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NEWTON'S METHOD ON THE COMPLEX EXPONENTIAL FUNCTION

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ABSTRACT. We show that when Newton's method is applied to the product of a polynomial and the exponential function in the complex plane, the basins of attraction of roots have finite area.

1. Introduction

Newton's method applied to polynomials has been studied extensively. However, little has been done on its application to entire functions. This paper examines the resulting dynamics when Newton's method is applied to the exponential function

$$F(z) = P(z)e^{Q(z)},$$

where P and Q are complex polynomials. In essence, this becomes an analysis of the dynamic behavior of a family of rational maps. Of course, one need only apply the numerical algorithm to the polynomial P to determine the roots of F. However, investigation of the dynamics of this model produces some unexpected and interesting results that differ considerably from results which follow from applying Newton's method to polynomials.

Sutherland proved in his thesis that the immediate basins of attraction for the roots of polynomials are large and, in fact, have a lower bound for their "width" [Sut]. The implications for the efficiency of this numerical method are that proper choice of initial values will guarantee convergence to the roots. In contrast, the immediate basins of roots of Pe^Q are small and in fact have finite area. The key to this result is the fact that infinity is a parabolic fixed point, as opposed to the case for polynomials where infinity is always a fixed repeller. The Fatou Flower Theorem provides a description of the local dynamics about the fixed point. We present a review of complex dynamics terms, a brief discourse on the Fatou Flower Theorem, and finally an example, followed by the proof of the main result.

The study of iterated maps is the study of the dynamics of orbits of points under repeated composition of a function with itself. In general, $f^n(x)$ represents the nth iterate of x. Under iteration, orbits of points either exhibit stable behavior or act unpredictably. We say that f has a *fixed point* at z_0 if $f(z_0) = z_0$. Similarly, f has a *periodic point of period* n at z_0 if $f^n(z_0) = z_0$ and $f^k(z_0) \neq z_0$ for $k = 1, \dots, n-1$. If z_0 is a periodic point of f, then $\lambda = [f^n(z_0)]'$ is called the *multiplier* or *eigenvalue* of z_0. A periodic point z_0 of f is classified as: *superattracting* if $\lambda = 0$; *attracting* if $|\lambda| < 1$; *repelling* if $|\lambda| > 1$; and *neutral* if $|\lambda| = 1$. A neutral fixed point is called

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parabolic or *rationally indifferent* if λ is a root of unity and *irrationally indifferent* if λ is irrational.

The Julia set is the set of points whose orbits have unpredictable or chaotic behavior. We define the family of functions $\{f^n\}$ to be *normal* on U if every sequence of f^n has a subsequence that either converges uniformly on compact subsets of U or converges uniformly to ∞ on compact subsets of U. The *Julia set*, J, of f is defined to be the set of all points for which the family of iterates $\{f^n\}$ is not normal at z. Equivalently, the Julia set of a rational map is equal to the closure of the set of repelling periodic points. The *Fatou set* or *stable set* is the complement of the Julia set, $F = \overline{\mathbb{C}} - J$.

Since the goal of Newton's method is convergence to roots, a natural area of interest is the study of regions of nonconvergence, their existence and estimates on their size. Sutherland produced estimates on the sizes of the convergent regions for Newton's method on polynomials. His results improved on Mannings [Ma] work by eliminating certain restrictions on the roots and showed that given an appropriately normalized complex polynomial, then for any root the area of the convergent regions is "large".

The main results of this paper focus on the sizes of the basins of attraction of fixed points. The *basin of attraction* of an attracting fixed point consists of the set of all points whose orbits converge to the point. The *immediate basin of attraction* is the component of the basin that contains the fixed point. A *parabolic basin* is the basin of attraction for a rationally indifferent fixed point. The basin is contained in the Fatou set and the parabolic point lies on the boundary of the basin and is in the Julia set.

A rational map R has a *critical point* at z_0 if $R'(z_0) = 0$. A rational map with degree d has $2d-2$ critical points, counted with multiplicity [MSS]. A *critical value* is the image of a critical point.

The family of functions generated by Newton's method applied to exponential functions is a family of rational maps as we shall subsequently see. The degree of a rational map equals max(deg p, deg q). Since the maps we deal with are all of degree two or greater, we will assume the same in the following discussion.

2. The Fatou Flower Theorem

The Fatou Flower Theorem provides an analytic description of the dynamics around a rationally indifferent fixed point.

Theorem 1 (Fatou Flower Theorem). *If M is a holomorphic map of the form*

$$M(z) = \lambda z + cz^{n+1} + O(z^{n+2}), \qquad c \neq 0,$$

defined in some neighborhood of the origin with $\lambda = 1$, then zero is a parabolic fixed point and there are exactly n attracting petals and n repelling petals for M at zero. Moreover, these petals alternate with one another. (*See Figures 1 and 2.*)

We will see later that Newton's method map studied in this paper has multiplier equal to 1 at the fixed point ∞. So, from here on we will assume that $\lambda = 1$. Furthermore, for simplicity we will assume that $c > 0$. An *attracting petal*, $\mathcal{P}^+$, for a map M at zero is an open simply connected forward invariant region with $0 \in \partial\mathcal{P}^+$, that shrinks down to the origin under iteration of M. More precisely, $\mathcal{P}^+$ is an attracting petal if $M(\overline{\mathcal{P}^+}) \subset \overline{\mathcal{P}^+} \cup \{0\}$ and $\bigcap_{n\geq 0} M^n(\overline{\mathcal{P}^+}) = \{0\}$. A *repelling petal*, $\mathcal{P}^-$, is an attracting petal for M^{-1} which exists locally since $M'(0) = 1$.

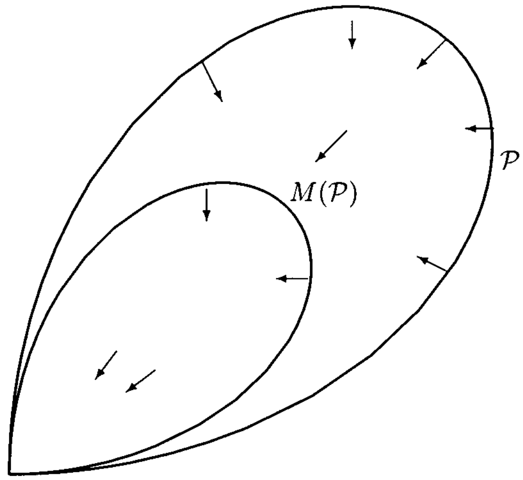

FIGURE 1. An attracting petal

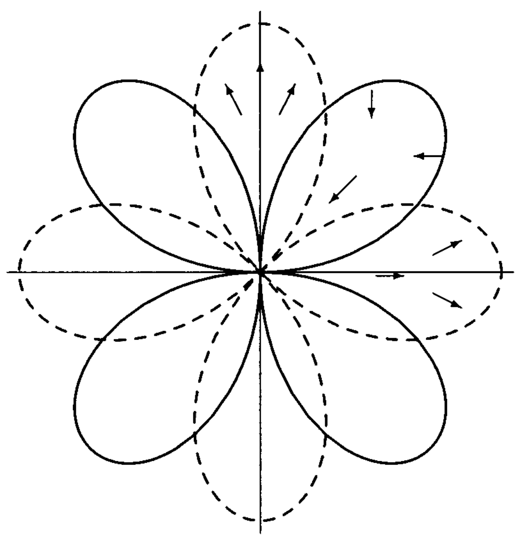

FIGURE 2. The attracting and repelling petals for $\lambda = 1$, $n = 4$

When $\lambda = 1$ there exist attracting and repelling directions, rays along which orbits tend to zero and infinity respectively. If $\theta_k = \frac{k\pi}{n}$, then k odd determines a repelling ray and k even an attracting one.

Corollary 1. *The fixed point zero is the only attracting orbit completely contained in the closure of the union of the attracting petals* [M].

Corollary 2. *Excepting preimages of zero, the orbit of z_0 converges to zero if and only if an image of z_0 lands in one of the attracting petals. It follows that z_0 is in the basin of attraction of zero* [M].

3. The Exponential Function

We now examine the resulting dynamics when Newton's method is applied to the exponential function, $F(z) = P(z)e^{Q(z)}$, where P and Q are polynomials.

Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree m and $Q : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n such that $F = Pe^Q$ is not constant. Then the Newton method function applied to $F = Pe^Q$ is the rational map

$$N(z) = z - \frac{F(z)}{F'(z)} = \frac{zP'(z) + zP(z)Q'(z) - P(z)}{P'(z) + P(z)Q'(z)},$$

and $N : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Fixed points of N coincide with roots of F. The nature of the fixed points is determined by the derivative of N,

$$N'(z) = \frac{F(z)F''(z)}{[F'(z)]^2}.$$

It is well known that simple roots of F are superattracting fixed points of N. If $z = a$ is a multiple root of F with multiplicity m then $N'(a) = \frac{m-1}{m}$. The implication is that the speed of convergence to a root depends on the multiplicity: the higher the multiplicity, the slower the convergence. If F has a critical point at $z = a$, then N has a pole at a if a is not also a root of F. In other words, a critical point that is not a root of F gets sent to infinity under a single iteration of N.

It is easy to see that infinity is a fixed point with derivative exactly equal to 1. We will use this to show that there is an upper bound on the area of the attractive basins for roots.

Proposition 1. *Infinity is a parabolic fixed point, with multiplier equal to* 1*, for Newton's method applied to* $F(z) = P(z)e^{Q(z)}$*, where* P *and* Q *are polynomials,* P *is not identically zero and* Q *is non-constant.*

Proof. Let P be a polynomial of degree m and Q be a polynomial of degree n. Inspection of N shows that the degree of the numerator is $m + n$, and the degree of the denominator is $m + n - 1$. Therefore, $\lim_{z\to\infty} N(z) = \infty$ and ∞ is a fixed point. To determine its nature, map ∞ to 0 via $g(z) = 1/z$. The conjugate function is

$$M(v) = v[1 + H(v)],$$

where

$$H(v) = \frac{vP(\frac{1}{v})}{P'(\frac{1}{v}) + Q'(\frac{1}{v})P(\frac{1}{v}) - vP(\frac{1}{v})}.$$

Evaluation of the derivative $M'(v) = 1 + H(v) + v\,H'(v)$ at 0 yields $M'(0) = 1$. Thus, ∞ is a parabolic fixed point. $\square$

From here on the additional assumption that the coefficients of the z^n and z^{n-1} terms of Q are equal to 1 is possible due to the following argument . Let $F = Pe^Q$. Then there exist polynomials $R(z) = P(az+b)$ and $S(z) = Q(az+b)$, $a, b \in \mathbb{C}$. Let $G = Re^S$. It is easily shown that N_F is conjugate to N_G via $z \mapsto az + b$. Since the conjugacy is an analytic map, the dynamics of N_G are equivalent to the dynamics of N_F. Moreoever, for some $a, b \in \mathbb{C}$ there exists a polynomial S with leading and second coefficients equal to 1.

Closer inspection of M yields results about the local dynamics near ∞ through application of the Fatou Flower Theorem. The series expansion of M is

$$M(v) \quad = \quad v + cv^{n+1} + O(v^{n+2}),$$

where $c = \frac{H^{(n)}(0)}{n!} = \frac{1}{n}$. Therefore, the degree of the exponent polynomial Q completely determines the number of petals at ∞. By the Fatou Flower Theorem, the following proposition is proved.

Proposition 2. *There are exactly n attracting and n repelling petals at the neutral fixed point ∞ if Q has degree n.*

4. An example

Before exploring the dynamics of the most general case let us first examine a simple example in which M is a polynomial.

Let $F(z) = ze^{z^n}$. Note that $M = v + \frac{1}{n}v^{n+1}$. By Proposition 2, there are n attracting and n repelling petals for N at ∞. Figures 3, 4, and 5 illustrate the attracting basins of ∞ as viewed from the origin and ∞.

Computer graphics illustrate how points behave under iteration of N. For each point in the complex plane, the algorithm tests to see whether $|N^k(z)| < 0.001$ for $k \leq 50$ and shades the point according to how soon that happens: lighter shades indicate quick convergence; darker shades indicate slower convergence. Points are colored black if they do not get within a distance of 0.1 of the root within fifty iterates.

In the pictures it appears that the immediate basin of attraction of 0 stretches out to ∞ in evenly spaced directions equal to the degree of the exponent in number. It also appears that the basin pinches down as it tends toward ∞, creating a starfish shape with n legs whose center is at the origin. In contrast, the immediate basins for Newton's method on polynomials open up as they extend to ∞.

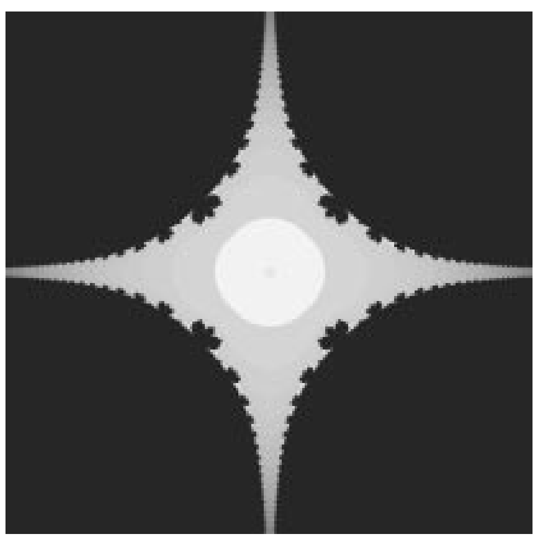

Figure 3. View from 0 for $n = 4$

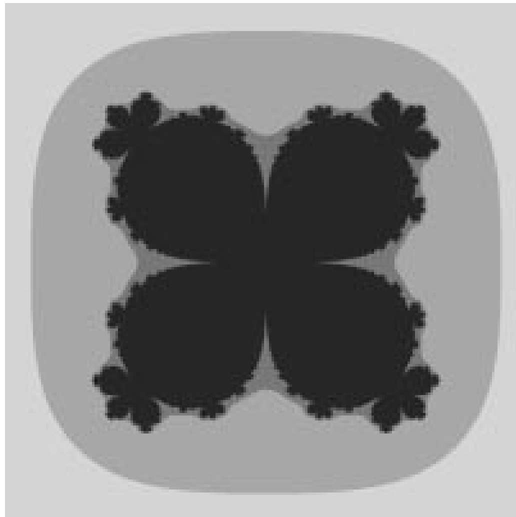

FIGURE 4. View from ∞ for $n = 4$

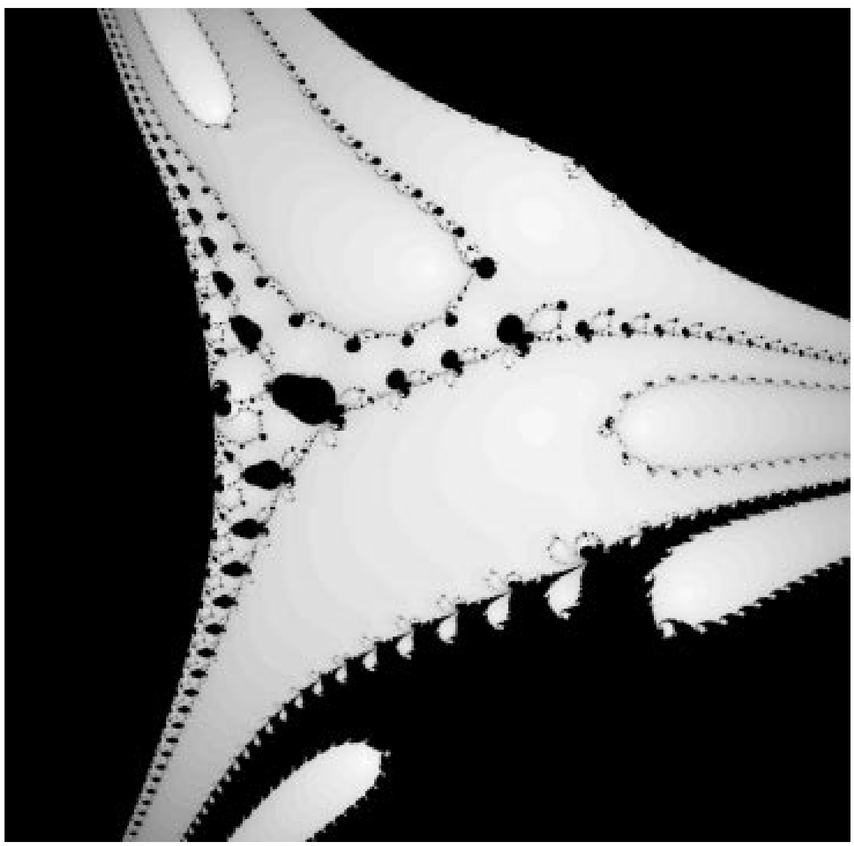

FIGURE 5. P with seven simple roots, $Q(z) = z^3$

5. The proof of finite area

Theorem 2. *If Newton's method is applied to $F = Pe^Q$, where P and Q are complex valued polynomials, P is not identically zero and Q is non-constant with* $\deg(Q) \geq 3$*, then the area of the attractive basin of a root of F has finite area.*

From here on we assume $n \geq 3$. In the case $n = 2$ the basin also pinches down, but area has been shown to be infinite for the case $F(z) = P(z)e^{z^2}$ [Kr]. The proof that the basin of a root has finite area, is accomplished by showing that the union of the attractive basins of all roots is completely contained within a region with finite area.

By Proposition 2 there are n attracting and n repelling petals, where $n = \deg Q$, and these petals are evenly spaced. Basin tails of roots extend to infinity and must lie between pairs of attracting petals for ∞. By constructing attracting petals with an appropriately high order of tangency at infinity we will show that these tails have finite area. The attracting petals determine parabolic basins of attraction for ∞. Moreover, the union of the basin of ∞ with the Julia set is the complement of the basins of the roots. It follows that the set of all basins of roots has finite area.

The computer generated pictures on the previous pages (Figures 3 and 5) illustrate this characteristic. The goal is to construct a region with finite area that contains the tail of a basin. Every axis ray is a repelling direction for ∞, so each axis ray to infinity lies within a basin tail. Since axis rays differ only by rotation we may consider a basin tail that extends to ∞ along the positive real axis. We need to find a curve γ that lies inside the attractive basin of ∞, and hence bounds the basin tail from above for values of t near infinity. By definition, an attracting petal must lie inside the immediate basin of ∞. Therefore, if γ lies inside of, or defines the boundary of, a petal near infinity then orbits of points on γ will tend to infinity under iteration. A symmetric argument shows that $-\gamma$ bounds the basin tail from below. Since the basin tail is contained in the region of bounded area, it has finite area.

To keep track of our location we will refer to the plane in which the Newton method function N acts as the $z-$plane. When we send ∞ to 0 via g in order to study the nature of the fixed point at ∞, we will be working with M and the attracting and repelling petals from the Fatou Flower Theorem in the $v-$plane. This conjugacy is merely a transitional phase. The next map, π, takes 0 back to ∞ and acts as a semi-conjugacy between M and a map G to be determined. We denote the plane in which G acts as the $w-$plane.

$$\begin{tikzcd}
z \arrow[r, "N"] \arrow[d, "g"'] & z \arrow[d, "g"] \\
v \arrow[r, "M"] \arrow[d, "\pi"'] & v \arrow[d, "\pi"] \\
w \arrow[r, "G"] & w
\end{tikzcd}$$

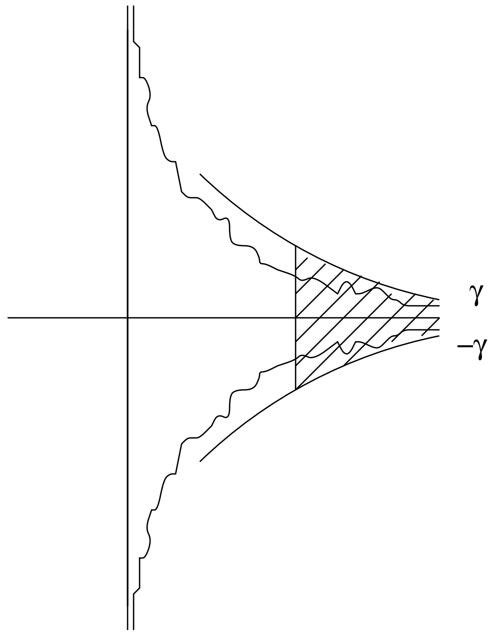

FIGURE 6. Basin tail bounded above by γ

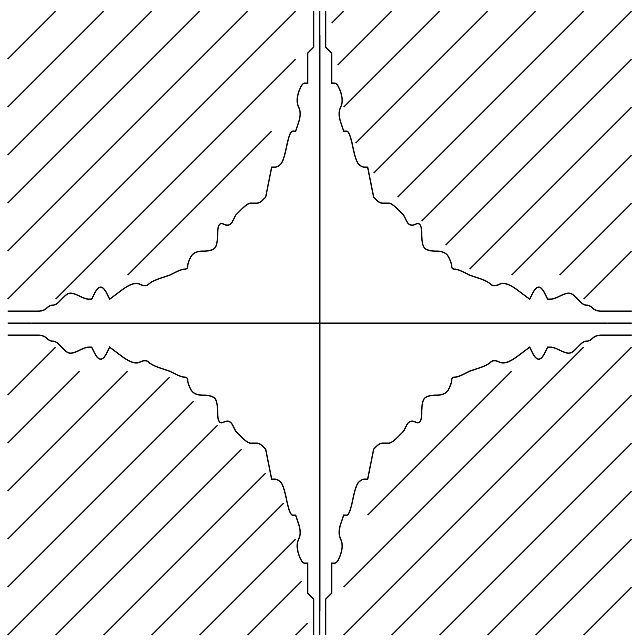

FIGURE 7. The $z-$plane

In a sense, $\pi \circ g$ "unfolds" the attracting petal at ∞ with attracting direction $\frac{\pi}{n}$ in the $z-$plane, into a wedge-shaped region defined near ∞ in the $w-$plane. The construction of the modified petal occurs in the $w-$plane.

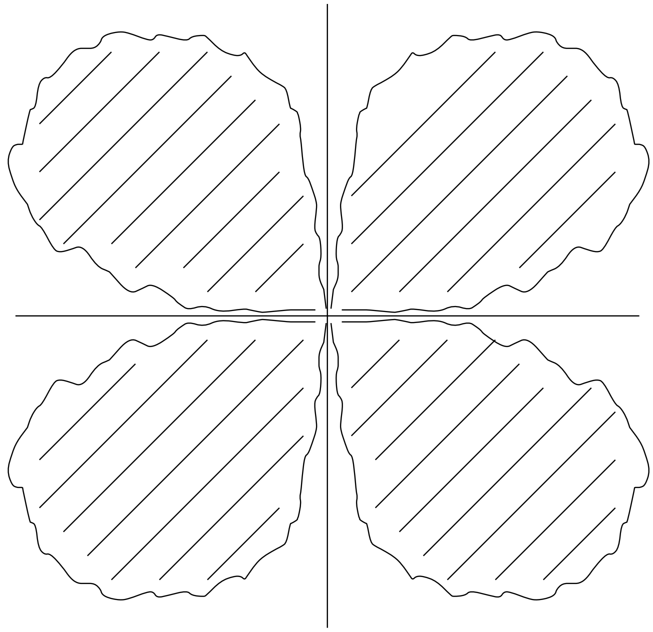

FIGURE 8. The v–plane

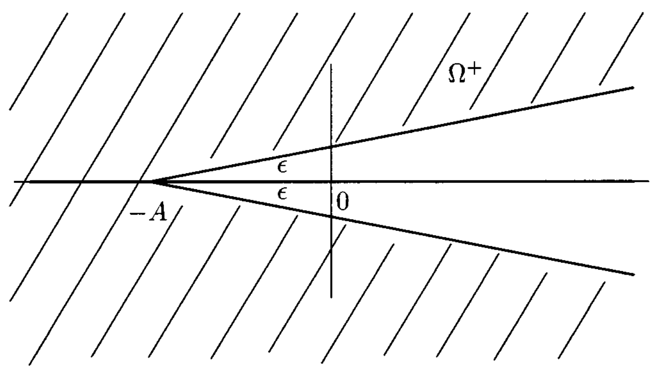

FIGURE 9. The w–plane

We begin with a standard construction of an attracting petal at ∞. Conjugate N in a neighborhood of ∞ via $g(z) = \frac{1}{z}$ to M near zero as before. Then

$$M(v) = v + cv^{n+1} + c_1 v^{n+2} + O(v^{n+3}),$$

where $c = \frac{1}{n}$ and $c_1 = -\frac{n-1}{n^2}$. We now move to the $w-$plane via the semi-conjugacy $w = \pi(v) = \frac{1}{v^n}$ which sends ∞ to 0. Choose the branch of the inverse associated with the attracting direction $\frac{\pi}{n}$ to obtain

$$G(w) = w - 1 + \frac{\alpha}{w^{\frac{1}{n}}} + \cdots,$$

where $\alpha = \frac{n-1}{n}$. Thus N and G are semi-conjugate.

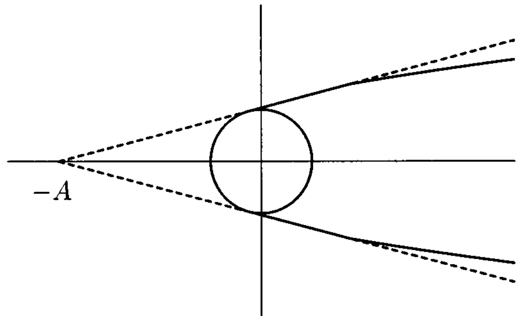

FIGURE 10. The parabolic envelope curve

Define an open wedge shaped region Ω in the $w-$plane by

$$\Omega = \{w \in \mathbb{C} \,|\, \epsilon < \arg(w + A) < 2\pi - \epsilon\},$$

$A \in \mathbb{R}^+$. (See Figure 10.) Note that near infinity iteration of G is effectively translation by one to the left. In fact, given $\epsilon \in (0, \pi)$, there exists $r_0 > 0$ such that

$$\left| w - 1 + \frac{\alpha}{w^{\frac{1}{n}}} - G(w) \right| \leq \left| \frac{\alpha}{w^{\frac{1}{n}}} \right| < \frac{1}{2} \sin \epsilon \ ,$$

for $|w| > r_0$. In particular, we obtain

$$|w - 1 - G(w)| < \sin \epsilon.$$

Given $\epsilon > 0$, we can choose $A \in \mathbb{R}^+$ large enough to satisfy $|w| > r_0$. Hence, the closure of Ω^+ maps inside itself under G. Taking one branch of the inverse map, $\pi^{-1}(\Omega^+)$ yields a single attracting petal at zero. The remaining $n - 1$ branches yield the rest of the attracting petals. The repelling petals are constructed in the same manner but with the change of variable $z^n = -\frac{1}{ncw}$.

Observe that an effort to increase the wedge size by decreasing the angle ϵ shifts Ω to the left since $A = \frac{\mu}{\tan \epsilon}$ for some $\mu > 0$. Fatou constructed a petal of greater area by noting that the envelope of the wedges limits to a parabolic curve near infinity.

We now give an estimate for the minimum value of A, with $\epsilon > 0$, that ensures that the closure of Ω maps inside itself. Without loss of generality we present the following arguments in the upper half plane. The proofs are easily extended to all of Ω due to the symmetry of Ω with respect to the real line.

In Figure 11, observe that $d = \sin \epsilon$. $G(w)$ lies inside Ω if $\left| \frac{\alpha}{w^{\frac{1}{n}}} \right| < \frac{\sin \epsilon}{2}$.

Since $\min |w| = \min |re^{i\theta}|$ occurs when $r = A \sin \epsilon$, we have

$$\left| \frac{\alpha}{w^{\frac{1}{n}}} \right| \leq \frac{\alpha}{(A \sin \epsilon)^{\frac{1}{n}}}.$$

Set $\frac{\alpha}{(A \sin \epsilon)^{\frac{1}{n}}} < \sin \epsilon$, then $A > \frac{\alpha^n}{(\sin \epsilon)^{n+1}}$. Since we required that $|w| > r_0$, the condition

$$A > \max \left\{ \frac{\alpha^n}{\sin^{n+1} \epsilon} \, , \, \frac{r_0}{\sin \epsilon} \right\}$$

produces a wedge Ω such that images of points in Ω under G remain in Ω.

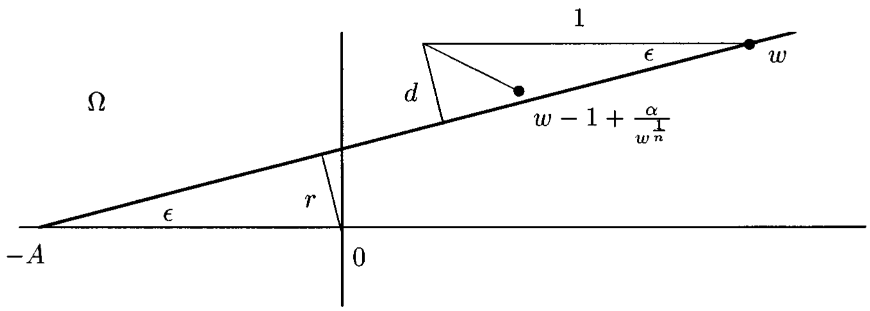

FIGURE 11. Location of $G(w)$

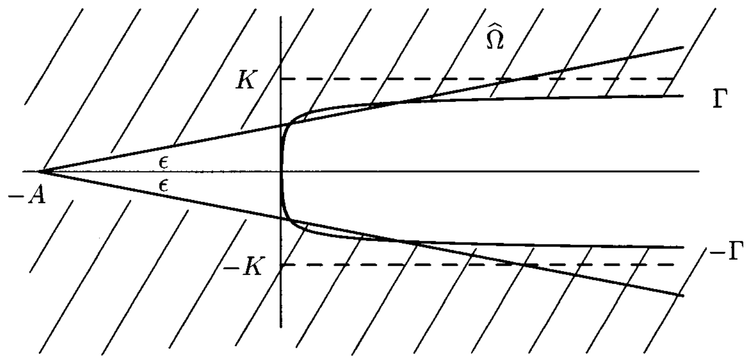

FIGURE 12. The modified wedge, $\widehat{\Omega}$

Let $\rho = \pi \circ g$. Then ρ conjugates N with G. Mapping Ω back to the $z-$plane by the appropriate branch of ρ^{-1} produces a single attracting petal at ∞ with attracting direction $\frac{\pi}{n}$.

Our next goal is to find a petal at ∞ that finitely bounds the area of a tail of an attracting basin. Insofar, we have constructed a region Ω whose image under a branch of ρ^{-1} yielded a single attracting petal at ∞. Unfortunately, ρ^{-1} takes Ω back to a petal at ∞ that is a subset of some sector at the origin. It is evident that $\rho^{-1}(\partial\Omega)$ does not bound a finite area near infinity. A second attempt can be made with horizontal lines defining the boundary of Ω in the right half plane. Although the resulting curve in the $z-$plane bounds finite area, this modification is unsatisfactory because the new "wedge" fails to map completely inside itself under G in the $w-$plane. The envelope curve constructed by Fatou is a natural alternative. However, this curve does not produce strong enough results. Under a branch of ρ^{-1}, the parabolic curve is sent to a curve of order $t^{1-\frac{n}{2}}$ near infinity in the $z-$plane. This approach yields finite area for $n \geq 6$, but says nothing about the cases where $n < 6$. Therefore, we will construct a modification of the wedge Ω, called $\widehat{\Omega}$, whose boundary near infinity is defined by some curve Γ. (See Figure 12.)

We will require that Γ be the image of another curve γ that lies in the $z-$plane where

$$\int_{t_0}^{\infty} \gamma \, dt < \infty$$

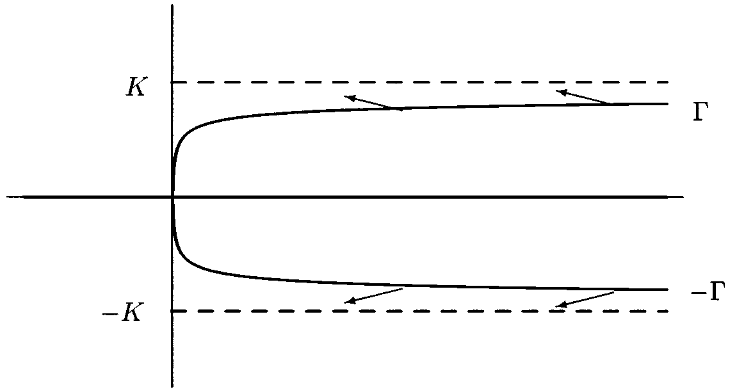

FIGURE 13. Points on Γ map above Γ

for some $t_0 > 0$. Finally, it is necessary that $\widehat{\Omega}$ satisfy

$$G\left(\overline{\widehat{\Omega}}\right) \subset \overline{\widehat{\Omega}}\,.$$

Standard calculus shows that the area under the curve

$$\gamma(t) = t + i\left(\frac{a}{t^{n-1}} - \frac{b}{t^n}\right)$$

is bounded for $n \geq 3$ and $a, b \in (0, +\infty)$ when t is large enough and positive. Now define

$$\widehat{\Omega} = \Omega \cup \{w \mid \text{Im } w > \text{Im } \Gamma(w),\ \text{Re } w > w_0\},$$

where $w_0 = \max\limits_{\Gamma \cap \delta\Omega} \{\text{Re } w\}$ and Γ is a curve in the $w-$plane satisfying the following four conditions.

(1) Γ is the image of γ under ρ.

(2) Points on Γ map above the curve Γ under G.

(3) Γ intersects the boundary of Ω.

(4) Points map above both Γ and $\partial\Omega$ under G.

To determine a precise expression for the curve Γ, apply $\rho(z) = z^n$ to γ, where $n \geq 3$ and $a, b \in \mathbb{R}^+$, to obtain

$$[\gamma(t)]^n \sim t^n + in\left(a - \frac{b}{t}\right).$$

Reparametrizing by setting $x = t^n$ results in

$$y \sim na - \frac{nb}{t^{\frac{1}{n}}}.$$

Therefore we will let

$$\Gamma(t) = t + i\left(K - \frac{k}{t^{\frac{1}{n}}}\right),$$

with $K = na$ and $k = nb$ to satisfy condition (1). Note that Γ is asymptotic to $y = K$.

The region $\widehat{\Omega}$ will map inside of itself under G as long as Γ satisfies conditions (2), (3), and (4) above. First we will show, as in Figure 13, that points on Γ are sent to points above Γ by

$$G(w) = w - 1 + \frac{\alpha}{w^{\frac{1}{n}}} + \cdots .$$

Denote the image of Γ under G by $\widetilde{\Gamma}$. It follows that

$$\widetilde{\Gamma}(t) = t + i\left(K - \frac{k}{t^{\frac{1}{n}}}\right) - 1 + \alpha\left(\frac{1}{t^2 + \left(K - \frac{k}{t^{\frac{1}{n}}}\right)^2}\right)^{\frac{1}{n}} \cdot \left[t^{\frac{1}{n}} - \frac{i}{n} t^{\frac{1}{n}-1}\left(K - \frac{k}{t^{\frac{1}{n}}}\right) - \cdots\right]$$

We require that

$$\operatorname{Im}\left(\widetilde{\Gamma}(t)\right) > \operatorname{Im}\left[\Gamma\left(\operatorname{Re}\left[\widetilde{\Gamma}(t)\right]\right)\right]. \tag{5}$$

Substituting in the real and imaginary parts of $\widetilde{\Gamma}$ from above gives

$$\frac{k}{t^{\frac{1}{n}}} + \frac{\frac{\alpha}{n} t^{\frac{1}{n}-1}\left(K - \frac{k}{t^{\frac{1}{n}}}\right)}{\left(t^2 + \left(K - \frac{k}{t^{\frac{1}{n}}}\right)^2\right)^{\frac{1}{n}}} + \ldots < \frac{k}{\left(t - 1 + \frac{\alpha t^{\frac{1}{n}}}{\left(t^2 + \left(K - \frac{k}{t^{\frac{1}{n}}}\right)^2\right)^{\frac{1}{n}}} + \cdots\right)^{\frac{1}{n}}}. \tag{6}$$

Denote the left side of the inequality by L. Since $\lim_{t\to\infty} t^{\frac{1}{n}} \cdot L = k$ and

$$\lim_{t\to\infty} t^{1+\frac{1}{n}} \cdot \left(L - \frac{k}{t^{\frac{1}{n}}}\right) = \frac{\alpha K}{n},$$

we have

$$L = \frac{k}{t^{\frac{1}{n}}} + \frac{\alpha K}{n t^{1+\frac{1}{n}}} + \cdots .$$

Denote the right side of the inequality by R and let $R = kB^{\frac{1}{n}}$, where

$$B = \frac{1}{t - 1 + \frac{\alpha t^{\frac{1}{n}}}{\left(t^2 + \left(K - \frac{k}{t^{\frac{1}{n}}}\right)^2\right)^{\frac{1}{n}}} + \cdots} .$$

Since $\lim_{t\to\infty} t \cdot B = 1$ and $\lim_{t\to\infty} t^2 \cdot \left(B - \frac{1}{t}\right) = 1$, we have

$$B = \frac{1}{t} + \frac{1}{t^2} + \cdots .$$

Thus,

$$R = \frac{k}{t^{\frac{1}{n}}} + \frac{k}{n t^{\frac{1}{n}+1}} + O\left(\frac{1}{t^{\frac{1}{n}+2}}\right).$$

We are now able to rewrite inequality (6) as

$$\frac{k}{t^{\frac{1}{n}}} + \frac{\alpha K}{n t^{1+\frac{1}{n}}} + \cdots < \frac{k}{t^{\frac{1}{n}}} + \frac{k}{n t^{1+\frac{1}{n}}} + \cdots$$

where $\alpha = \frac{n-1}{n}$, $K = na$ and $k = nb$. It is clear that this expression holds if $k > \alpha K$, or equivalently if $\frac{b}{a} > \frac{n-1}{n}$. Therefore, Γ provides a suitable boundary for $\widehat{\Omega}$ in that, for sufficiently large t and appropriate choices of a and b, condition (2) is satisfied.

More precisely, we may choose a number $\delta > 1$ and assume that $k = \alpha\delta K$. Then there exists a number T_0 depending only on P and Q such that (6) and subsequently (2) are satisfied. For the remainder of the proof, we shall work with these fixed values for k and T_0.

We are now prepared to show condition (3), that Γ and the boundary of Ω intersect for $t > 0$. See Figure 12. The approach is to find the point $t_0 > 0$ at which $\Gamma'(t_0)$ and the slope of the line defining $\delta\Omega$ are equal. If we determine that the curve lies above Ω at t_0 i.e.,

$$\tan\epsilon\,(t_0 + A) < \mathrm{Im}[\Gamma(t_0)], \tag{7}$$

then we are done since $\Gamma(t)$ lies below Ω for large values of t.

Fix $\epsilon > 0$ and subsequently A. Note that $k > 0$ and $K > 0$ when Γ lies in the upper half plane. The slope of the line is $\tan\epsilon$ and the derivative of the curve is $\Gamma'(t_0) = \frac{k}{n} t_0{}^{-\frac{1}{n}-1}$. Setting these equal and solving for t_0 yields

$$t_0 = \left(\frac{k}{n\tan\epsilon}\right)^{\frac{n}{n+1}}.$$

It remains to demonstrate that

$$\tan\epsilon\left(\left(\frac{k}{n\tan\epsilon}\right)^{\frac{n}{n+1}} + A\right) < K - \frac{k}{\left(\frac{k}{n\tan\epsilon}\right)^{\frac{1}{n+1}}},$$

or equivalently,

$$A\tan\epsilon + k^{\frac{n}{n+1}}\left[\tan\epsilon\left(\frac{1}{n\tan\epsilon}\right)^{\frac{n}{n+1}} + (n\tan\epsilon)^{\frac{1}{n+1}}\right] < K. \tag{8}$$

Recall that $A > \frac{\alpha}{(\sin\epsilon)^{n+1}}$ and $\alpha = -\frac{n-1}{n}$ are fixed by the polynomial Q and our choice of ϵ. However, for $K = na$, and $k = nb$ the value of n is predetermined, but the choices of a and b remain open. Therefore, in equation (8) we replace all fixed expressions by constants ν_1 and ν_2 and k by $\alpha\delta K$, resulting in

$$\nu_1 + (\alpha\delta K)^{\frac{n}{n+1}}\,\nu_2 < K, \tag{9}$$

or equivalently

$$\nu_1 < K\left(1 - \frac{(\alpha\delta)^{\frac{n}{n+1}}\,\nu_2}{K^{\frac{1}{n+1}}}\right) ,$$

which holds for K sufficiently large. Hence, (9), and in turn (7), holds, consequently satisfying condition (3).

To show condition (4), let t_1 and t_2 be the values for which Γ intersects the original wedge Ω with $T_0 \le t_1 \le t_2$ and $t_2 - t_1 \ge 2$. Define the boundary of $\widehat{\Omega}$ to be

$$\widehat{\Omega} = \begin{cases} \partial\Omega, & t < t_2, \\ \Gamma, & t \ge t_2. \end{cases}$$

Since $|\ \mathrm{Re}\,[\Gamma(t)] - \mathrm{Re}\,[G(\Gamma(t))]\ | < 2$ then $t \ge t_2$ implies that $t \ge T_0$. Thus for large enough values of K, condition (4) is satisfied.

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