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A COUNTEREXAMPLE CONCERNING THE RELATION BETWEEN DECOUPLING CONSTANTS AND UMD–CONSTANTS

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Abstract. For Banach spaces X and Y and a bounded linear operator $T : X \to Y$ we let $\rho(T) := \inf c$ such that

$$\left(AV_{\theta_l=\pm 1} \left\| \sum_{l=1}^{\infty} \theta_l \left(\sum_{k=\tau_{l-1}+1}^{\tau_l} h_k T x_k \right) \right\|_{L_2^Y}^2 \right)^{\frac{1}{2}} \leq c \left\| \sum_{k=1}^{\infty} h_k x_k \right\|_{L_2^X}$$

for all finitely supported $(x_k)_{k=1}^{\infty} \subset X$ and all $0 = \tau_0 < \tau_1 < \cdots$, where $(h_k)_{k=1}^{\infty} \subset L_1[0,1)$ is the sequence of Haar functions. We construct an operator $T : X \to X$, where X is superreflexive and of type 2, with $\rho(T) < \infty$ such that there is no constant $c > 0$ with

$$\sup_{\theta_k=\pm 1} \left\| \sum_{k=1}^{\infty} \theta_k h_k T x_k \right\|_{L_2^X} \leq c \left\| \sum_{k=1}^{\infty} h_k x_k \right\|_{L_2^X}.$$

In particular it turns out that the decoupling constants $\rho(I_X)$, where I_X is the identity of a Banach space X, fail to be equivalent up to absolute multiplicative constants to the usual UMD–constants. As a by-product we extend the characterization of the non–superreflexive Banach spaces by the finite tree property using lower 2–estimates of sums of martingale differences.

Introduction

A Banach space X is called a UMD–space, where UMD stands for 'unconditional martingale differences', whenever there is a constant $\beta > 0$ such that for all finitely supported sequences $(x_k)_{k=1}^{\infty} \subset X$ one has

$$\sup_{\theta_k=\pm 1} \left\| \sum_{k=1}^{\infty} \theta_k h_k x_k \right\|_{L_2^X[0,1)} \leq \beta \left\| \sum_{k=1}^{\infty} h_k x_k \right\|_{L_2^X[0,1)}, \tag{1}$$

where $(h_k)_{k=0}^{\infty} \subset L_1[0,1)$ is the sequence of Haar functions

$$h_0 = \chi_{[0,1)}, \quad h_1 = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)},$$

$$h_2 = \chi_{[0,\frac{1}{4})} - \chi_{[\frac{1}{4},\frac{1}{2})}, \quad h_3 = \chi_{[\frac{1}{2},\frac{3}{4})} - \chi_{[\frac{3}{4},1)},$$

$$h_4 = \chi_{[0,\frac{1}{8})} - \chi_{[\frac{1}{8},\frac{1}{4})}, \quad h_5 = \chi_{[\frac{1}{4},\frac{3}{8})} - \chi_{[\frac{3}{8},\frac{1}{2})}, \ldots$$

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and 'finitely supported' means that only finitely many of the x_k are non–zero. It can be easily seen that (1) is equivalent to the fact that, for some $\rho, \tau > 0$, one has simultaneously

$$\left(\int_0^1 \left\| \sum_{l=1}^{\infty} r_l(t) d_l \right\|^2_{L_2^X[0,1)} dt \right)^{\frac{1}{2}} \leq \rho \left\| \sum_{l=1}^{\infty} d_l \right\|_{L_2^X[0,1)} \tag{2}$$

and

$$\left\| \sum_{l=1}^{\infty} d_l \right\|_{L_2^X[0,1)} \leq \tau \left(\int_0^1 \left\| \sum_{l=1}^{\infty} r_l(t) d_l \right\|^2_{L_2^X[0,1)} dt \right)^{\frac{1}{2}} \tag{3}$$

for all $\tau_0 < \tau_1 < \tau_2 < \cdots$, finitely supported $(x_k)_{k=1}^{\infty} \subset X$, and $d_l := \sum_{k=\tau_{l-1}+1}^{\tau_l} h_k x_k$, where

$$r_l := \sum_{2^{l-1} \leq k < 2^l} h_k \qquad (l = 1, 2, ...)$$

is the l–th Rademacher function. The importance of the UMD–property (see [7] and [8] and the references therein) and the fact that there are applications of the UMD–property using only one of the 'decoupling' inequalities (2) and (3) (cotype q and (2) imply martingale cotype q in the notation of Section 1 and therefore convexity properties of X due to [22]—the same holds for (3), the type, and smoothness properties) justify a separate investigation of these decoupling inequalities as done by D.H.J. Garling [11]. For example there is shown that (3) is much weaker than the UMD–property since all subspaces of ℓ_1 satisfy this inequality. Besides the trivial implication (1) $\Rightarrow$ (2) almost nothing is known in the general vector valued case about the relation between (2) and (1). One subject of the present paper is to clarify the following basic 'quantitative' question:

(Q)
Let $\beta(X)$ and $\rho(X)$ be the best constants β in (1) and ρ in (2), respectively.
Is there some $c > 0$ such that for all X one has $\beta(X) \leq c\rho(X)$?

Before we start with our investigation let us mention a result of P. Hitczenko [15] saying that there is an absolute constant $c > 0$ (not depending on p !) such that for all dyadic martingale difference sequences $(d_l)_1^n \subset L_1[0,1)$ (see Section 1) and all $1 \leq p < \infty$ one has

$$\sup_{\theta_l = \pm 1} \left\| \sum_{l=1}^{n} \theta_l d_l \right\|_{L_p[0,1)} \leq c \left(\int_0^1 \int_0^1 \left| \sum_{l=1}^{n} r_l(t) d_l(s) \right|^p dt ds \right)^{\frac{1}{p}}.$$

Since the converse inequality is trivial we have in the scalar valued setting a very strong relation between the deterministic transforms $\sum_l \theta_l d_l$ and the 'random transforms' $\sum_l r_l d_l$. This could indicate a closer relation between (1) and (2) than between (1) and (3). Now let us start with

Definition 1. Assume that $T : X \to Y$ is a bounded linear operator between the Banach spaces X and Y and that $1 < q < \infty$. Then

(1) $\beta_q(T) := \inf \beta$, such that for all finitely supported $(x_k)_k \subset X$

$$\sup_{\theta_k \in \{-1,1\}} \left\| \sum_{k=1}^{\infty} \theta_k h_k T x_k \right\|_{L_q^Y[0,1)} \le \beta \left\| \sum_{k=1}^{\infty} h_k x_k \right\|_{L_q^X[0,1)},$$

(2) $\rho_q(T) := \inf \rho$, such that for all $0 = \tau_0 < \tau_1 < \cdots$ and finitely supported $(x_k)_k \subset X$

$$\left(\int_0^1 \left\| \sum_{l=1}^{\infty} r_l(t) \left(\sum_{k=\tau_{l-1}+1}^{\tau_l} h_k T x_k \right) \right\|_{L_q^Y[0,1)}^q dt \right)^{\frac{1}{q}} \le \rho \left\| \sum_{k=1}^{\infty} h_k x_k \right\|_{L_q^X[0,1)}.$$

For the case when such a constant $\beta > 0$ or $\rho > 0$ does not exist we set $\beta_q(T) = \infty$ and $\rho_q(T) = \infty$, respectively. In particular, let $\beta_q(X) = \beta_q(I_X)$ and $\rho_q(X) = \rho_q(I_X)$, where I_X is the identity of the Banach space X.

In Corollary A.2 we recall for $1 < q < \infty$

$$\frac{1}{c_q}\beta_q(\cdot) \le \beta_2(\cdot) \le c_q \beta_q(\cdot) \quad \text{and} \quad \frac{1}{c_q}\rho_q(\cdot) \le \rho_2(\cdot) \le c_q \rho_q(\cdot)$$

where $c_q > 0$ depends on q only. Therefore we set

$$\beta(\cdot) := \beta_2(\cdot) \quad \text{and} \quad \rho(\cdot) := \rho_2(\cdot)$$

but also use $\rho_q(\cdot)$ to apply interpolation techniques. The sequence $(d_l)_{l\ge 0}$ with $d_0 := 0$ used in (2) and (3) is a martingale difference sequence with respect to the filtration $(\mathcal{F}_l)_{l\ge 0}$ where $\mathcal{F}_l := \sigma(h_0, ..., h_{\tau_l})$. Hence, applying an approximation argument due to B. Maurey [20](Remarque 3) we obtain for an arbitrary martingale difference sequence $(d_l)_{l=0}^n \subset L_1^X(\Omega, \mathcal{F}, \mathbb{P})$ and $\theta_l = \pm 1$

$$\left\| \sum_{l=1}^{n} \theta_l T d_l \right\|_{L_2^Y} \le \beta(T) \left\| \sum_{l=1}^{n} d_l \right\|_{L_2^X}$$

and

$$\left(\int_0^1 \left\| \sum_{l=1}^{n} r_l(t) T d_l \right\|_{L_2^Y}^2 dt \right)^{\frac{1}{2}} \le \rho(T) \left\| \sum_{l=1}^{n} d_l \right\|_{L_2^X}.$$

Moreover, it is known that $\rho(T) \le \beta(T) = \beta(T')$ where $T' : Y' \to X'$ is the dual operator acting between the norm–duals Y' and X' of Y and X, respectively.

Let us turn to the question (Q) posed above. If one conjectures a negative answer one has to look for finite dimensional Banach spaces E_n with $\sup_n \frac{\beta(E_n)}{\rho(E_n)} = \infty$ which requires $\sup_n \beta(E_n) = \infty$. The spaces ℓ_1^N and ℓ_∞^N should be canonical candidates for this purpose. Using the validity of (3) for $X = \ell_1$ according to [11](Theorem 3) one gets in the first case

Example 2. There is some $c > 0$ such that for all $T : Y \to \ell_1$ one has

$$\rho(T) \le \beta(T) \le c\rho(T).$$

In the second case we obtain

Example 3. There is some constant $c > 0$ such that for all $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$ one has for $D_a : \ell_\infty \to \ell_\infty$ with $D_a\left((\xi_i)_i\right) := (\alpha_i \xi_i)_i$

$$\frac{1}{c} \sup_i (1+\log i)\alpha_i \leq \rho(D_a) \leq \beta(D_a) \leq c \sup_i (1+\log i)\alpha_i.$$

We shift the proof of Example 3 to the end of the introduction. In particular, choosing $\alpha_1 = \cdots = \alpha_n = 1$ and $0 = \alpha_{n+1} = \alpha_{n+2} = \cdots$ one gets $\frac{1}{c}(1+\log n) \leq \rho\left(\ell_\infty^n\right) \leq \beta\left(\ell_\infty^n\right) \leq c(1+\log n)$. The above examples show that the spaces ℓ_1^n and ℓ_∞^n do not provide a negative answer to question (Q). This leads in the next step to the investigation of the interpolation spaces generated by the operator of summation. The basic observation is the following: Although for the end points of this interpolation one has $\rho(\ell_1^{2^n}) \sim \rho(\ell_\infty^{2^n}) \sim n$ we obtain for the interpolation spaces a better estimate. So we can prove in Section 2 (for unexplained notation see Section 1)

Theorem 4. *There exist finite dimensional Banach spaces E_n $(n = 1, 2, ...)$ such that the ℓ_2–direct sum $X := \bigoplus_2 E_n$ is a superreflexive Banach space of type 2 and*

$$\sup_n \frac{\rho(E_n')}{\rho(E_n)} = \infty.$$

Hence there is some $T \in \mathcal{L}(X, X)$ with $\rho(T) < \infty$ and $\rho(T') = \infty$.

Recall that $\bigoplus_2 E_n$ is generated by the norm $\|(x_n)_{n=1}^\infty\| = \left(\sum_{n=1}^\infty \|x_n\|_{E_n}^2\right)^{\frac{1}{2}}$. Observing $\beta(S) = \beta(S') \geq \rho(S')$ for an operator S we deduce from the above theorem

Corollary 5. *There exist finite dimensional Banach spaces E_n $(n = 1, 2, ...)$ such that the ℓ_2–direct sum $X := \bigoplus_2 E_n$ is a superreflexive Banach space of type 2 and*

$$\sup_n \frac{\beta(E_n)}{\rho(E_n)} = \infty.$$

Hence there is some $T \in \mathcal{L}(X, X)$ with $\rho(T) < \infty$ and $\beta(T) = \infty$.

Hence the question (Q) from the beginning possesses a negative answer. By Proposition 2.7 we actually show more, namely that for all $1 \leq \alpha < 2$ there is no constant $c = c(\alpha) > 0$ such that for all Banach spaces X

$$\rho(X') \leq c\rho(X)^\alpha;$$

which implies the same for $\beta(X) \leq c\rho(X)^\alpha$ (concerning exponents $\alpha \geq 2$ no results in this direction are known, see Problem 4.2).

Let us comment on the type 2 property and the superreflexivity used in Theorem 4 and Corollary 5.

We begin with the type 2 property. On the one hand we get that the type 2 property, which is a fundamental property in the local theory of Banach spaces, does not allow a uniform estimate $\beta(X) \leq c\rho(X)$. On the other hand the occurrence of the type 2 property is not as surprising as it seems at first glance because of the following reason: A straightforward application of Fubini's theorem yields via

$$\left(\int_0^1 \left\| \sum_{l=1}^n r_l(t) \left(\sum_{k=\tau_{l-1}+1}^{\tau_l} h_k x_k \right) \right\|_{L_2^X}^2 dt \right)^{\frac{1}{2}} \leq 2\sqrt{n}\, t_2(X) \left\| \sum_{k=1}^{\tau_n} h_k x_k \right\|_{L_2^X} \tag{4}$$

an estimate closely related to (2). Unfortunately the estimate (4) requires (at least up to now) a control of the number n of 'blocks' used in the left–hand side so that we cannot use this observation (cf. Problem 4.1).

Let us turn to the superreflexivity. Denoting the best constant τ in inequality (3) by $\tau(X)$ we have $\beta(X) \leq \rho(X)\tau(X)$. Hence Corollary 5 implies that there is a superreflexive $X = \bigoplus_2 E_n$ of type 2 with

$$\tau(X) = \sup_n \tau(E_n) \geq \sup_n \frac{\beta(E_n)}{\rho(E_n)} = \infty.$$

The examples of G. Pisier [21] and D.H.J. Garling [11](Theorem 4) also yield superreflexive X (in fact of type 2 and a lattice with an upper 2–estimate, respectively) with $\tau(X) = \infty$ but do not include information about the relation between the quantities $\beta(\cdot)$ and $\rho(\cdot)$, which is the question of this paper. Nevertheless we will use Pisier's construction by exploiting additional information about the spaces involved in this construction.

We will proceed as follows. Basic results about interpolation and in particular about the interpolation spaces generated by the operator of summation due to G. Pisier and Q. Xu [23] are recalled in Section 1. In Section 2 we verify Theorem 4. The appendix contains the necessary material about the extrapolation techniques needed in this paper. In Section 3 a characterization of the non–superreflexive Banach spaces with the help of certain lower 2–estimates of sums of martingale differences is obtained as a byproduct of the considerations made in Section 2.

Proof of Example 3. (1) The upper estimate of $\beta(D_a)$ is known and can be deduced from Theorem A.1(a) if one uses $q = 2$:

$$A : L_1^{0,\mathbb{R}}\left([0,1), \mathcal{F}_K^h\right) \to L_1^+\left([0,1), \mathcal{F}_K^h\right) \text{ and } A\left(\sum_{k=1}^K h_k \xi_k\right)(t) := \left|\sum_{k=1}^K \theta_k h_k(t) \xi_k\right|,$$

where $\theta_k \in \{-1, 1\}$, such that $\|Af\|_2 = \|f\|_2$.

(2) To prove the lower estimate for $\rho(D_a)$ it is obviously sufficient to show that

$$\frac{1}{c}(1 + \log N) \leq \rho\left(\ell_\infty^N\right) \quad \text{for} \quad N = 1, 2, \ldots.$$

Moreover it is enough to do this for $N = 4^n$ with $n \geq 1$. For this purpose we construct i.i.d. $f^{(1)}, \ldots, f^{(N)} \in L_1[0,1)$ and i.i.d. $g^{(1)}, \ldots, g^{(N)} \in L_1([0,1) \times [0,1))$ by

$$f^{(i)}(s) := r_{\varphi_i(1)}(s) + \sum_{k=2}^n (-1)^{k-1} \prod_{l=1}^{k-1} \left(\frac{1 + r_{\varphi_i(l)}(s)}{2}\right) r_{\varphi_i(k)}(s)$$

and

$$g^{(i)}(t,s) := r_{\varphi_i(1)}(t) r_{\varphi_i(1)}(s) + \sum_{k=2}^n (-1)^{k-1} r_{\varphi_i(k)}(t) \prod_{l=1}^{k-1} \left(\frac{1 + r_{\varphi_i(l)}(s)}{2}\right) r_{\varphi_i(k)}(s)$$

where $\varphi_i(k) := (i-1)n + k$. It is easy to check that

$$\left\| f^{(i)} \right\|_\infty \leq 2 \quad \text{and} \quad \lambda \times \lambda\left(|g^{(i)}| \geq n\right) \geq \left(\frac{1}{4}\right)^n = \frac{1}{N}$$

where λ is the Lebesgue measure on $[0,1)$. Now we set

$$F := \left(f^{(1)}, \ldots, f^{(N)}\right) \in L_1^{\ell_\infty^N}[0,1) \text{ and } G := \left(g^{(1)}, \ldots, g^{(N)}\right) \in L_1^{\ell_\infty^N}\left([0,1)^2\right)$$

so that $\|F\|_{L_\infty^{\ell_\infty^N}} \leq 2$. On the other hand we obtain (cf. [2](Lemma 2.1))

$$\begin{aligned}
\|G\|_{L_1^{\ell_\infty^N}([0,1)\times[0,1))} &= \left\| \sup_{1\leq i\leq N} \left| g^{(1)}\left(t^i, s^i\right)\right| \right\|_{L_1\left(([0,1)\times[0,1))^N\right)} \\
&\geq n\,(\lambda\times\lambda)^N \left(\sup_{1\leq i\leq N} \left| g^{(1)}\left(t^i, s^i\right)\right| \geq n \right) \\
&= n\left[1 - \left[(\lambda\times\lambda)\left(\left|g^{(1)}\right| < n\right)\right]^N\right] \\
&= n\left[1 - \left[1-(\lambda\times\lambda)\left(\left|g^{(1)}\right| \geq n\right)\right]^N\right] \\
&\geq n\left[1 - \left[1 - \frac{1}{N}\right]^N\right] \geq \frac{n}{2}.
\end{aligned}$$

Since for $i = 1, ..., N$ and $k = 1, ..., n$ one has

$$\prod_{l=1}^{k-1} \left(\frac{1 + r_{\varphi_i(l)}}{2} \right) r_{\varphi_i(k)} \subseteq \operatorname{span}\left\{ h_{2^{(i-1)n+k-1}}, ..., h_{2^{(i-1)n+k}-1} \right\},$$

where we omit the product $\prod_{l=1}^{k-1}$ whenever $k = 1$, we observe that

$$G(t,s) = \sum_{l=1}^{Nn} r_l(t) \left(\sum_{k=2^{l-1}}^{2^l-1} h_k(s) x_k \right) \in L_1^{\ell_\infty^N}([0,1)\times[0,1))$$

if the $x_i \in \ell_\infty^N$ are taken from the representation $F = \sum_{k=1}^{2^{Nn}-1} h_k x_k$. □

Remark 6. There is also a duality argument for the estimate $\frac{1}{c}\log N \leq \rho(\ell_\infty^N)$ due to J. Wenzel [24] which is of interest if one does not need the independence of the coordinates of F and G, respectively. Using this argument one can choose F and G to be $\sum_{l=1}^n d_l(s)$ and $\sum_{l=1}^n r_l(t) d_l(s)$ where $(d_l)_0^n \subset L_1^{\ell_\infty^N}[0,1)$ is a dyadic martingale difference sequence and n is proportional to $\log N$.

1. Preliminaries

Basic notation. For simplicity all Banach spaces and random variables are assumed to be real. The Banach space of the linear and continuous operators $T : X \to Y$ from a Banach space X into a Banach space Y equipped with the operator norm $\|T\| := \sup\{\|Tx\| : x \in B_X\}$, where B_X is the closed unit ball of X, is denoted by $\mathcal{L}(X,Y)$. For integers $k, l \geq 0$ the σ–algebras $\mathcal{F}_k^h$ and $\mathcal{F}_l^{dyad}$ of Borel sets from $[0,1)$ are given by

$$\mathcal{F}_k^h := \sigma(h_0, ..., h_k) \quad \text{and} \quad \mathcal{F}_l^{dyad} := \sigma(h_0, ..., h_{2^l-1})$$

where $(h_k)_0^\infty \subset L_1[0,1)$ is the sequence of Haar functions. Given a martingale $f = (f_l)_{l\in I}$, where $I = \{0, ..., n\}$ or $I = \mathbb{N}$, we use $df_0 = f_0$ and $df_l = f_l - f_{l-1}$ for $l \geq 1$. A martingale with respect to $(\mathcal{F}_l^{dyad})_{l\in I}$ is called a dyadic martingale. To simplify the notation we will write $A \sim_c B$ instead of $\frac{1}{c}A \leq B \leq cA$. We shall often use the Khintchine–Kahane inequality for the Rademacher variables (see [18](Theorem 4.7)) which states $\|\sum_1^n r_l x_l\|_{L_q^X[0,1)} \sim_{c_q} \|\sum_1^n r_l x_l\|_{L_2^X[0,1)}$ for a Banach space X, $x_1,...,x_n \in X$, $0<q<\infty$, and the Rademacher variables $r_1,...,r_n$, where $c_q>0$ depends on q only.

Let $1\le p\le 2\le q<\infty$. A Banach space X is of type p (cotype q) if there is some $c>0$ such that for all finitely supported sequences $(x_l)_{l=1}^\infty \subset X$

$$\left\|\sum_{l=1}^{\infty} r_l x_l\right\|_{L_2^X} \le c\left(\sum_{l=1}^{\infty}\|x_l\|^p\right)^{\frac{1}{p}} \qquad \left(\left(\sum_{l=1}^{\infty}\|x_l\|^q\right)^{\frac{1}{q}} \le c\left\|\sum_{l=1}^{\infty} r_l x_l\right\|_{L_2^X}\right). \tag{5}$$

As usual $t_p(X):=\inf c$ ($c_q(X):=\inf c$). The Banach space X is of martingale type p (martingale cotype q) if there is some $c>0$ such that for all dyadic martingales $f=(f_l)_0^n \subset L_1^X[0,1)$ with $f_0=0$ one has

$$\|f_n\|_{L_2^X} \le c\left\|\left(\sum_1^n \|df_l\|_X^p\right)^{\frac{1}{p}}\right\|_2 \qquad \left(\left\|\left(\sum_1^n \|df_l\|_X^q\right)^{\frac{1}{q}}\right\|_2 \le c\|f_n\|_{L_2^X}\right).$$

As usual $Mt_p(X):=\inf c$ ($Mc_q(X):=\inf c$).

According to a result of G. Pisier [22] a Banach space X is superreflexive if and only if X is of martingale type p for some $p>1$ if and only if X is of martingale cotype q for some $q<\infty$. For convenience we take this equivalence as an alternative definition of superreflexivity.

Interpolation. For a compatible couple (E_0,E_1) of Banach spaces, $1\le q<\infty$, and $0<\theta<1$ we recall that the interpolation space $(E_0,E_1)_{\theta,q}$ is generated by the norm

$$\|x\|_{(E_0,E_1)_{\theta,q}} := \left(\int_0^\infty \left[t^{-\theta}K(x,t;E_0,E_1)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \qquad (x\in E_0+E_1)$$

where

$$K(x,t;E_0,E_1) := \inf\left\{\|x_0\|_{E_0}+t\|x_1\|_{E_1} \mid x=x_0+x_1\right\}$$

is the usual K–functional (see [4]). We will use

Lemma 1.1. *For all $0<\theta<1$ and $1<r<\infty$ one has*

$$\rho\left((E_0,E_1)_{\theta,r}\right) \le c\rho\left(E_0\right)^{1-\theta}\rho\left(E_1\right)^{\theta}$$

where $c>0$ depends on r only.

Proof. Fix $0=\tau_0<\tau_1<\cdots<\tau_L$ and define for $j=0,1$ and $\mathcal{G}_L=\mathcal{F}_L^{dyad}\times\mathcal{F}_{\tau_L}^h$ the operators $T_j : L_r^{E_j}\left([0,1),\mathcal{F}_{\tau_L}^h\right) \to L_r^{E_j}\left([0,1)^2,\mathcal{G}_L\right)$ by

$$T_j\left(\sum_0^{\tau_l} h_k x_k\right)(t,s) := \sum_1^L r_l(t)\left(\sum_{k=\tau_{l-1}+1}^{\tau_l} h_k(s)x_k\right).$$

It is known that

$$\left(L_r^{E_0}\left([0,1),\mathcal{F}_{\tau_L}^h\right), L_r^{E_1}\left([0,1),\mathcal{F}_{\tau_L}^h\right)\right)_{\theta,r} = L_r^{(E_0,E_1)_{\theta,r}}\left([0,1),\mathcal{F}_{\tau_L}^h\right)$$

and

$$\left(L_r^{E_0}\left([0,1)^2,\mathcal{G}_L\right), L_r^{E_1}\left([0,1)^2,\mathcal{G}_L\right)\right)_{\theta,r} = L_r^{(E_0,E_1)_{\theta,r}}\left([0,1)^2,\mathcal{G}_L\right)$$

where the constants involved in the norm equivalences are majorized by an absolute constant. Hence we get by interpolation for $T=(T_0,T_1)_{\theta,r}$

$$\left\|T : L_r^{(E_0,E_1)_{\theta,r}}\left([0,1),\mathcal{F}_{\tau_L}^h\right) \to L_r^{(E_0,E_1)_{\theta,r}}\left([0,1)^2,\mathcal{G}_L\right)\right\| \le c\|T_0\|^{1-\theta}\|T_1\|^{\theta}$$

where $c > 0$ is an absolute constant. Finally $\|T_j\| \le 2\rho_r(E_j)$ and Corollary A.2 imply the assertion. $\square$

Similarly, for $1 \le p \le 2$, $p \le r < \infty$, and $0 < \theta < 1$ the Khintchine–Kahane inequality and $\|\cdot\|_{(\ell_p^n(E_0),\ell_p^n(E_1))_{\theta,r}} \le c_0\|\cdot\|_{\ell_p^n((E_0,E_1)_{\theta,r})}$, where $c_0 > 0$ is an absolute constant, imply the basically known formula

$$t_p\left((E_0,E_1)_{\theta,r}\right) \le c\; t_p\left(E_0\right)^{1-\theta} t_p\left(E_1\right)^{\theta} \tag{6}$$

where $c > 0$ depends on r only (cf. [21](Lemma 4)). Finally, from [21](Lemma 4) and [13](Corollary 8.6) (cf. [22](Remark 3.3)) one gets for $1 \le p, p_0, p_1 \le 2$ and $0 < \theta < 1$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$

$$Mt_p\left((E_0,E_1)_{\theta,p}\right) \le cMt_{p_0}\left(E_0\right)^{1-\theta} Mt_{p_1}\left(E_1\right)^{\theta} \tag{7}$$

with $c > 0$ depending on p, p_0, and p_1 only.

The spaces $A_q^N(X)$ and $V_q[0,1)$.

Definition 1.2. For $1 \le q \le \infty$, a Banach space X, and $(x_i)_1^N \subset X$ let

$$\|(x_i)_i\|_{v_q^N(X)} := \sup \left\| \left(\|x_{\tau_0}\|, \|x_{\tau_1} - x_{\tau_0}\|, \cdots, \|x_{\tau_L} - x_{\tau_{L-1}}\|\right) \right\|_{\ell_q^{L+1}}$$

where the supremum is taken over $L = 1, 2, ...$ and $1 \le \tau_0 \le \tau_1 \le \cdots \le \tau_L \le N$. The spaces of N–tuples $(x_i)_{i=1}^N \subset X$ equipped with the norms $\|\cdot\|_{v_q^N(X)}$ are denoted by $v_q^N(X)$ and $v_q^N := v_q^N(\mathbb{R})$. The operator of summation

$$\sigma_N^q : \ell_1^N \to v_q^N \quad \text{is given by} \quad (\xi_i)_{i=1}^N \to \left(\sum_{i=1}^{j} \xi_i\right)_{j=1}^{N}.$$

We also use $\ell_\infty^N(X)$ the space of N–tuples $(x_i)_{i=1}^N \subset X$ endowed with the norm $\|(x_i)_i\|_{\ell_\infty^N(X)} := \sup_i \|x_i\|$ and set for $1 < p, q < \infty$ with $1 = \frac{1}{p} + \frac{1}{q}$

$$A_q^N(X) := \left(v_1^N(X), \ell_\infty^N(X)\right)_{\frac{1}{p},q} \quad \text{and} \quad A_q^N := \left(v_1^N, \ell_\infty^N\right)_{\frac{1}{p},q}.$$

In A_q^N we always take the coordinates arising from the standard coordinates of v_1^N and ℓ_∞^N. The spaces A_q^N are dual to each other in the following sense. Because of the map $(\xi_i)_1^N \to (\xi_N, \xi_N + \xi_{N-1}, ..., \xi_N + \cdots + \xi_1)$, which acts as an isometry between ℓ_1^N and v_1^N as well as $(v_1^N)'$ and ℓ_∞^N, we obtain (see [4](Theorem 3.7.1))

$$(A_q^N)' = (v_1^N, \ell_\infty^N)'_{\frac{1}{p},q} = \left(\ell_1^N, (v_1^N)'\right)_{\frac{1}{q},p} = \left(v_1^N, \ell_\infty^N\right)_{\frac{1}{q},p} = A_p^N$$

where the multiplicative constants involved in the norm equivalences are majorized by a constant depending on p only. Moreover $v_1 \subseteq (v_1, \ell_\infty)_{\frac{1}{q},p} \subseteq v_p$ ([5],[23] (Lemma 2)) gives

$$v_1^N \to A_p^N \to v_p^N \tag{8}$$

where the norms of the embeddings are again majorized by a constant depending on p only.

Theorem 1.3. ([23](Theorems 1 and 8)) *For $1 < q < \infty$ the following holds.*

(1) $\sup_N t_{q\wedge 2}\left(A_q^N\right) < \infty$ *for* $q \neq 2$.

(2) *There is some $c > 0$, depending on q only, such that for all $x_1, ..., x_L \in A_q^N$*

$$\left(\int_0^1 \left\|\sum_{l=1}^L r_l(t) x_l\right\|_{A_q^N}^2 dt\right)^{\frac{1}{2}} \sim_c \left\|\left((x_l^{(i)})_{l=1}^L\right)_{i=1}^N\right\|_{A_q^N(\ell_2^L)}$$

where $x_l = (x_l^{(i)})_{i=1}^N$.

(3) *There is some $c > 0$ such that for all $N = 1, 2, ...$ there is a Euclidean norm $\|\cdot\|$ on A_2^N such that for $H_2^N := \left[A_2^N, \|\cdot\|\right]$*

$$\left\|I : A_2^N \to H_2^N\right\| \left\|I : H_2^N \to A_2^N\right\| \leq c(1 + \log N).$$

Remark 1.4. In [23] the spaces $A_{\theta q}(X) := (v_1(X), \ell_\infty(X))_{\theta,q}$ are used where $v_p(X)$ and $\ell_\infty(X)$ consist of $(x_i)_1^\infty \subset X$ and are defined in the same way as $v_p^N(X)$ and $\ell_\infty^N(X)$. To get Theorem 1.3 one has to observe

$$\frac{1}{2}\|(x_1, ..., x_N, 0, 0, ...)\|_{v_p(X)} \leq \left\|(x_i)_{i=1}^N\right\|_{v_p^N(X)} \leq \|(x_i)_{i=1}^\infty\|_{v_p(X)}$$

so that the $A_q^N(X)$ are subspaces of $A_{\frac{1}{p}q}(X)$ via

$$\|(x_1, ..., x_N)\|_{A_q^N(X)} \leq \|(x_1, ..., x_N, 0, 0, ...)\|_{A_{\frac{1}{p}q}(X)} \leq 2\|(x_1, ..., x_N)\|_{A_q^N(X)}.$$

Definition 1.5. For $1 \leq q \leq \infty$ and $f : [0, 1) \to \mathbb{R}$ let

$$\|f\|_{V_q} := \sup \|(f(t_0), f(t_1) - f(t_0), ..., f(t_L) - f(t_{L-1}))\|_{\ell_q^{L+1}},$$

where the supremum is taken over $L = 1, 2, ...$ and $0 \leq t_0 \leq t_1 \leq \cdots \leq t_L < 1$. As usual

$$V_q[0, 1) := \left\{f \in L_\infty[0, 1) \mid \|f\|_{V_q} < \infty\right\}.$$

The operator of integration $I^q : L_1[0, 1) \to V_q[0, 1)$ is given by $(I^q f)(s) = \int_0^s f(t)dt$.

Remark 1.6. We shall use the following observation. Given a continuous function $f : [0, 1) \to \mathbb{R}$ linear on all $[s_{k-1}, s_k)$ for some $0 = s_0 < s_1 < \cdots < s_K = 1$ and if $f(1) := \lim_{s\to 1} f(s)$, then

$$\|f\|_{V_q} = \sup \|(f(t_0), f(t_1) - f(t_0), ..., f(t_L) - f(t_{L-1}))\|_{\ell_q^{L+1}}$$

where the supremum is taken over all $L = 1, 2, ...$ and $0 \leq t_0 \leq t_1 \leq \cdots \leq t_L \leq 1$ such that $\{t_0, ..., t_L\} \subseteq \{s_0, ..., s_K\}$.

2. Proof of Theorem 4

Before we prove Proposition 2.7 which immediately implies Theorem 4 we need a couple of lemmas.

Lemma 2.1. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^{m+1}$ such that for all $(\lambda_0, ..., \lambda_m) \in \mathbb{R}^{m+1}$ one has*

$$\|(0, ..., 0, \lambda_0, ..., \lambda_{m-l})\| \leq \|(\lambda_0, ..., \lambda_m)\| \quad \text{for} \quad l = 0, ..., m.$$

If $a_l := \lambda_l + \frac{\lambda_{l-1}}{2} + \cdots + \frac{\lambda_0}{2^l}$ for $l = 0, ..., m$, then

$$\frac{2}{3}\|(\lambda_0, ..., \lambda_m)\| \leq \|(a_0, ..., a_m)\| \leq 2\|(\lambda_0, ..., \lambda_m)\|.$$

Proof. The right–hand side follows from

$$\|(a_0, ..., a_m)\| \le \sum_{l=0}^{m} \frac{1}{2^l} \|(0, ..., 0, \lambda_0, ..., \lambda_{m-l})\| \le 2 \|(\lambda_0, ..., \lambda_m)\| .$$

For the left–hand side we consider $\lambda_l = a_l - \frac{a_{l-1}}{2}$ for $l = 0, ..., m$, where $a_{-1} = 0$, and

$$\|(\lambda_0, ..., \lambda_m)\| \le \|(a_0, ..., a_m)\| + \frac{1}{2} \|(0, a_0, ..., a_{m-1})\| \le \frac{3}{2} \|(a_0, ..., a_m)\| . \qquad \square$$

Now let us fix the dyadic martingale $(F_l)_{l\ge 0} \subset L_1^{L_1[0,1)}[0, 1)$ given by

$$F_l(t) := 2^l \sum_{k=1}^{2^l} \chi_{[\frac{k-1}{2^l}, \frac{k}{2^l})}(t)\chi_{[\frac{k-1}{2^l}, \frac{k}{2^l})}$$

so that $\|F_l(t)\|_{L_1[0,1)} = 1$ for $0 \le t < 1$.

Lemma 2.2. *For $1 \le p \le \infty$, $0 \le t < 1$, and $\xi_1, ..., \xi_n \in \mathbb{R}$ one has*

$$\left\|\sum_{l=1}^{n} \xi_l I^p dF_l(t)\right\|_{V_p} \ge \max\left\{ \frac{1}{12} \|(\xi_l)_1^n\|_{v_p^n} , \frac{1}{4} \left\|\sigma_n^p\Big((r_1(t)a_0, ..., r_n(t)a_{n-1})\Big)\right\|_{v_p^n} \right\}$$

with $a_l := \lambda_l + \frac{\lambda_{l-1}}{2} + \cdots + \frac{\lambda_0}{2^l}$ where $\lambda_0 := -\xi_1$, $\lambda_1 := \xi_1 - \xi_2$, ..., $\lambda_{n-1} := \xi_{n-1} - \xi_n$, $\lambda_n := \xi_n$, and $(r_l)_1^n$ is the sequence of Rademacher functions.

Proof. First note that $2^l a_l = 2^0\lambda_0 + \cdots + 2^l\lambda_l$. Now fix $0 \le t < 1$ and let $A_l \in \mathcal{F}_l^{dyad}$ be the atoms such that $t \in A_n \subset A_{n-1} \subset \cdots \subset A_0 = [0, 1)$. We obtain

$$\begin{aligned}
\sum_{l=1}^{n} \xi_l dF_l(t) &= \sum_{l=0}^{n} \lambda_l F_l(t) = \sum_{l=0}^{n} \left(2^l\lambda_l\right)\left(2^{-l}F_l(t)\right) = \sum_{l=0}^{n} \left(2^l\lambda_l\right)\chi_{A_l} \\
&= \chi_{A_n} 2^n a_n + \chi_{A_{n-1}\setminus A_n} 2^{n-1} a_{n-1} + \cdots + \chi_{A_0\setminus A_1} 2^0 a_0 \\
&= a_n \frac{\chi_{A_n}}{\lambda(A_n)} + \frac{1}{2}\left[a_{n-1}\frac{\chi_{A_{n-1}\setminus A_n}}{\lambda(A_{n-1}\setminus A_n)} + \cdots + a_0 \frac{\chi_{A_0\setminus A_1}}{\lambda(A_0\setminus A_1)}\right].
\end{aligned}$$

Furthermore, one easily sees that $A_{l-1}\setminus A_l$ lies to the left of A_n if $r_l(t) = -1$ and to the right of A_n if $r_l(t) = 1$. Hence

$$\begin{aligned}
\left\|\sum_{l=1}^{n} \xi_l I^p dF_l(t)\right\|_{V_p} = \Bigg\|\sigma_{2n+1}^p \Bigg(&\frac{1 - r_1(t)}{4} a_0, ..., \frac{1 - r_n(t)}{4} a_{n-1}, a_n, \\
&\frac{1 + r_n(t)}{4} a_{n-1}, ..., \frac{1 + r_1(t)}{4} a_0\Bigg)\Bigg\|_{v_p^{2n+1}}
\end{aligned}$$

where we have used Remark 1.6. Observing

$$\left\|\sigma_m^p\big((\eta_1, ..., \eta_m)\big)\right\|_{v_p^m} = \left\|\sigma_m^p\big((\eta_m, ..., \eta_1)\big)\right\|_{v_p^m}$$

we continue by Lemma 2.1 to

$$\begin{aligned}\left\|\sum_{l=1}^{n}\xi_l I^p dF_l(t)\right\|_{V_p} \quad &\geq \quad \max\left\{\left\|\sigma_n^p\left(\left(\frac{1-r_1(t)}{4}a_0,...,\frac{1-r_n(t)}{4}a_{n-1}\right)\right)\right\|_{v_p^n}, |a_n|,\right.\\ &\left.\left\|\sigma_n^p\left(\left(\frac{1+r_1(t)}{4}a_0,...,\frac{1+r_n(t)}{4}a_{n-1}\right)\right)\right\|_{v_p^n}\right\}\\ &\geq \frac{1}{4}\max\left\{\|\sigma_n^p((a_0,...,a_{n-1}))\|_{v_p^n}, \|\sigma_n^p((r_1(t)a_0,...,r_n(t)a_{n-1}))\|_{v_p^n}, 4|a_n|\right\}\\ &\geq \max\left\{\frac{1}{8}\left\|\sigma_{n+1}^p((a_0,...,a_n))\right\|_{v_p^{n+1}}, \frac{1}{4}\|\sigma_n^p((r_1(t)a_0,...,r_n(t)a_{n-1}))\|_{v_p^n}\right\}\\ &\geq \max\left\{\frac{1}{12}\left\|\sigma_{n+1}^p((\lambda_0,...,\lambda_n))\right\|_{v_p^{n+1}}, \frac{1}{4}\|\sigma_n^p((r_1(t)a_0,...,r_n(t)a_{n-1}))\|_{v_p^n}\right\}\\ &= \max\left\{\frac{1}{12}\|(\xi_l)_1^n\|_{v_p^n}, \frac{1}{4}\|\sigma_n^p((r_1(t)a_0,...,r_n(t)a_{n-1}))\|_{v_p^n}\right\}. \qquad \square\end{aligned}$$

Remark 2.3. (1) We apply this lemma to the restriction of I^p to the $\mathcal{F}_n^{dyad}$–measurable functions which will be considered as $\sigma_{2^n}^p : \ell_1^{2^n} \to v_p^{2^n}$. Nevertheless we have formulated the lemma for I^p to get a more transparent proof.

(2) Moreover, it is easy to see that one also has a converse inequality.

Lemma 2.4. *For all $1 < q < \infty$, $N = 1, 2, ...$, $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$, and all finitely supported sequences $(x_k)_{k=0}^{\infty} \subset \ell_1^N$ one has*

$$\left(\int_0^1 \left[\sum_{i=1}^{N}\left[\sum_{l=1}^{\infty}\left|\sum_{k=\tau_{l-1}+1}^{\tau_l} h_k(t)\langle x_k, e_i\rangle\right|^2\right]^{\frac{1}{2}}\right]^q dt\right)^{\frac{1}{q}} \leq c(1+\log N)\left\|\sum_{k=0}^{\infty} h_k x_k\right\|_{L_q^{\ell_1^N}}$$

where $c > 0$ depends on q only and $(e_i)_i$ is the unit vector basis of ℓ_∞^N.

Proof. Setting $d_l := \sum_{k=\tau_{l-1}+1}^{\tau_l} h_k x_k$ we derive with the help of the Khintchine–Kahane inequality for the Rademacher averages

$$\left\|\sum_{i=1}^{N}\left(\sum_{l=1}^{L}|\langle d_l, e_i\rangle|^2\right)^{\frac{1}{2}}\right\|_q \leq c\left\|\sum_{i=1}^{N}\int_0^1\left|\left\langle\sum_{l=1}^{L} r_l(t)d_l, e_i\right\rangle\right|dt\right\|_q$$

$$\leq c\int_0^1\left\|\sum_{l=1}^{L} r_l(t)d_l\right\|_{L_q^{\ell_1^N}} dt \leq c\rho_q\left(\ell_1^N\right)\left\|\sum_{l=1}^{L} d_l\right\|_{L_q^{\ell_1^N}} \leq 2c\rho_q\left(\ell_1^N\right)\left\|\sum_{k=0}^{\tau_L} h_k x_k\right\|_{L_q^{\ell_1^N}}$$

and conclude with $\rho_q\left(\ell_1^N\right) \leq c_q\rho\left(\ell_1^N\right) \leq c_q\beta\left(\ell_1^N\right) = c_q\beta\left(\ell_\infty^N\right) \leq c_q'(1+\log N)$ according to Corollary A.2 and Example 3. $\square$

Lemma 2.5. *For all $1 < q < \infty$, $N = 1, 2, ...$, $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$, and all finitely supported sequences $(x_k)_{k=0}^{\infty} \subset \ell_\infty^N$ one has*

$$\left(\int_0^1 \sup_i\left(\sum_{l=1}^{\infty}\left|\sum_{k=\tau_{l-1}+1}^{\tau_l} h_k(t)\langle x_k, e_i\rangle\right|^2\right)^{\frac{q}{2}} dt\right)^{\frac{1}{q}} \leq c\sqrt{1+\log N}\left\|\sum_{k=0}^{\infty} h_k x_k\right\|_{L_q^{\ell_\infty^N}}$$

where $c>0$ depends on q only and $(e_i)_i$ is the unit vector basis of ℓ_1^N.

Proof. We can assume that $x_0=0$. Fix $L\geq 1$, $0=\tau_0<\tau_1<\tau_2<\cdots<\tau_L$, and define the square–function operator $S_2 : L_1^{0,\mathbb{R}}\left([0,1),\mathcal{F}_{\tau_L}^h\right)\to L_1^+\left([0,1),\mathcal{F}_{\tau_L}^h\right)$ (for the notation of the spaces see the appendix) by

$$S_2 f(t) := \left(\sum_{l=1}^{L}\left(\sum_{k=\tau_{l-1}+1}^{\tau_l} h_k(t)\xi_k\right)^2\right)^{\frac{1}{2}}$$

for $f=\sum_1^{\tau_L} h_k\xi_k$ with $\xi_k\in\mathbb{R}$. It is clear that $A:=S_2$ satisfies the assumptions (1)–(4) of Theorem A.1. Moreover, the Burkholder–Davis–Gundy inequalities (see [12](Theorem II.1.1)) and Doob's maximal inequality give for $q\geq q_0:=2$

$$\|Af\|_q \leq \sqrt{2q}\left\|\sup_{1\leq l\leq L}\mathbb{E}\left(f\mid\mathcal{F}_{\tau_l}^h\right)\right\|_q \leq \sqrt{2q}\frac{q}{q-1}\|f\|_q \leq 2\sqrt{2q}\|f\|_q .$$

Now we apply Theorem A.1(b) to the case $r=2$, $\frac{A}{2\sqrt{2}}$, $q_0=2$, and $K=\tau_L$. □

Lemma 2.6. *For $1<q<\infty$ there is a constant $c>0$ depending on q only such that*

$$\rho(A_q^N) \leq c\,(1+\log N)^{\frac{1}{2}+\frac{1}{2q}} .$$

Proof. For $0=\tau_0<\tau_1<\cdots<\tau_L$ we define

$$T_0 : L_q^{v_1^N}\left([0,1),\mathcal{F}_{\tau_L}^h\right)\to L_q^{v_1^N(\ell_2^L)}\left([0,1),\mathcal{F}_{\tau_L}^h\right)$$

and

$$T_1 : L_q^{\ell_\infty^N}\left([0,1),\mathcal{F}_{\tau_L}^h\right)\to L_q^{\ell_\infty^N(\ell_2^L)}\left([0,1),\mathcal{F}_{\tau_L}^h\right)$$

by

$$T_j\left(\left(f^{(1)},...,f^{(N)}\right)\right) := \left(df_l^{(i)}\right)_{l=1,i=1}^{L,N}$$

where $df_l^{(i)}=\mathbb{E}\left(f^{(i)}\mid\mathcal{F}_{\tau_l}^h\right)-\mathbb{E}\left(f^{(i)}\mid\mathcal{F}_{\tau_{l-1}}^h\right)$. It is known that

$$\left(L_q^{v_1^N}\left([0,1),\mathcal{F}_{\tau_L}^h\right),L_q^{\ell_\infty^N}\left([0,1),\mathcal{F}_{\tau_L}^h\right)\right)_{\frac{1}{p},q} = L_q^{A_q^N}\left([0,1),\mathcal{F}_{\tau_L}^h\right)$$

and

$$\left(L_q^{v_1^N(\ell_2^L)}\left([0,1),\mathcal{F}_{\tau_L}^h\right),L_q^{\ell_\infty^N(\ell_2^L)}\left([0,1),\mathcal{F}_{\tau_L}^h\right)\right)_{\frac{1}{p},q} = L_q^{A_q^N(\ell_2^L)}\left([0,1),\mathcal{F}_{\tau_L}^h\right)$$

where the multiplicative constants involved in both norm equivalences are majorized by an absolute constant. Identifying ℓ_1^N and v_1^N via $\|(\xi_1,...,\xi_N)\|_{v_1^N} = \|(\xi_1,\xi_2-\xi_1,...,\xi_N-\xi_{N-1})\|_{\ell_1^N}$, Lemmas 2.4 and 2.5 imply some $c_1>0$, depending on q only, such that for $T=(T_0,T_1)_{\frac{1}{p},q}$ (note that $(1-\frac{1}{p})+\frac{1}{p}\frac{1}{2}=\frac{1}{2}+\frac{1}{2q}$)

$$\left\|T : L_q^{A_q^N}\left([0,1),\mathcal{F}_{\tau_L}^h\right)\to L_q^{A_q^N(\ell_2^L)}\left([0,1),\mathcal{F}_{\tau_L}^h\right)\right\| \leq c_1\,(1+\log N)^{\frac{1}{2}+\frac{1}{2q}} .$$

Now let $f=(f_l)_0^L \subset L_1^{A_q^N}[0,1)$ be a martingale with respect to $(\mathcal{F}_{\tau_l}^h)_{l=0}^L$ such that $f_0=0$. Theorem 1.3(2) and the Khintchine–Kahane inequality imply for $df_l=(df_l^{(i)})_{i=1}^N$ that

$$\begin{aligned}\left(\int_0^1\int_0^1\left\|\sum_{l=1}^L r_l(t)df_l(s)\right\|_{A_q^N}^q dtds\right)^{\frac{1}{q}} &\le c_2\left(\int_0^1\left\|\left((df_l^{(i)}(s))_l\right)_i\right\|_{A_q^N(\ell_2^L)}^q ds\right)^{\frac{1}{q}}\\ &\le c_1c_2\left(1+\log N\right)^{\frac{1}{2}+\frac{1}{2q}}\|f_L\|_{L_q^{A_q^N}[0,1)}.\end{aligned}$$

Now we can conclude with Corollary A.2. □

Proposition 2.7. *For all $\varepsilon>0$ there is a constant $c>0$ and a sequence of Banach spaces E_n with $\dim(E_n)=2^n$ such that*

(1) *$\bigoplus_2 E_n$ is superreflexive and of type 2,*

(2) *$\rho(E_n)\le cn^{\frac{1}{2}+\varepsilon}$,*

(3) *there is a dyadic martingale $g=(g_l)_0^n\subset L_\infty^{E_n'}[0,1)$ with $g_0=0$,*

$$\|g_n\|_{L_\infty^{E_n'}}\le n^\varepsilon, \quad \textit{and} \quad \frac{1}{c}\,\|(\xi_l)_1^n\|_{v_{1+\varepsilon}^n}\le \inf_{0\le t<1}\left\|\sum_1^n \xi_l dg_l(t)\right\|_{E_n'}$$

for all $(\xi_l)_1^n\subset\mathbb{R}$.

Proof. The construction of the spaces E_n and the considerations in (i) follow the ideas of [21]. First we apply Theorem 1.3(3) for $N=2^n$ to get

$$\left\|I:A_2^N\to H_2^N\right\|\left\|I:H_2^N\to A_2^N\right\|\le c_1 n \tag{9}$$

where $c_1>0$ is an absolute constant. Now for $1<p<2<q<\infty$ with $1=\frac{1}{p}+\frac{1}{q}$ let

$$G_p^N:=\left(A_p^N,H_2^N\right)_{\frac{2}{q},p} \quad \text{and} \quad E_n:=\left(G_p^N\right)'.$$

From now on all constants $c_2,c_3,\dots$ following below will depend (at most) on p.

(i) Since $\frac{1}{p}=\left(1-\frac{2}{q}\right)+\frac{2}{q}\frac{1}{2}$ we deduce from (7) and $Mt_1(A_p^N)=Mt_2(H_2^N)=1$

$$Mt_p\left(\bigoplus_2 G_p^N\right)=\sup_{N=2^n}Mt_p(G_p^N)\le c_2\sup_{N=2^n}Mt_1(A_p^N)^{1-\frac{2}{q}}Mt_2(H_2^N)^{\frac{2}{q}}=c_2.$$

Consequently $\bigoplus_2 G_p^N$, and by duality $\bigoplus_2 E_n$, is superreflexive. The space $\bigoplus_2 E_n$ is of type 2 since [4](Theorem 3.7.1), (6), and Theorem 1.3(1) give

$$\begin{aligned}t_2\left(\bigoplus_2 E_n\right) &= \sup_{N=2^n}t_2\left(\left(G_p^N\right)'\right)\le c_3\sup_{N=2^n}t_2\left(\left((A_p^N)',(H_2^N)'\right)_{\frac{2}{q},q}\right)\\ &\le c_4\sup_{N=2^n}t_2\left(A_q^N\right)^{1-\frac{2}{q}}<\infty.\end{aligned}$$

(ii) Lemma 1.1 and again [4](Theorem 3.7.1) imply that

$$\rho\left(E_n\right)\le c_5\rho\left(\left((A_p^N)',(H_2^N)'\right)_{\frac{2}{q},q}\right)\le c_6\rho\left(A_q^N\right)^{1-\frac{2}{q}}$$

so that according to Lemma 2.6

$$\rho\left(E_n\right)\le c_7 n^{\left(1-\frac{2}{q}\right)\left(\frac{1}{2}+\frac{1}{2q}\right)}\le c_7 n^{\left(\frac{1}{2}+\frac{1}{2q}\right)}.$$

(iii) Defining $1<\alpha,\beta<\infty$ such that $\frac{1}{\alpha}=\frac{1}{p}-\left(1-\frac{2}{q}\right)\frac{1}{q}$ and $\frac{1}{\beta}=\frac{1}{q}+\left(1-\frac{2}{q}\right)\frac{1}{q}$ and exploiting the reiteration theorem [4](Theorem 3.5.3) it follows that

$$\left(A_p^N,A_2^N\right)_{\frac{2}{q},p}=\left(v_1^N,\ell_\infty^N\right)_{\frac{1}{\beta},p}\subseteq\left(v_1^N,\ell_\infty^N\right)_{\frac{1}{\beta},\alpha}=A_\alpha^N\subseteq v_\alpha^N \tag{10}$$

where we used (8) and where the constants in the norm–estimates are majorized by constants depending on p only. Interpolating the identities $v_1^N\to A_p^N\to A_p^N$ and $v_1^N\to H_2^N\to A_2^N$ with parameters $(2/q,p)$ yields together with (9) and (10)

$$\left\|I:v_1^N\to E_n'\right\|\left\|I:E_n'\to v_\alpha^N\right\|\le c_8n^{\frac{2}{q}}. \tag{11}$$

Via the isometric embeddings $S_N:\ell_1^N\to L_1[0,1)$ and $T_N^\alpha:v_\alpha^N\to V_\alpha[0,1)$, where

$$S_N\left((\xi_i)_{i=1}^N\right)(t):=N\sum_{i=1}^N\xi_i\chi_{\left[\frac{i-1}{N},\frac{i}{N}\right)}(t)$$

and $T_N^\alpha\left((\eta_j)_{j=1}^N\right)$ is the continuous, on the intervals $\left[\frac{j-1}{N},\frac{j}{N}\right)$, linear function f satisfying $f(0)=0$ and $f\left(\frac{j}{N}\right)=\eta_j$ (with $f(1):=\lim_{t\to1}f(t)$), we see that σ_N^α is the restriction of I^α to the $\mathcal{F}_n^{dyad}$–measurable functions. Consequently Lemma 2.2 gives a dyadic martingale $f=(f_l)_{l=0}^n\subset L_1^{v_1^N}[0,1)$ with

$$\|f_n-f_0\|_{L_\infty^{v_1^N}}\le2\|f_n\|_{L_\infty^{v_1^N}}\le2\text{ and }\frac{1}{12}\|(\xi_l)_{l=1}^n\|_{v_\alpha^n}\le\inf_{0\le t<1}\left\|\sum_1^n\xi_ldf_l(t)\right\|_{v_\alpha^N}$$

for all $(\xi_l)_1^n\subset\mathbb{R}$, where we 'isometrically' pass from $\sigma_N^\alpha:\ell_1^N\to v_\alpha^N$ to the identity $I:v_1^N\to v_\alpha^N$. Using (11) we obtain a dyadic martingale $g=(g_l)_0^n\subset L_\infty^{E_n'}[0,1)$ with $g_0=0$,

$$\|g_n\|_{L_\infty^{E_n'}}\le n^{\frac{2}{q}},\quad\text{and}\quad\frac{1}{24c_8}\|(\xi_l)_1^n\|_{v_\alpha^n}\le\inf_{0\le t<1}\left\|\sum_1^n\xi_ldg_l(t)\right\|_{E_n'}.$$

Now we have to arrange p and q such that $\frac{2}{q}\le\varepsilon$, $\alpha=\frac{1}{\frac{1}{p}-\left(1-\frac{2}{q}\right)\frac{1}{q}}\le1+\varepsilon$, and $\frac{1}{2}+\frac{1}{2q}\le\frac{1}{2}+\varepsilon$. □

Remark 2.8. (1) The proof of Theorem 4 given below requires the above proposition for those $\varepsilon>0$ such that $\frac{1}{1+\varepsilon}>\frac{1}{2}+2\varepsilon$. For this purpose an upper estimate by $c(1+\log N)$ in Lemma 2.6 would not be sufficient.

(2) By duality it follows from assertion (3) of the above proposition that for all $n=1,2,...$ there is a dyadic martingale $(f_l)_0^n\subset L_1^{E_n}[0,1)$ with $f_0=0$,

$$\|f_n\|_{L_2^{E_n}[0,1)}\le1,\text{ and }\sup_{\theta_l=\pm1}\left\|\sum_1^n\theta_ldf_l\right\|_{L_2^{E_n}[0,1)}\ge\frac{n^{\frac{1}{1+\varepsilon}-\varepsilon}}{2c}.$$

Because of

$$\begin{aligned}\int_0^1\int_0^1\left\|\sum_1^nr_l(t)df_l(s)\right\|_{E_n}^2dtds&\ge\int_0^1\left[\frac{1}{\sqrt{n}}\sup_{\theta_l=\pm1}\left\|\sum_1^n\theta_ldf_l(s)\right\|_{E_n}\right]^2ds\\&\ge\left(\frac{1}{\sqrt{n}}\sup_{\theta_l=\pm1}\left\|\sum_1^n\theta_ldf_l\right\|_{L_2^{E_n}[0,1)}\right)^2\end{aligned}$$

we obtain $\rho(E_n) \geq \frac{n^{\frac{1}{1+\varepsilon}-\varepsilon-\frac{1}{2}}}{2c}$. Hence assertion (2) of the above proposition is nearly optimal (for small ε) if one supposes that assertion (3) is satisfied.

Proof of Theorem 4. Choosing $\varepsilon > 0$ such that $\frac{1}{1+\varepsilon} > \frac{1}{2} + 2\varepsilon$ we take the spaces E_n from Proposition 2.7 and get (note that $1+\varepsilon \leq 2$)

$$\begin{aligned}\sup_n \frac{\rho(E_n')}{\rho(E_n)} &\geq \sup_n \frac{1}{c^2 n^{\frac{1}{2}+2\varepsilon}} \left(\int_0^1 \left(|r_1(t)|^{1+\varepsilon} + \sum_{l=2}^n |r_l(t) - r_{l-1}(t)|^{1+\varepsilon} \right)^{\frac{2}{1+\varepsilon}} dt \right)^{\frac{1}{2}} \\ &\geq \sup_n \frac{1}{c^2 n^{\frac{1}{2}+2\varepsilon}} \left(1 + \sum_{l=2}^n \int_0^1 |r_l(t) - r_{l-1}(t)|\, dt \right)^{\frac{1}{1+\varepsilon}} = \infty.\end{aligned}$$

Now we consider the operator $T : E_1 \oplus_2 E_2 \oplus_2 \cdots \to E_1 \oplus_2 E_2 \oplus_2 \cdots$ given by

$$T(x_1, x_2, ...) := \left(\frac{x_1}{\rho(E_1)}, \frac{x_2}{\rho(E_2)}, \cdots \right)$$

and obtain $\rho(T) \leq 1$ and $\rho(T') \geq \sup_n \frac{\rho(E_n')}{\rho(E_n)} = \infty$. □

3. A characterization of superreflexivity

In this section we exploit the second term on the right–hand side of Lemma 2.2.

R. C. James ([16], [3](p. 231)) proved that the non–superreflexivity of a Banach space X is equivalent to the following finite tree property: There is some $c > 0$ such that for all $n = 1, 2, ...$ there is a dyadic martingale $f = (f_l)_{l=0}^n \subset L_1^X[0,1)$ with

$$\|f_n\|_{L_\infty^X} \leq 1 \quad \text{and} \quad \frac{1}{c} \leq \inf_{0 \leq t < 1} \|df_l(t)\|_X \tag{12}$$

for $l = 1, ..., n$. In Theorem 3.1 below we extend this characterization to a description of the non–superreflexivity which encloses the above finite tree property and is flexible enough to obtain lower estimates of the norms of transforms $\Phi : L_r^X[0,1) \to L_r^X[0,1)$ given by $\sum_0^n df_l \to \sum_1^n \xi_l df_l$ where $(df_l)_0^n \subset L_r^X[0,1)$ is a dyadic martingale difference sequence, $(\xi_l)_1^n \subset \mathbb{R}$, and X a non–superreflexive Banach space. The motivation of Theorem 3.1 lies in Corollary 3.3 which reproves $\rho(X) = \infty$ whenever X fails to be superreflexive, but here with the right order of magnitude of the lower estimates (see Remark 3.4).

Theorem 3.1. *Let* $0 < r < \infty$. *A Banach space* X *is not superreflexive if and only if there is a constant* $c > 0$ *such that for all* $n = 1, 2, ...$ *there is a dyadic martingale* $f = (f_l)_{l=0}^n \subset L_1^X[0,1)$ *with*

$$\|f_n\|_{L_\infty^X} \leq 1 \text{ and } \frac{1}{c} \left(|\xi_k|^2 + \sum_{l=k+1}^n |\xi_l - \xi_{l-1}|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{l=k}^n \xi_l df_l \right\|_{L_r^X(A,\lambda_A)} \tag{13}$$

for all $k = 1, ..., n$, $A \in \mathcal{F}_{k-1}^{dyad}$ *with* $\lambda(A) > 0$, *and* $(\xi_l)_k^n \subset \mathbb{R}$, *where* λ_A *is the normalized restriction of the Lebesgue measure* λ *to* A *and for* $k = n$ *one has to set* $\sum_{l=k+1}^n |\xi_l - \xi_{l-1}|^2 := 0$.

Remark 3.2. (1) Condition (13) implies (12) with the same $c > 0$. To see this one has to choose for $k \in \{1, ..., n\}$ the sequence $(\xi_l)_k^n = (1, 0, ..., 0)$ and then to check (13) for the atoms $A \in \mathcal{F}_{k-1}$.

(2) But martingales fulfilling (12) do not necessarily satisfy (13). For example the dyadic martingales $f^{(n)} = (f_l)_{l=0}^n \subset L_1^{\ell_\infty}[0,1)$ with $f_0 = 0$ and $f_l(t) = (r_1(t), ..., r_l(t), 0, 0, ...)$ satisfy (12) with $c = 1$ but there is no common constant $c > 0$ such that (13) holds true.

(3) The ℓ_2–norm in the right–hand side inequality of (13) cannot be replaced by an ℓ_p–norm for $p < 2$. This follows from the argument used in Remark 3.4(3) and the Khintchine–Kahane inequality.

Proof of Theorem 3.1. The 'if' part follows from Remark 3.2(1). To prove the 'only' part fix $n \in \mathbb{N}$ and set $N = 2^n$. Let $g = (g_l)_0^n \subset L_1^{\ell_1^N}[0,1)$ be the dyadic martingale such that $S_N g_l = F_l$ for $l = 0, ..., n$ where S_N is taken from the proof of Proposition 2.7 and the martingale $(F_l)_{l \geq 0}$ is introduced before Lemma 2.2. Now consider $k \in \{1, ..., n\}$, $A \in \mathcal{F}_{k-1}$ with $\lambda(A) > 0$, and a sequence $(\xi_l)_1^n$ with $\xi_1 = \cdots = \xi_{k-1} = 0$ (for $k = 1$ we have to use obvious modifications in the following). Because σ_N^∞ is the restriction of I^∞ to the $\mathcal{F}_n^{dyad}$–measurable functions of $L_1[0,1)$ (see the proof of Proposition 2.7) we can apply Lemma 2.2 in the situation $p = \infty$. Since in the notation of this lemma $a_0 = \cdots = a_{k-2} = 0$ we obtain

$$\begin{aligned} \left\| \sum_{l=k}^n \xi_l \sigma_N^\infty dg_l \right\|_{L_r^{v_\infty^N}(A,\lambda_A)} &\geq \frac{1}{4} \left(\int_A \left\| \sigma_{n-k+1}^\infty \big((r_l(t) a_{l-1})_{l=k}^n \big) \right\|_{v_\infty^{n-k+1}}^r \frac{dt}{\lambda(A)} \right)^{\frac{1}{r}} \\ &= \frac{1}{4} \left(\int_0^1 \left\| \sigma_{n-k+1}^\infty \big((r_k(t) a_{k-1}, ..., r_n(t) a_{n-1}) \big) \right\|_{v_\infty^{n-k+1}}^r dt \right)^{\frac{1}{r}} \\ &\geq \frac{1}{4} \left(\int_0^1 \sup_{k \leq l \leq n} \left| r_k(t) a_{k-1} + \cdots + r_l(t) a_{l-1} \right|^r dt \right)^{\frac{1}{r}}. \end{aligned}$$

Using the Khintchine–Kahane inequality and Lemma 2.1 we continue to

$$\begin{aligned} \left\| \sum_{l=k}^n \xi_l \sigma_N^\infty dg_l \right\|_{L_r^{v_\infty^N}(A,\lambda_A)} &\geq \frac{1}{4} \frac{1}{c_r} \left(\sum_{k-1}^{n-1} |a_l|^2 \right)^{\frac{1}{2}} = \frac{1}{4c_r} \left(\sum_0^{n-1} |a_l|^2 \right)^{\frac{1}{2}} \\ &\geq \frac{1}{6c_r} \left(\sum_0^{n-1} |\lambda_l|^2 \right)^{\frac{1}{2}} \geq \frac{1}{6c_r} \left(|\xi_k|^2 + \sum_{k+1}^n |\xi_l - \xi_{l-1}|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Now, if X is not superreflexive, then there are operators $U_N \in \mathcal{L}(\ell_1^N, X)$ and $V_N \in \mathcal{L}(X, v_\infty^N)$ with $\sigma_N^\infty = V_N U_N$, $\|U_N\| \leq 1$, and $\|V_N\| < 3$ ([17](Theorem 4)), where we have used $\|\cdot\|_{v_\infty^N} \leq 2 \|\cdot\|_{\ell_\infty^N}$. Taking $f_l(t) := U_N g_l(t) \in X$ we obtain the desired martingale $f = (f_l)_0^n \subset L_1^X[0,1)$ and $c = 18 c_r$. $\square$

Corollary 3.3. *If X is not superreflexive, then there is some $c > 0$ such that for all $n = 1, 2, ...$ there is a dyadic martingale $f = (f_l)_{l=0}^n \subset L_2^X[0,1)$ with*

$$\|f_n\|_{L_2^X} \leq 1 \quad \textit{and} \quad \frac{1}{c}\sqrt{n} \leq \left(\int_0^1 \int_0^1 \left\| \sum_{l=1}^n r_l(t) df_l(s) \right\|_X^2 dt ds \right)^{\frac{1}{2}}. \tag{14}$$

Remark 3.4. (1) If X is of type 2, then the converse of Corollary 3.3 holds true since

$$\frac{1}{c}\sqrt{n} \le \left(\int_0^1 \int_0^1 \left\| \sum_{l=1}^n r_l(t) df_l(s) \right\|_X^2 dtds \right)^{\frac{1}{2}} \le t_2(X) \left(\sum_{l=1}^n \|df_l\|_{L_2^X}^2 \right)^{\frac{1}{2}}$$

which contradicts $\|f_n\|_{L_2^X} \le 1$ and $Mc_q(X) < \infty$ for some $2 \le q < \infty$ (the superreflexivity implies finite martingale cotype, see [22]).

(2) With $\sqrt[q]{n}$ instead of $\sqrt{n}$ in the right–hand side inequality of (14) and under the assumption X is of cotype q ($2 \le q < \infty$) Corollary 3.3 can be found in [1] and [11] and is used to show that X is superreflexive whenever $\rho(X) < \infty$.

(3) The factor $\sqrt{n}$ is asymptotically best possible in Corollary 3.3 since there are non–superreflexive Banach spaces X of type 2. Indeed, continuing in (14) with the type 2 inequality as in the first item of this remark we arrive at

$$\frac{1}{c}\sqrt{n} \le 2\sqrt{n}\, t_2(X) \|f_n\|_{L_2^X} \le 2\sqrt{n}\, t_2(X).$$

(4) In general the converse of the above corollary turns out to be false. This is a consequence of an example due to J. Bourgain [6] which gives for all $1 < p < 2 < q < \infty$ a superreflexive Banach lattice $X_{p,q}$ of martingale type p and martingale cotype q and a constant $c > 0$ such that for all $n = 1, 2, ...$ there is a dyadic martingale $f = (f_l)_{l=0}^n \subset L_1^X[0,1)$ with

$$\|f_n\|_{L_\infty^X} \le 1 \quad \text{and} \quad \frac{1}{c} n^{\frac{1}{p}-\frac{1}{q}} \le \inf_s \int_0^1 \left\| \sum_{l=1}^n r_l(t) df_l(s) \right\|_X dt.$$

In [6] the example is not formulated in this way. In our setting we first replace the square function used in [6] by the Rademacher average with the help of the Khintchine–Kahane inequality. Second, one has to observe the estimate $\varepsilon \le 2n^{-1/p}$ for the $\varepsilon > 0$ occurring in [6](Lemma 4). Finally we switch from the upper and lower estimates to the moduli of smoothness and convexity by a result of T. Figiel and W.B. Johnson (cf. [19] (Theorem II.1.f.10)) and to the martingale type and cotype via Pisier's result [22](Proposition 2.4). In the latter step we additionally use Theorem A.1(a) for the martingale cotype and (for example) [13](Corollary 8.6) (cf. [22](Remark 3.3)) for the martingale type.

(5) The order of magnitude of the factor $n^{\frac{1}{p}-\frac{1}{q}}$ in Bourgain's example is optimal since martingale type p and martingale cotype q imply for $\theta_l = \pm 1$ that

$$\left\| \sum_{l=1}^n \theta_l df_l \right\|_{L_2^X} \le Mt_p(X) Mc_q(X) n^{\frac{1}{p}-\frac{1}{q}} \left\| \sum_{l=1}^n df_l \right\|_{L_2^X}.$$

4. Problems

Problem 4.1. *What classes $\mathcal{C}$ of finite–dimensional Banach spaces allow an estimate*

$$\rho(X) \le c\sqrt{1 + \log(\dim(X))}\, t_2(X) \qquad (X \in \mathcal{C}),$$

where $c > 0$ depends on $\mathcal{C}$ only?

The above problem is motivated by inequality (4) from the introduction. An investigation of this problem could provide an alternative approach to and improvement of Lemma 2.6 (in particular for $2 < q < \infty$). Up to now we prove this assertion without the usage of the type 2 properties of A_q^N.

Problem 4.2. *Is there a Banach space X with $\rho(X) < \infty$ but $\beta(X) = \infty$?*

Appendix A. Extrapolation techniques

Given $(f_k)_{k=1}^K \subset L_1[0,1)$ adapted with respect to $\left(\mathcal{F}_k^h\right)_{k=1}^K$ and $1 \leq r < \infty$ we set

$$\left\|(f_k)_{k=1}^K\right\|_{BMO} := \sup_{0\leq k\leq l\leq K} \sup_{C\in\mathcal{F}_k^h \text{atom}} \|f_l - f_{k-1}\|_{L_1(C,\lambda_C)}$$

and

$$\left\|(f_k)_{k=1}^K\right\|_{BMO_{exp_r}} := \sup_{0\leq k\leq l\leq K} \sup_{C\in\mathcal{F}_k^h \text{atom}} \|f_l - f_{k-1}\|_{exp_r(C,\lambda_C)},$$

where $f_{-1} = 0$, λ_C is the normalized restriction of the Lebesgue measure λ to C, and

$$\|g\|_{exp_r} := \inf\left\{c > 0 \mid \mathbb{E}e^{\left(\frac{|g|}{c}\right)^r} \leq 2\right\} \quad \text{for} \quad g \in L_1(\Omega,\mathcal{F},\mathbb{P})$$

and a probability space $[\Omega,\mathcal{F},\mathbb{P}]$. From A.M. Garsia [12](III.1.4) (see also [13] (Corollary 4.8)) it is known that there is some absolute $c > 0$ such that

$$\|\cdot\|_{BMO} \sim_c \|\cdot\|_{BMO_{exp_1}}. \tag{15}$$

Finally let

$$L_1^{0,X}\left([0,1),\mathcal{F}_k^h\right) := \left\{f \in L_1^X[0,1) \mid f \text{ is } \mathcal{F}_k^h\text{–measurable and of mean–zero}\right\}$$

and

$$L_1^{+}\left([0,1),\mathcal{F}_k^h\right) := \left\{f \in L_1[0,1) \mid f \text{ is } \mathcal{F}_k^h\text{–measurable and non–negative}\right\}.$$

The following extrapolation result has its origin in [10], [9], [14], and [13].

Theorem A.1. *Assume that $A : L_1^{0,X}\left([0,1),\mathcal{F}_K^h\right) \to L_1^{+}\left([0,1),\mathcal{F}_K^h\right)$, where $K \geq 1$ is fixed, satisfies the following properties:*

1. *$A(f) = A(-f)$.*
2. *$A(f+g)(t) \leq Af(t) + Ag(t)$ for all $t \in [0,1)$.*
3. *$Af(t) = 0$ for all $t \in [0,1)$ such that $\sum_{k=1}^K h_k^2(t)\|x_k\| = 0$, where $f = \sum_{k=1}^K h_k x_k$.*
4. *Af is $\mathcal{F}_k^h$–measurable whenever f is $\mathcal{F}_k^h$–measurable $(k = 0,...,K)$.*

(*a*) *If there is some $1 \leq q < \infty$ such that for all $f \in L_1^{0,X}\left([0,1),\mathcal{F}_K^h\right)$*

$$\|Af\|_q \leq \|f\|_{L_q^X},$$

then for all $1 < p < \infty$ there is a $c_p > 0$, depending on p only, with

$$\left\|\sup_{i\geq 1}\frac{Af^{(i)}}{1+\log i}\right\|_p \leq c_p \left\|\sup_{i\geq 1}\|f^{(i)}\|_X\right\|_p \quad \textit{for all } (f^{(i)})_{i=1}^\infty \subset L_1^{0,X}\left([0,1),\mathcal{F}_K^h\right).$$

(*b*) *If there are $1 \leq q_0, r < \infty$ such that for $q \geq q_0$ and $f \in L_1^{0,X}\left([0,1),\mathcal{F}_K^h\right)$*

$$\|Af\|_q \leq \sqrt[r]{q}\|f\|_{L_q^X},$$

then for all $1<p<\infty$ *there is some* $c>0$, *depending on* r, q_0, *and* p *only, with*

$$\left\| \sup_{i\geq 1} \frac{Af^{(i)}}{\sqrt[r]{1+\log i}} \right\|_p \leq c \left\| \sup_{i\geq 1} \|f^{(i)}\|_X \right\|_p \quad \textit{for all } (f^{(i)})_{i=1}^\infty \subset L_1^{0,X}\left([0,1),\mathcal{F}_K^h\right).$$

Proof. We give the main details. Let $0\leq k\leq l\leq K$, $C\in\mathcal{F}_k^h$ be an atom, and $\widetilde{C}\in\mathcal{F}_{k-1}^h$ be the unique atom containing C if $k\geq 1$ and $\widetilde{C}:=C$ if $k=0$. For $f_k:=\mathbb{E}\left(f\mid\mathcal{F}_k^h\right)$ and $f_{-1}:=0$ we get $\left[A(f_l-f_{k-1})\right]\chi_{\widetilde{C}}=A\left([f_l-f_{k-1}]\,\chi_{\widetilde{C}}\right)$ from

$$\left|A(f_l-f_{k-1})-A\left([f_l-f_{k-1}]\,\chi_{\widetilde{C}}\right)\right|\leq A\left([f_l-f_{k-1}]\left[1-\chi_{\widetilde{C}}\right]\right)$$

and (3). Hence for $1\leq q<\infty$ we deduce

$$\begin{aligned}
\|Af_l-Af_{k-1}\|_{L_q(C,\lambda_C)} &\leq \|A\,(f_l-f_{k-1})\|_{L_q(C,\lambda_C)}\\
&\leq 2\,\|A\,(f_l-f_{k-1})\|_{L_q\left(\widetilde{C},\lambda_{\widetilde{C}}\right)}\\
&\leq 2\lambda\left(\widetilde{C}\right)^{-\frac{1}{q}}\left\|A\left((f_l-f_{k-1})\chi_{\widetilde{C}}\right)\right\|_{L_q[0,1)}
\end{aligned}$$

as well as $\lambda\left(\widetilde{C}\right)^{-\frac{1}{q}}\left\|(f_l-f_{k-1})\chi_{\widetilde{C}}\right\|_{L_q^X[0,1)}\leq 2\,\|f\|_{L_\infty^X[0,1)}$. We obtain in (a)

$$\|Af_l-Af_{k-1}\|_{L_1(C,\lambda_C)}\leq 4\,\|f\|_{L_\infty^X[0,1)}$$

and in (b)

$$\sup_{q\geq q_0}\frac{\|Af_l-Af_{k-1}\|_{L_1(C,\lambda_C)}}{\sqrt[r]{q}}\leq 4\,\|f\|_{L_\infty^X[0,1)}\,.$$

Because of (15) we deduce for (a)

$$\left\|(Af_k)_{k=0}^K\right\|_{BMO_{exp_1}}\leq c_1\|f\|_{L_\infty^X},$$

where $c_1>0$ is an absolute constant, and since $\sup_{q\geq q_0}\frac{\|\cdot\|_{L_q(\Omega,\mathcal{F},\mathbb{P})}}{\sqrt[r]{q}}\sim_{c_{q_0,r}}\|\cdot\|_{exp_r}$ with $c_{q_0,r}>0$ depending on q_0 and r only, we deduce in (b)

$$\left\|(Af_k)_{k=0}^K\right\|_{BMO_{exp_r}}\leq c_r\|f\|_{L_\infty^X},$$

where $c_r>0$ depends on r and q_0 only. Now we simultaneously treat assertions (a) and (b) and introduce $\mathcal{A},\mathcal{B}:L_1^{0,\ell_\infty^N(X)}\left([0,1),\mathcal{F}_K^h\right)\to L_1^+\left([0,1),\mathcal{F}_K^h\right)$ by

$$\mathcal{A}F(t):=\sup_{1\leq i\leq N}\frac{Af^{(i)}(t)}{\sqrt[r]{1+\log i}}$$

and

$$\mathcal{B}F(t):=\sup_{1\leq k<K}\left[\|F_k(t)\|_{\ell_\infty^N(X)}+\|dF_{k+1}(t)\|_{\ell_\infty^N(X)}\right]$$

where $F=(f^{(1)},...,f^{(N)})$ and $F_k=\mathbb{E}\left(F\mid\mathcal{F}_k^h\right)$ (for $K=1$ simply use $\mathcal{B}F=\|F_1\|_{\ell_\infty^N(X)}$). Applying [13](Theorem 1.5) yields

$$\begin{aligned}
\left\|(\mathcal{A}F_k)_{k=0}^K\right\|_{BMO_{exp_r}} &= \left\|\left(\sup_{1\leq i\leq N}\frac{Af_k^{(i)}}{\sqrt[r]{1+\log i}}\right)_{k=0}^K\right\|_{BMO_{exp_r}}\\
&\leq c_r'\sup_{1\leq i\leq N}\left\|\left(Af_k^{(i)}\right)_{k=0}^K\right\|_{BMO_{exp_r}},
\end{aligned}$$

where $c'_r > 0$ depends on r only, so that

$$\left\|(\mathcal{A}F_k)_{k=0}^K\right\|_{BMO_{exp_r}} \le c'_r c_r \sup_{1\le i\le N} \|f^{(i)}\|_{L_\infty^X[0,1)} \le c'_r c_r \|\mathcal{B}F\|_{L_\infty[0,1)}.$$

Identifying $L_1^{0,\ell_\infty^N(X)}\left([0,1),\mathcal{F}_K^h\right)$ with the set of all $(h_k x_k)_0^K$ where $x_k \in \ell_\infty^N(X)$ with $x_0 = 0$ we can apply [13](Theorem 1.7, Proposition 7.3) on $\mathcal{A}$ and $\mathcal{B}$ (the point is that $\mathcal{B}$ is monotone and predictable in the sense of [13]) and get

$$\|\mathcal{A}F\|_p \le c_{p,r} c'_r c_r \|\mathcal{B}F\|_p$$

where $c_{p,r} > 0$ depends on p and r only. Now the assertion follows from

$$\|\mathcal{B}F\|_p \le 3\left\|\sup_k \|F_k\|_{\ell_\infty^N(X)}\right\|_p \le \frac{3p}{p-1}\|F\|_{L_p^{\ell_\infty^N(X)}}$$

(which is a consequence of Doob's maximal inequality) and $N \to \infty$. □

Corollary A.2. *For all $1 < q < \infty$ there is a constant $c_q > 0$ such that*

$$\frac{1}{c_q}\beta_q(\cdot) \le \beta_2(\cdot) \le c_q \beta_q(\cdot) \quad \textit{and} \quad \frac{1}{c_q}\rho_q(\cdot) \le \rho_2(\cdot) \le c_q \rho_q(\cdot).$$

For the quantities $\beta_q(X)$ this is proved in [20]. This also follows from characterizations of the UMD–spaces proved by D.L. Burkholder; see [7]. For $\rho_q(X)$ this is stated in [11]. The reader can easily deduce Corollary A.2 from Theorem A.1(a), where one has to use for the quantities $\beta_q(\cdot)$

$$A\left(\sum_{k=1}^K h_k x_k\right)(s) := \frac{1}{\beta_2(T)}\left\|\sum_{k=1}^K \theta_k h_k(s) T x_k\right\|$$

and for $\rho_q(\cdot)$

$$A\left(\sum_{k=1}^{\tau_L} h_k x_k\right)(s) := \frac{1}{\rho_2(T)}\left(\int_0^1 \left\|\sum_{l=1}^L r_l(t)\left(\sum_{k=\tau_{l-1}+1}^{\tau_l} h_k(s) T x_k\right)\right\|^2 dt\right)^{\frac{1}{2}}$$

and the Khintchine–Kahane inequality.

References

1. D.J. Aldous. Unconditional bases and martingales in $L_p(F)$. *Math. Proc. Camb. Phil. Soc.*, 85:117–123, 1979. MR **80b:**60009
2. N.H. Asmar and S.J. Montgomery-Smith. On the distribution of Sidon series. *Ark. Math.*, 31(1):13–26, 1993. MR **94i:**43006
3. B. Beauzamy. *Introduction to Banach spaces and their geometry*, volume 68 of *Mathematics Studies*. North–Holland, Amsterdam, New York, Oxford, 1985. MR **88f:**46021
4. J. Bergh and J. Lőfstrőm. *Interpolation spaces. An introduction.* Springer, 1976. MR **58:**2349
5. J. Bergh and J. Peetre. On the spaces v_p $(0 < p < \infty)$. *Bollettino della Unione Matematica Italiana*, 10:632–648, 1974. MR **52:**1289
6. J. Bourgain. Some remarks on Banach spaces in which martingale difference sequences are unconditional. *Ark. Mat.*, 21:163–168, 1983. MR **85a:**46011
7. D.L. Burkholder. Martingales and Fourier analysis in Banach spaces. In *C.I.M.E. Lectures, Varenna (Como), Italy, 1985*, volume 1206 of *Lect. Notes in Math.*, pages 61–108. Springer, 1986. MR **88c:**42017
8. D.L. Burkholder. Explorations in martingale theory and its applications. In *Ecole d'Eté de Probabilités de Saint–Flour XIX–1989*, volume 1464 of *Lect. Notes Math.*, pages 1–66. Springer, 1991. MR **92m:**60037

9. D.L. Burkholder, B.J. Davis, and R.F. Gundy. Integral inequalities for convex functions of operators on martingales. In *Proc. 6th Berkeley Sympos*, volume 2 of *Math. Statist. Probab.*, pages 223–240, 1972. MR **53:**4214
10. D.L. Burkholder and R.F. Gundy. Extrapolation and interpolation of quasilinear operators on martingales. *Acta Math.*, 124:249–304, 1970. MR **55:**13567
11. D.H.J. Garling. Random martingale transform inequalities. In *Probability in Banach spaces 6. Proceedings of the sixth international conference, Sandbjerg, Denmark 1986*, pages 101–119. Birkhäuser, 1990. MR **92a:**60111
12. A.M. Garsia. *Martingale inequalities.* Seminar Notes on Recent Progress. Benjamin, Reading, 1973. MR **56:**6844
13. S. Geiss. BMO_ψ–spaces and applications to extrapolation theory. *Studia Math.*, 122(3):235–274, 1997. MR **98a:**46037
14. P. Hitczenko. Upper bounds for the L_p-norms of martingales. *Probab. Th. Rel. Fields*, 86:225–238, 1990. MR **92a:**60112
15. P. Hitczenko. Domination inequality for martingale transforms of a Rademacher sequence. *Israel J. Math.*, 84:161–178, 1993. MR **94i:**60061
16. R.C. James. Some self dual properties of normed linear spaces. *Ann. of Math. Studies*, 69:159–175, 1972. MR **56:**12849
17. R.C. James. Super-reflexive Banach spaces. *Can. J. Math.*, 5:896–904, 1972. MR **47:**9248
18. M. Ledoux and M. Talagrand. *Probability in Banach spaces.* Springer, 1991. MR **93c:**60001
19. J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces I/II.* Springer, 1977/1979. MR **58:**17766 MR **81c:**46001
20. B. Maurey. Système de Haar. *Seminaire Maurey–Schwartz, Ecole Polytechnique, Paris*, 1974–1975. MR **54:**8851
21. G. Pisier. Un exemple concernant la superréflexivité. *Seminaire Maurey–Schwartz, Ecole Polytechnique, Paris*, 1974–1975. MR **53:**14090
22. G. Pisier. Martingales with values in uniformly convex spaces. *Israel J. Math.*, 20:326–350, 1975. MR **52:**14940
23. G. Pisier and Q. Xu. Random series in the real interpolation spaces between the spaces v_p. In *Israel Functional Seminar, GAFA*, volume 1267 of *Lect. Notes Math.*, pages 185–209. Springer, 1987. MR **89d:**46011
24. J. Wenzel. Personal communication.

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