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WITTEN-HELFFER-SJÖSTRAND THEORY FOR S^1-EQUIVARIANT COHOMOLOGY

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ABSTRACT. Given an S^1-invariant Morse function f and an S^1-invariant Riemannian metric g, a family of finite dimensional subcomplexes $(\widetilde{\Omega}^*_{inv,sm}(M,t), D(t))$, $t \in [0,\infty)$, of the Witten deformation of the S^1-equivariant de Rham complex is constructed, by studying the asymptotic behavior of the spectrum of the corresponding Laplacian $\widetilde{\Delta}^k(t) = D^*_k(t)D_k(t)+D_{k-1}(t)D^*_{k-1}(t)$ as $t \to \infty$. In fact the spectrum of $\widetilde{\Delta}^k(t)$ can be separated into the small eigenvalues, finite eigenvalues and the large eigenvalues. Then one obtains $(\widetilde{\Omega}^*_{inv,sm}(M,t), D(t))$ as the complex of eigenforms corresponding to the small eigenvalues of $\widetilde{\Delta}(t)$. This permits us to verify the S^1-equivariant Morse inequalities. Moreover suppose f is self-indexing and (f,g) satisfies the Morse-Smale condition, then it is shown that this family of subcomplexes converges as $t \to \infty$ to a geometric complex which is induced by (f,g) and calculates the S^1-equivariant cohomology of M.

1. INTRODUCTION

In this paper, we shall use Witten's method of proving Morse inequality to verify the S^1-equivariant Morse inequality ([B]) for an S^1-equivariant Morse function on M. This is accomplished by constructing a family of finite dimensional subcomplexes $(\widetilde{\Omega}^*_{small}(M,t), D(t))$, $t \in [0,\infty)$, of the Witten deformation of the S^1-equivariant de Rham complex. To obtain this family of subcomplexes, we consider the asymptotic behavior of the spectrum of the corresponding Laplacians $\widetilde{\Delta}^k(t)$ as $t \to \infty$. The spectrum of $\widetilde{\Delta}^k(t)$ can be separated into the small eigenvalues, finite eigenvalues and large eigenvalues. While the small eigenvalues tend to zero as $t \to \infty$, the finite eigenvalues and the large eigenvalues tend to a positive constant and infinity respectively. Then one obtains $(\widetilde{\Omega}^*_{inv,sm}(M,t), D(t))$ as the complex of eigenforms corresponding to the small eigenvalues of $\widetilde{\Delta}(t)$. Suppose furthermore that f is self-indexing and (f,g) satisfies the Morse-Smale condition, then this family of subcomplexes is shown to converge as $t \to \infty$ to a geometric complex which is induced by (f,g) and calculates the S^1-equivariant cohomology of M.

The paper is organized as follows: in §1.1 we review the Witten-Helffer-Sjöstrand theory for a Morse function. In §1.2, we discuss S^1-equivariant cohomology, S^1-equivariant Hodge theory, equivariant Morse theory, Witten deformation as well as the definition of '*localized*' *operators* which we need in the formulation of results. In §1.3, we formulate the results. In §§2, 3 and 4 we prove these results.

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1.1. **Review of the Witten-Helffer-Sjöstrand theory for a Morse function.**

(a) *Classical Morse theory and Witten's proof of Morse inequality.*

Let M^n be a compact orientable Riemannian manifold, f be a Morse function on M, m_i be the number of critical points of index i.

Define

$$\mathcal{P}(M,t) = \sum_{i=0}^{\infty} t^i dim H^i(M) = \sum_{i=0}^{\infty} t^i \beta_i,$$

$$\mathcal{M}(M,f,t) = \sum_{i=0}^{\infty} t^i m_i.$$

Then we have

Theorem (Morse Inequality)**.**

Formulation I:

$$\mathcal{M}(M,f,t) - \mathcal{P}(M,t) = (1+t)\mathcal{Q}(t)$$

where $\mathcal{Q}(t) = \sum_{i=0}^{\infty} q_i t^i$ *with* $q_i \geq 0$.

Formulation II:

$$\sum_{i=0}^{k} (-1)^i m_i - \sum_{i=0}^{k} (-1)^i \beta_i \begin{cases} \geq 0 & \textit{if } k \textit{ is even}, \\ \leq 0 & \textit{if } k \textit{ is odd}. \end{cases}$$

Remark. The above two formulations are in fact equivalent.

Witten's idea of proving Morse inequality ([W], [S]) is to use Witten deformation of de Rham complex to construct a complex of finite dimensional vector spaces which calculates the cohomology of M and whose dimension in degree k equals the number of critical points of index k. The Morse inequality will follow from the following algebraic lemma:

Lemma. *Suppose* (C^k, d) *is a complex of finite dimensional vector spaces, let* $m_i = dim C^i$, $\beta_i = dim H^i(C,d)$, *then*

$$\sum_{i=0}^{k} (-1)^i m_i - \sum_{i=0}^{k} (-1)^i \beta_i \begin{cases} \geq 0 & \textit{if } k \textit{ is even}, \\ \leq 0 & \textit{if } k \textit{ is odd}. \end{cases}$$

To construct the complex of finite dimensional vector spaces, let $c_1^k, \ldots, c_{m_k}^k$ be the critical points of f of index k, $x = (x_1, \ldots, x_n)$ be a coordinate system in a neighbourhood of c_j^k given by the Morse Lemma, i.e.

$$f(x) = f(0) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$

Let g be a Riemannian metric on M and suppose that in a neighbourhood of the critical points, the metric g is given by the canonical Euclidean metric when represented in the above coordinate system. (Such a pair (f,g) will be called compatible.) Consider the de Rham complex $(\Omega^*(M), d)$ which calculates the cohomology of M.

Let

$$d(t) = e^{-tf} d e^{tf},$$

$$d^*(t) = e^{tf} d^* e^{-tf},$$

$$\Delta(t) = d(t)d^*(t) + d^*(t)d(t).$$

Since $d(t)$ and d are conjugated by e^{tf}, $(\Omega^*(M), d(t))$ also calculates the cohomology of M, which is also given by the harmonic forms of $\Delta(t)$.

In the above coordinate system, $\Delta(t)$ is given by

$$\Delta(t) = dd^* + d^*d + 4t^2x^2 + t\left\{-\sum_{i=1}^{k}[dx_i, i(\partial_i)] + \sum_{i=k+1}^{k}[dx_i, i(\partial_i)]\right\}$$

where dx_i is the exterior multiplication by dx_i, while $i(\partial_i)$ is the interior multiplication by ∂_i. (Recall that the interior multiplication by a vector field X is a zero order operator $i_X : \Omega^*(M) \to \Omega^{*-1}(M)$.)

Now define the '*localized*' operator $\overline{\Delta_j^k}(t) : \Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$ to be given by the above expression and notice that

$$\overline{\Delta_j^k}(t) = U(t^{1/2})t\overline{\Delta_j^k}(1)U(t^{-1/2})$$

where $(U(\lambda)\omega)(x) = \lambda^{n/2}\omega(\lambda x)$. Since $U(\lambda)$ is an isometry, then $\overline{\Delta_j^k}(t)$ and $t\overline{\Delta_j^k}(1)$ are conjugated by an isometry.

Let $K_j^k = \overline{\Delta_j^k}(1)$ and $\{c_j\}_{1\le j\le r}$ be all the critical points of f.

Let

$$0 \le e_1^{(k)} \le e_2^{(k)} \le \cdots \le e_l^{(k)} \le \cdots$$

be all the eigenvalues of $\oplus_{j=1}^k K_j^k : \oplus_{j=1}^k \Omega^k(\mathbb{R}^n) \to \oplus_{j=1}^k \Omega^k(\mathbb{R}^n)$.

Then

$$0 \le te_1^{(k)} \le te_2^{(k)} \le \cdots \le te_l^{(k)} \le \cdots$$

are all the eigenvalues of $\bigoplus_{j=1}^r \overline{\Delta_j^k}(t) : \bigoplus_{j=1}^r \Omega^k(\mathbb{R}^n) \to \bigoplus_{j=1}^r \Omega^k(\mathbb{R}^n)$.

Let

$$0 \le E_1^{(k)}(t) \le E_2^{(k)}(t) \le \cdots \le E_l^{(k)}(t) \le \cdots$$

be all the eigenvalues of $\Delta^k(t) : \Omega^k(M) \to \Omega^k(M)$. Then we have

Theorem (Witten, Simon)**.**

$$\lim_{t\to\infty} \frac{E_l^{(k)}(t)}{t} = e_l^{(k)}.$$

Let us consider those $E_l^{(k)}(t)$ s.t. $\lim_{t\to\infty} \frac{E_l^{(k)}(t)}{t} = 0$.

Observe that K_j^k has exactly one zero eigenvalue (whose corresponding eigenvector is a k-form) iff *index* $c_j = k$.

So we have

$$0 = \lim_{t\to\infty} \frac{E_1^{(k)}}{t} = \lim_{t\to\infty} \frac{E_2^{(k)}}{t} = \cdots = \lim_{t\to\infty} \frac{E_{m_k}^{(k)}}{t} = 0 < \lim_{t\to\infty} \frac{E_{m_k+1}^{(k)}}{t} \le \cdots.$$

In fact, one can show that

$$\begin{cases} \lim_{t\to\infty} E_1^{(k)}(t) = \lim_{t\to\infty} E_2^{(k)}(t) = \cdots = \lim_{t\to\infty} E_{m_k}^{(k)}(t) = 0 \\ \lim_{t\to\infty} E_{m_k+1}^{(k)}(t) = \lim_{t\to\infty} E_{m_k+2}^{(k)}(t) = \cdots = +\infty. \end{cases}$$

Corollary 1. *For any* $0 < a < b$, *there exists* $t_0 > 0$ *s.t. for* $t \ge t_0$,

$$0 \le E_1^{(k)}(t) \le \cdots \le E_{m_k}^{(k)}(t) < a < b \le E_{m_k+1}^{(k)}(t) \le E_{m_k+2}^{(k)}(t) \le \cdots.$$

We say that $E(t)$ is a small (resp. large) eigenvalue of $\Delta^k(t)$ if $\lim_{t\to\infty} E(t) = 0$ (resp. ∞).

Define

$$\Omega^k_{small}(M,t) = Span\{\Psi(t) \in \Omega^k(M) | \Delta^k(t)\Psi(t) = E(t)\Psi(t) \\ \text{where } E(t) \text{ is a small eigenvalue}\}.$$

Similarly, $\Omega^k_{large}(M,t)$ is defined to be the linear span of the eigenforms corresponding to the large eigenvalues of $\Delta^k(M)$.

Then we have

Corollary 2.

(i)

$$(\Omega^*(M), d(t)) = (\Omega^*_{small}(M,t), d(t)) \oplus (\Omega^*_{large}(M,t), d(t))$$

where $(\Omega^*_{large}(M,t), d(t))$ *is acyclic.*

(ii) $(\Omega^*_{small}(M,t), d(t))$ *calculates the cohomology of* M *with* $dim\Omega^k_{small}(M,t) = m_k < \infty$.

Corollary 3 (Morse Inequality)**.**

$$\sum_{i=0}^{k}(-1)^i m_i - \sum_{i=0}^{k}(-1)^i \beta_i \begin{cases} \geq 0 & \text{if } k \text{ is even,} \\ \leq 0 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Apply the algebraic lemma to the complex $(\Omega^*_{small}(M,t), d(t))$.

(b) *Review of Helffer-Sjöstrand theory for a Morse Function.*

Definition. 1. Suppose f is a Morse function, g a Riemannian metric on M, the pair (f,g) is said to satisfy the Morse-Smale condition if for any two critical points x and y, the ascending manifold W^+_x and the descending manifold W^-_y, w.r.t. $-Grad_g f$, intersect transversally.

2. The pair (f,g) is said to be compatible if for any critical point c, one can choose a coordinate system about c such that

$$f(x) = f(c) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2,$$
$$dg = d^2x_1 + \cdots + d^2x_n$$

when represented in the above coordinate system. Such a coordinate system will be called a compatible coordinate system for f and g.

3. A Morse function f is said to be self-indexing if for any critical point x,

$$f(x) = indexk.$$

Proposition 1. (i) (*cf.* [Sm]) *For any pair* (f,g), *there is a* C^1 *approximation* g' *such that* $g = g'$ *in a neighbourhood of the critical points of* f *and* (f,g') *satisfies the Morse-Smale condition.*

(ii) (*cf.* [M], §4) *For any Morse function* f, *there exists a self-indexing Morse function* f' *such that* f *and* f' *have the same critical points with the same indices.*

Theorem. *Suppose* f *is a self-indexing Morse function and* g *a metric such that* (f,g) *is compatible and satisfies the Morse-Smale condition, then*

(*i*) $\{W^-_{x^k_j}\}_{0\leq k\leq n; 1\leq j\leq m_k}$ *is a CW-complex, where* m_k *is the number of critical points of* f *of index* k.

(ii) Let $(C_*(M,f),\partial)$ be the cellular chain complex of the above CW-complex, $(C^*(M,f),\delta)$ be its dual cochain complex.

Then $Int : (\Omega^*(M),d) \to (C^*(M,f),\delta)$

$$Int(\omega_k)(W^-_{x_j^k}) = \int_{W^-_{x_j^k}} \omega_k \quad \text{for } \omega_k \in \Omega^k(M)$$

is a morphism of cochain complexes.

Proof. A proof of this theorem can be found in [L].

Hence the composition

$$(\Omega^*_{small}(M,f,t),d(t)) \xrightarrow{e^{tf}} (\Omega^*(M),d) \xrightarrow{Int} (C^*(M,f),\delta)$$

is also a morphism of cochain complexes.

Suppose further that (f,g) is compatible. For any critical point x_j^k choose a compatible coordinate system about x_j^k so that the chart contains $B(0,\varepsilon)$ the unit ball of radius ε in $\mathbb{R}^n$ for some $\varepsilon > 0$. Choose $\rho \in C_0^\infty(\mathbb{R})$ so that

$$\rho(x) = \begin{cases} 1 & \text{if } |x| \le \frac{\varepsilon}{2}, \\ 0 & \text{if } |x| \ge \varepsilon. \end{cases}$$

Define

$$\Psi_{x_j^k}(t) = \beta(t)\rho(|x|)\left(\frac{2t}{\pi}\right)^{n/4} e^{-t(x_1^2+\cdots+x_n^2)} dx_1 \wedge \cdots \wedge dx_k$$

where $\beta(t)$ is chosen s.t. $\|\Psi_{x_j^k}(t)\| = 1$.

Define $J_k(t) : C^k(M,f) \to \Omega^k(M)$ by

$$J_k(t)(e_{x_j^k}) = \Psi_{x_j^k}(t)$$

where $\{e_{x_j^k}\}$ is the dual basis of $\{W^-_{x_j^k}\}$.

Let $Q_k(t)$ be the orthogonal projection onto $\Omega^k_{small}(M,t)$.

Define

$$H_k(t) = (Q_k(t)J_k(t))^*(Q_k(t)J_k(t)),$$
$$\widetilde{J}_k(t) = Q_k(t)J_k(t)(H_k(t))^{-1/2}.$$

Then $\widetilde{J}_k(t) : C^k(M,f) \to \Omega^k_{small}(M,t)$ is an isometry.

Define

$$E_{x_j^k}(t) = \widetilde{J}_k(t)(e_{x_j^k}).$$

Then $\{E_{x_j^k}(t)\}_{1\le j\le m_k}$ forms an orthonormal basis in $\Omega^k_{small}(M,t)$.

Proposition 2 (cf. [HS], [BZ])**.** (i) *There exists a neighbourhood $U_{x_j^k}$ of x_j^k contained in the chart of compatibility s.t. on $U_{x_j^k}$,*

$$E_{x_j^k}(t) = \left(\frac{2t}{\pi}\right)^{n/4} e^{-tx^2}(dx_1 \wedge \cdots \wedge dx_k + O(t^{-1})),$$

(ii)

$$Int_k e^{tf}(E_{x_j^k}(t)) = \left(\frac{2t}{\pi}\right)^{\frac{n-2k}{4}} e^{tk}(e_{x_j^k} + O(t^{-1})).$$

Hence, define $F^k(t) : \Omega^k_{small}(M,t) \to C^k(M,f)$ s.t.

$$F^k(t)(E_{x_j^k}(t)) = \left(\frac{\pi}{2t}\right)^{\frac{n-2k}{4}} e^{-tk} Int_k e^{tf}(E_{x_j^k}(t))$$

and let

$$(\Omega^*_{small}(M,t), \tilde{d}(t)) = (\Omega^*_{small}(M,t), e^t(\pi/2t)^{1/2} d(t)),$$

then we have

Theorem (cf. [BZ], [HS])**.** $F^*(t) : (\Omega^*_{small}(M,t), \tilde{d}(t)) \to (C^*(M,f), \delta)$ *is a morphism of cochain complexes s.t. w.r.t. the bases* $\{E_{x_j^k}(t)\}$ *and* $\{e_{x_j^k}\}$

$$F^*(t) = I + O(t^{-1}).$$

As a consequence, we have

Theorem (Helffer-Sjöstrand)**.**

$$\langle E_{x_j^{k+1}}(t), d(t)E_{x_i^k}(t)\rangle = e^{-t}\sqrt{\frac{2t}{\pi}}\left(\sum_\gamma \varepsilon_\gamma + O(t^{-1})\right)$$

where $i(x_j^{k+1}, x_i^k) = \sum_\gamma \varepsilon_\gamma$ *is the incidence number between* x_j^{k+1} *and* x_i^k *as is defined in the Witten-Morse theory.*

Proof. Let

$$d(t)E_{x_i^k}(t) = \sum_j \lambda_{ji}(t) E_{x_j^{k+1}}(t)$$

for some real $\lambda_{ji}(t)$, $1 \le j \le m_{k+1}$.

Since $\delta Int_k e^{tf} = Int_{k+1} e^{tf} d(t)$, we have

$$\delta(Int_k e^{tf} E_{x_i^k}(t)) = \sum_j \lambda_{ji}(t) Int_{k+1} e^{tf} E_{x_j^{k+1}}(t).$$

By Proposition 2(ii),

$$\delta e_{x_i^k} = \left(\frac{2t}{\pi}\right)^{-1/2} e^t \left(\sum_j \lambda_{ji}(t) e_{x_j^{k+1}} + O(t^{-1})\right).$$

Using the fact that $\delta e_{x_i^k} = \sum_j i(x_j^{k+1}, x_i^k) e_{x_j^{k+1}}$, the theorem follows by comparing coefficients in the above equation.

1.2. **S^1-equivariant cohomology, S^1-equivariant Hodge theory, equivariant Morse theory, Witten deformation and 'localized' operators.** (a) S^1-*equivariant Cohomology.* Let M^n be a compact manifold, $\mu : S^1 \times M \to M$ be a smooth action. Let X be the infinitesimal generator of the S^1-action and i_X be the contraction along the vector field X.

Then $d : \Omega^k(M) \to \Omega^{k+1}(M)$ and $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$.

Define

$$\widetilde{\Omega}^k(M) \equiv \Omega^k(M) \oplus \Omega^{k-2}(M) \oplus \cdots .$$

So $d + i_X : \widetilde{\Omega}^k(M) \to \widetilde{\Omega}^{k+1}(M)$. Note that

$$(d + i_X)^2 = di_X + i_X d = L_X$$

where L_X is the Lie derivative along the vector field X. Define

$$\Omega^*_{inv}(M) = \{\omega \in \Omega^*(M) | L_X\omega = 0\},$$
$$\widetilde{\Omega}^k_{inv}(M) = \Omega^k_{inv}(M) \oplus \Omega^{k-2}_{inv}(M) \oplus \cdots .$$

Then $D \equiv d + i_X : \widetilde{\Omega}^k_{inv}(M) \to \widetilde{\Omega}^{k+1}_{inv}(M)$ and $(\widetilde{\Omega}^*_{inv}(M), D)$ is a differential complex.

Define the S^1-equivariant cohomology of M to be

$$H^*_{S^1}(M) \equiv H^*(\widetilde{\Omega}^*_{inv}(M), D).$$

The complex $(\widetilde{\Omega}^*_{inv}(M), D)$ will be referred to as S^1-equivariant de Rham complex. A different description of the same complex will be given later.

(b) *S^1-equivariant Cohomology with Coefficient in an Orientation Line Bundle.*

More generally, let $E \to M$ be a vector bundle over M of rank k, $o(E) = \Lambda^{rank\mathrm{E}}(E) \to M$ be its orientation line bundle. This bundle has a natural flat connection and therefore if $\Omega^*(M, o(E))$ denotes the space of $o(E)$-valued forms on M, one can define $d : \Omega^k(M, o(E)) \to \Omega^{k+1}(M, o(E))$ by $d(\omega \otimes s) = d\omega \otimes s$ where s is a locally constant section of $o(E)$. Then $(\Omega^*(M, o(E)), d)$ is a cochain complex.

Also one can define $i_X : \Omega^k(M, o(E)) \to \Omega^{k-1}(M, o(E))$ by

$$i_X(\omega \otimes s) = i_X\omega \otimes s$$

and consider $L_X = di_X + i_X d : \Omega^k(M, o(E)) \to \Omega^k(M, o(E))$.

Let

$$\Omega^k_{inv}(M, o(E)) = \{\omega \in \Omega^k(M, o(E)) | L_X\omega = 0\},$$
$$\widetilde{\Omega}^k_{inv}(M, o(E)) = \Omega^k_{inv}(M, o(E)) \oplus \Omega^{k-2}_{inv}(M, o(E)) \oplus \cdots .$$

Since $D = d + i_X : \widetilde{\Omega}^k_{inv}(M, o(E)) \to \widetilde{\Omega}^{k+1}_{inv}(M, o(E))$ with $D^2 = 0$, one defines

$$H^*_{S^1}(M, o(E)) \cong H^*(\widetilde{\Omega}^*_{inv}(M, o(E)), D).$$

(c) *S^1-equivariant Hodge Theory.*

Suppose further that g is an S^1-invariant metric on M. Using the inner product induced by g on $\Omega^*(M)$ and hence on $\widetilde{\Omega}^*(M)$, we have

$$D = d + i_X : \widetilde{\Omega}^k(M) \to \widetilde{\Omega}^{k+1}(M),$$
$$D^* = d^* + i^*_X : \widetilde{\Omega}^{k+1}(M) \to \widetilde{\Omega}^k(M)$$

and $D(\widetilde{\Omega}^k_{inv}(M)) \subset \widetilde{\Omega}^{k+1}_{inv}(M), D^*(\widetilde{\Omega}^{k+1}_{inv}(M)) \subset \widetilde{\Omega}^k_{inv}(M)$.

Define

$$\widetilde{\Delta}^k = DD^* + D^*D : \widetilde{\Omega}^k(M) \to \widetilde{\Omega}^k(M).$$

Since $\widetilde{\Delta}^k = dd^* + d^*d + (di^*_X + i^*_X d + d^* i_X + i_X d^*) + i_X i^*_X + i^*_X i_X$ is elliptic,

$$\widetilde{\mathcal{H}}^k(M) \equiv \{\omega \in \widetilde{\Omega}^k(M) | \widetilde{\Delta}^k\omega = 0\}$$

is finite dimensional. Note that we have

$$I = P + \widetilde{\Delta}^k G \text{ on } \widetilde{\Omega}^k(M)$$

where P is the projection onto the harmonic forms and G is the Green's operator (or the parametrix) of $\widetilde{\Delta}^k$.

Therefore,

$$I = P + D_k(D^*_k G) + D^*_k(D_k G).$$

Let $\omega \in \widetilde{\Omega}^k_{inv}(M)$, then

$$\omega = P\omega \oplus D_k(D_k^* G\omega) \oplus D_k^*(D_k G\omega).$$

Since $D(\widetilde{\Omega}^k_{inv}(M)) \subset \widetilde{\Omega}^{k+1}_{inv}(M)$ and $D^*(\widetilde{\Omega}^{k+1}_{inv}(M)) \subset \widetilde{\Omega}^k_{inv}(M)$, every term in the above equation is in $\widetilde{\Omega}^*_{inv}(M)$, one can easily verify that

$$\widetilde{\Omega}^k_{inv}(M) = \widetilde{\mathcal{H}}^k_{inv}(M) \perp D_k(\widetilde{\Omega}^{k-1}_{inv}(M)) \perp D_k^*(\widetilde{\Omega}^{k+1}_{inv}(M)).$$

Consequently, $H^*_{S^1}(M) \cong \widetilde{\mathcal{H}}^k_{inv}(M)$ and $dim H^*_{S^1}(M) < \infty$.

(d) *Equivariant Morse theory.* Let M be a compact manifold with a smooth S^1-action, f be an S^1-invariant function on M. For $x \in M$, let $O_x = \{gx | g \in S^1\}$ be the orbit of x. A submanifold O of M is called an orbit if $O = O_x$ for some $x \in M$.

Let $d_x^2 f : T_xM \times T_xM \to \mathbb{R}$ be the Hessian of f at $x \in O$. Since f is invariant, it induces a symmetric bilinear form on T_xM/T_xO:

$$d_x^{\tilde{2}} f : (T_xM/T_xO) \times (T_xM/T_xO) \to \mathbb{R}.$$

Definition. $O \subset M$ is called a non-degenerate critical orbit of f if

(i) O is an orbit which consists of critical points of f.

(ii) $d_x^{\tilde{2}} f$ is non-degenerate for some x (and hence for any $x \in O$).

Remark. The above definition is independent of the choice of $x \in O$.

We shall consider only invariant functions whose critical orbits are all non-degenerate. Since M is compact, and since non-degenerate critical orbits are isolated, f has only a finite number of critical orbits.

Now let O be a critical orbit of f, $x \in O$ and $\nu(O)$ be the normal bundle of O in M. Since $d_x^{\tilde{2}} f$ is symmetric and non-degenerate, let $\nu^-(O)$ be the subbundle spanned by the negative eigenvectors of $d_x^{\tilde{2}} f$ where $x \in O$. More precisely, using the metric g, we identify $T_x(M)/T_x(O)$ with $T_X(O)^\perp$. With the above identification, we regard $d_x^{\tilde{2}} f$ as a non-degenerate, symmetric bilinear form on $T_x(O)^\perp$. Let $H_x : T_x(O)^\perp \to T_x(O)^\perp$ be the linear map associated with the bilinear form $d_x^{\tilde{2}} f$ w.r.t. the metric g. Then $\nu^-(O)$ is the subbundle spanned by the eigenvectors corresponding to the negative eigenvalues of H_x where $x \in O$.

Let θ^- be the orientation line bundle of $\nu^-(O)$.

Definitions.

1. *index* $O =$ *index* $d_x^{\tilde{2}} f$ for any $x \in O$,
2. $$\mathcal{P}_{S^1}(M,t) \equiv \sum_{i=0}^{\infty} t^i dim H^i_{S^1}(M),$$
3. $$\mathcal{M}_{S^1}(M,f,t) \equiv \sum_{O \in \text{ Crit Orbits}} t^{index\ O} \mathcal{P}_{S^1}(O,\theta^-,t),$$

where $\mathcal{P}_{S^1}(O,\theta^-,t) = \sum_{i=0}^{\infty} t^i dim H^i_{S^1}(O,\theta^-)$.

Then following Bott [B], we formulate the S^1-equivariant Morse Inequality as follows:

Theorem (S^1-equivariant Morse Inequality)**.**

$$\mathcal{M}_{S^1}(M,f,t) - \mathcal{P}_{S^1}(M,t) = (1+t)\mathcal{Q}(t)$$

where $\mathcal{Q}(t) = \sum_{i=0}^{\infty} t^i q_i$ *with* $q_i \geq 0$.

Our approach of proving the S^1-equivariant Morse inequality is to apply the Witten deformation of the S^1-equivariant de Rham complex, which is explained below, to obtain a subcomplex of finite dimensional vector spaces. The Morse inequality is proved once again by applying the algebraic lemma.

(e) *Witten deformation of S^1-equivariant de Rham complex.*

Let M be an S^1-manifold and g be an S^1-invariant metric on M.

Let $D(t) : \widetilde{\Omega}^k_{inv}(M) \to \widetilde{\Omega}^{k+1}_{inv}(M)$.

$D(t) \equiv e^{-tf}(d+i_X)e^{tf} = e^{-tf}de^{tf} + i_X = d(t) + i_X$.

Then $D(t)^* = e^{tf}(d^* + i_X^*)e^{-tf} = d(t)^* + i_X^*$.

Define

$$\begin{aligned}\widetilde{\Delta}^k(t) &= D(t)D(t)^* + D(t)^*D(t) : \widetilde{\Omega}^k_{inv}(M) \to \widetilde{\Omega}^k_{inv}(M)\\ &= d(t)d(t)^* + d(t)^*d(t) + (i_X i_X^* + i_X^* i_X) + (d^* i_X + i_X d^*) + (d i_X^* + i_X^* d).\end{aligned}$$

Clearly for any t, $(\widetilde{\Omega}^*_{inv}(M), D(t))$ calculates the S^1-equivariant cohomology of M. Since $\widetilde{\Delta}^k(t)$ (with domain and range $\widetilde{\Omega}^*(M)$) is elliptic, the Hodge Decomposition Theorem remains true for any t; moreover for any t, the $\widetilde{\Delta}^k(t)$-harmonic forms calculate $H^k_{S^1}(M)$.

(f) *Morse Lemma.*

Let G be a compact Lie group, H a closed subgroup, $\rho_i : H \to O(\mathbb{R}^{k_i})$, $i = 1,2$, be two orthogonal representations. Denote by $\rho = \rho_1 \oplus \rho_2 : H \to O(\mathbb{R}^{k_1+k_2})$ the direct sum of ρ_1 and ρ_2. With the diagonal action of H on $\mathbb{R}^{k_1+k_2} \times G$, the quotient space $\mathbb{R}^{k_1+k_2} \times_H G$ becomes a bundle E over $H \setminus G$.

$$E(\rho_1,\rho_2) = \mathbb{R}^{k_1+k_2} \times_H G \to H \setminus G.$$

Note that the zero section of E is an orbit of G which has the isotropy group H. Let $\mu : E \times G \to G$ be the smooth action given by

$$\mu([v,g_1],g) = [v,g_1 g].$$

Also, let $h : E \to \mathbb{R}$ be the map defined by

$$h([v_1 \oplus v_2, g]) = -|v_1|^2 + |v_2|^2.$$

The above example is called the standard model.

Morse Lemma. *Let M be a G-manifold of dimension n, f an invariant Morse function, x a critical point of f with orbit O_x, G_x be the isotropy group of x. Suppose that*

$$d^{\tilde{2}}_x f : T_xM/T_xO \times T_xM/T_xO \to \mathbb{R}$$

is a symmetric non-degenerate bilinear form of index k. Then there exist two orthogonal representations $\rho_1 : G_x \to O(\mathbb{R}^k)$ and $\rho_2 : G_x \to O(\mathbb{R}^{n-k-dim(G/G_x)})$ and a G-equivariant diffeomorphism

$$\phi : D(E(\rho_1,\rho_2)) \to U$$

where $D(E(\rho_1,\rho_2))$ is the unit disc bundle of E and U an open neighbourhood of O_x, so that

(i) *The zero section of* $D(E(\rho_1,\rho_2)) \to G_x \setminus G$ *is mapped onto* O_x.
(ii) $(f - f(x)) \circ \phi = h$.

Remark. Note that the product metric on $\mathbb{R}^{n-dim(G/G_x)} \times G$ induces a metric on $E = \mathbb{R}^{n-dim(G/G_x)} \times_{G_x} G$ which in turn, by using the above G-equivariant diffeomorphism ϕ, induces a metric on U.

(g) *Local expression of* $\widetilde{\Delta}^k(t)$ *near critical orbits.*

In the case of $G = S^1$, $G_x \cong 1, \mathbb{Z}_m$, or S^1. The following cases exhaust all the possibilities of the standard model.

Case 1: $G_x \cong 1, O_x \cong S^1$ and $U \cong D(E) \cong D^{n-1} \times S^1$ where D^{n-1} is the unit disc in $\mathbb{R}^{n-1}$. Let *index* $O_x = l$.

Let $x = (x_1, \ldots, x_{n-1}), \theta$ be the coordinates in D^{n-1} and S^1 respectively. Then the function f and the canonical metric g can be expressed as

$$f \circ \phi(x,\theta) = f(0) - x_1^2 - \cdots - x_l^2 + x_{l+1}^2 + \cdots + x_{n-1}^2, \tag{1.1}$$

$$dg = d^2x_1 + \cdots + d^2x_{n-1} + d^2\theta. \tag{1.2}$$

Recall that $i_X : \widetilde{\Omega}^{k-1}(M) \to \widetilde{\Omega}^k(M), i_X^* : \widetilde{\Omega}^k(M) \to \widetilde{\Omega}^{k-1}(M)$
with

$$\begin{aligned} i_X^*(\omega_k, \omega_{k-2}, \omega_{k-4}, \ldots) &= (i_X^*\omega_{k-2}, i_X^*\omega_{k-4}, \ldots) \\ &= (d\theta \wedge \omega_{k-2}, d\theta \wedge \omega_{k-4}, \ldots) \text{ in } U. \end{aligned}$$

For $\omega \in \widetilde{\Omega}^k_{inv}(M)$, write

$$\omega = (\omega_k \oplus \omega_{k-1} \wedge d\theta) \oplus (\tilde{\omega}_{k-2} \oplus \tilde{\omega}_{k-3} \wedge d\theta) \text{ in } U,$$

where $\tilde{\omega}_{k-2}$, respectively $\tilde{\omega}_{k-3}$, is the pullback of a form in $\widetilde{\Omega}^{k-2}(D^{n-1})$, respectively in $\widetilde{\Omega}^{k-3}(D^{n-1})$, by $\pi : D^{n-1} \times S^1 \to D^{n-1}$ and ω_k, respectively ω_{k-1} is the pullback of a form in $\Omega^k(D^{n-1})$, respectively $\Omega^{k-1}(D^{n-1})$, by the map π.

Since

$$\begin{aligned} (di_X^* + i_X^*d)(\tilde{\omega}_{k-2} \oplus \tilde{\omega}_{k-3} \wedge d\theta) &= d(i_X^*\tilde{\omega}_{k-2}) + i_X^*d\tilde{\omega}_{k-2} + i_X^*d(\tilde{\omega}_{k-3} \wedge d\theta) \\ &= d(d\theta \wedge \tilde{\omega}_{k-2}) + d\theta \wedge d\tilde{\omega}_{k-2} + 0 \\ &= -d\theta \wedge d\tilde{\omega}_{k-2} + d\theta \wedge d\tilde{\omega}_{k-2} \\ &= 0, \end{aligned}$$

$$(di_X^* + i_X^*d)(\omega_k \oplus \omega_{k-1} \wedge d\theta) = 0.$$

Hence

$$\begin{cases} di_X^* + i_X^*d = 0, \\ d^*i_X + i_Xd^* = (di_X^* + i_X^*d)^* = 0. \end{cases}$$

Therefore,

$$\begin{aligned} \widetilde{\Delta}^k(t) &= d(t)d(t)^* + d(t)^*d(t) + i_Xi_X^* + i_X^*i_X \text{ in } U \\ &= \widetilde{P}^k(t) + i_Xi_X^* + i_X^*i_X \end{aligned} \tag{1.3}$$

where $\widetilde{P}^k(t) = d(t)d(t)^* + d(t)^*d(t)$.

Case 1′: $G_x \cong \mathbb{Z}_m$, $O_x \cong S^1$ and $U \cong D^{n-1} \times_{\mathbb{Z}_m} S^1$.

Let p be the canonical projection

$$p : D^{n-1} \times S^1 \to D^{n-1} \times_{\mathbb{Z}_m} S^1 \cong U.$$

Note that the metric on $D^{n-1}\times_{\mathbb{Z}_m}S^1$ is induced by the product metric on $D^{n-1}\times S^1$. Since p is locally a diffeomorphism, we can use on $D^{n-1}\times_{\mathbb{Z}_m}S^1$ the coordinates $(x_1,\dots,x_{n-1},\theta)$ of $D^{n-1}\times_{\mathbb{Z}_m}S^1$. With respect to this coordinate system, $\widetilde{\Delta}^k(t)$ is given by the same expression as in Case 1.

Case 2: $G_x\cong S^1, O_x=x$ and $U\cong D^n\times_{S^1}S^1\cong D^n$.

In this case,

$$\widetilde{\Delta}^k(t)=\widetilde{P}^k(t)+(i_Xi_X^*+i_X^*i_X)+(di_X^*+i_X^*d)+(d^*i_X+i_Xd^*).$$

(h) '*Localized*' *operators.*

We define the corresponding Laplace operators in the standard models $\mathbb{R}^{n-1}\times S^1$, $E=\mathbb{R}^{n-1}\times_{\mathbb{Z}_m}S^1$ and $\mathbb{R}^n$ respectively.

Case 1: Define $\overline{\widetilde{\Delta}}{}^k_j(t):\widetilde{\Omega}^k_{inv}(\mathbb{R}^{n-1}\times S^1)\to\widetilde{\Omega}^k_{inv}(\mathbb{R}^{n-1}\times S^1)$ given by the same expression in (1.3), where the '$-$' signifies an operator in the standard model, j corresponds to the critical orbit O_j, with index $O_j=l$ and

$$\widetilde{P}^k(t)=dd^*+d^*d+4t^2x^2+t\left[-\sum_{i=1}^{l}[dx_i,i_{\partial_i}]+\sum_{i=l+1}^{n-1}[dx_i,i_{\partial_i}]\right],$$
$$i_Xi_X^*+i_X^*i_X=\varepsilon Id,$$

where

$$\varepsilon(\omega)=\begin{cases}0 & \text{if }\omega\in\Omega^k_{inv}(\mathbb{R}^{n-1}\times S^1),\\ \omega & \text{if }\omega\in\widetilde{\Omega}^{k-2}_{inv}(\mathbb{R}^{n-1}\times S^1).\end{cases}$$

Define $\mathcal{U}^{(n-1)}(\lambda):\Omega^*(\mathbb{R}^{n-1})\to\Omega^*(\mathbb{R}^{n-1})$ by

$$(\mathcal{U}^{(n-1)}(\lambda)\omega)(x)=\lambda^{n-1/2}\omega(\lambda x).$$

Then

$$\overline{\widetilde{\Delta}}{}^k_j(t)=\mathcal{U}^{(n-1)}(t^{1/2})[t\widetilde{K}^k_j+(i_Xi_X^*+i_X^*i_X)]\mathcal{U}^{(n-1)}(t^{-1/2})$$

where $\widetilde{K}^k_j=\widetilde{P}^k(1)$.

Case 1′: Define $\overline{\widetilde{\Delta}}{}^k_j(t):\widetilde{\Omega}^k_{inv}(E)\to\widetilde{\Omega}^k_{inv}(E)$ where $E=\mathbb{R}^{n-1}\times_{\mathbb{Z}_m}S^1$ is a vector bundle over S^1. The operator is given by the same expression in Case 1.

Case 2: Define $\overline{\widetilde{\Delta}}{}^k_j(t):\widetilde{\Omega}^k_{inv}(\mathbb{R}^n)\to\widetilde{\Omega}^k_{inv}(\mathbb{R}^n)$ by

$$\overline{\widetilde{\Delta}}{}^k_j(t)=\widetilde{P}^k(t)+(i_Xi_X^*+i_X^*i_X)+(di_X^*+i_X^*d)+(d^*i_X+i_Xd^*).$$

Since S^1 acts by isometry, the S^1-action is given by an orthogonal representation $\mathcal{R}$ of S^1 on $\mathbb{R}^n$ of the form

$$\mathcal{R}(e^{i\theta})=e^{im_1\theta}I_2\oplus e^{im_2\theta}\oplus\cdots\oplus e^{im_q\theta}I_2\oplus I_{n-2q}$$

for some $q\in\mathbb{Z}$ and some $m_i\in\mathbb{Z}$ for $1\le i\le q$. Here I_k denotes the identity on $\mathbb{R}^k$.

Then

$$X=(-m_1x_2,m_1x_1,-m_2x_4,m_2x_3,\dots,-m_qx_{2q},m_qx_{2q-1},0,\dots,0)$$

and $i_Xi_X^*+i_X^*i_X=\varepsilon|X|^2$ where ε is defined as in Case 1 above.

In this case,

$$\overline{\widetilde{\Delta}}{}^k_j(t)=\mathcal{U}^{(n)}(t^{1/2})\left[t\widetilde{K}^k_j+\frac{1}{t}\varepsilon|X|^2+(i_X^*d+di_X^*+i_Xd^*+d^*i_X)\right]\mathcal{U}^{(n)}(t^{-1/2}).$$

1.3. **Formulation of results.**

(a) *Witten theory.*

Let $O_1, \dots, O_r$ be all the critical orbits of f,

$$0 \le E_1^{(k)}(t) \le E_2^{(k)}(t) \le \cdots \le E_l^{(k)}(t) \le \dots \text{ be all the eigenvalues of } \widetilde{\Delta}^k(t),$$

$$0 \le \overline{e}_1^{(k)} \le \overline{e}_2^{(k)} \le \cdots \le \overline{e}_l^{(k)} \le \dots \text{ be all the eigenvalues of } \bigoplus_{j=1}^r \overline{\widetilde{K}}{}_j^k,$$

$$0 \le \overline{e}_1^{(k)}(t) \le \overline{e}_2^{(k)}(t) \le \cdots \le \overline{e}_l^{(k)}(t) \le \dots \text{ be all the eigenvalues of } \bigoplus_{j=1}^r \overline{\widetilde{\Delta}}{}_j^k(t).$$

Here $\widetilde{\Delta}^k(t)$ acts on $\widetilde{\Omega}_{inv}^k(M)$ and the localized operators $\overline{\widetilde{\Delta}}{}_j^k(t)$ act on the corresponding spaces of invariant forms, namely $\widetilde{\Omega}_{inv}^k(\mathbb{R}^{n-1} \times S^1), \widetilde{\Omega}_{inv}^k(E)$ or $\widetilde{\Omega}_{inv}^k(\mathbb{R}^n)$. Note that $\bigoplus_{j=1}^r \overline{\widetilde{K}}{}_j^k$ acts on the space

$$\Big(\bigoplus \widetilde{\Omega}_{inv}^k(\mathbb{R}^{n-1} \times S^1)\Big) \oplus \Big(\bigoplus \widetilde{\Omega}_{inv}^k(E)\Big) \oplus \Big(\bigoplus \widetilde{\Omega}_{inv}^k(\mathbb{R}^n)\Big)$$

with as many copies in the individual summands as the number of critical orbits whose local structure is described by the corresponding standard model.

Theorem 1.

$$\lim_{t\to\infty} \frac{E_l^{(k)}(t)}{t} = \lim_{t\to\infty} \frac{e_l^{(k)}(t)}{t} = \overline{e}_l^{(k)}.$$

Now consider those eigenvalues of $\widetilde{\Delta}^k(t)$ so that $\lim_{t\to\infty} \frac{E_l^{(k)}(t)}{t} = 0$. That is we need to count the zero eigenvalues of $\bigoplus_{j=1}^r \overline{\widetilde{K}}{}_j^k$. It suffices to count in each case separately.

Case 1: $\overline{\widetilde{K}}{}_j^k : \widetilde{\Omega}_{inv}^k(\mathbb{R}^{n-1} \times S^1) \to \widetilde{\Omega}_{inv}^k(\mathbb{R}^{n-1} \times S^1)$.

For any $\omega \in \widetilde{\Omega}_{inv}^k(\mathbb{R}^{n-1} \times S^1)$, there exists $\tilde{\omega}_k, \tilde{\omega}_{k-1}$ which are pullback of forms in $\widetilde{\Omega}^k(\mathbb{R}^{n-1}), \widetilde{\Omega}^{k-1}(\mathbb{R}^{n-1})$ respectively by the projection $p : \mathbb{R}^{n-1} \times S^1 \to \mathbb{R}^{n-1}$ such that

$$\omega = \tilde{\omega}_k \otimes 1 + \tilde{\omega}_{k-1} \otimes d\theta$$

that is

$$\tilde{\Omega}_{inv}^k(\mathbb{R}^{n-1} \times S^1) = (\widetilde{\Omega}^k(\mathbb{R}^{n-1}) \otimes \mathbb{R} \cdot 1) \oplus (\widetilde{\Omega}^{k-1}(\mathbb{R}^{n-1}) \otimes \mathbb{R} \cdot d\theta).$$

Let Δ_M denote the Laplace operator on a manifold M. Then

$$\begin{aligned} \overline{\widetilde{K}}{}_j^k &= \Delta_{\mathbb{R}^{n-1} \times S^1} + 4x^2 + A_j \\ &= (\Delta_{\mathbb{R}^{n-1}} \otimes id \oplus id \otimes \Delta_{S^1}) + 4x^2 + A_j \\ &= (\Delta_{\mathbb{R}^{n-1}} + 4x^2 + A_j) \otimes id + id \otimes \Delta_{S^1} \end{aligned}$$

where $A_j = -\sum_{i=1}^{index\ O_j} [dx_i, i_{\partial_i}] + \sum_{i=index\ O_j+1}^{n-1} [dx_i, i_{\partial_i}]$.

When regarded as an operator on the space of all forms in $\mathbb{R}^{n-1}$, then $\Delta_{\mathbb{R}^{n-1}} + 4x^2 + A_j$ has exactly one zero eigenvalue with corresponding eigenform $\omega_{l_j} \in \Omega^{l_j}(\mathbb{R}^{n-1})$, where $l_j = index\ O_j$. This implies that $\omega_{l_j} \otimes 1, \omega_{l_j} \otimes d\theta$ are eigenforms corresponding the eigenvalue zero in $\widetilde{\Omega}_{inv}^*(\mathbb{R}^{n-1} \times S^1)$. Since $\overline{\widetilde{K}}{}_j^k$ acts on this space, we have

(i) If $l_j \leq k$, then $\overline{\overline{\widetilde{K}}}{}^k_j$ has exactly one zero eigenvalue. This is because

$$\begin{cases} \text{If } l_j \equiv k (\operatorname{mod} 2), \text{ then } \omega_{l_j} \otimes 1 \text{ is the corresponding eigenform,} \\ \text{If } l_j \not\equiv k (\operatorname{mod} 2), \text{ then } \omega_{l_j} \otimes d\theta \text{ is the corresponding eigenform.} \end{cases}$$

(ii) If $l_j > k$, then $\overline{\overline{\widetilde{K}}}{}^k_j$ has no zero eigenvalue.

Case 1′: $\overline{\overline{\widetilde{K}}}{}^k_j : \widetilde{\Omega}^k_{inv}(E) \to \widetilde{\Omega}^k_{inv}(E)$.

The situation is similar, with the only difference that we have to restrict to those O_j whose θ^- is trivial. (Indeed, if there exists such an eigenform $\omega_{l_j} = g(x)dx_1 \wedge \cdots \wedge dx_{l_j}$, since it is invariant, $g(x) \neq 0$ for any $x \in O_j$, which implies that θ^- is trivial.)

Case 2: $\overline{\overline{\widetilde{K}}}{}^k_j : \widetilde{\Omega}^k_{inv}(\mathbb{R}^n) \to \widetilde{\Omega}^k_{inv}(\mathbb{R}^n)$ where $\overline{\overline{\widetilde{K}}}{}^k_j = \Delta + 4x^2 + A_j$.

In this case, any eigenform corresponding to eigenvalue zero is automatically invariant. Hence

(i) If $l_j \leq k$ and $l_j \equiv k(\operatorname{mod} 2)$, then $\overline{\overline{\widetilde{K}}}{}^k_j$ has exactly one zero eigenvalue.

(ii) If $l_j \leq k$ and $l_j \not\equiv k(\operatorname{mod} 2)$, then $\overline{\overline{\widetilde{K}}}{}^k_j$ has no zero eigenvalue.

(iii) If $l_j > k$, then $\overline{\overline{\widetilde{K}}}{}^k_j$ has no zero eigenvalue.

Definition.

$$\widetilde{\Omega}^k_{inv,0}(M,t) = Span\left\{\Psi(t) \in \widetilde{\Omega}^k_{inv}(M) | \widetilde{\Delta}^k(t)\Psi(t) = E(t)\Psi(t) \text{ and } \lim_{t\to\infty} \frac{E(t)}{t} = 0\right\}.$$

Corollary. (i)

$$dim \widetilde{\Omega}^k_{inv,0}(M,t) = m_k + m_{k-1} + \cdots + m_0 + m^f_k + m^f_{k-2} + m^f_{k-4} + \cdots$$

where

$$\begin{cases} m_i = \textit{ number of critical orbits of index } i \textit{ whose } \theta^- \textit{ is trivial,} \\ m^f_i = \textit{ number of critical fixed points of index } i. \end{cases}$$

(ii) $(\widetilde{\Omega}^*_{inv,0}(M,t), D(t))$ *is a complex of finite dimensional vector spaces which calculates the* S^1*-equivariant cohomology of* M.

Using the complex $(\widetilde{\Omega}^*_{inv,0}(M,t), D(t))$, one can verify the S^1-equivariant Morse Inequality. However, one can also verify the S^1-equivariant Morse Inequality by using another subcomplex

$$(\widetilde{\Omega}^*_{inv,sm}(M,t), D(t))$$

which not only calculates the S^1-equivariant cohomology of M, but also is better connected with the geometric complex which is induced by (f, g) and calculates the S^1-equivariant cohomology.

To introduce this subcomplex, note that the eigenvalue $E(t)$ such that

$$\lim_{t\to\infty} \frac{E(t)}{t} = 0$$

have the property

$$\lim_{t\to\infty} E(t) = a$$

for some $a \geq 0$ which depends on the one parameter of eigenvalues $E(t)$.

Definition.

1. $E(t)$ is a small eigenvalue iff $\lim_{t\to\infty} E(t) = a = 0$. Otherwise, i.e. if $a > 0$, $E(t)$ is of finite type.

2.
$$\widetilde{\Omega}^k_{inv,sm}(M,t) = Span\{\Psi(t) \in \widetilde{\Omega}^k_{inv}(M) | \widetilde{\Delta}^k(t)\Psi(t) = E(t)\Psi(t) \text{ and } E(t) \text{ is a small eigenvalue}\}.$$

Clearly,
$$\widetilde{\Omega}^*_{inv,sm}(M,t) \subset \widetilde{\Omega}^*_{inv,0}(M,t).$$

Lemma.
$$dim\, \widetilde{\Omega}^k_{inv,sm}(M,t) = m_k + m^f_k + m^f_{k-2} + m^f_{k-4} + \cdots .$$

Using the complex $(\widetilde{\Omega}^*_{inv,sm}(M,t), D(t))$, the Morse inequality follows by applying the algebraic lemma mentioned in §1.1.

(b) *Helffer-Sjöstrand Theory for S^1-equivariant Cohomology.*

Suppose f is an S^1-invariant function on M, g an S^1-invariant metric on M.

Definition. (f,g) is said to satisfy the Morse-Smale condition if for any two critical orbits O_x and O_y, W^-_x and W^+_y intersect transversally, where W^-_x, W^+_y are the descending and ascending manifold of O_x and O_y respectively.

Definition. An S^1-invariant Morse function is said to be self-indexing if for any critical orbit $O_x \ni x$,
$$f(x) = index\ x.$$

Definition. The pair (f,g) is said to be compatible if for any critical orbit O_x of f, one can choose local coordinate system about O_x such that

(a) If $O_x \cong S^1$, then f and g are given by (1.1) and (1.2) respectively when represented in the above coordinate system.

(b) If $O_x \cong$ *point*, then when represented in the above coordinate system $(x_1, \ldots, x_n)$,
$$f(x_1,\ldots,x_n) = f(x) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2,$$
$$dg = d^2dx_1 + \cdots + d^2x_n.$$

Let f be a self-indexing invariant Morse function such that (f,g) is compatible and satisfies the Morse-Smale condition. Then a geometric complex, which calculates the S^1-equivariant cohomology of M, can be described as follows (see §4.1 for details).

Let $E = E_{S^1}$ be the universal principal bundle of S^1. $M_{S^1} = E \times M/S^1$ be the homotopy quotient of M.

Define $\tilde{f} : E \times M \to \mathbb{R}$ by $\tilde{f}(e,x) = f(x)$. Then $\tilde{f}$ descends to a function on the infinite dimensional manifold M_{S^1} which is denoted by f_{S^1}. Let $X_k = f^{-1}_{S^1}((-\infty, k+\frac{1}{2}])$, then we have
$$X_0 \subset X_1 \subset \cdots \subset X_n = M_{S^1}.$$

Define $\partial : H_*(X_k, X_{k-1}) \to H_{*-1}(X_{k-1}, X_{k-2})$ by
$$\begin{array}{ccccc} H_*(X_k,X_{k-1}) & \to & H_{*-1}(X_{k-1}) & \xrightarrow{i_{*-1}} & H_{*-1}(X_{k-1},X_{k-2}) \\ [\sigma] & \to & [\partial\sigma] & \to & i_*[\partial\sigma] \end{array}$$

where i_* is induced from the inclusion $X_{k-2} \subset X_{k-1}$

$$H_*(X_{k-2}) \to H_*(X_{k-1}) \xrightarrow{i_*} H_*(X_{k-1}, X_{k-2}) \to H_{*-1}(X_{k-2}).$$

Define

$$C_k(M,f) = \bigoplus_{i=0}^{n} H_k(X_i, X_{i-1}).$$

Also the above boundary map induces the map

$$\partial : C_k(M,f) \to C_{k-1}(M,f)$$

with $\partial^2 = 0$. Therefore, we obtain the complex $(C_*(M,f), \partial)$ and its dual cochain complex

$$(C^*(M,f), \delta).$$

The complex $(\widetilde{\Omega}^*_{inv,sm}(M,t), D(t))$ can be interpreted as a complex of differential forms on M_{S^1} which can be explained as follows (see [AB] and §4.2 for details):

Let $g \cong \mathbb{R}$ be the Lie algebra of S^1, g^* be the dual of g.

Let Sg^* be the symmetric algebra generated by g^* whose generator is denoted by u, Λg^* be the exterior algebra generated by g^* whose generator is denoted by θ with $\deg u = 2, \deg \theta = 1$.

Let $W(g) = \Lambda g^* \otimes Sg^*$, called the Weil algebra, which is the algebra generated freely by θ and u as a commutative graded algebra i.e.

$$\omega_p \omega_q = (-1)^{pq} \omega_q \omega_p.$$

Define $D_0 : W(g) \to W(g)$ by

$$\begin{cases} D_0\theta + u = 0, \\ D_0 u = 0 \end{cases}$$

and is extended to $W(g)$ as an anti-derivation. Observe that $D_0^2 = 0$ and the complex $(W(g), D_0)$ is a subcomplex of $(\Omega^*(E_{S^1}), D_0)$ where D_0 is the exterior derivative. $(W(g), D_0)$ is usually referred to as the de Rham model for E_{S^1} whose homology calculates the cohomology of E_{S^1}.

Consider the principal S^1-bundle over B_{S^1}, $S^1 \dashrightarrow E_{S^1} \xrightarrow{\pi} B_{S^1}$. Since S^1 acts on E_{S^1}, let X be its generating vector field. Note that $\pi^* : \Omega^*(B_{S^1}) \to \Omega^*(E_{S^1})$ is an injection and $\omega \in \pi^*(\Omega^*(B_{S^1}))$ can be characterized by

$$\begin{cases} i_X(\omega) = 0, \\ L_X(\omega) = (i_X D_0 + D_0 i_X)(\omega) = 0 \end{cases}$$

where D_0 is the exterior derivative. Hence define the basic subcomplex Bg of $W(g)$

$$Bg = \{\omega \in W(g) | i_X(\omega) = L_X(\omega) = 0\}.$$

Then $Bg = \mathbb{R}[u] \cong H^*(B_{S^1})$ and is called the de Rham model of B_{S^1}.

For the de Rham model for $M_{S^1} = E \times_{S^1} M$, consider

$$(W(g) \otimes \Omega^*(M), \mathbb{D} = D_0 \otimes id + (-1)^{\deg \omega} id \otimes d)$$

which is the de Rham model for $E_{S^1} \times M$. Since S^1 acts on $E_{S^1} \times M$ by diagonal action, let X be its generating vector field on $E_{S^1} \times M$.

Define the basic subcomplex $(\Omega_g^*(M), \mathbb{D})$ of $W(g) \otimes \Omega^*(M)$ by

$$\Omega_g^*(M) = \{\omega \in W(g) \otimes \Omega^*(M) | i_X(\omega) = L_X(\omega) = 0\}.$$

It is well known (cf. [AB]) that $H^*(\Omega^*_g(M), \mathbb{D}) = H^*_{S^1}(M)$. $(\Omega^*_g(M), \mathbb{D})$ is referred to as the de Rham model for M_{S^1} and can be interpreted as a subcomplex of $(\Omega^*(M_{S^1}), \mathbb{D})$.

In fact, there exists an isomorphism

$$\tilde{\lambda} : (\widetilde{\Omega}^*_{inv}(M), D) \to (\Omega^*_g(M), \mathbb{D})$$

between the two cochain complexes, and therefore it induces for any t an isomorphism

$$\tilde{\lambda}(t) : (\widetilde{\Omega}^*_{inv}(M), D(t) = e^{-tf} D e^{tf}) \to (\Omega^*_g(M), \mathbb{D}(t) = (e^{-tf})_{S^1} \mathbb{D} (e^{tf})_{S^1})$$

between the two cochain complexes. Since $(\widetilde{\Omega}^*_{inv,sm}(M,t), D(t)) \subset (\widetilde{\Omega}^*_{inv}(M), D(t))$, $\tilde{\lambda}(t)$ induces a corresponding '*small*' subcomplex $(\Omega^*_{g,small}(M,t), \mathbb{D}(t))$ which calculates the S^1-equivariant cohomology of M.

Proposition.

$$\begin{array}{rcl} Int : (\Omega^*_g(M), \mathbb{D}) & \to & (C^*(M,f), \delta), \\ \omega & \to & \int_\sigma \omega \end{array}$$

is a morphism of cochain complexes.

As a consequence, the composition

$$(\Omega^*_{g,small}(M,t), \mathbb{D}(t)) \xrightarrow{(e^{tf})_{S^1}} ((e^{tf})_{S^1} \Omega^*_{g,small}(M,t), \mathbb{D}) \xrightarrow{Int} (C^*(M,f), \delta)$$

is a morphism of cochain complexes. Finally, we can state one of the main results in this paper (see §4.3):

Theorem 2. *Suppose f is a self-indexing invariant Morse function such that (f,g) satisfies the Morse-Smale condition. Then there exists a morphism of cochain complexes*

$$F^*(t) = Int(e^{tf})_{S^1} : (\widetilde{\Omega}^*_{g,small}(M,t), D(t)) \to (C^*(M,f), \delta)$$

such that w.r.t. some suitably chosen bases (*see* §4.3 *for the description of these bases*)*,*

$$F^*(t) = I + O(t^{-1}).$$

The following is an outline of this paper. In §2 we prove Theorem 1, and in §3, we apply Theorem 1 to obtain $dim\, \widetilde{\Omega}^k_{inv,sm}(M,t)$ as described in the above lemma, hence finish the proof of the Morse inequality. In §4 we show that $(\widetilde{\Omega}^*_{inv,sm}(M,t), D(t))$ converges to $(C^*(M,f), \delta)$ as $t \to \infty$, i.e. we prove Theorem 2.

This paper is originally written as part of my Ph.D. thesis under the guidance of Professor Dan Burghelea. The author would like to thank him for suggesting the problem and his help throughout this work. Also the author plans to extend the results to the case of an arbitrary compact connected Lie group in future work.

2. Proof of Theorem 1

We begin by recalling and introducing some notations. We have already defined the Laplacians $\widetilde{\Delta}^k(t), \overline{\widetilde{\Delta}}{}^k_j(t), \overline{\widetilde{K}}{}^k_j$ in §1. (Here, the bar above an operator designates an operator on the standard model.) In this section, k will be a fixed integer. For simplicity of notation, the superscript k for eigenvalues and eigenvectors will be dropped.

Let

$$\begin{aligned}
0 \le E_1(t) \le E_2(t) \le \cdots \le E_l(t) \le \cdots &\text{ be all the eigenvalues of } \widetilde{\Delta}^k(t),\\
\Psi_1(t), \Psi_2(t), \ldots, \Psi_l(t), \ldots &\text{ be the corresponding normalized}\\
&\text{ eigenvectors,}\\
0 \le \overline{e}_1(t) \le \overline{e}_2(t) \le \cdots \le \overline{e}_l(t) \le \cdots &\text{ be all the eigenvalues of } \bigoplus_{j=1}^{r} \overline{\widetilde{\Delta}}{}_j^k(t),\\
0 \le \overline{e}_1 \le \overline{e}_2 \le \cdots \le \overline{e}_l \le \cdots &\text{ be all the eigenvalues of } \bigoplus_{j=1}^{r} \overline{\widetilde{K}}{}_j^k.
\end{aligned}$$

Theorem 1.

$$\lim_{t\to\infty} \frac{E_l(t)}{t} = \overline{e}_l.$$

Proof. We shall follow the argument of B. Simon (cf. [S], pp. 219–222) and separate the proof into two parts. In Part I, we shall prove

$$\overline{\lim_{t\to\infty}} \frac{E_l(t)}{t} \le \overline{e}_l \tag{2.1}$$

and in Part II,

$$\underline{\lim_{t\to\infty}} \frac{E_l(t)}{t} \ge \overline{e}_l. \tag{2.2}$$

Part I. For any n, let $\rho_n \in C_c^\infty(\mathbb{R}^n)$, $0 \le \rho_n \le 1$, such that

$$\rho_n(x) = \begin{cases} 1 & \text{if } |x| \le 1/2, \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

For any critical orbit O_j, $1 \le j \le r$, define $J_j \in C^\infty(M)$ such that

$$J_j(x) = \begin{cases} \rho_{n-1}(x) & \text{if } O_j \cong S^1 \text{ where } (x_1, \ldots, x_{n-1}, \theta) \text{ are the} \\ & \text{coordinates in } U_j \text{ as defined in §1.2(g)}, \\ \rho_n(x) & \text{if } O_j \text{ is a critical fixed point.} \end{cases}$$

Define $J_0 = \sqrt{1 - \sum_{j=1}^r J_j^2}$.

It is clear that $\overline{\widetilde{\Delta}}{}_j^k(t)$ acting on L^2-forms have discrete spectrum in Cases 1 and 1′, and hence has a complete orthonormal basis of eigenvectors $\{\overline{\Psi}_{l,j}(t)\}_{l\in\mathbb{N}}$.

In proving Part I, we need several lemmas concerning the operator $\overline{\widetilde{\Delta}}{}_j^k(t)$ corresponding to a critical fixed point O_j, which will be proved in the Appendix.

Lemma 2.1. *For a critical fixed point O_j, the 'localized' operator $\overline{\widetilde{\Delta}}{}_j^k(t)$ has discrete spectrum and has a complete orthonormal basis of eigenvectors $\{\overline{\Psi}_{l,j}(t)\}_{l\in\mathbb{N}}$ corresponding to the eigenvalues*

$$0 \le \overline{e}_{1,j}(t) \le \overline{e}_{2,j}(t) \le \cdots \le \overline{e}_{l,j}(t) \le \cdots.$$

Let $0 \le \overline{e}_{1,j} \le \overline{e}_{2,j} \le \cdots \le \overline{e}_{l,j} \le \cdots$ be all the eigenvalues of $\overline{\widetilde{K}}{}_j^k$.

Lemma 2.2.

$$\lim_{t\to\infty}\frac{\overline{e}_{l,j}(t)}{t}=\overline{e}_{l,j}.$$

In all cases, define $\Phi_{l,j}(t)\in C^\infty(M)$ by

$$\Phi_{l,j}(t)=J_j\overline{\Psi}_{l,j}(t).$$

Note that $\Phi_{l,j}(t)$ is localized at the critical orbit O_j. Also, using the identification of U_j with a neighbourhood of zero section of the standard model, $\Phi_{l,j}(t)$ can be considered as a form in $\widetilde{\Omega}^k_{inv}(E)$ or $\widetilde{\Omega}^k_{inv}(\mathbb{R}^n)$.

Lemma 2.3. *For a critical fixed point O_j,*

$$\lim_{t\to\infty}\langle\Phi_{l,j}(t),\Phi_{m,j}(t)\rangle=\delta_{lm}.$$

The above lemmas will be proved in the Appendix. Also we need (cf. [S]).

IMS Localization Formula.

$$J_j\widetilde{\Delta}^k(t)J_j=\frac{1}{2}(J_j^2\widetilde{\Delta}^k(t)+\widetilde{\Delta}^k(t)J_j^2)+|dJ_j|^2.$$

Hence

$$\widetilde{\Delta}^k(t)=\sum_{j=0}^{r}J_j\widetilde{\Delta}^k(t)J_j-\sum_{j=0}^{r}|dJ_j|^2.$$

Based on the above lemmas, we establish

Proposition 2.4. *For any critical orbit O_j, suppose $\overline{\Psi}_{l,j},\overline{\Psi}_{m,j}$ are two eigenvectors of $\overline{\widetilde{\Delta}}{}^k_j(t)$, then $\langle\Phi_{l,j}(t)\overline{\widetilde{\Delta}}{}^k_j(t)\Phi_{m,j}(t)\rangle=\overline{e}_{l,j}(t)\delta_{lm}+o(t)$.*

Proof. In Cases 1 and 1′, we have (cf. [S])

$$\lim_{t\to\infty}\langle\Phi_{l,j}(t),\Phi_{m,j}(t)\rangle=\delta_{lm}.$$

By Lemma 2.3, we have in all cases,

$$\langle\Phi_{l,j}(t),\Phi_{m,j}(t)\rangle=\delta_{lm}+\varepsilon_{lm}(t)$$

where $\lim_{t\to\infty}\varepsilon_{lm}(t)=0$

$$\begin{aligned}\langle\Phi_{l,j}(t),\overline{\widetilde{\Delta}}{}^k_j(t)\Phi_{m,j}(t)\rangle&=\langle\overline{\Psi}_{l,j}(t),J_j\overline{\widetilde{\Delta}}{}^k_j(t)J_j\overline{\Psi}_{m,j}(t)\rangle\\&=\frac{1}{2}\langle\overline{\Psi}_{l,j}(t),(J_j^2\overline{\widetilde{\Delta}}{}^k_j(t)+\overline{\widetilde{\Delta}}{}^k_j(t)J_j^2)\overline{\Psi}_{m,j}(t)\rangle\\&\quad+\langle\overline{\Psi}_{l,j}(t),|dJ_j|^2\overline{\Psi}_{m,j}(t)\rangle\\&\qquad\text{(by IMS Localization Formula)}\\&=\frac{1}{2}(\overline{e}_{l,j}(t)+\overline{e}_{m,j}(t))\langle\Phi_{l,j}(t),\Phi_{m,j}(t)\rangle+O(1)\\&=\frac{1}{2}(\overline{e}_{l,j}(t)+\overline{e}_{m,j}(t))\delta_{lm}+\frac{1}{2}(\overline{e}_{l,j}(t)+\overline{e}_{m,j}(t))\varepsilon_{lm}(t)+o(t).\end{aligned}$$

By Lemma 2.2,

$$\lim_{t\to\infty}\frac{1}{2}\frac{(\overline{e}_{l,j}(t)+\overline{e}_{m,j}(t))\varepsilon_{lm}(t)}{t}=\frac{1}{2}\lim_{t\to\infty}\left(\frac{\overline{e}_{l,j}(t)}{t}+\frac{\overline{e}_{m,j}(t)}{t}\right)\lim_{t\to\infty}\varepsilon_{lm}(t)=0.$$

This proves the proposition.

Now let $\{\overline{\Psi}_l(t)\}_{l\in\mathbb{N}}$ be the eigenvectors corresponding to the eigenvalues $\{\overline{e}_l(t)\}_{l\in\mathbb{N}}$. Then there exist j_l, n_l so that

$$\overline{\Psi}_l(t) = \overline{\Psi}_{n_l,j_l}.$$

Define $\Phi_l(t) = J_{j_l}\overline{\Psi}_{n_l,j_l}(t)$.

The above proposition leads to

Proposition 2.5.

$$\langle \Phi_l(t), \widetilde{\Delta}^k(t)\Phi_m(t)\rangle = \overline{e}_l(t)\delta_{lm} + o(t).$$

Proof. If $j_l \neq j_m$, then for sufficient large t, $\Phi_l(t), \Phi_m(t)$ have disjoint support. Hence

$$\langle \Phi_l(t)\widetilde{\Delta}^k(t)\Phi_m(t)\rangle = 0.$$

If $j_l = j_m = j$, then for sufficient large t, $\Phi_l(t), \Phi_m(t)$ have support in U_j and since $\widetilde{\Delta}^k(t), \overline{\widetilde{\Delta}}{}^k_j(t)$ agree on U_j, we have

$$\begin{aligned} \langle \Phi_l(t), \widetilde{\Delta}^k(t)\Phi_m(t)\rangle &= \langle \Phi_{n_l,j}(t), \overline{\widetilde{\Delta}}{}^k_j(t)\Phi_{n_m,j}(t)\rangle \\ &= \overline{e}_{n_l,j}(t)\delta_{lm} + o(t) \text{ by Proposition 2.4} \\ &= \overline{e}_l(t)\delta_{lm} + o(t). \end{aligned}$$

This proves the proposition.

Since $\widetilde{\Delta}^k(t)$ operates on forms on a compact manifold, it has a discrete spectrum. The min-max principle, Proposition 2.5 and Lemma 2.2 imply

$$\overline{\lim_{t\to\infty}} \frac{E_l(t)}{l} \leq \lim_{t\to\infty} \frac{\overline{e}_l(t)}{t} = \overline{e}_l.$$

Part II.

$$\underline{\lim}_{t\to\infty} \frac{E_l(t)}{t} \geq \overline{e}_l. \tag{2.3}$$

Proof. To prove (2.3), it suffices to show that for any $e \in (\overline{e}_l, \overline{e}_{l+1})$, there exists a symmetric operator $R(t)$ of rank at most l such that

$$\widetilde{\Delta}^k(t) \geq te + R(t) + o(t). \tag{2.4}$$

If such operator exists, in order to derive (2.3) from (2.4), choose $0 \neq \Psi \in \widetilde{\Omega}^k_{inv}(M)$ such that

$$\Psi(t) \in Span\{\Psi_1(t), \ldots, \Psi_{l+1}(t)\} \cap KerR(t) \text{ and } \|\Psi(t)\| = 1.$$

Then

$$\begin{aligned} (2.4) &\Rightarrow \langle \Psi(t), \widetilde{\Delta}^k(t)\Psi(t)\rangle \geq te + o(t) \\ &\Rightarrow E_{l+1}(t) \geq \langle \Psi(t), \widetilde{\Delta}^k(t)\Psi(t)\rangle \geq te + o(t) \\ &\Rightarrow \underline{\lim}_{t\to\infty} \frac{E_{l+1}(t)}{t} \geq e \quad \forall e \in (\overline{e}_l, \overline{e}_{l+1}) \\ &\Rightarrow \underline{\lim}_{t\to\infty} \frac{E_{l+1}(t)}{t} \geq \overline{e}_{l+1}. \end{aligned}$$

To construct $R(t)$, let f_j be an invariant function on M such that

$$\begin{cases} f_j(x) = \|df(x)\|^2 & \text{if } x \in U_j, \\ f_j(x) \geq c_j > 0 & \text{if } x \text{ is outside } U_j. \end{cases}$$

Define $\widetilde{\Delta}_j^k(t) : \widetilde{\Omega}_{inv}^k(M) \to \widetilde{\Omega}_{inv}^k(M)$ such that

$$\widetilde{\Delta}_j^k(t) = dd^* + d^*d + t^2 f_j + tA + (i_X^* i_X + i_X i_X^*) + (i_X^* d + d i_X^*) + (i_X d^* + d^* i_X).$$

Observe that $\widetilde{\Delta}_j^k(t) = \widetilde{\Delta}^k(t)$ on U_j.

In order to show that $R(t)$ has rank at most l, we need

Lemma 2.6.

$$\lim_{t\to\infty} \frac{E_{l,j}(t)}{t} = \overline{e}_{l,j}$$

where

$$0 \leq E_{1,j}(t) \leq E_{2,j}(t) \leq \cdots \leq E_{l,j}(t) \leq \cdots$$

are all the eigenvalues of $\widetilde{\Delta}_j^k(t)$.

Proof of Lemma 2.6. By the proof of Part I,

$$\overline{\lim_{t\to\infty}} \frac{E_{l,j}(t)}{t} \leq \overline{e}_{l,j}.$$

Therefore, it suffices to show

$$\underline{\lim}_{t\to\infty} \frac{E_{l,j}(t)}{t} \geq \overline{e}_{l,j}$$

let $\Psi_{l,j}(t)$ be a normalized eigenvector of $\widetilde{\Delta}_j^k(t)$ corresponding to the eigenvalue $E_{l,j}(t)$. Recall that

$$U_j \cong \begin{cases} D^{n-1} \times_{G_x} S^1 & \text{if } O_j \cong S^1, \\ D^n & \text{if } O_j \text{ is a critical fixed point.} \end{cases}$$

Define

$$U_j' \cong \begin{cases} D^{n-1}(1/2) \times_{G_x} S^1 & \text{if } O_j \cong S^1, \\ D^n(1/2) & \text{if } O_j \text{ is a critical fixed point,} \end{cases}$$

where $D^n(1/2)$ is the disc of radius $1/2$ in $\mathbb{R}^n$.

Claim 1.

$$\lim_{t\to\infty} \|\Psi_{l,j}(t)\|_{M\setminus U_j'} = 0.$$

Proof of Claim 1. Suppose the above is false, then there exists $\varepsilon > 0$, $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \nearrow \infty$ such that

$$\langle \Psi_{l,j}(t_n), \Psi_{l,j}(t_n)\rangle_{M\setminus U_j'} \geq \varepsilon.$$

Since $\widetilde{\Delta}_j^k(t)\Psi_{l,j}(t) = E_{l,j}(t)\Psi_{l,j}(t)$, we have

$$\begin{aligned} E_{l,j}(t)\langle \Psi_{l,j}(t), \Psi_{l,j}(t)\rangle &= \langle (dd^* + d^*d)\Psi_{l,j}(t), \Psi_{l,j}(t)\rangle + t^2\langle f_j \Psi_{l,j}(t), \Psi_{l,j}(t)\rangle \\ &\quad + \langle (tA + (B + B^*) + i_X i_X^* + i_X^* i_X)\Psi_{l,j}(t), \Psi_{l,j}(t)\rangle \end{aligned}$$

where $B = i_X^* d + d i_X^*$.

Since $dd^* + d^*d \geq 0$ and $\langle f_j \Psi_{l,j}(t), \Psi_{l,j}(t)\rangle \geq c_j \|\Psi_{l,j}(t)\|^2_{M\setminus U'_j}$, therefore

$$\begin{aligned}&E_{l,j}(t)\|\Psi_{l,j}(t)\|^2\\ &\qquad \geq c_j t^2 \|\Psi_{l,j}(t)\|^2_{M\setminus U'_j} + \langle (tA + (B+B^*) + i_X i_X^* + i_X^* i_X)\Psi_{l,j}(t), \Psi_{l,j}(t)\rangle.\end{aligned}$$

Clearly, $A, i_X i_X^* + i_X^* i_X$ are bounded operators. It will be shown in the Appendix that B and B^* are also bounded. Hence,

$$\begin{aligned}E_{l,j}(t)\|\Psi_{l,j}(t)\|^2 &\geq c_j t^2 \|\Psi_{l,j}(t)\|^2_{M\setminus U'_j}\\ &\quad - (t\|A\| + \|B + B^* + i_X i_X^* + i_X^* i_X\|)\|\Psi_{l,j}(t)\|^2.\end{aligned}$$

Since $\|\Psi_{l,j}(t_n)\|^2_{M\setminus U'_j} \geq \varepsilon > 0$, that is the inequality should be: $\|\Psi_{l,j}(t_n)\|^2_{M\setminus U'_j} \geq \varepsilon > 0$,

$$\frac{E_{l,j}(t_n)}{t_n} \geq c_j t_n \varepsilon - \|A\| - \frac{1}{t_n}\|B + B^* + i_X i_X^* + i_X^* i_X\|.$$

This contradicts

$$\overline{\lim_{n\to\infty}} \frac{E_{l,j}(t_n)}{t_n} \leq \overline{e}_{l,j}.$$

Claim 1 is then proved.

Recall that J_j has support in U_j and it is equal to 1 on U'_j.
Define $\phi_{l,j}(t) \equiv J_j \Psi_{l,j}(t)$.

Claim 2.

$$\lim_{t\to\infty} \langle \phi_{l,j}(t), \phi_{m,j}(t)\rangle = \delta_{lm}.$$

Proof of Claim 2.

$$\begin{aligned}\langle \phi_{l,j}(t), \phi_{m,j}(t)\rangle &= \int_{U'_j} \phi_{l,j}(t) \wedge *\phi_{m,j}(t) + \int_{M\setminus U'_j} \phi_{l,j}(t) \wedge *\phi_{m,j}(t)\\ &\leq \int_{U'_j} \Psi_{l,j}(t) \wedge *\Psi_{m,j}(t) + \|\phi_{l,j}(t)\|_{M\setminus U'_j} \|\phi_{m,j}(t)\|_{M\setminus U'_j}\\ &= \langle \Psi_{l,j}(t), \Psi_{m,j}(t)\rangle - \langle \Psi_{l,j}(t), \Psi_{m,j}(t)\rangle_{M\setminus U'_j}\\ &\quad + \|J_j\Psi_{l,j}(t)\|_{M\setminus U'_j} \|J_j\Psi_{m,j}(t)\|_{M\setminus U'_j}\\ &\leq \delta_{lm} + 2\|\Psi_{l,j}(t)\|_{M\setminus U'_j} \|\Psi_{m,j}(t)\|_{M\setminus U'_j}.\end{aligned}$$

Similarly, one shows that

$$\langle \phi_{l,j}(t), \phi_{m,j}(t)\rangle \geq \delta_{lm} - 2\|\Psi_{l,j}(t)\|_{M\setminus U'_j} \|\Psi_{m,j}(t)\|_{M\setminus U'_j}.$$

Claim 2 then follows from Claim 1.

Observe that since $\phi_{l,j}(t)$ has support in U_j it can be regarded as a form on the standard model.

Claim 3.

$$\langle \phi_{l,j}(t), \overline{\widetilde{\Delta}}{}^k_j(t)\phi_{m,j}(t)\rangle = E_{l,j}(t)\delta_{lm} + o(t).$$

Proof of Claim 3.

$$\begin{aligned}\langle\phi_{l,j}(t),\overline{\widetilde{\Delta}}{}_j^k(t)\phi_{m,j}(t)\rangle &= \langle\phi_{l,j}(t),\widetilde{\Delta}{}_j^k(t)\phi_{m,j}(t)\rangle\\ &= \langle\Psi_{l,j}(t),J_j\widetilde{\Delta}_j^k(t)J_j\Psi_{m,j}(t)\rangle\\ &= \frac{1}{2}\langle\Psi_{l,j}(t),(J_j^2\widetilde{\Delta}_j^k(t)+\widetilde{\Delta}_j^k(t)J_j^2)\Psi_{m,j}(t)\rangle\\ &\quad+\langle\Psi_{l,j}(t),\|dJ_j\|^2\Psi_{m,j}(t)\rangle\\ &= \frac{1}{2}(E_{l,j}(t)+E_{m,j}(t))\langle\phi_{l,j}(t),\phi_{m,j}(t)\rangle+O(1)\\ &= E_{l,j}(t)\delta_{lm}+\frac{1}{2}(E_{l,j}(t)+E_{m,j}(t))\varepsilon(t)+o(t)\end{aligned}$$

where $\lim_{t\to\infty}\varepsilon(t)=0$.

Since $\overline{\lim}_{t\to\infty}\frac{E_{l,j}(t)}{t}\le\overline{e}_{l,j}$, Claim 3 is proved.

By Lemma 2.1, $\overline{\widetilde{\Delta}}{}_j^k(t)$ has discrete spectrum. Lemma 2.2 together with the min-max principle imply

$$\overline{e}_{l,j}=\varliminf_{t\to\infty}\frac{\overline{e}_{l,j}(t)}{t}\le\varliminf_{t\to\infty}\frac{E_{l,j}(t)}{t}$$

which in turn proves the lemma.

Now we show (2.4). By the IMS Localization Formula,

$$\begin{aligned}\widetilde{\Delta}^k(t)&=\sum_{j=0}^r J_j\widetilde{\Delta}^k(t)J_j-\sum_{j=0}^r\|dJ_j\|^2\\ &=J_0\widetilde{\Delta}^k(t)J_0+\sum_{j=1}^r J_j\widetilde{\Delta}^k(t)J_j+O(1).\end{aligned}$$

Denote $i_X^*d+di_X^*$ by B. Since $dd^*+d^*d, i_Xi_X^*+i_X^*i_X$ are positive, we have

$$J_0\widetilde{\Delta}^k(t)J_0\ge J_0(t^2|df|^2+tA+B+B^*)J_0.$$

Also since J_0 has support away from $\bigcup_{j=1}^r U_j, |df(x)|^2\ge c$ for some $c>0$ on support of J_0. Hence,

$$\begin{aligned}J_0\widetilde{\Delta}^k(t)J_0&\ge J_0(ct^2+tA+B+B^*)J_0\\ &\ge teJ_0^2+tJ_0\left(ct+A+\frac{1}{t}(B+B^*)-e\right)J_0\\ &\ge teJ_0^2 \text{ for sufficiently large } t.\end{aligned}$$

Since $\widetilde{\Delta}_j^k(t)=\widetilde{\Delta}^k(t)$ on U_j,

$$\widetilde{\Delta}^k(t)\ge teJ_0^2+\sum_{j=1}^r J_j\widetilde{\Delta}_j^k(t)J_j+O(1).$$

Recall that $E_{l,j}(t)$ are eigenvalues of $\widetilde{\Delta}_j^k(t)$ with corresponding eigenvector $\Psi_{l,j}(t)$. Note that in general $E_{l,j}(t)\ne E_m(t)$ for any l,m.

For any j, define l_j as follows. Recall that

$$\lim_{t\to\infty}\frac{E_{l,j}(t)}{t}=\lim_{t\to\infty}\frac{\overline{e}_{l,j}(t)}{t}=\overline{e}_{l,j}.$$

l_j is chosen such that

$$\overline{e}_{l_j,j} < e < \overline{e}_{l_j+1,j}.$$

Define $R_j(t) : \widetilde{\Omega}^k_{inv}(M) \to \widetilde{\Omega}^k_{inv}(M)$ such that on $Span\{\Psi_{l,j}(t)\}_{1\leq l\leq l_j}$ and w.r.t. $\{\Psi_{l,j}(t)\}_{1\leq l\leq l_j}$,

$$R_j(t) = \begin{pmatrix} E_{1,j}(t) - te & 0 & \cdots & 0 \\ 0 & E_{2,j}(t) - te & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{l_j,j}(t) - te \end{pmatrix}$$

and on $(Span\{\Psi_{l,j}(t)\}_{1\leq l\leq l_j})^{\perp}$, $R_j(t)$ is zero.

Then $RankR_j(t) = l_j$ and $\widetilde{\Delta}^k_j(t) \geq R_j(t) + te$ for sufficiently large t.

Therefore,

$$\begin{aligned} \widetilde{\Delta}^k(t) &\geq teJ_0^2 + \sum_{j=1}^{r} J_j R_j(t) J_j + te\sum_{j=1}^{r} J_j^2 + O(1) \\ &= te + R(t) + o(t) \end{aligned}$$

where

$$R(t) \stackrel{def}{=} \sum_{j=1}^{r} J_j R_j(t) J_j.$$

Then $RankR(t) \leq \sum_{j=1}^{r} RankR_j(t) = \sum_{j=1}^{r} l_j = l$ and this proves (2.4) and completes the proof of Theorem 1.

Appendix.

Lemma 2.0. *$B = i_X^* d + di_X^*$ is a zero order operator. Hence it is a bounded operator in*

(i) $L^2(\widetilde{\Omega}^k_{inv}(M))$ *where M is a compact S^1-manifold.*

(ii) $L^2(\widetilde{\Omega}^k_{inv}(\mathbb{R}^n))$ *where $\mathbb{R}^n$ is the standard model associated to a critical fixed point where $L^2(H)$ denotes the L^2 completion of the space H.*

Proof. By direct computations.

Lemma 2.1.

$$\overline{\widetilde{\Delta}}^k_j(t) = dd^* + d^*d + 4t^2x^2 + tA + (i_X i_X^* + i_X^* i_X) + (i_X^* d + di_X^*) + (i_X d^* + d^* i_X)$$

has a complete orthonormal basis $\{\overline{\Psi}_{l,j}(t)\}_{l\in\mathbb{N}}$ of eigenvectors corresponding to the eigenvalues

$$\overline{e}_{1,j}(t) \leq \overline{e}_{2,j}(t) \leq \cdots \leq \overline{e}_{l,j}(t) \leq \cdots$$

and $\lim_{l\to\infty} \overline{e}_{l,j}(t) = \infty$.

Proof. Observe that $i_X^* i_X + i_X i_X^* = |X|^2 = \sum_{i=1}^{q} m_i^2(x_{2i-1}^2 + x_{2i}^2)$. Then,

$$\begin{aligned} L(t) &\stackrel{def}{=} dd^* + d^*d + 4t^2x^2 + tA + (i_X^* i_X + i_X i_X^*) \\ &= dd^* + d^*d + 4\sum_{i=1}^{q}(t^2 + m_i^2)(x_{2i-1}^2 + x_{2i}^2) + 4t^2 \sum_{i=2q+1}^{n} x_i^2 + tA. \end{aligned}$$

Since $L(t)$ is a quantum harmonic oscillator, it has a compact resolvent, i.e.

$$(L(t)-\lambda)^{-1} \text{ is compact } \forall\lambda \in \rho(L(t)).$$

By Lemma 2.0, $\overline{\widetilde{\Delta}}{}_j^k(t)$ is a perturbation of $L(t)$ by a bounded operator.

Since $(L(t)-\lambda)^{-1}$ is compact $\forall\lambda \in \rho(L(t))$, choose $i\lambda_0 \in \rho(L(t))$ where $\lambda_0 \in \mathbb{R}$ such that $|\lambda_0| \gg \|B+B^*\|$, then

$$(\overline{\widetilde{\Delta}}{}_j^k(t) - i\lambda_0)^{-1} = ((L(t)+B+B^*) - i\lambda_0)^{-1}$$

exists and is compact. Lemma 2.1 follows in view of the standard theorem ([RS], Theorem XIII.64 p. 245).

Lemma 2.2.

$$\lim_{t\to\infty} \frac{\overline{e}_{l,j}(t)}{t} = \overline{e}_{l,j}.$$

Proof. Recall that $(\mathcal{U}^{(n)}(\lambda)\omega)(x) = \lambda^{n/2}\omega(\lambda x)$, $\overline{\widetilde{P}}{}^k(t) = dd^* + d^*d + 4t^2x^2 + tA$, $\overline{\widetilde{K}}{}_j^k = \overline{\widetilde{P}}{}_j^k(1)$ and $\varepsilon : \widetilde{\Omega}_{inv}^k(\mathbb{R}^n) \to \widetilde{\Omega}_{inv}^k(\mathbb{R}^n)$ is such that

$$\varepsilon(\omega) = \begin{cases} 0 & \text{if } \omega \in \Omega_{inv}^k(\mathbb{R}^n), \\ \omega & \text{if } \omega \in \widetilde{\Omega}_{inv}^{k-2}(\mathbb{R}^n). \end{cases}$$

A direct computation shows that

$$\overline{\widetilde{\Delta}}{}_j^k(t) = \mathcal{U}^{(n)}(t^{1/2})t\left[\overline{\widetilde{K}}{}_j^k + \frac{1}{t^2}\varepsilon X^2 + \frac{1}{t}(i_X^*d + di_X^* + i_Xd^* + d^*i_X)\right]\mathcal{U}^{(n)}(t^{-1/2}).$$

Let $\beta = \frac{1}{t}$, then $t\to\infty \Rightarrow \beta \to 0^+$.

Let

$$T(\beta) = \overline{\widetilde{K}}{}_j^k + \beta^2\varepsilon X^2 + \beta(i_X^*d + di_X^* + i_Xd^* + d^*i_X)$$

for $\beta \in R \subset \mathbb{C}$ such that $\mathbb{R}^+ \cup \{0\} \subset R$. Note that for $\beta \in \mathbb{R}$, $T(\beta)$ is self-adjoint.

Our strategy is to think of $T(\beta)$ as a perturbation of $\overline{\widetilde{K}}{}_j^k$ and apply the analytic perturbation theory to study the asymptotic behavior of the spectrum of $\overline{\widetilde{\Delta}}{}_j^k(t)$. (For introduction and proofs of the following statements, see [RS], [K]).

Definition. An operator-valued function $T(\beta)$ defined on a complex domain R is called an analytic family in the sense of Kato if (i) $\forall\beta \in R$, $T(\beta)$ is closed and $\rho(T(\beta)) \neq \varnothing$ where $\rho(T(\beta))$ is the resolvent set of $T(\beta)$, (ii) for any $\beta_0 \in R$, there exists a $\lambda_0 \in \rho(T(\beta_0))$ such that $\lambda_0 \in \rho(T(\beta))$ for β near β_0, and $(T(\beta)-\lambda_0)^{-1}$ is an analytic (i.e. holomorphic) operator-valued function of β near β_0.

Theorem (Kato-Rellich) (cf. [RS], p. 22)**.** *Let $T(\beta)$ be an analytic family in the sense of Kato that is self-adjoint for β real. Let E_0 be a discrete eigenvalue of $T(\beta_0)$ of multiplicity m. Then for β near β_0, there exists m not necessarily distinct single-valued functions, analytic near β_0, $E^{(1)}(\beta),\dots,E^{(m)}(\beta)$ of eigenvalues of $T(\beta)$ near β_0 with $E^{(i)}(\beta_0) = E_0$. Also these are all the eigenvalues near E_0.*

Suppose that we have shown that $T(\beta)$ is an analytic family in the sense of Kato, by the above Theorem of Kato-Rellich, for any $l \geq 1$, there exists l' with $l \leq l'$ s.t.

$$\lim_{t\to\infty} \frac{\overline{e}_{l',j}(t)}{t} = \overline{e}_{l,j}.$$

In fact, one can show that

$$\lim_{t\to\infty}\frac{\overline{e}_{l,j}(t)}{t}=\overline{e}_{l,j}.$$

To show that $T(\beta)$ is an analytic family in the sense of Kato, one can show that $T(\beta)$ is an analytic family of type (B), hence an analytic family in the sense of Kato (cf. [K], pp. 322–323).

Lemma 2.3. *For a critical fixed point O_j, $\Phi_{l,j}(t)=J_j\Psi_{l,j}(t)$,*

$$\lim_{t\to\infty}\langle\Phi_{l,j}(t),\Phi_{m,j}(t)\rangle=\delta_{lm}.$$

Proof. Recall that

$$\begin{aligned}\overline{\overline{\Delta}}{}^k_j(t)&=\mathcal{U}^{(n)}(t^{1/2})t\left[\overline{\overline{K}}{}^k_j+\frac{1}{t^2}\varepsilon X^2+\frac{1}{t}(i_X^*d+di_X^*+i_Xd^*+d^*i_X)\right]\mathcal{U}^{(n)}(t^{-1/2})\\&=\mathcal{U}^{(n)}(t^{1/2})\left[tT\left(\frac{1}{t}\right)\right]\mathcal{U}^{(n)}(t^{-1/2}),\\T(\beta)&=\overline{\overline{K}}{}^k_j+\beta^2\varepsilon X^2+\beta(i_X^*d+di_X^*+i_Xd^*+d^*i_X).\end{aligned}$$

Let $\{\Gamma_l(\beta)\}_{l\in\mathbb{N}}$ be the eigenvectors of $T(\beta)$.

Define a 2-parameter family of operators

$$\overline{\overline{\Delta}}{}^k_j(t,\beta)=\mathcal{U}^{(n)}(t^{1/2})[tT(\beta)]\mathcal{U}^{(n)}(t^{-1/2})$$

with eigenvectors $\{\mathcal{U}^{(n)}(t^{1/2})\Gamma_l(\beta)\}_{l\in\mathbb{N}}$. Then

$$\begin{cases}\overline{\overline{P}}{}^k_j(t)=\mathcal{U}^{(n)}(t^{1/2})[t\overline{\overline{K}}{}^k_j]\mathcal{U}^{(n)}(t^{-1/2})\\ \qquad\quad=\overline{\overline{\Delta}}{}^k_j(t,0)\text{ with eigenvector }\mathcal{U}^{(n)}(t^{1/2})\Gamma_l(0),\\ \overline{\overline{\Delta}}{}^k_j(t)=\overline{\overline{\Delta}}{}^k_j\left(t,\frac{1}{t}\right)\text{ with eigenvector }\overline{\Psi}_{l,j}(t)=\mathcal{U}^{(n)}(t^{1/2})\Gamma_l\left(\frac{1}{t}\right).\end{cases}$$

Theorem (Kato-Rellich) (cf. [RS], p. 15)**.** *Let $T(\beta)$ be an analytic family in the sense of Kato. Let E_0 be a non-degenerate discrete eigenvalue of $T(\beta_0)$. Then for β near β_0, there is exactly one eigenvalue $E(\beta)\in\sigma(T(\beta))$ near E_0 and this eigenvalue is isolated and non-degenerate. $E(\beta)$ is an analytic function of β near β_0, and there is an analytic eigenvector $\Gamma(\beta)$ for β near β_0.*

Case 1: For simplicity, assume E_0 is a non-degenerate eigenvalue of $T(0)=\overline{\overline{K}}{}^k_j$. By the above theorem,

$$\lim_{\beta\to 0}\Gamma_l(\beta)=\Gamma_l(0).$$

Observe that $\Gamma_l(0)$ is an eigenvector of $\overline{\overline{K}}{}^k_j$, so

$$\lim_{t\to\infty}\langle J_j\mathcal{U}^{(n)}(t^{1/2})\Gamma_l(0),J_j\mathcal{U}^{(n)}(t^{1/2})\Gamma_m(0)\rangle=\delta_{lm}.$$

The above equality in fact is the corresponding statement for the Witten deformation of de Rham complex on $\mathbb{R}^n$ and can be shown directly.

Note that

$$\begin{aligned}\langle \Phi_{l,j}(t), \Phi_{m,j}(t)\rangle &= \langle J_j \overline{\Psi}_{l,j}(t), J_j \overline{\Psi}_{m,j}(t)\rangle \\ &= \langle J_j \mathcal{U}^{(n)}(t^{1/2})\Gamma_l(\beta), J_j \mathcal{U}^{(n)}(t^{1/2})\Gamma_m(\beta)\rangle \text{ where } \beta = \frac{1}{t} \\ &= \langle J_j \mathcal{U}^{(n)}(t^{1/2})\Gamma_l(0), J_j \mathcal{U}^{(n)}(t^{1/2})\Gamma_m(0)\rangle \\ &\quad - \langle J_j \mathcal{U}^{(n)}(t^{1/2})\Gamma_l(0), J_j \mathcal{U}^{(n)}(t^{1/2})(\Gamma_m(0) - \Gamma_m(\beta))\rangle \\ &\quad - \langle J_j \mathcal{U}^{(n)}(t^{1/2})(\Gamma_l(0) - \Gamma_l(\beta)), J_j \mathcal{U}^{(n)}(t^{1/2})(\Gamma_m(\beta)\rangle.\end{aligned}$$

But

$$\begin{aligned}&|\langle J_j \mathcal{U}^{(n)}(t^{1/2})\Gamma_l(0), J_j \mathcal{U}^{(n)}(t^{1/2})(\Gamma_m(0) - \Gamma_m(\beta))\rangle| \\ &\quad \leq \|J_j \mathcal{U}^{(n)}(t^{1/2})\Gamma_l(0)\| \, \|J_j \mathcal{U}^{(n)}(t^{1/2})(\Gamma_m(0) - \Gamma_m(\beta))\| \\ &\quad \leq \|\Gamma_l(0)\| \, \|\Gamma_m(0) - \Gamma_m(\beta)\| \xrightarrow{t\to\infty} 0.\end{aligned}$$

Similarly,

$$\begin{aligned}&|\langle J_j \mathcal{U}^{(n)}(t^{1/2})(\Gamma_l(0) - \Gamma_l(\beta)), J_j \mathcal{U}^{(n)}(t^{1/2})\Gamma_m(\beta)\rangle| \\ &\quad \leq \|\Gamma_l(0) - \Gamma_l(\beta)\| \, \|\Gamma_m(\beta)\| \overset{t\to\infty}{\to} 0.\end{aligned}$$

Hence,

$$\lim_{t\to\infty} \langle \Phi_{l,j}(t), \Phi_{m,j}(t)\rangle = \lim_{t\to\infty} \langle J_j \mathcal{U}^{(n)}(t^{1/2})\Gamma_l(0), J_j \mathcal{U}^{(n)}(t^{1/2})\Gamma_m(0)\rangle = \delta_{lm}.$$

Case 2: More generally, suppose E_0 is an eigenvalue of multiplicity m. By the calculation in Case 1, it is clear that it suffices to show that $\Gamma_l(\beta)$ is an analytic family and hence $\lim_{\beta\to 0} \Gamma_l(\beta) = \Gamma_l(0)$. The analyticity of $\Gamma_l(\beta)$ is essentially a consequence of the Kato-Rellich Theorem. Also it can be seen by applying Theorem XII.2 ([RS], p. 22) to

$$P(\beta) = \frac{-1}{2\pi i} \int_{|E-E_0|=\varepsilon} \frac{1}{T(\beta) - E} dE$$

and [RS], p. 71, Ex 17, to $\widehat{T}(\beta) = U^{-1}(\beta)T(\beta)U(\beta)|_{Range P(0)}$.

3. Proof of the Morse inequality

In §2, we proved the theorem that the first term of the asymptotic expansion of the eigenvalues of $\widetilde{\Delta}^k(t)$ can be expressed in terms of the eigenvalues of the '*localized*' operators $\overline{\widetilde{\Delta}}{}^k_j(t)$, namely, $\lim_{t\to\infty} \frac{E_l(t)}{t} = \overline{e}_l$. As a consequence, we obtain the complex $(\widetilde{\Omega}^*_{inv,0}(M,t), D(t))$ which is spanned by the eigenvectors of $\widetilde{\Delta}(t)$ corresponding to the eigenvalues $E(t)$ so that

$$\lim_{t\to\infty} \frac{E(t)}{t} = 0. \tag{3.1}$$

This complex calculates the S^1-equivariant cohomology of M with

$$dim\, \widetilde{\Omega}^k_{inv,0}(M,t) = m_k + m_{k-1} + \cdots + m_0 + m^f_k + m^f_{k-2} + \cdots < \infty$$

where m_i is the number of critical orbits of index i whose θ^- is trivial and m^f_i is the number of critical fixed points of index i.

We want to show further that such eigenvalues are bounded. This can be accomplished by refining the arguments in the proof of Theorem 1 for the eigenvalues satisfying (3.1).

Recall that $\overline{\tilde{\Delta}}{}_j^k(t)$ is the '*localized*' Laplace operator on the standard model $\mathbb{R}^{n-1} \times_{\mathbb{Z}_m} S^1$ or $\mathbb{R}^n$. Consider those critical orbits with normalized eigenvector $\overline{\Psi}_{1,j}(t)$ corresponding to the smallest eigenvalue $\overline{e}^k_{1,j}(t)$ of $\overline{\tilde{\Delta}}{}_j^k(t)$ which satisfies $\lim_{t\to\infty} \frac{\overline{e}_{1,j}(t)}{t} = 0$. (Such eigenvector exists for the critical orbit O_j iff *index* $O_j \leq k$ and θ^- trivial in Cases 1 and $1'$; in Case 2 in addition *index* $O_j \equiv k (\mathrm{mod}\, 2)$.)

Lemma 3.1.

(i) *If $O_j = x_j$ is a critical fixed point with index $x_j \leq k$, index $x_j \equiv k (\mathrm{mod}\, 2)$, then*

$$\lim_{t\to\infty} \overline{e}^k_{1,j}(t) = 0.$$

(ii) *If $O_j \cong S^1$, index $O_j = k, \theta_j^-$ trivial, then*

$$\lim_{t\to\infty} \overline{e}^k_{1,j}(t) = 0.$$

(iii) *If $O_j \cong S^1$, index $O_j < k, \theta_j^-$ trivial, then*

$$\lim_{t\to\infty} \overline{e}^k_{1,j}(t) = b_j > 0$$

for some $b_j > 0$.

Proof. (i) Recall that

$$\overline{\tilde{\Delta}}{}_j^k(t) = \mathcal{U}^{(n)}(t^{1/2}) t [\overline{\tilde{K}}{}_j^k + \beta^2 \varepsilon X^2 + \beta(i_X^* d + d i_X^* + i_X d^* + d^* i_X)] \mathcal{U}^{(n)}(t^{1/2})$$

where $\beta = \frac{1}{t}$. Let

$$T(\beta) = \overline{\tilde{K}}{}_j^k + \beta^2 \varepsilon X^2 + \beta(i_X^* d + d i_X^* + i_X d^* + d^* i_X).$$

Then $\overline{e}^k_{1,j}(t)/t$ is the smallest eigenvalue of $T(\beta)$ with corresponding eigenvector $\Gamma^k_{1,j}(\beta)$.

Let

$$\begin{cases} \frac{\overline{e}^k_{1,j}(t)}{t} = \sum_{i=1}^{\infty} a_i \beta^i, \\ \Gamma^k_{1,j}(\beta) = \sum_{i=0}^{\infty} \phi_i \beta^i. \end{cases}$$

Claim. $a_1 = 0$.

Proof. Let $B = i_X^* d + d i_X^* + i_X d^* + d^* i_X$.

Then

$$\begin{aligned} T(\beta)\Gamma^k_{1,j}(t) &= \frac{\overline{e}^k_{1,j}(t)}{t} \Gamma^k_{1,j}(t) \\ &\Rightarrow \overline{\tilde{K}}{}_j^k \phi_0 + (\overline{\tilde{K}}{}_j^k \phi_1 + B\phi_0)\beta + \cdots = (a_1\phi_0)\beta + \cdots \\ &\Rightarrow \begin{cases} \overline{\tilde{K}}{}_j^k \phi_0 = 0 \\ \overline{\tilde{K}}{}_j^k \phi_1 + B\phi_0 = a_1 \phi_0 \end{cases} \\ &\Rightarrow \langle \phi_0, \overline{\tilde{K}}{}_j^k \phi_1 \rangle + \langle \phi_0, B\phi_0 \rangle = a_1 \langle \phi_0, \phi_0 \rangle. \end{aligned}$$

But $\|\Gamma^k_{1,j}(\beta)\| = 1$, this implies that $\langle \phi_0, \phi_1 \rangle = 0$. But also

$$\overline{\tilde{K}}{}_j^k (\langle \phi_0 \rangle^\perp) \subset \langle \phi_0 \rangle^\perp$$

we have $\langle \phi_0, \overline{\tilde{K}}{}_j^k \phi_1 \rangle = 0$.

Therefore,

$$a_1 = \frac{\langle \phi_0, B\phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle} = \langle \phi_0, B\phi_0 \rangle.$$

Since $\phi_0 \in \Omega^k(\mathbb{R}^n)$, then $B\phi_0 \in \Omega^{k-2}(\mathbb{R}^n) \oplus \Omega^{k+2}(\mathbb{R}^n)$. This implies $\langle \phi_0, B\phi_0 \rangle = 0$.

Therefore $a_1 = 0$ and this finishes the proof of the Claim.

Now the above Claim implies that

$$\overline{e}^k_{1,j}(t) = t \sum_{i=1}^{\infty} a_i \beta^i = \frac{a_2}{t} + \frac{a_3}{t^2} + \cdots \overset{t\to\infty}{\to} 0.$$

(ii) Recall that

$$\overline{\widetilde{\Delta}}{}^k_j(t) = \widetilde{P}^k(t) + i_X i_X^* + i_X^* i_X$$

where $\widetilde{P}^k(t) = d(t)d(t)^* + d(t)^* d(t)$. By using the canonical coordinates $(x_1, \ldots, x_{n-1}, \theta)$, the eigenvector $\overline{\Psi}^k_{1,j}(t)$ corresponding to the smallest eigenvalue is given by

$$\overline{\Psi}^k_{1,j}(t) = \left(\frac{2t}{\pi}\right)^{n-1/4} e^{-t(x_1^2+\cdots+x_{n-1}^2)}\, dx_1 \wedge \cdots \wedge dx_{index\ O_j}.$$

Since $\overline{\Psi}^k_{1,j}(t)$ is a k-form, $(i_X i_X^* + i_X^* i_X)(\Psi^k_{1,j}(t)) = 0$ and we have $\overline{\widetilde{\Delta}}{}^k_j(t)(\Psi^k_{1,j}(t)) = 0$. Hence $\overline{e}^k_{1,j}(t) = 0$.

(iii) In this case, we have the same formula for the eigenvector $\Psi^k_{1,j}(t)$. Since *index* $O_j < k$,

$$\overline{\widetilde{\Delta}}{}^k_j(t)(\Psi^k_{1,j}(t)) = (i_X i_X^* + i_X^* i_X)(\Psi^k_{1,j}(t)) = |X|^2 \Psi^k_{1,j}(t) = b_j \Psi^k_{1,j}(t)$$

where $b_j = |X|^2 > 0$. Hence $\overline{e}^k_{1,j}(t) = b_j > 0$.

Proposition 3.2. *There exists a constant M s.t. for t large enough, we have*

$$E_l(t) \le M$$

where $1 \le l \le dim\widetilde{\Omega}^k_0(M,t)$.

Proof. Let us refine the proof of (2.1) in Theorem 1 for the eigenvalues s.t.

$$\lim_{t\to\infty} \frac{E_l(t)}{t} = 0$$

where $1 \le l \le dim\widetilde{\Omega}^k_0(M,t)$. By Proposition 2.5, we have

$$\langle \Phi_l(t), \widetilde{\Delta}^k(t)\Phi_m(t) \rangle = \overline{e}_l(t)\delta_{lm} + o(t).$$

It is clear from Lemma 3.1 that there exists a constant M s.t.

$$\overline{e}_l(t) \le M \tag{3.2}$$

for $1 \le l \le dim\widetilde{\Omega}^k_0(M,t)$. The min-max principle together with (3.2) imply

$$\overline{\lim_{t\to\infty}}\, E_l(t) \le \overline{\lim_{t\to\infty}}\, \overline{e}_l(t) \le M$$

for $1 \le l \le dim\widetilde{\Omega}^k_0(M,t)$.

Lemma 3.3.

(i) $\lim_{t\to\infty} E_l(t) = 0$ *if* $1 \le l \le M_k = m_k + m_k^f + m_{k-2}^f + \cdots$.

(ii) $\underline{\lim}_{t\to\infty} E_{M_k+1}(t) > 0$.

Proof. (i) Recall that we have

$$\overline{\lim_{t\to\infty}}\, E_l(t) \le \overline{\lim_{t\to\infty}}\, \overline{e}_l(t).$$

By Lemma 3.1, for $1 \le l \le M_k$ we have,

$$\lim_{t\to\infty} \overline{e}_l(t) = 0.$$

Hence, (i) follows.

(ii) This is proved by refining the argument in the proof of (2.2) in Theorem 1 for the eigenvalue $E_{M_k+1}(t)$. Observe that using similar argument in the proof of Lemma 2.3, one can show that $\lim_{t\to\infty} E_{1,j}(t) = 0$ if (a) $O_j = x_j$ is a critical fixed point of f s.t. $index x_j \equiv k (\bmod 2)$ and $index x_j \le k$ or (b) $O_j \cong S^1$ is a critical orbit of index k s.t. θ^- is trivial. Also one can show that $\lim_{t\to\infty} E_{1,j}(t) = \lim_{t\to\infty} \overline{e}_{1,j}(t) = b_j > 0$ if $O_j \cong S^1$ is a critical orbit of f whose index differs from k and θ^- is trivial.

Let $0 < e < \min_j b_j$. By the IMS Localization Formula, we have

$$\begin{aligned}\widetilde{\Delta}^k(t) &= J_0\widetilde{\Delta}^k(t)J_0 + \sum_{j=0}^{r} J_j\widetilde{\Delta}_j^k(t)J_j + O(1)\\ &\ge eJ_0^2 + \sum_{j=1}^{r} J_j\widetilde{\Delta}_j^k(t)J_j.\end{aligned}$$

For any critical orbit O_j s.t. $\lim_{t\to\infty} E_{1,j}(t) = 0$, define $R_j(t) : \widetilde{\Omega}^k_{inv}(M) \to \widetilde{\Omega}^k_{inv}(M)$ to be

$$R_j(t) = (E_{1,j}(t) - e)P_{\Psi_{1,j}(t)}$$

where $\Psi_{1,j}(t)$ is the eigenvector corresponding to the smallest eigenvalue $E_{1,j}(t)$ of $\widetilde{\Delta}_j^k(t)$ and $P_{\Psi_{1,j}(t)}$ is the orthogonal projection onto the eigenspace spanned by $\Psi_{1,j}(t)$.

For any other critical orbit s.t. $\lim_{t\to\infty} E_{1,j}(t) \ne 0$, define

$$R_j(t) = 0.$$

Then we have

$$\widetilde{\Delta}_j^k(t) \ge R_j(t) + e.$$

Therefore

$$\widetilde{\Delta}^k(t) \ge eJ_0^2 + \sum_{j=1}^{r} J_jR_j(t)J_j + e\sum_{j=1}^{r} J_j^2.$$

Define

$$R(t) = \sum_{j=1}^{r} J_jR_j(t)J_j.$$

Then $\widetilde{\Delta}^k(t) \ge e + R(t)$ and $Rank R(t) \le M_k$.

Hence,

$$\underline{\lim}_{t\to\infty}\, E_{M_k+1}(t) \ge e > 0.$$

From the previous lemmas, it is clear that the eigenvalues of $\widetilde{\Delta}^k(t)$ in $\widetilde{\Omega}^k_{inv,0}(M,t)$ are divided into two classes. Namely, those which are bounded from below by a positive constant, and those which tend to zero as $t \to \infty$.

Definitions. 1. $E(t)$ is a small eigenvalue of $\widetilde{\Delta}^k(t)$ if $\lim_{t\to\infty} E(t) = 0$.
2.

$$\widetilde{\Omega}^k_{inv,sm}(M,t) = Span\{\Psi(t) \in \widetilde{\Omega}^k_{inv}(M) | \widetilde{\Delta}^k(t)\Psi(t) = E(t)\Psi(t) \\ \text{and } E(t) \text{ is a small eigenvalue}\}.$$

Corollary 3.4.

$$dim\widetilde{\Omega}^k_{inv,sm}(M,t) = m_k + m^f_k + m^f_{k-2} + \cdots .$$

Lemma 3.5. *Let* $M_k = dim\widetilde{\Omega}^k_{inv,sm}(M,t)$, *then*

$$\sum_{i=0}^{k}(-1)^i M_i - \sum_{i=0}^{k}(-1)^i \beta_i \begin{cases} \geq 0 & \text{if } k \text{ is even,} \\ \leq 0 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. This proof follows from the finite dimensional analogue of Hodge decomposition theorem.

Proof of the S^1-equivariant Morse Inequality.

Lemma 3.6.
(i) *If* $O_j \cong S^1$, *then*

$$\mathcal{P}_{S^1}(O_j,t) = \begin{cases} 1 & \text{if } \theta^- \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If* $O_j \cong x_j$ *and* $Stab\, O_j \cong S^1$, *then*

$$\mathcal{P}_{S^1}(O_j,t) = 1 + t^2 + t^4 + \cdots .$$

Proof. We prove the lemma for $O_j \cong S^1$ and θ^- non-trivial. The other cases can be proved similarly.

Let $O_j \cong S^1, Stab\ O_j \cong \mathbb{Z}_m$. Then $H^*_{S^1}(O_j,\theta^-)$ can be calculated from the cochain complex

$$0 \to \widetilde{\Omega}^0_{inv}(S^1,o(E)) \xrightarrow{d+i_X} \widetilde{\Omega}^1_{inv}(S^1,o(E)) \xrightarrow{d+i_X} \widetilde{\Omega}^0_{inv}(S^1,o(E)) \to \cdots .$$

But

$$\begin{cases} \widetilde{\Omega}^{2k}_{inv}(S^1,o(E)) \cong \Omega^0_{inv}(S^1,o(E)) \cong 0, \\ \widetilde{\Omega}^{2k+1}_{inv}(S^1,o(E)) \cong \Omega^1_{inv}(S^1,o(E)) \cong 0. \end{cases}$$

Hence, $H^*_{S^1}(O_j,\theta^-) \cong 0$ and $\mathcal{P}_{S^1}(O_j,\theta^-,t) = 0$.

By using Lemma 3.5 and the calculations of $\mathcal{P}_{S^1}(O_j,\theta^-,t)$ in Lemma 3.6, the Morse inequality follows immediately.

4. Helffer-Sjöstrand theory for S^1-equivariant cohomology

Recall that from the previous sections, we have the complex of finite dimensional vector spaces

$$(\widetilde{\Omega}^*_{inv,sm}(M,t), D(t))$$

which calculates the S^1-equivariant cohomology of M. Here $D(t) = e^{-tf}(d+i_X)e^{tf}$.

We want to show that $(\widetilde{\Omega}^*_{inv,sm}(M,t), D(t))$ converges to a geometric complex $(C^*(M,f),\delta)$ as $t \to \infty$, where this geometric complex also calculates the S^1-equivariant cohomology of M. Our strategy is as follows: In §4.1, we construct a geometric chain complex $(C_*(M,f),\partial)$ (and its dual $(C^*(M,f),\delta)$) from a filtration in the homotopy quotient M_{S^1} induced by f. In §4.2 we then interpret $(\widetilde{\Omega}^*_{inv,sm}(M,t), D(t))$ as a cochain complex of differential forms in M_{S^1}. In §4.3, we deduce the Helffer-Sjöstrand theory by integration of differential forms in M_{S^1}. We begin with some preliminaries in §4.0.

4.0. **Preliminaries.** Suppose f is an S^1-invariant function in M^n, g an S^1-invariant metric on M.

Definition. (f,g) is said to satisfy the Morse-Smale condition if for any two critical orbits O_x and O_y

$$W_x^- \pitchfork W_y^+$$

where W_x^-, W_y^+ are the descending and ascending manifold of O_x and O_y respectively.

Definition. An S^1-invariant Morse function is said to be self-indexing if for any critical orbit O_x, $f(x) = \textit{index } O_x$.

Proposition 4.0.1. *Suppose f is an S^1-invariant Morse function s.t. (f,g) satisfies the Morse-Smale condition, let $\mathcal{C} = \{W_O^- | O$ is a critical orbit of $f\}$, then $(M,\mathcal{C})$ is a Whitney pre-stratification of M.*

By a theorem of M. Ferrarotti [F], we have

Corollary 4.0.2. *Suppose (f,g) satisfies the Morse-Smale condition, then*

$$\int_{W_0^-} d\omega = \int_{\partial W_0^-} \omega.$$

4.1. **Construction of $(C_*(M,f),\partial)$.**

For simplicity, let f be a self-indexing S^1-invariant Morse function in M.

Define $\tilde{f} : E \times M \to \mathbb{R}$ by

$$\tilde{f}(e,x) = f(x)$$

where E is the universal principal bundle of S^1. Recall that S^1 acts on $E \times M$ by diagonal action, the homotopy quotient of M is defined as the quotient space

$$M_{S^1} = E \times_{S^1} M \equiv E \times M/S^1.$$

Since f is S^1-invariant, $\tilde{f}$ descends to a function on M_{S^1} which is denoted by f_{S^1}.

Proposition 4.1.1. (i) *If f is non-degenerate on M, then f_{S^1} is non-degenerate on M_{S^1}.*

(ii) *If O is a non-degenerate critical orbit of f in M, then f_{S^1} has corresponding non-degenerate critical manifold $E \times_{S^1} O = O_{S^1}$ and index O = index O_{S^1}.*

Consequently, let $O_1^k, \ldots, O_{m_k}^k$ be the critical orbits of f of *index* k, then $(O_1^k)_{S^1}, \ldots, (O_{m_k}^k)_{S^1}$ are the critical manifolds of f_{S^1} of *index* k.

Let $X_k = f_{S^1}^{-1}((-\infty, k+\frac{1}{2}])$, then $X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = M_{S^1}$.

Define $\partial : H_*(X_k, X_{k-1}) \to H_{*-1}(X_{k-1}, X_{k-2})$ by

$$\begin{array}{ccccc} H_*(X_k, X_{k-1}) & \longrightarrow & H_{*-1}(X_{k-1}) & \xrightarrow{i_*-1} & H_{*-1}(X_{k-1}, X_{k-2}) \\ [\sigma] & \longrightarrow & [\partial\sigma] & \longrightarrow & i_*[\partial\sigma] \end{array}$$

where i_* is induced from the inclusion $X_{k-2} \subset X_{k-1}$

$$H_*(X_{k-2}) \to H_*(X_{k-1}) \xrightarrow{i_*} H_*(X_{k-1}, X_{k-2}) \to H_{*-1}(X_{k-2}).$$

Since the above sequence is exact, it is easy to see that $\partial^2 = 0$.

Define

$$C_k(M, f) = \bigoplus_{i=0}^{n} H_k(X_i, X_{i-1}).$$

Then the above map ∂ induces the boundary homomorphism

$$\partial : C_k(M, f) \to C_{k-1}(M, f)$$

with $\partial^2 = 0$ and therefore $(C_*(M, f), \partial)$ is a chain complex.

Proposition 4.1.2. *The chain complex $(C_*(M, f), \partial)$ calculates the S^1-equivariant homology of M.*

Proof. Associated to the filtration

$$X_0 \subset X_1 \subset \cdots \subset X_n = M_{S^1}$$

is the filtration on the singular chain complex

$$\Delta(X_0) \subset \Delta(X_1) \subset \cdots \subset \Delta(X_n).$$

Therefore, there is a convergent E^1 spectral sequence with

$$E^1_{s,t} = H_{s+t}(X_s, X_{s-1})$$

and d^1 corresponds to the boundary operator of the triple (X_s, X_{s-1}, X_{s-2}) (cf. [Sp], §9.1). The limit term of the spectral sequence calculates the S^1-equivariant homology of M. But we have

$$d^r : E^r_{s,t} \to E^r_{s-r,t+r-1} \text{ with } d^r = 0 \text{ for } r \geq 2.$$

This can be explained as follows (cf. [ABK], pp. 2–3, [Sp], §9.1). Let c_s be the set of critical orbits of f of *index* s, $C_s = (c_s)_{S^1} \subset M_{S^1}$. Let the set of all points lying on the gradient lines that originate from C_s be denoted by $W^-_{C_s}$, which is a bundle over C_s:

$$\mathbb{R}^s \dashrightarrow E^-_{C_s} \xrightarrow{\pi_s} C_s.$$

Let $\mathcal{M}(C_s, C_{s-r})$ be the set of points lying on the gradient lines that originate from C_s and end at C_{s-r} in M_{S^1}, denote $\widetilde{\mathcal{M}}(C_s, C_{s-r}) = \mathcal{M}(C_s, C_{s-r})/\mathbb{R}$. Using the gradient flow, define the initial and end point maps:

$$i : \widetilde{\mathcal{M}}(C_s, C_{s-r}) \to C_s,$$

$$e : \widetilde{\mathcal{M}}(C_s, C_{s-r}) \to C_{s-r}.$$

Let x be a critical fixed point of f of index s, $x_{S^1} = x \times_{S^1} E \cong \mathbb{C}P^\infty$. The points that lie on the gradient lines which originate from x_{S^1} form a bundle $W^-_{x_{S^1}}$ over x_{S^1}:

$$W^-_x \dashrightarrow W^-_{x_{S^1}} \xrightarrow{\pi_x} x_{S^1} \cong \mathbb{C}P^\infty.$$

Let $\sigma = \pi_x^{-1}(\mathbb{C}P^{\frac{t}{2}}) \in H_{s+t}(X_s, X_{s-1})$, then

$$d^r(\sigma) = \pi_{s-r}^{-1} e_* i^{-1} \pi_x(\sigma) \in E^r_{s-r,s+r-1}.$$

Observe that if $r \geq 2$, then $e_* i^{-1}\pi_x(\sigma) \in H_{t+r-1}(\bigcup \mathbb{C}P^t) = 0$, where the union is the disjoint union of a number of copies of $\mathbb{C}P^t$. The number equals the number of critical fixed points of f of index $s-r$. Hence $d^r = 0$ for $r \geq 2$ and $(C_*(M,f),\partial) = (E^1, d^1)$ calculates the S^1-equivariant homology of M.

Note that

$$\begin{aligned} H_*(X_k, X_{k-1}) &\cong \bigoplus_{index\, O=k} H_*(DN^-(O) \times_{S^1} E, SN^-(O) \times_{S^1} E) \\ &\cong \bigoplus_{index\, O=k} H_*^{S^1}(DN^-(O), SN^-(O)) \end{aligned}$$

where $N^-(O)$ is the negative normal bundle of the critical orbit O.

Recall that (i) if the orientation line bundle of $N^-(O)$ is non-trivial, then

$$H_*^{S^1}(DN^-(O), SN^-(O)) \cong 0.$$

(ii) If $O \cong S^1$ with orientation line bundle trivial, then

$$H_*^{S^1}(DN^-(O), SN^-(O)) \cong \begin{cases} \mathbb{R} & \text{if } * = \textit{index } O, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) If $O \cong$ point, then

$$H_*^{S^1}(DN^-(O), SN^-(O)) \cong \begin{cases} \mathbb{R} & \text{if } * = \textit{index } O + 2k,\ k = 0,1,2,\dots, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we have

Proposition 4.1.3.

$$\begin{aligned} dim C_k(M,f) &= m_k + m_k^f + m_{k-2}^f + m_{k-4}^f + \cdots \\ &= dim \widetilde{\Omega}^k_{inv,sm}(M,t) < \infty. \end{aligned}$$

4.2. **Interpretation of $\widetilde{\Omega}^*_{inv,sm}(M,t)$ as a complex of differential forms on M_{S^1}.** Recall that $H^*_{S^1}(M) = H^*(\widetilde{\Omega}^*_{inv}(M), d + i_X)$. First define $(\Omega^*_{inv}(M)[u], d_X)$ by

$$\Omega^k_{inv}(M)[u] = \left\{ \sum_{\deg \varphi_i + 2i = k} \varphi_i u^i | \varphi_i \in \Omega^*_{inv}(M) \right\},$$

$$\begin{cases} d_X \varphi = d\varphi + i_X(\varphi) u, \\ d_X u = 0. \end{cases}$$

Therefore,

$$d_X\left(\sum_{\deg\varphi_i+2i=k}\varphi_i u^i\right)=\sum_{\deg\varphi_i+2i=k}(d_X\varphi_i)u^i$$
$$=\sum_{\deg\varphi_i+2i=k}(d\varphi_i+i_X(\varphi_i)u)u^i.$$

Clearly

$$(\Omega^*_{inv}[u],d_X)\cong(\widetilde{\Omega}^*_{inv}(M),d+i_X).$$

Our aim in this section is to introduce a complex $(\Omega^*_g(M),D)$ of differential forms on M_{S^1} and an isomorphism of cochain complexes

$$\lambda:(\Omega^*_{inv}[u],d_X)\to(\Omega^*_g(M),D).$$

Using the above isomorphism, we can regard $\Omega^*_{inv}(M)[u]$ (hence $\widetilde{\Omega}^*(M)$) as differential forms on M_{S^1}.

Let $g\cong\mathbb{R}$ be the Lie algebra of $G=S^1$, g^* be the dual of g.

Let Sg^* be the symmetric algebra generated by g^* whose generator is denoted by u, Λg^* be the exterior algebra generated by g^* whose generator is denoted by θ with $\deg u=2,\deg\theta=1$.

Define $W(g)=\Lambda g^*\otimes Sg^*$, called the Weil algebra, which is the algebra generated freely by θ and u as a commutative graded algebra i.e.

$$\omega_p\omega_q=(-1)^{pq}\omega_q\omega_p.$$

Define $D_0:W(g)\to W(g)$ by

$$\begin{cases}D_0\theta+u=0\\ D_0u=0\end{cases}$$

and is extended to $W(g)$ as an anti-derivation. Observe that $D_0^2=0$ and

$$H^*(W(g),D_0)\cong\mathbb{R}.$$

Consider

$$(W(g)\otimes\Omega^*(M),\mathbb{D}=D_0\otimes id+(-1)^{\deg\omega}id\otimes d)$$

and define the basic subcomplex of $W(g)\otimes\Omega^*(M)$

$$\Omega^*_g(M)=\{\omega\in W(g)\otimes\Omega^*(M)|i_X(\omega)=L_x(\omega)=0\}.$$

For $\omega\in W(g)\otimes\Omega^*(M)$, let $\omega=\sum_k u^ka_k+\sum_k u^k\theta b_k$ where $a_k,b_k\in\Omega^*(M)$. Then

$$\begin{cases}i_X(\omega)=\sum_k u^ki_X(a_k)+\sum_k u^k(b_k-\theta i_X(b_k))=0\\ L_X(\theta)=\sum_k u^kL_X(a_k)+\sum_k u^kL_X(b_k)=0\end{cases}$$
$$\Leftrightarrow\begin{cases}b_k=-i_X(a_k)\\ L_X(a_k)=0.\end{cases}$$

Therefore we have

$$\Omega^*_g(M)=\left\{\omega=\sum_k u^k(a_k+\theta b_k)\in W(g)\otimes\Omega^*(M)|L_X(a_k)=0,b_k=-i_X(a_k)\right\}.$$

Define $\lambda : (\Omega^*_{inv}[u], d_X) \to (\Omega^*_g(M), \mathbb{D})$ such that

$$\begin{cases} \lambda(\varphi) = \varphi - \theta i_X(\varphi) \\ \lambda(u) = u \end{cases}$$

and extend as a ring homomorphism. Observe that it is also a ring isomorphism.

Proposition 4.2.1.

$$\lambda : (\Omega^*_{inv}[u], d_X) \to (\Omega^*_g(M), \mathbb{D})$$

is an isomorphism between the two cochain complexes.

Proof. It suffices to show that $\lambda\, d_X(\sum_k u^k a_k) = \mathbb{D}\lambda(\sum_k u^k a_k)$.

$$\begin{aligned} \lambda\, d_X\left(\sum_k u^k a_k\right) &= \lambda\left(\sum_k u^k(da_k + u i_X(a_k))\right) \\ &= \lambda\left(\sum_k u^k(da_k + i_X(a_{k-1}))\right) \\ &= \sum_k u^k[(da_k + i_X(a_{k-1})) - \theta i_X(da_k + i_X(a_{k-1}))] \\ &= \sum_k u^k(da_k + i_X(a_{k-1})) - \sum_k u^k \theta i_X(da_k). \end{aligned}$$

Also

$$\begin{aligned} \mathbb{D}\lambda\left(\sum_k u^k a_k\right) &= \mathbb{D}\left(\sum_k u^k(a_k - \theta i_X(a_k))\right) \\ &= \mathbb{D}\left(\sum_k u^k a_k\right) - \mathbb{D}\left(\sum_k u^k \theta i_X(a_k)\right) \\ &= \sum_k u^k\, da_k + \sum_k u^{k+1} i_X(a_k) + \sum_k u^k \theta\, d i_X(a_k) \\ &= \sum_k u^k(da_k + i_X(a_{k-1})) - \sum_k u^k \theta i_X(da_k). \end{aligned}$$

Since $a_k \in \Omega^*_{inv}(M)$, we have $L_X(a_k) = (di_X + i_X d)(a_k) = 0$. Hence, $\lambda\, d_X = \mathbb{D}\lambda$.

It is possible to interpret $(\Omega^*_g(M), \mathbb{D})$ (hence $(\widetilde{\Omega}^*_{inv}(M), d + i_X)$) as a subcomplex of differential forms on M_{S^1}. For this purpose assume that one can describe M_{S^1} as an infinite dimensional Hilbert manifold, which is the base of the principal fibration

$$S^\infty \times M \to S^\infty \times M/S^1.$$

Here S^∞ is the unit sphere in the separable Hilbert space $l_2(\mathbb{C})$. $S^1 = \{e^{i\alpha} \in \mathbb{C} | \alpha \in \mathbb{R}\}$ acts freely on S^∞ by

$$\mu(e^{i\alpha}, (z_1, z_2, \dots)) = (e^{i\alpha} z_1, e^{i\alpha} z_2, \dots)$$

and consider the diagonal action on $S^\infty \times M$.

Identify the element $\theta \in W^1(g)$, respectively $u \in W^2(g)$, with the restriction of the form $\sum_i z_i\, d\overline{z}_i$, respectively $\sum_i dz_i \wedge d\overline{z}_i$ to S^∞. Then one can regard $W(g) \otimes \Omega^*(M)$ as a subcomplex of differential forms on $S^\infty \times M$. Via this interpretation, $\Omega_g(M)$ consists of invariant forms on $S^\infty \times M$ which are pullback of smooth forms on $S^\infty \times M/S^1 = M_{S^1}$.

Proposition 4.2.2. *The following diagram is commutative*

$$\begin{CD} \Omega^*_{inv}[u] @>{\lambda}>> \Omega^*_g(M) \\ @V{e^{tf}}VV @VV{(e^{tf})_{S^1}}V \\ \Omega^*_{inv}[u] @>{\lambda}>> \Omega^*_g(M) \end{CD}$$

where $(e^{tf})_{S^1}$ *is defined as follows: let* $e^{\tilde{t}f} : W(g) \otimes \Omega^*(M) \to W(g) \otimes \Omega^*(M)$

$$e^{\tilde{t}f}\left(\sum_k u^k(a_k + \theta b_k)\right) = \sum_k u^k(e^{tf}a_k + \theta e^{tf} b_k).$$

Then

$$(e^{tf})_{S^1} = e^{\tilde{t}f}|_{\Omega^*_g(M)} : \Omega^*_g(M) \to \Omega^*_g(M).$$

Proof.

$$\begin{aligned} \lambda e^{tf}\left(\sum_k u^k a_k\right) &= \lambda(e^{tf})\lambda\left(\sum_k u^k a_k\right) \\ &= e^{tf}\lambda\left(\sum_k u^k a_k\right) \\ &= (e^{tf})_{S^1}\lambda\left(\sum_k u^k a_k\right). \end{aligned}$$

Remark. The map $(e^{tf})_{S^1}$ is reminiscent of the multiplication of forms by the function $(e^{tf})_{S^1}$ on M_{S^1} induced by e^{tf}.

Corollary 4.2.3. *For any* t,

$$\lambda(t) : (\Omega^*_{inv}[u], d_X(t) = e^{-tf} d_X e^{tf}) \to (\Omega^*_g(M), \mathbb{D}(t) = (e^{-tf})_{S^1}\mathbb{D}(e^{tf})_{S^1})$$

is an isomorphism between the two cochain complexes.

Corollary 4.2.4. *Since* $(\widetilde{\Omega}^*_{small}(M,t), D(t)) \subset (\Omega^*_{inv}[u], e^{-tf}\, d_X e^{tf})$, $\lambda(t)$ *induces a corresponding small complex* $(\Omega^*_{g,small}(M,t), \mathbb{D}(t))$ *whose homology is the* S^1-*equivariant cohomology of* M.

Next, let us describe the elements in $\Omega^*_{g,small}(M,t)$. Since $M \dashrightarrow M_{S^1} \xrightarrow{p} B_{S^1} = \mathbb{C}P^\infty$, let $H^*(\mathbb{C}P^\infty) = \mathbb{R}[u]$, then

$$\lambda(u) = p^*(u). \tag{4.1}$$

By abuse of notation, we write $\lambda(u) = u$.

Recall that $\lambda(\varphi) = \varphi - \theta i_X(\varphi)$, $\varphi \in \Omega^k_{inv}(M)$.

Let $x \in M$, $v_1, \ldots, v_k \in T_xM$, $\gamma_i(t)$ be smooth curves in M such that

$$\begin{cases} \gamma_i(0) = x, \\ \dot{\gamma}_i(0) = v_i, \ 1 \le i \le k. \end{cases}$$

Let $[(e,x)] \in E \times_{S^1} M$, then $[(e, \gamma_i(t))]$ $1 \le i \le k$, are smooth curves in M_{S^1} passing through $[(e,x)]$ with tangent vectors V_i at $t = 0$.

Let $\pi : E \times M \to E \times_{S^1} M$, then $(\pi_*)_{(e,x)}(0, v_i) = V_i$. One can check that

$$\lambda(\varphi)(V_1, \ldots, V_k) = \varphi(v_1, \ldots, v_k). \tag{4.2}$$

Now, let $\sum_{\deg \varphi_i + 2i = k} u^i \varphi_i(t) \in \widetilde{\Omega}^k_{small}(M,t)$ be a small eigenvector localized at a critical orbit O of index 1. Then we have

$$\begin{cases} \|\varphi_l(t)\| \xrightarrow{t\to\infty} 1 \\ \|\varphi_i(t)\| \xrightarrow{t\to\infty} 0 \quad \text{if } i \neq l. \end{cases}$$

But $\lambda(\sum_i u^i \varphi_i(t)) = \sum_i u^i \lambda(\varphi_i(t))$. (4.1), (4.2) show that $\lambda(\sum_i u^i \varphi_i(t))$ is a '*small*' eigenvector of differential form on M_{S^1} which is localized at $E \times_{S^1} O = O_{S^1}$ as $t \to \infty$.

4.3. **Helffer-Sjöstrand theory.** In the previous sections, we have described the complexes $(\Omega^*_{g,small}(M,t), \mathbb{D}(t))$ and $(C^*(M,f), \delta)$ (which is the dual cochain complex of $(C_*(M,f), \partial)$).

Recall that

$$\begin{aligned} C_k(M,f) &= \bigoplus_{i=0}^{n} H_k(X_i, X_{i-1}) \\ &= \bigoplus_{O \in Crit(f)} \dot{H}_k(DN^-(O) \times_{S^1} E, SN^-(O) \times_{S^1} E) \end{aligned}$$

where $Crit(f)$ is the set of critical orbits of f.

Consider $H_k(DN^-(O) \times_{S^1} E, SN^-(O) \times_{S^1} E)$.

Case 1: $O \cong S^1, stab\, O = 1, index\, O = l$.

In this case, $N^-(O)$ is a trivial bundle over O.

$$\begin{aligned} H_k(DN^-(O) \times_{S^1} E, SN^-(O) \times_{S^1} E) &\cong H_{k-index\, O}(O \times_{S^1} E) \\ &\cong H_{k-index\, O}(point). \end{aligned}$$

Let $y \in O \times_{S^1} E, D^l \dashrightarrow DN^-(O) \times_{S^1} E \xrightarrow{\pi_0} O \times_{S^1} E$, then $[\pi_0^{-1}(y)]$ generates $H_{index\, O}(DN^-(O) \times_{S^1} E, SN^-(O) \times_{S^1} E)$.

Case 1′: $O \cong S^1, stab\, O \cong \mathbb{Z}_m$ for some $m > 1$ with $N^-(O)$ orientable (otherwise, $H_*(DN^-(O) \times_{S^1} E, SN^-(O) \times_{S^1} E) = 0$).

In this case, $H_*(DN^-(O) \times_{S^1} E, SN^-(O) \times_{S^1} E)$ is similarly described as in Case 1.

Case 2: $O = point, stab\, O = S^1, N^-(O)$ orientable (otherwise $H_*(DN^-(O) \times_{S^1} E, SN^-(O) \times_{S^1} E) = 0$).

$$\begin{aligned} H_k(DN^-(O) \times_{S^1} E, SN^-(O) \times_{S^1} E) &\cong H_{k-index\, O}(O \times_{S^1} E) \\ &= H_{k-index\, O}(\mathbb{C}P^\infty). \end{aligned}$$

But $H_*(\mathbb{C}P^\infty) = \bigoplus_{i=0}^\infty H_{2i}(\mathbb{C}P^\infty) = \bigoplus_{i=0}^\infty \mathbb{R}[\mathbb{C}P^i]$.

Let $D^l \dashrightarrow N^-(O) \times_{S^1} E \xrightarrow{\pi_0} O \times_{S^1} E$. Then $[\pi_0^{-1}(\mathbb{C}P^i)]$ generates $H_k(DN^-(O) \times_{S^1} E, SN^-(O) \times_{S^1} E)$ where $2i + index\, O = k$.

Proposition 4.3.1. *Suppose (f,g) satisfies the Morse-Smale condition, then*

$$Int : (\Omega^*_g(M), \mathbb{D}) \to (C^*(M,f), \delta)$$
$$\omega \to \int_\sigma \omega$$

is a morphism of cochain complexes.

Proof. Recall that W_x^- denotes the descending manifold associated with the critical orbit O_x of $x \in M$. Then we have

$$W_x^- \dashrightarrow W_x^- \times_{S^1} E \xrightarrow{\pi_x} O_x \times_{S^1} E.$$

Also we can use

$$\begin{cases} [\pi_x^{-1}(\mathbb{C}P^i)] & \text{if } O_x = \text{point} \\ [\pi^{-1}(y)] & \text{if } O_x \cong S^1 \text{ and } y \in O_x \times_{S^1} E \end{cases}$$

to represent the relative homology classes in $H_*(X_k, X_{k-1})$. The proposition follows from Corollary 4.0.2.

As a consequence, the composition

$$(\Omega^*_{g,small}(M,t), \mathbb{D}(t)) \xrightarrow{(e^{tf})_{S^1}} ((e^{tf})_{S^1}\Omega^*_{g,small}(M,t), \mathbb{D}) \xrightarrow{Int} (C^*(M,f), \delta)$$

is also a morphism of cochain complexes.

Next we define $\Psi_{O_j^k}(t)$.

Case 1: $O_j^k \cong S^1, stab\, O_j^k \cong 1, index\ O_j^k = k.$

Let $U_j \cong D^{n-1} \times S^1$ be an open neighbourhood of O_j^k s.t. $(x_1, \ldots, x_{n-1}, \theta)(\in D^{n-1} \times S^1)$ is a compatible coordinate system about O_j^k.

Define

$$\Psi_{O_j^k}(t) = \beta(t)\rho(|x|)\left(\frac{2t}{\pi}\right)^{n-1/4} e^{-t(x_1^2+\cdots+x_{n-1}^2)}\, dx_1 \wedge \cdots \wedge dx_k$$

where $|x| = \sqrt{x_1^2 + \cdots + x_{n-1}^2}, \rho \in C_0^\infty(\mathbb{R})$, such that

$$\rho(x) = \begin{cases} 1 & \text{if } |x| \le \frac{\varepsilon}{2}, \\ 0 & \text{if } |x| \ge \varepsilon \end{cases}$$

for some $\varepsilon > 0$ small enough and $\beta(t)$ is chosen so that $\|\Psi_{O_j^k}(t)\| = 1$.

Case 1′: $O_j^k \cong S^1, stab\, O_j^k \cong \mathbb{Z}_m$ for some $m > 1$.

Let U_j be an open neighbourhood of O_j^k s.t.

$$U_j \cong D^{n-1} \times_{\mathbb{Z}_m} S^1 \text{ (an orientable bundle over } O_j^k).$$

Let $p : D^{n-1} \times S^1 \to D^{n-1} \times_{S^1} S^1$ be the canonical projection. Since $\mathbb{Z}_m$ acts by isometry, the standard metric on $D^{n-1} \times S^1$ in Case 1 induces a metric on $D^{n-1} \times_{S^1} S^1$ via p. Let $\overline{\widetilde{\Delta}}{}^k_{j,i}(t)$ denote the '*localized*' operator in Case i, $i = 1, 1'$. Then

$$p\overline{\widetilde{\Delta}}{}^k_{j,1}(t) = \overline{\widetilde{\Delta}}{}^k_{j,1'}(t)p.$$

Hence, if $\Psi(t)$ is the small eigenvector of $\overline{\widetilde{\Delta}}{}^k_{j,1'}(t)$, then $p^*(\Psi)$ is the small eigenvector of $\overline{\widetilde{\Delta}}{}^k_{j,1}(t)$, which is $(\frac{2t}{\pi})^{n-1/4} e^{-t(x_1^2+\cdots+x_{n-1}^2)}\, dx_1 \wedge \cdots \wedge dx_k$.

Hence, define

$$\Psi_{O_j^k}(t) = \beta(t)\rho(|x|)\left(\frac{2t}{\pi}\right)^{n-1/4} e^{-t(x_1^2+\cdots+x_{n-1}^2)}\, dx_1 \wedge \cdots \wedge dx_k$$

as in Case 1.

Case 2: $O_j^{l_j} = x_j^{l_j}$ a critical fixed point of index l_j where $l_j \le k$ and $l_j \equiv k (\bmod 2)$.

Let $U_j \cong D^n$ be an open neighbourhood of $x_j^{l_j}$ s.t. $(x_1,\dots,x_n)(\in D^n)$ is a compatible coordinate system about $x_j^{l_j}$.

Recall that in Case 2, the '*localized*' operator

$$\overline{\overline{\Delta}}{}_j^k(t) = \Delta^k + 4t^2x^2 + tA + i_X i_X^* + i_X^* i_X + (di_X^* + i_X^* d) + (d^* i_X + i_X d^*).$$

Let $\overline{\Psi}_{1,j}^k(t)$ be the normalized ground state of $\overline{\overline{\Delta}}{}_j^k(t)$.

Define

$$\Psi_{x_j^k}(t) = \beta_1(t)\rho(|x|)\overline{\Psi}_{1,j}^k(t)$$

where $|x| = \sqrt{x_1^2+\cdots+x_n^2}$ and $\beta_1(t)$ is chosen s.t. $\|\Psi_{x_j^k}(t)\| = 1$.

Let $C_k(M,f)$ be generated by

$$\{\sigma_{O_j^l}^k | \text{Case 1 and } 1' : O_j^l \cong S^1, l = k,$$
$$\text{Case 2: } O_j^l = x_j^l, l \equiv k (\bmod 2), l \le k\}.$$

Let $\{e_{O_j^l}^k\}$ be the dual basis of $\{\sigma_{O_j^l}^k\}$.

Define $J_k(t) : C^k(M,f) \to \widetilde{\Omega}^k(M)$ s.t.

$$J_k(t)(e_{O_j^l}^k) = \Psi_{O_j^l}(t).$$

Define

$$Q_k(t) : \widetilde{\Omega}^k(M) \to \widetilde{\Omega}_{small}^k(M,t)$$

to be the orthogonal projection onto $\widetilde{\Omega}_{small}^k(M,t)$.

Let

$$H_k(t) = (Q_k(t)J_k(t))^*(Q_k(t)J_k(t)),$$
$$\widetilde{J}_k(t) = Q_k(t)J_k(t)H_k^{-\frac{1}{2}}(t).$$

Then $\widetilde{J}_k(t) : C^k(M,f) \to \widetilde{\Omega}_{small}^k(M,t)$ is an isometry.

Define

$$E_{O_j^l}^k(t) = \widetilde{J}_k(t)(e_{O_j^l}^k).$$

Proposition 4.3.2. *There exists neighbourhood $U_{O_j^k}$ of O_j^k contained in the chart of compatibility s.t.*

(i) $O_j^k \cong S^1$

$$E_{O_j^k}^k(t) = \left(\frac{2t}{\pi}\right)^{n-1/4} e^{-t(x_1^2+\cdots+x_{n-1}^2)}(dx_1\wedge\cdots\wedge dx_k + O(t^{-1})) \text{ on } U_{O_j^k}.$$

(ii) $O_j^l = x_j^l$ $(l \equiv k (\bmod 2), l \le k)$

$$E_{x_j^l}^k(t) = \left(\frac{2t}{\pi}\right)^{n/4} e^{-t(x_1^2+\cdots+x_n^2)}(dx_1\wedge\cdots\wedge dx_l + O(t^{-1})) \text{ on } U_{x_j^k}.$$

Proposition 4.3.3.

(i) $O_j^k \cong S^1$

$$Int_k(e^{tf})_{S^1}(\lambda E_{O_j^k}^k(t)) = \left(\frac{2t}{\pi}\right)^{\frac{n-1-2k}{4}} e^{tk}(e_{O_j^k}^k + O(t^{-1})).$$

(ii) $O_j^l = x_j^l$ $(l \equiv k (\bmod 2), l \leq k)$

$$Int_k(e^{tf})_{S^1}(\lambda E^k_{O^l_j}(t)) = \left(\frac{2t}{\pi}\right)^{\frac{n-2l}{4}} e^{tl} \left(e^k_{x^l_j} + \sum_{l' \leq k} \beta_{ji}(t) e^k_{O^{l'}_i} + O(t^{-1}) \right)$$

where $\beta_{ji}(t)$ is defined s.t.

$$\left(\frac{2t}{\pi}\right)^{\frac{n-2l}{4}} e^{tl}\beta_{ji}(t) = \int_{\sigma^k_{O^{l'}_i}} (e^{tf})_{S^1}(\lambda E^k_{x^l_j}(t)).$$

Proof. We prove (ii), (i) can be proved similarly. To show (ii), it suffices to integrate on $\sigma^k_{x^l_j}$. With the identification of $\widetilde{\Omega}^*(M) \cong \Omega^*_{inv}[u]$, we have

$$\begin{aligned}
\int_{\sigma^k_{x^l_j}} (e^{tf})_{S^1}(\lambda E^k_{x^l_j}(t)) &\sim \int_{\sigma^k_{x^l_j}} (e^{tf})_{S^1}\lambda\left[\left(\frac{2t}{\pi}\right)^{n/4} e^{-t(x_1^2+\cdots+x_n^2)}\, dx_1\wedge\cdots\wedge dx_l u^{\frac{k-l}{2}}\right] \\
&= \left(\frac{2t}{\pi}\right)^{n/4} \int_{\sigma^k_{x^l_j}} (e^{tf})_{S^1}\lambda(e^{-t(x_1^2+\cdots+x_n^2)}\, dx_1\wedge\cdots\wedge dx_l)u^i \\
&= \left(\frac{2t}{\pi}\right)^{n/4} \int_{\pi_0^{-1}(\mathbb{C}P^i)} \lambda(e^{tf}e^{-t(x_1^2+\cdots+x_n^2)}\, dx_1\wedge\cdots\wedge dx_l)u^i \\
&= \left(\frac{2t}{\pi}\right)^{n/4} \int_{\pi_0^{-1}(\mathbb{C}P^i)} \lambda(e^{tl}e^{-2t(x_1^2+\cdots+x_l^2)}\, dx_1\wedge\cdots\wedge dx_l)u^i \\
&= \left(\frac{2t}{\pi}\right)^{n/4} e^{tl}\int_{\mathbb{C}P^i} u^i \int_{\pi_0^{-1}(y), y\in\mathbb{C}P^i} \lambda(e^{-2t(x_1^2+\cdots+x_l^2)}\, dx_1\wedge\cdots\wedge dx_l) \\
&\sim \left(\frac{2t}{\pi}\right)^{n/4} e^{tl}\int_{\mathbb{C}P^i} u^i \int_{D^l\cap U_j} e^{-2t(x_1^2+\cdots+x_l^2)}\, dx_1\wedge\cdots\wedge dx_l \\
&\sim \left(\frac{2t}{\pi}\right)^{n/4} e^{tl}\left(\frac{2t}{\pi}\right)^{-l/2} \\
&= \left(\frac{2t}{\pi}\right)^{\frac{n-2l}{4}} e^{tl}
\end{aligned}$$

where in the above computation, we let $i = \frac{k-l}{2}$.

Note that if $l' = index\ O^{l'}_i \leq index\ O^l_j = l$, then

$$|\beta_{ji}(t)| \leq e^{t[(l'-l)-\varepsilon_0]} \tag{4.3}$$

for some $\varepsilon_0 > 0$. This is due to the decrease of the function f by $e^{t(l'-l)}$ and the decay of the $E^k_{x^l_j}(t)$. If $l' > l$, then

$$|\beta_{ji}(t)| \leq e^{\varepsilon t} \tag{4.4}$$

for any $\varepsilon > 0$. This is due to the exponential decay of $E^k_{x^l_j}(t)$ (cf. [HS], p. 265, [HS1], p. 138).

If one calculates the matrix $M^k(t)$ of the linear map $f^k(t) = int_k(e^{tf})_{S^1}$ in terms of the following ordered bases:

$$\{E^k_{O^k_j}(t), E^k_{x^k_{j_1}}(t), \ldots, E^k_{x^k_{j_{s_1}}}(t), E^k_{x^{k-2}_{j_{s_1+1}}}(t), \ldots, E^k_{x^{k-2}_{j_{s_1+s_2}}}(t), \ldots\}$$

and

$$\{e^k_{O^k_j}, e^k_{x^k_{j_1}}, \ldots, e^k_{x^k_{j_{s_1}}}, e^k_{x^{k-2}_{j_{s_1+1}}}, \ldots, e^k_{x^{k-2}_{j_{s_1+s_2}}}, \ldots\}$$

then using (4.3) and (4.4), it is seen that $M^k(t)$ is invertible.

Hence, define

$$\begin{cases} \lambda(\widehat{E}^k_{O^k_j}(t)) = (M^k(t))^{-1}\lambda(E^k_{O^k_j}(t)) & \text{if } O^k_j \cong S^1, \\ \lambda(\widehat{E}^k_{x^l_j}(t)) = (M^k(t))^{-1}\lambda(E^k_{x^l_j}(t)) & \text{if } O^l_j = x^l_j. \end{cases}$$

Finally, we have proved

Theorem 2. *Suppose f is a self-indexing invariant Morse function such that (f, g) satisfies the Morse-Smale condition. Then*

$$F^*(t) = Int(e^{tf})_{S^1} : (\Omega^*_{g,small}(M,t), \mathbb{D}(t)) \to (C^*(M,f), \delta)$$

is a morphism of cochain complexes such that

$$F^*(t) = I + O(t^{-1}).$$

w.r.t. the bases $\{\lambda(\widehat{E}^k_{O^k_j}(t)), \lambda(\widehat{E}^k_{x^l_j}(t))\}$ *and* $\{e^k_{O^k_j}, e^k_{x^l_j}\}$.

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