product) of a set of copies of L indexed by S, as discussed in the Introduction, we may take U_s to be the subgroup generated by the copies of L indexed by the elements of $S - \{s\}$.

Proposition 7. *Let G be a group, S a G-set, N a group on which G acts by automorphisms, and U_s $(s \in S)$ a family of subgroups of N indexed by S in such a way that for $g \in G$, $s \in S$ we have ${}^gU_s = U_{({}^gs)}$. Then if $d\colon G \to N$ is a derivation, the function $\delta\colon G \to \mathbf{P}(S)$ defined by*

$$\delta(g) \;=\; \{s \in S \mid d(g) \notin U_s\} \tag{16}$$

is a derivation support function.

Proof. (8) follows from (3), so it suffices to show (12). If $fgh \;=\; 1$, then

$$1 \;=\; d(fg\,h) \;=\; d(f)\,{}^fd(g)\,{}^{fg}d(h);$$

hence if any of the three factors on the right fails to lie in U_s, at least one other must fail to lie in that subgroup. Further, the condition ${}^fd(g) \notin U_s$ can be written $d(g) \notin {}^{f^{-1}}U_s = U_{({}^{f^{-1}}s)}$, i.e., ${}^{f^{-1}}s \in \delta(g)$, equivalently, $s \in {}^f\delta(g)$. In the same way, the conditions $d(f) \notin U_s$ and ${}^{fg}d(h) \notin U_s$ are equivalent to $s \in \delta(f)$ and $s \in {}^{fg}\delta(h)$. Thus the statement that if one of these conditions holds, at least one other must hold is precisely (12). $\square$

We noted at the end of the last section that if δ was the derivation support function obtained, as in that section, from a derivation $d : G \to L^S$, one had $\ker(\delta) = \ker(d)$. In the more general situation of Proposition 7, we see that we will have $\ker(\delta) \supseteq \ker(d)$, with equality if $\bigcap_S U_s = \{1\}$.

As noted in §1, if we view a derivation d as describing a section σ_1 of the projection homomorphism $N \rtimes G \to G$, then $\ker(d)$ is the *equalizer* in G of σ_1 and the trivial section σ_0; that is, $\{g \in G \mid \sigma_0(g) = \sigma_1(g)\}$. If, however, we wish to study the equalizer of an arbitrary pair of sections σ_1 and σ_2 to f, we need to compare *two* derivations d_1 and d_2. It turns out that we can get a derivation support function which measures their difference essentially as above, if our subgroups U_s are normal:

Proposition 8. *Let G be a group, S a G-set, N a group on which G acts by automorphisms, and U_s $(s \in S)$ a family of normal subgroups of N, again indexed in such a way that for $g \in G$, $s \in S$ we have ${}^gU_s = U_{({}^gs)}$. Then if $d_1,\, d_2 : G \to N$ are derivations, the function $\delta : G \to \mathbf{P}(S)$ defined by*

$$\delta(g) \ = \ \{s \in S \mid d_1(g)\ d_2(g)^{-1} \notin U_s\} \tag{17}$$

is a derivation support function.

Proof. As in the proof of Lemma 7, (8) is clear, and we will prove (12). If $fgh = 1$, we have

$$1 \ = \ d_i(fg\,h) = d_i(f)\,{}^f d_i(g)\,{}^{fg} d_i(h) \quad \text{for} \quad i \ = \ 1,\, 2.$$

Thus, taking $s \in S$, if any of the congruences

$$d_1(f) \equiv d_2(f), \qquad {}^f d_1(g) \equiv {}^f d_2(g), \qquad {}^{fg} d_1(h) \equiv {}^{fg} d_2(h) \qquad (\mathrm{mod}\ U_s)$$

fails to hold, at least two must fail. As in the proof of Proposition 7, the failure of each of these congruences is equivalent to one of the conditions $s \in \delta(f)$, $s \in {}^f\delta(g)$, $s \in {}^{fg}\delta(h)$, establishing (12). $\square$

4. Equalizers of sections and free factors

If k is a ring (here always assumed associative with 1), then a left k-module on which G acts by k-module automorphisms is equivalent to a left module over the group ring $k\,G$. By a derivation d from G to such a module M, we shall mean a derivation into M as an abelian group with G-action (ignoring the k-module structure). Note that if S is a left G-set, then the function-space k^S and its subgroup kS of elements with finite support become left $k\,G$-modules, and that in particular, if the G-set S is free, then kS becomes a free left $k\,G$-module.

Having seen in the preceding section how to go from derivations to derivation support functions, let us now note that if S is a free left G-set, we can, conversely, realize any derivation support function $\delta : G \to S$ by a derivation of a very convenient sort. For Theorem 5 says we may express δ in terms of an equivalence relation $\sim$ on S as in (13); so if we let k be any ring of cardinality at least as large as the number of equivalence classes in $S/{\sim}$, and $x \in k^S$ be any map which takes on equal values at two points if and only if they are equivalent under $\sim$, then the derivation support function associated with the inner derivation $d_x : G \to k^S$ will be precisely δ.

In the above situation, if δ has the property of carrying every element of G to a *finite* subset of S, then the inner derivation d_x we have constructed will take values in the free submodule $kS \subseteq k^S$. However, it may not be possible to take x itself to lie in that subgroup; that is, though d_x is inner as a derivation to k^S, it may not be inner as a derivation to kS.

Strong results are known about the kernels of derivations into free $k\,G$-modules; in particular,

Corollary 9 (to [17, Theorem 5.1]). *Let G be a finitely generated torsion-free group, and d a derivation from G to a free left $k\,G$-module, for some commutative ring k. Then $H = \ker(d)$ is a free factor of G (i.e., a member of a coproduct decomposition $G = H * K$).*

Proof (W. Dicks, personal communication). The theorem from [17] cited says that for G as above, if a *non-inner* derivation d from G to a free $k\,G$-module restricts to an *inner* derivation on a subgroup H, then H is contained in a proper free factor of G.

Now let H be the kernel of an arbitrary derivation d from G to a free $k\,G$-module. If $H = \{1\}$ or G, the conclusion is trivial; in the contrary case, d is a nonzero derivation with nontrivial kernel. But it follows from the torsion-freeness of G that a nonzero *inner* derivation to a free module has trivial kernel. Hence d is not inner, but its restriction to H, being zero, is; so by the theorem cited, H lies in a proper free factor $G_1 < G$. Now our free $k\,G$-module is also a free $k\,G_1$-module, and H is still the kernel of d as a derivation from G_1 to this module, so if H is a proper subgroup of G_1 we may repeat the process with G_1 in place of G. Since a proper free factor requires strictly fewer generators than the original group [14] [6, Theorem I.10.6], after finitely many steps we must get a free factor G_n of G that coincides with H. □

Using the observations preceding the above corollary on realizing derivation support functions by k^S-valued, and in favorable cases, kS-valued derivations when S is a free G-set, we get the first assertion of the next result. The second assertion is easily deduced by applying the first assertion to finite unions of orbits of our free G-set, and then using the descending chain condition on free factors, noted in the last sentence of the above proof.

Proposition 10. *Let G be a finitely generated torsion-free group. Then the kernel of any derivation support function δ from G to the* finite *subsets of a free G-set S is a free factor in G.*

More generally, this is true of the kernel of any derivation support function from G to the subsets of a free G-set S which have finite intersection with each orbit of S.

Remark. If one deletes from the hypotheses of Corollary 9 the condition that G be torsion-free, one can still get strong information on $\ker(\delta)$: The Almost Stability Theorem [6, Theorem III.8.5] reduces one to Bass-Serre Theory ([15], [6, Theorem I.4.1]), which yields a decomposition of G in terms of fundamental groups of trees of groups. In fact, in applying this argument to get a generalization of Proposition 10, one can skip the realization of δ by a kS-valued derivation, and apply the Almost Stability Theorem directly to the equivalence classes of the relation $\sim$ of Theorem 5 on each orbit of S, since the condition that δ be finite-subset valued on each orbit shows that these classes are "almost stable" (differ from each of their G-translates at only finitely many points). However, I leave the application of this more general approach to the experts; we will use below only consequences of Proposition 10 as stated.

Suppose now that G and L are any groups, and we form their coproduct $E = G * L$, and consider the canonical map

$$f:\ E \to G,$$

which acts as the identity on G and annihilates L. This has a right inverse, the inclusion $\sigma_0\colon G \to E$; hence E can be written as a semidirect product of G with the kernel N of f. This kernel is easily shown to be the coproduct of the conjugates ${}^gL = gLg^{-1}$ $(g \in G)$ of L, which are permuted freely by G.

If for each $g \in G$ we let U_g be the subgroup of N freely generated by all ${}^hL$ with $h \neq g$, then since each nonidentity element of N involves factors from a well-defined nonempty but finite subset of the ${}^gL$, these subgroups U_g have trivial intersection, and each element of N lies in all but finitely many of them. Hence if $\sigma_1\colon G \to G * L$ is any other section of f, and we apply Proposition 7 to the derivation $d\colon G \to N$ defined by $d(g) = \sigma_1(g)\,\sigma_0(g)^{-1}$, we get a derivation support function $\delta\colon G \to \mathbf{P}(G)$ which is finite-subset valued, and whose kernel is the kernel of d, i.e., the equalizer of σ_0 and σ_1. Hence by Proposition 10, if G is finitely generated and torsion-free, this equalizer is a free factor in G.

The above argument required one of our two sections to be the trivial section σ_0. Can we get the same result for the equalizer of two arbitrary sections σ_1 and σ_2 of f? We might hope to do this using the decomposition of E as a semidirect product of G and N via the splitting by σ_1 rather than σ_0. But if we do this, we find that we don't know enough about the structure of N under the resulting action of G (i.e., under conjugation by elements $\sigma_1(g)$). So let us continue to regard G as acting on N under the action induced by σ_0. Then the equalizer of σ_1 and σ_2 may be described as the equalizer of two derivations $d_1(g) = \sigma_1(g)\,\sigma_0(g)^{-1}$ and $d_2(g) = \sigma_2(g)\,\sigma_0(g)^{-1}$, and we can try to apply Proposition 8. However, that proposition calls for *normal* subgroups U_s, and the subgroups used in the preceding paragraph were not normal. If we write V_g for the normal subgroup of N generated by U_g, i.e., the kernel of the natural map from N as a coproduct of the groups ${}^hL$ to the particular free factor ${}^gL$, we find that $\bigcap V_g$ is nontrivial; e.g., given ${}^hx \in {}^hL - \{1\}$ and ${}^{h'}x' \in {}^{h'}L - \{1\}$ where $x, x' \in L$ and $h \neq h'$, the commutator $({}^hx,\, {}^{h'}x')$ has trivial image under the natural maps to all the ${}^gL$. Thus, the map $x \mapsto \sigma_1(x)\,\sigma_2(x)^{-1}$ may be nontrivial, but have its values in $\bigcap V_g$, in which case the derivation support function induced by the subgroups V_g is trivial, and gives us no information on the kernel of this map.

Note, however, that the commutator $({}^hx,\, {}^{h'}x')$ of the above example will have *nontrivial* image under the natural map $N \to {}^hL * {}^{h'}L$. This suggests that we should take for our normal subgroups the kernels of the natural maps of N onto coproducts of finite families of the ${}^gL$.

Hence, let S denote the set of all finite nonempty strings of distinct elements of G,

$$S \;=\; \{(g_1,\dots,g_n) \mid n>0,\ g_i \in G,\ g_i \neq g_j \ \text{ for } \ i \neq j\}, \tag{18}$$

and let us make this a G-set by taking ${}^g(g_1,\dots,g_n) = (gg_1,\dots,gg_n)$. This action is clearly free. For each $s = (g_1,\dots,g_n) \in S$, let V_s denote the kernel of the homomorphism $N \to {}^{g_1}L * \dots * {}^{g_n}L$ taking members of each ${}^{g_i}L$ to themselves, and annihilating all the other groups ${}^hL$. If x is a nonidentity element of N, we can choose $g_1,\dots,g_n$ such that x lies in ${}^{g_1}L * \dots * {}^{g_n}L \subseteq N$; hence for $s = (g_1,\dots,g_n)$ we have $x \notin V_s$; hence $\bigcap V_s = \{1\}$. On the other hand, for $x \in {}^{g_1}L * \dots * {}^{g_n}L$ as above, and any $t = (h_1,\dots,h_m) \in S$, we see that x will lie in all but finitely many of the subgroups $V_{({}^gt)}$, namely, it will lie in $V_{({}^gt)}$ except for some subset of the $\leq mn$ values of g for which $\{g_1,\dots,g_n\} \cap \{gh_1,\dots,gh_m\} \neq \varnothing$.

Hence our pair of sections σ_1, σ_2 yields a derivation support function $G \to \mathbf{P}(S)$ whose values have finite intersection with each orbit, and whose kernel is the equalizer of σ_1 and σ_2. Applying Proposition 10 (and, if we are interested in more than two sections, using induction to handle finite families of sections and descending chain condition on free factors to pass to the general case), we get

Theorem 11. *Let G be a finitely generated torsion-free group, and L any group. Then the equalizer of any two group-theoretic sections, and hence, more generally, of any set of such sections of the natural map $G * L \to G$, is a free factor in G.* □

(My apologies here to the reader who would prefer to have gotten to the above result by the shortest possible route. The introduction of derivation support functions could, of course, have been replaced by the minimal combinatorial manipulations needed to get from two sections σ_1 and σ_2 the almost-stable equivalence relation $\sim$ underlying the proof of Theorem 11. However, I felt the above development more enlightening. The next section moves faster.)

5. Fixed subgroups of sets of endomorphisms of free groups

If $f : E \to G$ is any surjective homomorphism of free groups, it is known (though nontrivial) that there exists a section σ of f whose image is a free factor of E; i.e., that we can write $E \cong G * L$ for some free group L, in such a way that f corresponds to the map $G * L \to G$ which is the identity on G and the trivial homomorphism on L ([8, Theorem 6.3, with $X' = U' = \varnothing$]; cf. [6, Theorem I.10.5]). Thus, Theorem 11 yields

Corollary 12. *If $f : E \to G$ is a surjective homomorphism of free groups with G finitely generated, then the equalizer of any family of sections of f is a free factor in G.* □

We can now obtain the following result, assuming the case where all members of B are one-to-one, proved by Dicks and Ventura [7, Corollary IV.5.8]. (See the Introduction to [7] for a summary of the many results of earlier authors that led up to theirs.) As in [7], the fixed group of a set B of endomorphisms of a group G will be denoted $\mathrm{Fix}(B)$. The *rank* of a free group G, written $\mathrm{rank}(G)$, will mean the cardinality of a free generating set for G.

Theorem 13. *Let F be a free group of finite rank, and B a set of endomorphisms of F. Then*

$$\mathrm{rank}(\mathrm{Fix}(B)) \ \leq \ \mathrm{rank}(F).$$

Proof. Assume without loss of generality that B is closed under composition and contains the identity endomorphism, and choose $\beta \in B$ so as to minimize $\mathrm{rank}(\beta(F))$. Thus, all elements of B act injectively on $\beta(F)$. Note that all elements of $\beta B = \{\beta\gamma \mid \gamma \in B\}$ carry $\beta(F)$ into itself. The subgroup $\mathrm{Fix}(\beta B)$ (not in general equal to the subgroup $\mathrm{Fix}(B)$ we are interested in, but a step in that direction) can thus be described as the fixed group of the action of βB on $\beta(F)$; hence, since this action is by injective endomorphisms, the result of [7] cited above says that $\mathrm{rank}(\mathrm{Fix}(\beta B)) \leq \mathrm{rank}(\beta(F)) \leq \mathrm{rank}(F)$.

Let us now look at how the general element of B acts on $\mathrm{Fix}(\beta B)$. To do this, let

$$E \ = \ \beta^{-1}\,\mathrm{Fix}(\beta B)$$

so that

$$\beta:\ E \to \mathrm{Fix}(\beta B) \tag{19}$$

is a surjective homomorphism of free groups. Then for every $\gamma \in B$, the restriction of γ to $\mathrm{Fix}(\beta B)$ is a section of (19), since $\beta\gamma \in \beta B$ acts as the identity on $\mathrm{Fix}(\beta B)$.

The equalizer of this set of sections of (19) is precisely $\mathrm{Fix}(B)$ (since B contains the identity map). By Corollary 12, this equalizer is a free factor in $\mathrm{Fix}(\beta B)$; thus, its rank is $\leq \mathrm{rank}(\mathrm{Fix}(\beta B))$, which we have seen is $\leq \mathrm{rank}(F)$. □

In the remaining five sections, we return to some of the ideas introduced above, making additional observations, and noting questions for further investigation.

6. More on derivation support functions

We begin with some observations on the conditions (8)-(12) which we used, in various combinations, to characterize the class of derivation support functions.

Condition (8) cannot be dropped from any of these characterizations, for the function that takes every element of G to the whole set S satisfies (9)-(12), but not (8). To show that the combinations of (8)-(12) listed in Lemma 2 are the only minimal conjunctions of those conditions that characterize derivation support functions, we need only show that in the presence of (8), neither (9) nor (10) implies the other. The first nonimplication is easy because (8) and (9) only use the structure of G as a set with identity element and inverse operation, a small part of the group structure; the reader can easily supply the required counterexample. For an example of a function satisfying (8) and (10) but not (9), we may take $G = \mathbf{Z}$ and any nonempty G-set S, and define $\delta(n)$ to be S if n is positive, empty otherwise.

We note that the class of derivation support functions is closed under taking unions (where by the union of a family of functions $\delta_i : G \to \mathbf{P}(S)$ we mean the function $g \mapsto \bigcup_i \delta_i(g)$); such unions correspond to intersections of the groupoids of Lemma 4, and to derivations into direct products of groups N_i. It is also closed under taking images and inverse images under maps of G-sets: that is, if δ is a derivation support function from G to subsets of a G-set S, and f, respectively f', is a map of G-sets out of, respectively into, S, then the operation $g \mapsto f(\delta(g))$, respectively $g \mapsto {f'}^{-1}(\delta(g))$, is a derivation support function, since such images and inverse images respect unions (including the empty union) and the action of G. We may call these constructions the push-forward of δ along f and the pullback of δ along f'.

We saw in §2 why we should not expect derivation support functions on a general G-set S to be induced by equivalence relations on S; let us note here an explicit class of counterexamples. If S is a 1-element G-set, then the unique equivalence relation on S induces the trivial derivation support function. But for every proper subgroup $H < G$, the function $\delta\colon G \to \mathbf{P}(S)$ which takes elements of H to $\varnothing$ and elements of $G - H$ to S is a nontrivial derivation support function.

Note, however, that if δ is a derivation support function from a group G to subsets of any G-set S, we may write S as a surjective image of a free G-set T, pull δ back to a $\mathbf{P}(T)$-valued derivation support function δ', express δ' in terms of an equivalence relation on T, and observe that δ is the push-forward of this induced derivation support function δ' under the map $T \to S$. Although δ will in general be the push-forward of more than one derivation support function to subsets of T, it is easy to add conditions that make δ' unique, giving the following result. For brevity we refer only to the case of transitive G-sets $S = G/H$.

Lemma 14. *Let G be a group, and H a subgroup. Then the pullback/push-forward construction and the construction of Theorem 5 yield a bijection between derivation support functions from G to subsets of the G-set G/H, and equivalence relations on G that are invariant under right translation by all members of H (i.e., invariant as equivalence relations, though the separate equivalence classes need not be invariant).*

Example. Let $G = \mathbf{Z}$, let n be any positive integer, let $\sim_0$ be the equivalence relation on the free G-set $\mathbf{Z}$ with two equivalence classes, the negative and the nonnegative integers, and let $\delta_0 \colon \mathbf{Z} \to \mathbf{P}(\mathbf{Z})$ be the derivation support function on $\mathbf{Z}$ induced by $\sim_0$. The action of the subgroup $n\mathbf{Z}$ does not respect $\sim_0$, hence does not respect δ_0. However, we may push δ_0 forward to a derivation support function $\delta \colon \mathbf{Z} \to \mathbf{P}(\mathbf{Z}/n\mathbf{Z})$, then pull δ back to a derivation support function $\delta_1 \colon \mathbf{Z} \to \mathbf{P}(\mathbf{Z})$, which *is* $n\mathbf{Z}$-invariant. We find that δ_1 corresponds to the equivalence relation $\sim_1$ on $\mathbf{Z}$ with equivalence classes $\{rn,\, rn+1,\, \dots\,,\, rn+(n-1)\}$ $(r \in \mathbf{Z})$. So this is the equivalence relation corresponding to δ under the bijection of the above lemma.

A few directions for possible investigation:

We have considered general-subset valued and finite-subset valued derivation support functions on groups. If S is a *totally ordered* G-set on which G acts in an order-preserving fashion (for instance, if G is an ordered or left ordered group, and $S = G$), it might be of interest to consider derivation support functions whose values are *well-ordered* subsets of S, since formal sums with such supports arise in the theory of generalized formal power series. (Cf. [3, §2.4] and references noted there; and for a sketch of a more general class of constructions, [1, §10].)

We can also generalize the concept of a derivation support function by replacing $\mathbf{P}(S)$ in conditions (8)-(10) by any lattice (or upper semilattice) Λ with a least element, given with an action of G on Λ by automorphisms. (An easy class of examples where Λ is not of the form $\mathbf{P}(S)$ is gotten by starting with a derivation d from a group G into a group N with G-action, letting Λ be the *subgroup lattice* of N, and defining $\delta(g)$ to be the cyclic subgroup generated by $d(g)$.) Actually, any upper semilattice with G-action can be embedded in one of the form $\mathbf{P}(S)$ for S a G-set, so in a sense, the increased generality of considering (semi)lattices is illusory. Nonetheless, it may be heuristically useful.

A derivation on a group can be described as a 1-cocycle. This suggests that one might look more generally at the "support functions" associated with n-cocycles. These would be maps $G^n \to \mathbf{P}(S)$ satisfying conditions similar to (8)-(10).

7. Is there a noncommutative analog of Hochschild's Theorem?

In the paragraph following Theorem 5 we mentioned Hochschild's Theorem, which asserts that every derivation from a group G to a group of the form L^G with the natural G-action is inner. In particular, any derivation $G \to \bigoplus_G L$, though it may not be inner as a derivation into $\bigoplus_G L$, becomes inner as a derivation into L^G, the completion of $\bigoplus_G L$ in an appropriate topology. It would be interesting to know whether there is an analogous natural completion of the *coproduct* of a G-tuple of copies of L, i.e., the kernel N of the natural map $G * L \to G$, likewise having the property that every derivation from G to it is inner. Analogy with the $\bigoplus_G L$ case suggests as a candidate the inverse limit of the system of factor-groups

$N/V_s = {}^{g_1}L * \cdots * {}^{g_n}L$ used in proving Theorem 11 (which we would make an inverse system by preordering S so that $(g_1, \dots, g_m) \leq (h_1, \dots, h_n)$ if and only if $\{g_1, \dots, g_m\} \subseteq \{h_1, \dots, h_n\}$). Let us note that the desired result is at least true in the case where G is finite, so that the inverse limit is N itself:

Lemma 15. *Let G be a finite group, or more generally, any group which is neither a coproduct of proper subgroups, nor a nontrivial free group. Then for any group L, any derivation d from G to the kernel of the natural map $f\colon G * L \to G$ is inner.*

Proof. Such a derivation d corresponds to a section σ to the map f. By the Kurosh Subgroup Theorem [12, Corollary 4.9.1, p.243] [6, Theorem I.7.8], $\sigma(G)$, as a subgroup of the coproduct group $G * L$, will be a coproduct of subgroups of conjugates of G, subgroups of conjugates of L, and a free group. But since $\sigma(G) \cong G$, which is neither a proper coproduct nor a nontrivial free group, $\sigma(G)$ must itself be a subgroup of a conjugate of G or of a conjugate of L. Since f annihilates all conjugates of L, and maps $\sigma(G)$ on the one hand, and all conjugates of G on the other, isomorphically to G, the only possibility is that $\sigma(G)$ is a full conjugate of G. Writing the conjugating element as xg with $x \in N$ and $g \in G$, we see that $\sigma(G)$ is also the conjugate of G by x. It follows that σ is the conjugate of the trivial section of f by x, so d is the inner derivation d_x. $\square$

8. Are the results of §4 sharp?

The results of §4 all assumed the group G finitely generated and torsion free. Are these hypotheses needed?

Torsion freeness is certainly needed in some of these statements; for consider any group G with a finite subgroup H. The equivalence relation on G having two equivalence classes, H and its complement, induces a finite-subset valued derivation support function $\delta\colon G \to \mathbf{P}(G)$ whose kernel is H. Thus, if we choose G and H such that H is not a free factor in G, the conclusion of Proposition 10 fails, and from this we easily get a counterexample to the conclusion of Corollary 9.

Dicks and Dunwoody [5] have recently shown that Theorem 11 is true without torsion-freeness. Let us note here a special case that is easy to prove, namely that the equalizer of a section σ_1 with the *trivial* section σ_0 is always a free factor in G. To get this, let us first note that the Kurosh Subgroup Theorem implies

(20) The intersection of a free factor A of a group $A * B$ with any subgroup $H < A * B$ is a free factor in H.

Indeed, using the formulation of the Kurosh Subgroup Theorem in [6, Theorem I.7.8], we may take the system of conjugates of A and B indicated therein to include A itself. Then H is expressed as a coproduct of subgroups, one of which is $H \cap A$.

Now given arbitrary groups G and L, and a section σ_1 of the natural map $G * L \to G$, the equalizer K of σ_0 and σ_1 can be identified with the intersection in $G * L$ of $\sigma_1(G)$ with $\sigma_0(G) = G$, and as the latter is a free factor in this group, K is a free factor in $\sigma_1(G)$. But $f\colon G * L \to G$ maps $\sigma_1(G)$ isomorphically to G and fixes elements of K; hence K is a free factor in G, as claimed.

The results of §4 also assume G finitely generated. This is known not to be needed for Corollary 9 (e.g., see [6, Theorem IV.2.5], noting that $(AG)_d$ is defined at [6, Definition IV.2.1]); hence it is also not needed for Proposition 10. However, the proof of Theorem 11 from that proposition twice uses descending chain condition on free factors: once because the only finiteness condition we have on our derivation support function is that its values have finite intersection with each of the *infinitely many* orbits of our G-set (18), and once in passing from the equalizer of a pair of sections to the equalizer of an arbitrary family. We shall see in a moment that if the hypothesis that G be finitely generated is deleted from Theorem 11, the conclusion about equalizers of arbitrary families of sections fails. I do not know whether the equalizer of a single pair of sections can fail to be a free factor.

The counterexample for infinite equalizers will be based on

Example 16 (W. Dicks and J. Stallings, personal communication)**.** *A descending chain of free factors H_n of a free group G of countable rank, whose intersection is not a free factor of G.*

Let G be free on $\{x_i,\, y_i \mid i \geq 0\}$, and for each $n \geq 0$, let $H_n < G$ be the subgroup generated by all x_i and y_i with $i > n$, and the elements $y_i(x_i, y_{i+1})$ with $i \leq n$. It is easy to see that for each n, H_n is a free factor, having for complement the subgroup generated by the x_i with $i \leq n$. On the other hand, the intersection of the H_n is the subgroup H_∞ generated by the elements $y_i(x_i, y_{i+1})$ alone. This is not a free factor, since if it were, the factor-group of G by the normal subgroup generated by H_∞ would be free, whereas in fact, using the chain of relations $\bar{y}_i(\bar{x}_i, \bar{y}_{i+1}) = 1$ $(i \geq 0)$ in that factor-group, we see that the intersection of the lower central series of this factor-group contains $\bar{y}_0 \neq 1$, which cannot happen in a free group. This establishes the asserted properties.

I claim also that if G and L are any nontrivial groups, then any free factor G_1 of G can be represented as the equalizer of the trivial section σ_0 of the natural map $G * L \to G$ and some other section σ_1. Indeed, if $G = G_1 * G_2$, and we take σ_1 to agree with σ_0 on G_1, while behaving on G_2 as the conjugate of σ_0 by any nonidentity element of L, then that equalizer is precisely G_1. Thus, for G as in Example 16 and L any nontrivial group, we can get for each $n > 0$ a section $\sigma_n : G \to G * L$ whose equalizer with the trivial section σ_0 is H_n. The equalizer of the section σ_0 and all these sections σ_n is thus H_∞, giving the required counterexample.

(The group H_∞ of the above example *is* a retract of G, by the retraction taking every x_i to 1 and each y_i to $y_i(x_i, y_{i+1})$. But the class of retracts of a free group of rank > 1 is much larger than its class of free factors; we will say more about this in the next section.)

We record below some properties of intersections of free factors, which in particular apply to the above example.

Lemma 17. *Let G be a group and H a subgroup. Then the following two conditions are equivalent:*

(a) *H is the intersection of a set of free factors of G.*
(a$'$) *H is the intersection of a downward directed system of free factors of G.*

If G is countable, these are also equivalent to the (*formally stronger*) *condition*

(a$''$) *H is the intersection of a chain of free factors of G.*

Moreover, each of the above conditions implies

(b) *For every finitely generated subgroup $K < G$, the intersection $K \cap H$ is a free factor in K.*

If G is the directed union of its finitely generated free factors (*i.e., if every finitely generated subgroup of G lies in a finitely generated free factor of G*), *then* (b) *is equivalent to* (b′) *H is a directed union of free factors of G.*

If H is countable, then (b′) *is equivalent to*

(b″) *H is the union of a chain of free factors of G.*

Sketch of Proof. By (20), the intersection of any two free factors of G is a free factor in G; so if H is the intersection of a family of free factors H_i of G ($i \in I$), we may also write it as the intersection of the directed system of free factors $H_{i(1)} \cap \cdots \cap H_{i(n)}$ $(i(1), \ldots, i(n) \in I)$ giving the implication (a) $\Rightarrow$ (a′); the reverse implication is trivial. If G is countable, we can enumerate the elements of $G - H$ as $g_1, g_2, \ldots$, choose for each m a free factor H_m not containing g_m, and use the chain of subgroups $H_1 \cap \cdots \cap H_n$ $(n = 1, 2, \ldots)$ to get (a″).

(a′) $\Rightarrow$ (b) follows immediately from (20) and the descending chain condition on free factors of a finitely generated group, noted in §3 above. The arguments giving the equivalence of (b) with (b′) and (b″) under appropriate hypotheses are straightforward. ☐

9. Retracts and free factors in free groups – compared and contrasted

In any variety of algebras (in the sense of universal algebra), a homomorphism onto a free algebra admits a section. The class of free *groups* shares with the class of free *abelian groups* the property that every surjective homomorphism f among them has a section whose image is a factor in a coproduct decomposition. (This is the result recalled in the first sentence of §5.) However, for free abelian groups, the image of *any* section of f has this property (with the kernel of the given surjection as the complementary summand), while this is not so for free groups; thus the study of retractions of free groups is not the same as the study of free factorizations. (This is why, to get Theorem 13, we needed to study equalizers of arbitrary pairs of sections of a map $G * L \to G$, rather than just equalizers of the canonical section and another section.)

In fact, let us show that the following classes of subgroups of a free group F of finite rank form, in general, a chain of four distinct terms, though in a free *abelian* group of finite rank, the classes corresponding to the first, third and fourth of these are equal.

$$\{H < F \mid a^n \in H \Rightarrow a \in H \text{ for all } n > 0\} \tag{21}$$

$$\supset \ \{H < F \mid a,\, bab^{-1} \in H - \{1\} \ \Rightarrow \ b \in H\} \tag{22}$$

$$\supset \ \{\text{retracts of } F\} \tag{23}$$

$$\supset \ \{\text{free factors of } F\}. \tag{24}$$

To see that (21) $\supseteq$ (22), let $H \in$ (22) (sometimes called the class of "malnormal" subgroups of F), and note that if $a^n \in H$, then a^n and $a(a^n)a^{-1}$ lie in H, so assuming $a \neq 1$, the condition defining (22) implies that $a \in H$; while if $a = 1$ this is trivial. An example of a subgroup H in (21) but *not* (22) (taken in the free group on two generators, $F = \langle x, y\rangle$, as are all subsequent examples if the contrary is not stated) is $\langle x,\, yxy^{-1}\rangle$.

To show that (22) $\supseteq$ (23), note that if f is a retraction of F onto H which fixes nonidentity elements a and bab^{-1}, we get $bab^{-1} = f(bab^{-1}) = f(b)\,a\,f(b)^{-1}$; comparing the first and last terms, we see that $b^{-1}f(b)$ centralizes a; note also that that element lies in the kernel of f. An element centralizing $a \neq 1$ must be a power of the smallest-length element $\alpha \in F$ of which a is a power, and since nth roots are unique in free groups, and a is fixed by f, α must also be; so the only power of α in the *kernel* of f is $\alpha^0 = 1$; hence $b^{-1}f(b) = 1$, i.e., f fixes b, as claimed.

There are many sorts of examples showing this inclusion strict. For instance, if H is a retract of F, then applying the abelianization functor, we see that the induced map $H/H' \to F/F'$ is also left invertible, hence one-to-one; the subgroup $H = \langle xyx^{-1}y^{-1}\rangle \subseteq F = \langle x, y\rangle$ is in (22) but obviously does not have this property. Another example is $H = \langle xy,\, x^3y^3, \dots,\, x^{2^n-1}y^{2^n-1}\rangle$, for any $n > 2$. Here one can verify that there is sufficiently little partial cancellation among products of these generators and their inverses so that, from the reduced expression in x and y for a word or a "cyclic word" in these generators, one can "read out" the factors. Hence on the one hand, H is free on the given generators, hence has rank n, so its abelianization can't embed in that of $\langle x, y\rangle$, hence it cannot lie in (23), while on the other hand, by the "cyclic words" observation, two products of these generators that are conjugate in F are conjugate in this subgroup H, and the conjugating element lies in H, so (22) does hold. An example of an H which lies in (22) and whose abelianization *is* a retract of the abelianization of G, but which is still not itself a retract, is $\langle xy,\, x^2y^3\rangle$. Indeed, its abelianization maps isomorphically to the abelianization of F, hence is a retract thereof, and H lies in (22) by the same sort of non-cancellation argument as the preceding example; but if it were a retract, then having the same rank as F, it would have to equal F. However, its indicated generators do not show sufficient cancellation for any product of them and their inverses that involves the second of these generators to reduce to a word in x and y not having a subword $y^{\pm 2}$, so this group does not contain x or y, and so is not F. (One may note that the arguments used in these examples implicitly define several additional classes of subgroups between (22) and (23).)

The inclusion (24) $\subseteq$ (23) is clear. Some known examples of subgroups of $\langle x, y\rangle$ in (23) but not (24) are the cyclic subgroups $\langle x^2y^3\rangle$ [8, Remark, 5.3] and $\langle x^2yx^{-1}y^{-1}\rangle$ [2, and references noted there]. Indeed, in each of these cases let us write u for the indicated cyclic generator; then in the first case, $\langle u\rangle$ is a retract of $\langle x, y\rangle$ via the endomorphism taking x to u^{-1} and y to u, and in the second case, via the endomorphism taking x to u and y to 1. In each case, one can easily prove that $\langle u\rangle$ is not a free factor by showing that the factor-group of $\langle x, y\rangle$ by the normal subgroup generated by u is not free. In fact, this is known to be a necessary and sufficient condition for a cyclic subgroup of a free group to be a free factor [11, Proposition II.5.10]. Alternatively, one can in each case use the results of [19], which express the rank of the least free factor of a free group containing a given set S of elements in terms of the rank of the $\mathbf{Z}G$-module spanned by the Fox derivatives of the elements of S, to show that u is not contained in a free factor of rank 1.

Let us note that fixed groups of *automorphisms* behave rather differently from retracts; in particular, though they clearly satisfy (21), they need not satisfy (22). A standard example is the automorphism of $\langle x, y\rangle$ carrying x to itself and y to yx, which has fixed group $\langle x,\, yxy^{-1}\rangle$. In [4] an elegant description is given of all automorphisms β of a free group F of finite rank which, like this example, satisfy $\mathrm{rank}(\mathrm{Fix}(\beta)) = \mathrm{rank}(F)$.

The study of the fixed group of a single endomorphism is to a large extent the join of the theory of retracts and the theory of the fixed group of an automorphism, in view of the following results of Turner and Imrich:

(25) ([18, Theorem 1] and [10].) If β is an endomorphism of a free group F, and we define $\beta^\infty(F) = \bigcap_i \beta^i(F)$, then $\beta^\infty(F)$ is a retract of F, and the restriction of β to this retract is an automorphism. If β is one-to-one, then $\beta^\infty(F)$ is in fact a free factor of F.

Note that by (20), the intersection of two free factors of any group is a free factor. Consequently, if G is a group having descending chain condition on free factors, e.g., a finitely generated group, then the free factors of G will form a lattice, and every subgroup of G will lie in a unique least free factor of G, its "free factor closure".

On the other hand, the analog of (20) with retracts in place of free factors is false. For instance, let G be a group and H any subgroup which is not a retract of G, and let us form the coproduct $G_1 *_H G_2$ of two isomorphic copies of G with amalgamation of H. Then G_1 is a retract of this group (by the map induced by the identity on G_1 and the given isomorphism $G_2 \to G_1$ on G_2), but its intersection with G_2 is H, which is not a retract in G_2. Moreover, H is an intersection of two retracts G_1 and G_2 of $G_1 *_H G_2$, but is not itself a retract.

However, the latter situation cannot occur in a free group of finite rank:

Lemma 18. *Let F be a free group of finite rank. Then any intersection of retracts of F is a retract.*

Proof. Note that if a retract A of any algebra C is contained in a subalgebra $B \subseteq C$, then A is also a retract of B. Hence for any proper inclusion of retracts of our free group F, we have strict inequality of ranks; in particular, such retracts satisfy descending chain condition. Hence, to prove the lemma, it suffices to consider an intersection of two retracts.

If we had a counterexample, we would have idempotent endomorphisms α and β of F such that $\alpha(F) \cap \beta(F)$ was not a retract. In this situation, assume β chosen to minimize $\mathrm{rank}(\beta(F))$. This minimality implies

(26) no proper retract of $\beta(F)$ contains $\alpha(F) \cap \beta(F)$.

Now if the restriction of the composite map $\beta\alpha$ to a map $\beta(F) \to \beta(F)$ is not an automorphism of $\beta(F)$, its fixed group, which clearly contains $\alpha(F) \cap \beta(F)$, must, by (25), be contained in a proper retract of $\beta(F)$, contradicting (26); so $\beta\alpha$ acts by an automorphism on $\beta(F)$. If we precompose α with the inverse γ of this automorphism, we get a map $\alpha\gamma : \beta(F) \to F$ whose image, intersected with $\beta(F)$, still contains $\alpha(F) \cap \beta(F)$, and which, by construction, is a *section* of $\beta : F \to \beta(F)$. By Corollary 9, the equalizer of this section with the identity of $\beta(F)$, i.e., $\alpha\gamma(\beta(F)) \cap \beta(F)$, is a free factor in $\beta(F)$. In view of (26), this intersection must be all of $\beta(F)$; hence in particular, $\alpha(F) \supseteq \beta(F)$, so $\alpha(F) \cap \beta(F) = \beta(F)$, contradicting our assumption that this intersection was not a retract of F. □

On the other hand, I do not know whether the analog of (20) holds for retracts in free groups. Restricting attention to the finite rank case, this asks

Question 19. *If F is a free group of finite rank, R a retract of F, and H a subgroup of F of finite rank, will $H \cap R$ be a retract of H?*

10. Some open questions on ranks of fixed groups

For B a set of injective endomorphisms of a free group F of finite rank, Dicks and Ventura show not only that $\mathrm{rank}(\mathrm{Fix}(B)) \leq \mathrm{rank}(F)$, but [7, Theorem IV.5.7] that for every subgroup $H < F$, we have

$$\mathrm{rank}(H \cap \mathrm{Fix}(B)) \ \leq \ \mathrm{rank}(H),$$

a condition they express by saying that $\mathrm{Fix}(B)$ is *inert* in F. One may ask whether fixed groups of sets B of not-necessarily-injective endomorphisms of F are also inert. (Cf. [7, Problems 2 and 5], which also ask this question for fixed groups of sets of automorphisms or injective endomorphisms of a free group F of possibly infinite rank.) In this connection, let us recall ([12, Exercise 33, p.118], or for a more general result, [7, Theorem I.4.11]) that if the intersection of a chain $H_1 > H_2 > \dots$ of subgroups of a free group has rank $\geq n$, then almost all terms of the chain have rank $\geq n$. Thus, the intersection of a chain of rank-n subgroups of a free group has rank $\leq n$. It follows that the intersection of a chain of inert subgroups of a free group is inert. It is also clear that the intersection of two inert subgroups is inert; hence the class of inert subgroups is closed under arbitrary intersections; hence to show that the fixed subgroup $\mathrm{Fix}(B)$ of an arbitrary family of endomorphisms is inert, it suffices to show this for the fixed subgroup $\mathrm{Fix}(\{\beta\})$ of a singleton. Now (25) reduces this question to the case where β is an automorphism, handled in [7], and the case where it is a retraction. Thus, our inertness question reduces to

Question 20 (cf. [7, Problems 2 and 5 and following commentary]). *Is every retract R of a free group F of finite rank inert in F?*

Clearly, an affirmative answer to this question would follow from an affirmative answer to Question 19. Let me show that it would also follow from an affirmative answer to the following question, or even to part (b) thereof.

Question 21. *Suppose $K < H < F$ are free groups of finite rank, such that* $\mathrm{rank}(H) < \mathrm{rank}(K)$, *but all proper subgroups of H which contain K have ranks* $\geq \mathrm{rank}(K)$. *Then is the inclusion of H in F the only homomorphism $H \to F$ fixing all elements of K?*

This question may be broken into two parts:

(a) *Is the inclusion of H in F the only* one-to-one *homomorphism $H \to F$ fixing all elements of K?*

(b) *Is every homomorphism $H \to F$ fixing all elements of K one-to-one?*

Now suppose we had a counterexample to Question 20, i.e., suppose β were an idempotent endomorphism of a free group F of finite rank, and H a subgroup of finite rank meeting $\beta(F)$ in a subgroup $K = H \cap \beta(F)$ of rank $> \mathrm{rank}(H)$. Replacing H if necessary by a minimal subgroup between K and H having rank less than that of K (which exists by the preceding observation on ranks of intersections of chains), we would have the situation of the first sentence of Question 21, and the restriction to H of the retraction $\beta : F \to \beta(F)$ would be a map $H \to F$ agreeing with the identity on K. Now assuming an affirmative answer to (b) above, β must be one-to-one on H. Hence it gives an isomorphism $H \to \beta(H)$, and the inverse of this isomorphism can be looked at as a section of the retraction $\beta^{-1}(\beta(H)) \to \beta(H)$ given by β. But K is the equalizer of this section and the identity section, and an application of Corollary 12 now gives a contradiction to the inequality $\mathrm{rank}(K) > \mathrm{rank}(H)$. Hence an affirmative answer to Question 21 (b) would indeed imply an affirmative answer to Question 20.

The condition on ranks of subgroups between K and H in Question 21 is expressed in [7, Problem 1] by saying that K is a *compressed* subgroup of H. Using this language, let us pose a slight generalization of the above question. Given a free group H of finite rank and a compressed subgroup K with $\mathrm{rank}(H) < \mathrm{rank}(K)$, suppose we write a set of free generators of K as words $w_1, \dots, w_n$ in a set of free generators of H. Then we may ask whether, if $y_1, \dots, y_n$ are independent elements of a free group F (i.e., elements satisfying no relations), and if the system of equations determined by the above family of words:

$$y_1 \;=\; w_1(x_1, \dots, x_m), \quad \dots\;, \quad y_n \;=\; w_n(x_1, \dots, x_m) \tag{27}$$

has a solution $x_1, \dots, x_m$ in F, then this solution must be unique. (The answer is negative if we do not assume the above "no relations" condition on the y's. E.g., let H be free on generators z_1, z_2, and K the subgroup generated by $w_1(z_1, z_2) = z_1$, $w_2(z_1, z_2) = z_2 z_1 z_2^{-1}$, $w_3(z_1, z_2) = z_2^2 z_1 z_2^{-2}$. Then taking $y_1 = y_2 = y_3 = 1$ in any nontrivial free group F, the system (27) clearly has nonunique solution.) An affirmative answer would say in particular that all fixed groups of endomorphisms of F that contain $y_1, \dots, y_n$ also contain $x_1, \dots, x_m$, a result reminiscent of (21) and (22).

Stallings asked in [16, §8.3] whether for any two homomorphisms h_1, h_2 from a free group H of finite rank into a free group F, with h_1 injective, the equalizer K of h_1 and h_2 must be of finite rank. This was answered affirmatively in [9]. Dicks and Ventura [7, Problem 6] ask whether K must have rank $\leq \mathrm{rank}(H)$. Note that if we had a counterexample, we could pass to one in which H was minimal among subgroups containing K but having smaller rank than K, from which it is easily deduced that the above question is equivalent to Question 21.

A possible approach to questions on fixed groups and equalizers in free groups quite different from those we have been considering would be to embed the given free groups $\langle x_1, \dots, x_n \rangle$ in the multiplicative groups of noncommuting formal power series rings $k\,\langle\langle\, \xi_1, \dots, \xi_n \,\rangle\rangle$ by sending each generator x_i to the power series $1 + \xi_i$, then note that homomorphisms among free groups induce homomorphisms of these power series rings, and ask whether equalizers of homomorphisms of power series rings have good properties, which might carry over to their intersections with the embedded free groups.

References

1. George M. Bergman, *Embedding rings in completed graded rings. 4. Commutative algebras*, J. Alg. **84** (1983), 62-106. MR **85i:**16001d
2. M. Cohen, W. Metzler and B. Zimmermann, *What does a basis of $F(a, b)$ look like?*, Math. Ann. **257** (1981), 435-445. MR **82m:**20028
3. P. M. Cohn, *Skew fields. Theory of general division rings*, Encyclopedia of Mathematics and its Applications, vol. 57, Cambridge University Press, 1995. MR **97d:**12003
4. D. J. Collins and E. C. Turner, *All automorphisms of free groups with maximal rank fixed groups*, Math. Proc. Cambridge Philos. Soc. **119** (1996), 615-630. MR **96j:**20037
5. Warren Dicks, *On equalizers of sections*, J. Alg. (to appear).
6. Warren Dicks and M. J. Dunwoody, *Groups acting on graphs*, xvi+283 pp., Cambridge Studies in Advanced Mathematics, vol. 17, Cambridge University Press, 1989. MR **91b:**20001
7. Warren Dicks and Enric Ventura, *The group fixed by a family of injective endomorphisms of a free group*, Contemporary Mathematics, vol. 195, 1996, pp. x+81. MR **97h:**20030
8. Herbert Federer and Bjarni Jónsson, *Some properties of free groups*, Trans. Amer. Math. Soc. **68** (1950), 1-27. MR **11:**323a
9. R. Z. Goldstein and E. C. Turner, *Fixed subgroups of homomorphisms of free groups*, Bull. London Math. Soc. **18** (1986), 468-470. MR **87m:**20096
10. W. Imrich and E. C. Turner, *Endomorphisms of free groups and their fixed points*, Math. Proc. Cambridge Phil. Soc. **105** (1989), 421-422. MR **90f:**20037

11. Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Ergebnisse der Math. u. ihrer Grenzgebiete, vol. 89, Springer-Verlag, 1977. MR **58:**28182
12. Wilhelm Magnus, Abraham Karrass and Donald Solitar, *Combinatorial group theory*, Interscience Publishers, 1966; Dover Publications, 1976. MR **34:**7617; MR **54:**10423
13. A. I. Mal'cev, *On the embedding of group algebras in division algebras*, (Russian), Doklady Akad. Nauk SSSR **60** (1948), 1499-1501. MR **10:**8c
14. B. H. Neumann, *On the number of generators of a free product*, J. London Math. Soc. **18** (1943), 12-20. MR **5:**58t
15. Jean-Pierre Serre, *Arbres, amalgames, SL_2*, Astérisque, no. 46, 1977, 1983; translated as *Trees*, Springer-Verlag, 1980. MR **57:**16426
16. John R. Stallings, *Topology of finite graphs*, Inventiones Math. **71** (1983), 551-565. MR **85m:**05037a
17. Richard G. Swan, *Groups of cohomological dimension one*, J. Alg. **12** (1969), 585-601. MR **39:**1531
18. Edward C. Turner, *Test words for automorphisms of free groups*, Bull. London Math. Soc. **28** (1996), 255-263. MR **96m:**20039
19. U. U. Umirbaev, *On the ranks of elements of free groups*, Fundamental'naya i Prikladnaya Matematika **2** (1996), 313-315 (Russian).

Department of Mathematics, University of California, Berkeley, California 94720-3840

E-mail address: gbergman@math.berkeley.edu