

FAREY POLYTOPES AND CONTINUED FRACTIONS ASSOCIATED WITH DISCRETE HYPERBOLIC GROUPS

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ABSTRACT. The known definitions of Farey polytopes and continued fractions are generalized and applied to diophantine approximation in n -dimensional euclidean spaces. A generalized Remak-Rogers isolation theorem is proved and applied to show that certain Hurwitz constants for discrete groups acting in a hyperbolic space are isolated. The approximation constant for the imaginary quadratic field of discriminant -15 is found.

1. INTRODUCTION

Let V be the n -dimensional Euclidean space \mathbf{R}^n . The upper half-space $H^{n+1} = \{(z, t) : z \in V, t > 0\}$ with the metric $ds^2 = t^{-2}(|dz|^2 + dt^2)$ can be used as a model of the $(n + 1)$ -dimensional hyperbolic space. Here $|\cdot|$ is Euclidean length in V . Let $\text{Con}(n)$ denote the group of orientation-preserving isometries of H^{n+1} . Let G be a geometrically finite discrete subgroup of $\text{Con}(n)$ (see [1]). A geodesic in H^{n+1} is a semicircle or a ray which is orthogonal to V . An element $g \in \text{Con}(n)$ extends to a conformal transformation of $V \cup H^{n+1}$, the closure of H^{n+1} . Hence, g will fix a point either in H^{n+1} or on its boundary V . The type of g is *elliptic*, *parabolic* or *loxodromic* depending on whether it has a fixed point in H^{n+1} , a single fixed point in V , or exactly two fixed points in V (see e.g. [1]). If g is loxodromic, the geodesic connecting its fixed points is called the *axis* of g . The transformation g is *hyperbolic* if it is loxodromic and every plane containing its axis is g -invariant. We denote by \mathcal{P} the set of parabolic fixed points (*cusps*) of G . In the sequel, we assume that $\infty \in \mathcal{P}$.

Let P be a Dirichlet polygon of $G_\infty = \text{Stab}(\infty, G)$ in V . Denote $P_\infty = \{(z, t) \in H^{n+1} : z \in P\}$. The region

$$(1) \quad D = P_\infty \cap \{x \in H^{n+1} : |g'(x)| < 1, g \in G\}$$

is an *isometric* fundamental domain for G in H^{n+1} (see [1] or [2]). Here $g'(x)$ stands for the Jacobian of the transformation g . Denote

$$(2) \quad K = K(\infty) = G_\infty \bar{D}, \quad K(u) = gK(\infty),$$

where $u = g(\infty)$. Let ∂K be the boundary of K . We shall say that $\partial K \cap \bar{D}$ is the *floor* of D . In the sequel, we shall be mainly concerned with the components of ∂K (and D) of dimensions 0, 1, and n . We shall call them *vertices* (or *cusps*), *edges*, and *faces* of K respectively. The vertices (and edges) of K which belong

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to \bar{D} will be called the vertices (and edges) of D . For any region R in H^{n+1} , the components of the boundary of R of dimension n which lie in vertical planes will be called the *vertical faces* of R . (Note that, in general, according to these definitions, the components of the boundary of D of dimension 0 (or 1) which lie in the vertical faces of D are not vertices (or edges) of D .)

Let α be a real irrational number. In 1891 A. Hurwitz [17] showed that the inequality

$$(3) \quad |\alpha - ac^{-1}| < \frac{1}{k|c|^2}$$

has infinitely many solutions in coprime integers a and c when $k = \sqrt{5}$, and $\sqrt{5}$ is the best constant possible. The first geometric proof of this result was obtained by L. Ford in [12] where he makes use of properties of the modular group. Let O_d be the ring of integers of the imaginary quadratic number field $\mathbf{Q}(\sqrt{-d})$. In 1925 Ford [13] applied his approach to the Picard group to show that for any $\alpha \notin \mathbf{Q}(\sqrt{-1})$ (3) has infinitely many solutions in coprime $a, c \in O_1$ when $k = \sqrt{3}$, and $\sqrt{3}$ is the best constant possible. A modification of the Ford geometric approach is developed in [35] where the approximation constants for the fields $\mathbf{Q}(\sqrt{-5})$ and $\mathbf{Q}(\sqrt{-6})$ are found.

Let $\alpha \in V - \mathcal{P}$. Assume that $\infty \in \bar{D}$, where \bar{D} is the closure of D . It is known (see e.g. [1] or [3]) that there is a constant $k > 0$, depending only on G , such that the inequality

$$(4) \quad |\alpha - g(\infty)| < r^2(g)/k$$

holds for infinitely many left cosets of G_∞ in G . Here $r(g)$ is the radius of the *isometric sphere* $I(g) = \{x \in H^{n+1} : |g'(x)| = 1\}$ of $g \in G$. If $n = 1$ and G is the modular group, (4) is reduced to (3).

For a fixed $\alpha \in V - \mathcal{P}$, we denote by $k(\alpha)$ the supremum of all such k in (4). The set of numbers

$$\mathcal{L}(G) = \{1/k(\alpha), \alpha \in V - \mathcal{P}\}$$

is called the *Lagrange spectrum* for G and $C(G) = \sup \mathcal{L}(G)$ the *Hurwitz constant* for G .

For an oriented geodesic L in H^{n+1} with the initial and terminal endpoints $\eta', \eta \in V$ respectively we write $L = (\eta', \eta)$. In particular, (∞, η) is a vertical ray in H^{n+1} through η . We shall say that $ht(L) = |\eta' - \eta|/2 = h_L/2$ is the *height* of L and denote

$$k(L) = \sup h_{gL} = \sup |g(\eta) - g(\eta')|, \quad g \in G.$$

If L is the axis of a loxodromic element in G , then $k(L) = k(\eta') = k(\eta)$. Otherwise, this is not always true (see e.g. [11]). The set of numbers

$$\mathcal{M}(G) = \{1/k(L), L \subset H^{n+1}\}$$

is called the *Markov spectrum* for G .

Let D be an isometric fundamental domain of G . Let an edge σ of D lie on a geodesic L which is not a vertical ray. The point of L farthest from V is called the *summit* of σ .

Define k_G to be the largest value of k such that the connected parts of D lying below $t = k/2$ are pyramidal regions bounded by the faces of D which meet at a vertex or cusp of D and the Euclidean plane $t = k/2$. If the summit of every edge of D belongs to the closure of D , then k_G is twice the distance from V to the set

of summits of D . The following theorem is proved in [36] (see also Theorems 9 and 17).

Theorem 1. *Let G be a geometrically finite group acting in the $(n+1)$ -dimensional hyperbolic space H^{n+1} . Let the n -dimensional Euclidean space V be the limit set and \mathcal{P} the set of parabolic fixed points of G . Suppose that $\infty \in \mathcal{P}$. Let $\alpha \in V - \mathcal{P}$. Then there are infinitely many left cosets of G_∞ in G whose members g satisfy (4) with $k = k_G$. Thus, $C(G) \leq 1/k_G$.*

In §2, we introduce the fundamental notion of a v -cell $N(v)$ for every vertex or cusp of a fundamental domain for G . The k -neighborhood of v introduced in [34], [35], and [36] is a subset of $N(v)$, so that $N(v)$ is the ∞ -neighborhood of the vertex. The Farey tessellation is a tessellation of H^{n+1} by v -cells. It is G -invariant. For the modular group, this definition was introduced by C. Series in [26]. A Farey polytope in V is the projection of a non-vertical face of a v -cell from ∞ into V . For some Bianchi groups, Farey polygons with three, four and six sides were introduced by A. L. Schmidt in [21] and [24] (see also [22]) using another definition. Here, the basic properties of the Farey polytopes are established (Theorems 5 and 9).

In §3, Algorithm I generating the best possible approximations of $\alpha \in V$ by the cusps $g\infty$, $g \in G$, is introduced. It is analogous to but different from the standard continued fraction algorithm and the algorithms introduced by A. L. Schmidt in [23], [24], and [25]. Here, some of the known applications of the continued fraction algorithms are discussed. In particular, Algorithm I is applied to develop the reduction theory for geodesics in H^{n+1} which is similar to the reduction theory of binary indefinite quadratic forms.

In §4, the notion of an extremal geodesic is introduced. (A geodesic $L = (\eta', \eta)$ is extremal if $k(L) = |\eta' - \eta|$.) If an extremal geodesic exists in the G -orbit of a geodesic, Algorithm I can be used to find it since the set of reduced geodesics contains an extremal one in that case. Simple criteria of extremality of a geodesic are given in Corollary 24 and Lemma 25. In Examples 27, 28, and 31, they are applied to Bianchi groups $PGL_2(O_d)$, $d \equiv 3 \pmod{4}$, $d \leq 19$, and to the extended Bianchi group B_{15} . In all the known cases (see e.g. [35]), among the extremal geodesics found there is a geodesic L such that the Hurwitz constant $C(G) = 1/k(L)$. For the group B_{15} , we find that $1/\sqrt{2} \leq C(B_{15}) < 2/\sqrt{7}$. (It is proved in §5 that $C(B_{15}) = 1/\sqrt{2}$ so that in the only unknown case of $d = 15$ such an extremal geodesic has been found too.) Applying Algorithm I to the extremal geodesics found, we also find the set of reduced geodesics. In conclusion, we prove that if the endpoints v and v' of a critical edge σ of D are cusps and there is a reflection $R \in G$ such that $v' = R(v)$, then the Hurwitz constant $C(G) = k(\sigma) = |v' - v|$ and it is an accumulation point in the Lagrange and Markov spectra (Corollary 30). These results are applied to show that the inequality (3) has infinitely many solutions in coprime $a, c \in O_{15}$ when $k = 1/2$, and $1/2$ is the best constant possible. Thus, the Hurwitz constant $C(G) = 2$, $G = PGL_2(O_{15})$, and it is not isolated in $\mathcal{L}(G)$ (Example 31). In Example 32, a similar result is obtained for the set of integral quaternions whose approximation constant is 1 (see [36]).

It is shown in [35] that the strict inequality $C(G) < 1/k_G$ holds in Theorem 1 for a Bianchi group $G = B_d$ when $d \equiv 3 \pmod{4}$, $d \leq 19$, $d \neq 15$, though $C(G) = 1/k_G$ when $d = 1, 2, 5, 6$. In Example 28, it is found that $1/\sqrt{2} \leq C(G) < 1/k_G = 2/\sqrt{7}$ when $d = 15$. In §5, applying the results obtained in §2 and §4 we find that

$C(B_{15}) = 1/\sqrt{2}$ which is the approximation constant for the field $\mathbf{Q}(\sqrt{-15})$. The result obtained can be represented as follows.

Theorem 2. *Let $\alpha \in \mathbf{C} - \mathbf{Q}(\sqrt{-15})$. The inequality*

$$\left| \alpha - \frac{a}{c} \right| < \frac{n(a, c)}{k|c|^2}$$

has infinitely many solutions in $a, c \in \mathcal{O}_{15}$ when $k = \sqrt{2}$, whereas for $\alpha = (\omega + \sqrt{\omega})/2$ and $(\omega + \sqrt{-\omega})/2$ the inequality holds only for a finite number of $a/c \in \mathbf{Q}(\sqrt{-15})$ if $k > \sqrt{2}$. Here \mathcal{O}_{15} stands for the ring of integers in $\mathbf{Q}(\sqrt{-15})$ and $n(a, c)$ for the norm of the ideal generated by a and c .

In §6, we prove an isolation theorem (Theorem 36) which generalizes to n -dimensional euclidean spaces the isolation theorems proved in [8], p. 25, and [31] for $n = 1$ and 2. In contrast to the case mentioned above when the endpoints of σ are cusps of D and the Hurwitz constant $C(G)$ is a limit point in the Lagrange and Markov spectra of G , the isolation theorem implies, in all the known cases, that $C(G)$ is isolated provided the critical edges σ of D lie on the extremal geodesics which are the axes of loxodromic elements in G . Here it is shown for the first time that the approximation constants for the imaginary quadratic fields $\mathbf{Q}(\sqrt{-5})$ and $\mathbf{Q}(\sqrt{-6})$, which are found in [35], and for the three-dimensional euclidean space (see [36], Example 3) are isolated.

2. FAREY TESSELLATION AND FAREY POLYTOPES

In this section, we first introduce the notion of a v -component and a v -cell for every vertex and cusp v of D . A k -neighborhood of v (see [34], [35], [36]) is a subset of the v -cell, so that the v -cell is the ∞ -neighborhood of v . Then the definition of a Farey polytope in V as the projection of a non-vertical face of a v -cell from ∞ into V is given. It is shown (see Examples 7 and 11) that the Farey polygons introduced by A. L. Schmidt in [21] and [24] are covered by this definition. We also introduce the definition of the Farey tessellation of H^{n+1} by v -cells. For the modular group it was first introduced by C. Series in [26].

Let ϕ_i be a face of K (see (2)) which lies on the isometric sphere of $g_i \in G$ with center $u_i = g_i(\infty)$. The projection of ϕ_i from ∞ into V is a polytope $p(u_i)$. The polytopes $p(u_i)$ form a tessellation of V . Let v_V be the projection of a vertex v of K . Suppose that it is the common vertex of the polytopes $p(u_i)$, $i = 1, \dots, r$. Denote by $A(v_V)$ the convex hull of the points u_i , $i = 1, \dots, r$, in V . (Note that if all the faces of K are congruent modulo G_∞ , then the polytope $p(u_i)$ is the *Voronoi cell* of u_i and $A(v_V)$, as defined above, is the *Delaunay cell* of v_V (see [9], 33-35).) Let $P(v_V) = \{(z, t) \in H^{n+1} : z \in A(v_V)\}$. The sets $\mathcal{A}(v) = P(v_V) \cap K$ and their images $\mathcal{A}(gv) := g\mathcal{A}(v)$ will be called the v -components. The v -components form a tessellation of H^{n+1} since the sets gK , $g \in G/G_\infty$, do. We shall say that the union of all $\mathcal{A}(gu)$, $g \in G$, $u \in \bar{D}$, such that $gu = v$ is a v -cell $N(v)$. For any $g \in G$, the set $N(gv) := gN(v)$ will be also called a v -cell. The tessellation of H^{n+1} by the v -cells $N(gv)$ will be called the *Farey tessellation* of H^{n+1} associated with G . Note that the Farey tessellation is G -invariant and it is independent of the choice of P_∞ (see (1)).

Let v be a vertex of K and let $G_{\infty, v}$ be the stabilizer of v in G_∞ . Denote $A'(v_V) = A(v_V)/G_{\infty, v}$ and $\mathcal{A}'(v) = \mathcal{A}(v)/G_{\infty, v}$. The union of non-congruent modulo G_∞ polytopes $p(u_i)$ (and $A'(v_V)$) is a fundamental domain of G_∞ in V .

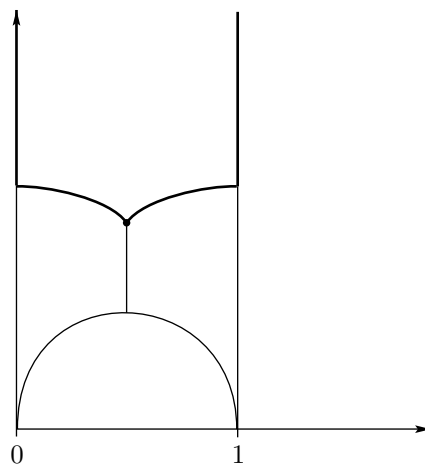


FIGURE 1

Hence if $v_j, j = 1, \dots, m$, is a complete set of vertices of K which are not congruent modulo G_∞ , then the union of the sets $\mathcal{A}'(v_j)$ is the closure of a fundamental domain D of G in H^{n+1} . By taking if necessary the union of all $\mathcal{A}'(v)$, where v runs through the set of all the vertices of the fundamental domain defined by (1) which are not congruent modulo G_∞ , we can always assume that D is connected.

Assume that β is a vertical face of $\mathcal{A}(v)$ which lies in the vertical plane in H^{n+1} through the points $u_i = g_i(\infty), i = 1, \dots, k$. Then the geodesic σ which is the intersection of the isometric spheres of $g_i, i = 1, \dots, k$, is orthogonal to β , and β passes through the summit of σ . It follows that any face B of $N(gv)$ consists of the images of vertical faces of v -components. Hence B is the image of some vertical face of $N(v), v \in \bar{D}$. A face B of $N(gv)$ and the projection of B from ∞ into V will be called a *hyperbolic Farey polytope* and *Farey polytope* respectively. A *Farey set of order m* is the union of the sets of vertices of v -cells $N(gv), g \in G$, such that any vertical line which passes through D passes through at most m v -cells between ∞ and $N(gv)$.

When the isometric fundamental domain D of G has only one vertex $v \neq \infty, \mathcal{A}'(v) = \bar{D}$, and the v -cell $N(v)$ is a fundamental domain of some subgroup G_F of G of index $|\text{Stab}(v, G)|$.

Example 3. Let $n = 1$ and $G = SL_2(\mathbf{Z})$ (cf. [26]). Then v is an elliptic fixed point of G of order 3, the v -component $\mathcal{A}(v) = \bar{D}$, and G_F is the subgroup of G of index 3 whose fundamental domain $N(v)$ is the triangle in H^2 with vertices at 0, 1, ∞ (see Figure 1). The Farey sets are the standard Farey sets for the set of rational numbers (see e.g. [16]).

Example 4. Let $n = 1$ and G be the Hecke group G_q (cf. [15], [34], [36]). Then v is an elliptic fixed point of G of order q , the set $\mathcal{A}(v) = \bar{D}$, and G_F is the subgroup of G of index q whose fundamental domain $N(v)$ is the polygon in H^2 with vertices at $R^i(\infty), i = 0, 1, \dots, q - 1$, where $R \in G_q$ is a generator of $\text{Stab}(v, G_q)$ (see Figures 1, 2 and 3 where $q = 3, 4$ and 6 respectively). These groups G_F were introduced by A. Haas and C. Series in [15].

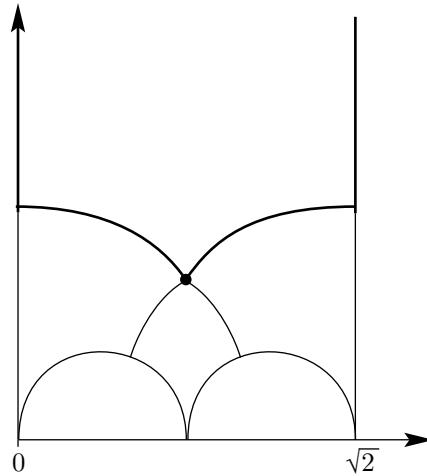


FIGURE 2

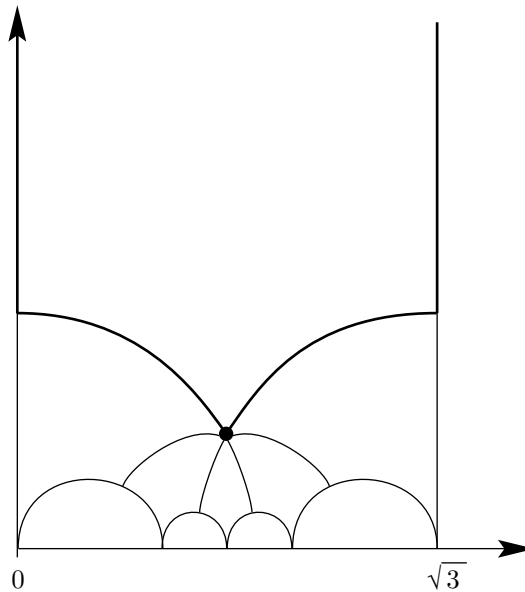


FIGURE 3

Let $g \in G$. For any $k > 0$, let $\mathcal{R}(g, k)$ be the open Euclidean ball in H^{n+1} tangent to V at $g(\infty)$ having radius r^2/k where $r = r(g)$ is the radius of the isometric sphere $I(g)$ of $g \in G$. We have $\mathcal{R}(g, k) = g(\mathcal{R}_k)$ where $\mathcal{R}_k = \mathcal{R}(id, k) = \{(z, t) \in H^{n+1} : t \geq k/2\}$. Denote by $Q(g, k)$ and Q_k the boundaries of the horoballs $\mathcal{R}(g, k)$ and \mathcal{R}_k respectively, and by $\mathcal{N}(k)$ the region in $H^{n+1} \cup \mathcal{P}$ which is exterior to all $\mathcal{R}(g, k)$, $g \in G$.

Let $\alpha \in V - \mathcal{P}$ and $L = (\infty, \alpha)$. Then the inequality (4) holds if and only if L cuts the horosphere $Q(g, k)$.

A subset $N(gv, k)$ of $N(gv)$ bounded by the horospheres $Q(h, k)$ such that $h(\infty)$ is a vertex of $N(gv)$ is the k -neighborhood of gv (cf. [34], [35], [36]). Thus, $N(gv) = N(gv, \infty)$. The region $\mathcal{N}(k)$ is covered by the images of the k -neighborhoods $N(v, k), v \in \bar{D}$. There are two kinds of faces of $N(v, k)$: parts of horospheres $Q(g, k), g \in G$, which will be called the *horospherical* faces of $N(v, k)$, and the *geodesic* faces which are the images of the parts of the vertical faces of $N(v), v \in \bar{D}$.

Let σ be the geodesic through v which is perpendicular to a vertical face B of a v -cell $N(v)$. Denote $h_B = 2ht(\sigma)$. By definition,

$$k_G = \inf h_B,$$

the infimum being taken over the set of vertical faces B of all v -cells $N(v), v \in \bar{D}$. In the following statement, some of the properties of the sets $N(v)$ and $N(v, k)$ are enumerated. All of them follow from the definitions of these sets.

Theorem 5. 1. *Adjacent horospherical and geodesic faces of $N(v, k)$ are orthogonal.*

2. *The bases of horospherical faces of $N(v, k)$ adjacent to its geodesic face B are the vertices of the face of the v -cell $N(v)$ which contains B . The converse is also true.*

3. *Let σ be a geodesic through v perpendicular to a vertical face B of $N(v)$. Let $\sigma' = g(\sigma), v' = g(v)$, and $B' = g(B)$ for some $g \in G$. Then all the horospherical faces of $N(v', h_B)$ adjacent to B' are tangent to the geodesic σ' at the point of intersection of σ' and B' .*

4. *Every face of a v -cell $N(gv), g \in G$, is the image of a vertical face of some v -cell $N(v'), v' \in \bar{D}$.*

5. *Let B be a vertical face of a v -cell $N(v)$. Let $k < h_B$. If a geodesic L in H^{n+1} cuts a face $B' = h(B), h \in G$, then L cuts a horosphere $Q(g, k)$, where $g(\infty)$ is a vertex of B' .*

Proof. 4. If $h(\infty)$ is a vertex of a face B of $N(gv)$, then $h^{-1}B$ is a vertical face of $h^{-1}N(gv)$.

5. Horoballs $\bar{\mathcal{R}}(g, h_B)$ cover B' . □

Remark. It follows from Theorem 5.3 that all the faces of $N(gv, k_G)$ are horospherical which implies Theorem 1 since the vertical line $L = (\infty, \alpha)$ cuts infinitely many v -cells $N(gv)$.

Since the boundary of a Farey polytope F is the projection of the boundary of a face of some v -cell $N(gv)$, which lies in an $(n - 1)$ -dimensional hemisphere orthogonal to V , the faces of F lie in Euclidean $(n - 1)$ -planes in V . Assume that ∞ is not a vertex of $N(gv)$. Since $N(gv)$ is convex and bounded by hemispheres with centers in V or vertical planes, it is bounded from above by one face and all other non-vertical faces of $N(gv)$ form its lower boundary. This leads to the natural subdivision of a Farey polytope into other Farey polytopes (cf. [21], Theorem 4).

Lemma 6. *Let Farey polytope F be the projection of the upper boundary of a v -cell $N(gv)$. Let m and m_v be the numbers of faces and of vertical faces of $N(gv)$ respectively. Then F is subdivided into $m - m_v - 1$ Farey polytopes, the projections of faces of $N(gv)$ which form its lower boundary.*

Example 7. Let $n = 2$ and G be the Bianchi group $GL_2(O_d)$ where O_d is the ring of integers of the imaginary quadratic number field $\mathbf{Q}(\sqrt{-d})$, or extended Bianchi

group (see [35], [33]). For $d = 1, 2, 3, 7$, and 11 , there is only one vertex v and $\mathcal{A}'(v) = \bar{D}$. If $d = 19$, then there are two vertices. For these values of d , the spherical groups $G_v = \text{Stab}(v, G)$ and the orbits $G_v\infty$, which are the sets of the vertices of the v -cell $N(v)$, are given in [14]. For $d = 1, 2, 3$, and 7 , the projections of $N(v)$ into $V = \mathbf{C}$, which are the Farey polygons, can be found in [21], Figures 1–4, and for $d = 11$, in [24]. Note that in [21] and [24] a different definition of the Farey polygons is given.

Denote by k_F the smallest k such that any geodesic of height $k/2$ in H^{n+1} cuts a vertical face of some v -cell $N(hv)$, $v \in \bar{D}$, $h \in G_\infty$.

Lemma 8. *If a geodesic L in H^{n+1} cuts a horosphere $Q(g, k_F)$ then $g(\infty)$ is a vertex of a face of some v -cell $N(hv)$, $h \in G_\infty$, which is cut by L .*

Proof. Assume that L cuts a horosphere $Q(g, k_F)$. Denote $L' = g^{-1}(L)$. Then $ht(L') > k_F/2$ and, by definition, L' cuts a vertical face B of some v -cell $N(hv)$, $h \in G_\infty$. Thus, $L = g(L')$ cuts the face $g(B)$ having $g(\infty)$ as one of its vertices. \square

Let $\alpha \in V - \mathcal{P}$. Let $L = (\infty, \alpha)$. Applying Lemma 8 to the geodesic L we obtain the following statement which contains the basic property of Farey polytopes (cf. [21], Theorem 3).

Theorem 9. *Let $\alpha \in V - \mathcal{P}$. If the inequality (4) holds with $k = k_F$, then α belongs to a Farey polytope having $g(\infty)$ as one of its vertices.*

Let B be a vertical face of a v -cell $N(v)$ and let a Farey polytope F be the projection of $h(B)$, $h \in G$, into V . If $\alpha \in F$, then (4) holds with $k = h_B$ for some vertex $g(\infty)$ of F .

Proof. Let the face B be the same as in the proof of Lemma 8. The Farey polytope which is the projection of $g(B)$ into V contains α .

The second statement follows from Theorem 5.5. \square

Remark. If $\alpha \in V - \mathcal{P}$, then the vertical geodesic $L = (\infty, \alpha)$ in H^{n+1} passes through infinitely many v -cells $N(gv)$. Hence there is a sequence of Farey polytopes F_i which contain α and such that $F_{i+1} \subset F_i$, $i = 1, 2, \dots$, and $\lim_{i \rightarrow \infty} F_i = \alpha$. Since, by Theorem 9, at least one of the vertices of every F_i satisfies the inequality (4) with $k = k_G = \inf h_B$, the infimum being taken over the set of vertical faces B of all v -cells $N(v)$, $v \in \bar{D}$, this inequality has infinitely many solutions with $k = k_G$. Thus, Theorem 9 implies Theorem 1.

In applications, it is easier to use another definition of k_F which is equivalent to the one given above: k_F is the largest value of k such that there is a geodesic of height $k/2$ in H^{n+1} which does not cut a vertical face of any v -cell $N(hv)$, $v \in \bar{D}$, $h \in G_\infty$.

Example 10. Let $n = 1$ and G be the Hecke group G_q . In that case, the geodesic with endpoints at 0 and $2 \cos(\pi/q)$ is the highest geodesic which does not cut the vertical faces of $N(v)$ (see Figures 1, 2, and 3). Thus, $k_F = 2 \cos(\pi/q)$.

Example 11. Let $n = 2$ and G be a Bianchi group $PGL_2(O_d)$. First suppose that the fundamental domain D of G has only one vertex $v \neq \infty$ (see Example 4). If $d = 3, 7$, or 11 , the projections of the vertical faces of the v -cell $N(v)$ form the triangle with vertices at $0, 1$, and ω , where $\omega = (1 + \sqrt{-d})/2$. Figure 1 shows the vertical face of $N(v)$ over the side $[0, 1]$ of the triangle. (Note that, for $d = 1$,

the vertical cross section of $N(v)$ over $(0, 1 + i)$ is shown in Figure 2.) It is easily seen that k_F is the diameter of the circumscribed circle of the triangle. Hence, $k_F = 2/\sqrt{3}, 4/\sqrt{7}, 6/\sqrt{11}$ for $d = 3, 7, 11$ respectively. If $d = 1$ or 2 , then the projections of the vertical faces of $N(v)$ form the rectangle with vertices at $0, 1, \sqrt{-d}$ and $1 + \sqrt{-d}$. Again, k_F is the diameter of the circumscribed circle of the rectangle. Thus, $k_F = \sqrt{2}, \sqrt{3}$ for $d = 1, 2$ respectively (cf. [21], Theorem 3). If $d = 19$, then D has two vertices v and v' (see [14] or [28]). The projections of vertical boundaries of $N(v)$ and $N(v')$ form the triangle with vertices at $\omega, \omega/2$, and $(1 + \omega)/2$ and trapezoid with vertices at $0, 1, \omega/2$, and $(1 + \omega)/2$. Thus, k_F is the diameter of the circumscribed circle for the trapezoid. Therefore, $k_F = \sqrt{35/19}$.

Example 12. Let G be the discrete subgroup $SV(\mathbf{Z}^N)$, $N = 2^{n-1}$, of the Vahlen's group of Clifford matrices which is considered in [36] (see also [18]). If $n = 1, 2, 3$, or 4 , D has only one vertex v , the projections of the vertical boundaries of $N(v)$ are the boundary of the unit cube, and k_F equals the length of the diagonal of the cube. Thus, $k_F = \sqrt{n}$ for $n \leq 4$. For $n = 1$ and 2 , these values are found in Examples 10 and 11.

3. CONTINUED FRACTIONS

In this section, we introduce Algorithm I which can be used to generate the best possible approximations of elements of the n -dimensional euclidean space V by the cusps $g(\infty)$, $g \in G$. This algorithm is similar to the standard continued fraction algorithm and the algorithms used by A. L. Schmidt in [23], [24], and [25]. The properties of Algorithm I are studied and it is applied to develop the reduction theory for geodesics in H^{n+1} which is similar to the reduction theory of binary indefinite quadratic forms.

Let $u = g(\infty)$ be a cusp of the fundamental domain $g(D)$. The set $K(u)$ (see (2)) is closely related to the horoballs $\mathcal{R}(g, k)$. Evidently, if a horoball \mathcal{R}_k , $t = k/2$, belongs to $K(\infty)$, then $\mathcal{R}(g, k) \subset K(g\infty)$ for any $g \in G$. And if ∞ is the only cusp of D , then there is $k' > 0$ such that $K(g\infty) \subset \mathcal{R}(g, k')$ for any $g \in G$. It is clear that $\bigcup K(u) = H^{n+1}$, $u \in G\infty$, and that $\dim(K(u) \cap K(u')) \leq n$ if $u \neq u'$.

Lemma 13. *Assume that a geodesic L passes through the intersection of $K(u')$, $u' = g'(\infty)$ and a horoball $\mathcal{R}(g, k)$. Then L cuts $Q(g', k)$.*

Proof. Let $u' \neq u = g(\infty)$. Let $z \in K(u') \cap \mathcal{R}(g, k)$. Suppose that the geodesic interval M with endpoints at u and z passes through K_1, K_2, \dots, K_m , where $K_1 = K(u)$ and $K_m = K(u')$, in the indicated order. Denote by S_1 the sphere which contains the common boundary of K_1 and $K_2 = K(g_2\infty)$. Then $\mathcal{R}(g_2, k)$ is the image of $\mathcal{R}(g, k)$ under reflection with respect to S and it contains the part of $\mathcal{R}(g, k)$ which is outside the sphere S . Hence, $\mathcal{R}(g_2, k)$ contains the part of M with the endpoints at z and $z_1 = M \cap S$. This argument can be continued to show that $z \in \mathcal{R}(g', k)$. □

Let $\alpha \in V - \mathcal{P}$ and let $L = (\infty, \alpha)$. Suppose that the geodesic L passes through the sets $K(\infty), K(u_1), \dots, K(u_i), \dots, u_i = g_i(\infty)$, $g_i \in G$, in the indicated order.

Corollary 14. *Let $\alpha \in V - \mathcal{P}$ and let L be the vertical ray through α in H^{n+1} . Then*

$$(5) \quad k(\alpha) = 2 \limsup_{i \rightarrow \infty} ht(g_i^{-1}L) = \limsup_{i \rightarrow \infty} |g_i^{-1}(\infty) - g_i^{-1}(\alpha)|$$

where $g_i \in G$ are defined as above.

Denote

$$(6) \quad \lambda_i = L \cap K(u_i), \quad u_i = g_i(\infty).$$

The (continued fraction) Algorithm I can be used to find the sequence $\{g_i\} \subset G$ mentioned above explicitly. The corresponding shift operator is defined on the sequence

$$(7) \quad \lambda_1, \lambda_2, \dots, \lambda_i, \dots$$

Since for any orbit Gz , $z \in H^{n+1}$, a point of the largest height in the orbit belongs to the fundamental domain D , we can confine ourself to the geodesics which pass through D . Recall that the *floor* of D consists of the non-vertical faces of D .

We now introduce the natural *orientation* of a geodesic $L' = g(L)$ (from $g(\infty)$ to $g(\alpha)$). The partition of L' into arks λ'_i is defined by (6). It is clear that this partition is invariant under the action of G , that is, $\lambda'_i = g(\lambda_i)$ for all i .

We shall say that a geodesic L' is *reduced* if it passes through D and the initial point of $\lambda' = L' \cap K(\infty)$ lies in the floor of D .

Suppose that the floor of D consists of faces ϕ_1, \dots, ϕ_r . Let transformation $R_j \in G$ be such that

$$\phi_j = \bar{D} \cap R_j^{-1}(\bar{D}), \quad j = 1, \dots, r.$$

Algorithm I.

Step 0. Suppose that $\alpha \in U_0(P)$, $U_0 \in G_\infty$. Denote $L'_0 = U_0^{-1}(L)$. Clearly, L'_0 cuts the floor of D and it is not reduced.

Step 1. Let $t_1 \in \phi_j$ be the point of intersection of L'_0 with the floor of D . Denote $S_0 = R_j$, $g_0 = T_0 = U_0 S_0$, and $L_1 = S_0^{-1} L'_0$. (If L passes through the boundary of two or more faces in the floor of D , S_0 is not unique.) Then $L = g_0 L_1$ where L_1 is reduced.

Assume that the elements T_1, \dots, T_{i-1} in G are determined. Let $g_k = g_{k-1} T_k$ and $L_k = T_k L_{k+1}$, $k = 1, \dots, i-1$. Then $L = T_0 \dots T_{i-1} L_i$.

Step $i + 1$. Let $\lambda_i = (t_i, t_{i+1}) = L_i \cap K(\infty)$. Let $L'_i = U_i^{-1}(L_i)$ where $U_i \in G_\infty$ is determined so that $U_i^{-1} t_{i+1}$ lies in a face ϕ_k of the floor of D . Denote $S_i = R_k$, $T_i = U_i S_i$, and $L_{i+1} = S_i^{-1} L'_i$ where a unique $S_i \in G$ is such that $S_i A'_1 \in \bar{D}$. (If L_i passes through the boundary of two or more faces of D , S_i is not unique.) Then

$$g_i := g_{i-1} T_i$$

and

$$L_i = T_i L_{i+1}, \quad L = T_0 \dots T_i L_{i+1}.$$

It is clear that Algorithm I enumerates $g_i \in G$ in the same order as L passes through the sets $K(g_i \infty)$, and that there is a 1-1 correspondence between the arcs λ_i of L and $T_i \in G$ as defined by Algorithm I. The cusps $g_i(\infty)$ as defined by Algorithm I will be called the *convergents* of α .

All the geodesics L_i , as well as L'_i with orientation changed for the opposite, generated by Algorithm I are reduced. Thus, there are two reduced geodesics in the G -orbit of L whose arc $\lambda_i = (t_i, t_{i+1})$ lies in $K(\infty)$: one with t_i and the second with t_{i+1} in the floor of D . (If $\lambda_i \in D$, then these two geodesics are different only in orientation.) Since the partition of L is invariant with respect to G , any reduced geodesic in the G -orbit of L belongs to one of these two sequences generated

by Algorithm I. When applying Algorithm I, to reduce the number of different transformations R_i used, we choose D so that the number of faces in the floor of D is as small as possible.

Example 15. Suppose that there is an isometric fundamental domain D for G such that the floor of D lies in the unit sphere $|z|^2 + t^2 = 1$. Such a fundamental domain exists, for example, when $n = 1$ and G is a Hecke group, or $n = 2$ and G is a Bianchi group $PGL_2(\mathcal{O}_d)$, $d = 1, 2, 3, 7, 11$, or G is the discrete subgroup $SV(\mathbf{Z}^N)$, $N = 2^{n-1}$, $n = 1, 2, 3$, or 4 , of the Vahlen group (see Example 12). In all these cases, we choose P_∞ to be the Dirichlet polygon for G_∞ in V with center at the origin.

In that case we can choose $S_i = S$ to be the reflection with respect to this unit sphere. Thus, $Sx = (\bar{x})^{-1}$ for $x \in H^{n+1}$. Here $\bar{}$ is the conjugation in the Clifford algebra so that $x\bar{x} = |x|^2$. Since $U_i = U(a_i) = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}$ where a_i belongs to the lattice $\Lambda \in V$, we have

$$T_i = T(a_i) = \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} \theta,$$

where θ is the conjugation in the Clifford algebra mentioned above. Let

$$g_i = T_0 \dots T_i = \begin{pmatrix} p_i & p_{i-1} \\ q_i & q_{i-1} \end{pmatrix}.$$

By definition, the endpoints of the geodesic L_i are $g_i^{-1}(\infty) = -q_{i-1}q_i^{-1}$ and $\alpha_{i+1} = g_i^{-1}(\alpha)$. Since L_i is reduced,

$$|q_{i-1}| < |q_i| \text{ and } |\alpha_{i+1}| > 1.$$

It follows that, for any i , the inequality

$$(8) \quad |\alpha - p_i q_i^{-1}| < \frac{1}{k|q_i|^2}$$

holds with $k = 2 \sin \frac{\pi}{q}$ if $G = G_q$, the Hecke group, and with $k = \sqrt{2}$ when G is the Picard group $PGL_2(O_1)$. In particular, if G is the modular group G_3 , (8) holds with $k = \sqrt{3}$ for any i .

Suppose that a geodesic L' passes through a v -cell $N(v)$. Let $u = g(\infty)$ be a vertex of $N(v)$. If the cusp u does not belong to a face of $N(v)$ which is cut by L' , then, by Theorem 9, the geodesic $g^{-1}(L')$ does not cut a vertical face of any $N(hv)$, $h \in G_\infty$. Hence $2ht(g^{-1}L') < k_F < k_G$ and geodesic $g^{-1}(L')$ is not extremal (see § 4 for the definition of an extremal geodesic).

Assume that a geodesic L'_i enters a v -cell $N(v)$ through a vertical face B and exits through face B' . Let A_0, A_1, \dots, A_r ($A_0 \in \bar{D}$) be all the v -components in $N(v)$ which are passed through by L'_i in the indicated order. Let convergent $u_j = g_{i+j}(\infty)$ be the cusp of A_j , $j = 0, \dots, r$. We say that u_j is a *convergent of type I* if it belongs neither to the face B nor to B' . Otherwise, u_j is a *convergent of type II*. Thus, any convergent of type I does not satisfy the inequality (8) with $k = k_F$. Similarly, we shall say that a geodesic L_i which passes through $N(v)$ is of type I if it does not cut a vertical face of $N(v)$. Otherwise, L_i is of type II.

Since the union of all the v -components in $N(v)$ whose cusps belong to B is a convex set, the set of convergents u_0, \dots, u_r can be divided into two subsets: the set of convergents of type I u_k, \dots, u_m , $0 < k \leq m < r$ (which can be empty), and

the set of convergents of type II. Any convergent of type II satisfies the inequality (8) with $k = k_F$. Moreover, the convergents u_0 and u_r satisfy (8) with $k = k_G$ since L'_i and L'_{i+r} cut the vertical faces of A_0 and $g_{i+r}^{-1}(A_r)$ respectively. We have the following.

Theorem 16. *Let $\alpha \in V - \mathcal{P}$. Then $k(\alpha)$ can be found from (5) where the sequence of $g_i \in G$ is generated by Algorithm I and $g_i(\infty)$ runs through the subsequence of convergents of type II.*

Remark. Assume that α lies in the projection of a vertical face B of a v -cell $N(v)$ into V . Then any convergent $p_i q_i^{-1}$ for α is of type II and it satisfies (8) with $k = h_B \geq k_G$ since an arc of L_i lies in a vertical face of a v -component in D . For example, let $n = 2$ and G be a Bianchi group B_d . If the projection of B into $V = \mathbf{C}$ is the interval $[0, 1]$ on the real axis (see Figure 1), then $h_B = \sqrt{3}$. Thus, if α is a real number, then any convergent of α is real and satisfies the inequality (8) with $k = \sqrt{3}$ (cf. Example 15). If the projection of B into V is the interval $[0, \omega]$, then $h_B = \sqrt{3}, \sqrt{2}, 1$, for $d = 3, 7, 11$ respectively (see Figures 2 and 3). Thus, the convergents of $\alpha \in \omega\mathbf{R}$ satisfy (8) with $k = h_B$. These approximation constants coincide with the constants $k = 2 \sin \frac{\pi}{q}$ obtained for the Hecke groups G_q for $q = 3, 4, 6$ in Example 15. It happens because, in these cases, $\text{Stab}(P_B, G)$ is isomorphic to the corresponding Hecke group. Here P_B is the vertical plane in H^3 spanned by B .

For a v -cell $N(v)$, denote by $m(v)$ the largest number of v -components in $N(v)$ passed through by a geodesic, and let $m(G) = \max m(v)$, the maximum being taken over all vertices $v \in \bar{D}$. (If D has a cusp in V , then $m(G) = \infty$.) Since L passes through infinitely many v -cells, the following result, which is an analog of Vahlen's [29] (and/or Borel's [6]) theorem on regular continued fractions, provides an alternative proof of Theorem 1.

Theorem 17. *Let $u_i, \dots, u_{i+m(G)-1}$ be a set of consecutive convergents for $\alpha \in V - \mathcal{P}$ where $m(G)$ is defined as above. Then at least one of them satisfies the inequality (8) with $k = k_G$.*

Example 18. Let $n = 1$ and let G be a Hecke group G_q . Then $k_G = 2$, $m(G_q) = [(q+1)/2] + 1$, and no more than four convergents u_0, u_1, u_{r-1}, u_r out of $r+1$ convergents u_0, \dots, u_r mentioned above can be of type II. By Theorem 17, for any $\alpha \in V - \mathcal{P}$ at least one of $[(q+1)]$ consecutive convergents satisfies (8) with $k = 2$. When $q = 3$ or 4 , $[(q+1)] = 2$. For $q = 3$ it is an analog of Vahlen's theorem [29]; for $q = 4$, of Borel's theorem [6] (cf. [36], Example 1).

Remark. Let r_i be the radius of the isometric sphere of $g_i \in G$. Since $|g_i^{-1}(u) - g_i^{-1}(u')| = O(r_i^{-2})$ for any $u, u' \in V$, one can expect that the geodesics $L = (u, \alpha)$ and $L' = (u', \alpha)$ will eventually have the same tails. It is easily seen that if α lies on the hemisphere which contains a common boundary H of $K(\infty)$ and $K(S_i(\infty))$, u lies inside and u' outside of H , then L and L' pass through different sequences of v -components. (But if $S_i^2 = id$, then $S_i L$ and L' pass through the same sequences. Hence they have the same tails.)

Now let $\eta, \theta \in V - \mathcal{P}$ and $L = (\eta, \theta)$. The Algorithm I can be defined as above but the sequences of reduced geodesics L_i as well as the sequences of T_i and g_i in G are infinite in both directions as $i \rightarrow \infty$ and as $i \rightarrow -\infty$. Now we have

$$L_{-k} = T_{-k} \cdots T_0 \cdots T_i L_{i+1}$$

for any non-negative integers k and i . If the orientation is changed for the opposite, then geodesics L_i are not reduced any more but the geodesics L'_i , defined by Algorithm I, are.

Example 19. Let $n = 1$ and let G be the Hecke group G_4 . Let $L = (-1, 1)$. We apply Algorithm I to $L' = U(\sqrt{2})L = (\sqrt{2} - 1, \sqrt{2} + 1)$. If, as in Example 8, $S_i = S$ is the reflection with respect to the unit circle with center at the origin, the sequence of reduced geodesics generated by Algorithm I is periodic, and the period consists of two reduced geodesics $L' = T(2\sqrt{2})T(-2\sqrt{2})L'$ and $L'' = T(-2\sqrt{2})L'$.

If, as in the Rosen algorithm [20], we choose $S_i = W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then the period consists of only one geodesic $L' = TL'$ where $T = \begin{pmatrix} -2\sqrt{2} & -1 \\ 1 & 0 \end{pmatrix}$. Thus, there are exactly two reduced geodesics in the G -orbit of L .

Let $n = 1$ and $G = GL_2(\mathbf{Z})$. Let $L = ((1 - \sqrt{5})/2, (1 + \sqrt{5})/2)$. Choosing S as in Example 15, we have $L = T(1)L$. There are two reduced geodesics L and $S(L)$. If we choose $S_i = W$, then $L = T_0T_1L$ and $L' = T_1L$, where $T_0 = U(1)W$, $T_1 = U(-1)W$, are reduced.

For $L = (2 - \sqrt{3}, 2 + \sqrt{3})$, there are only two reduced geodesics in the G -orbit of L : $L = T(4)T(-4)L$ and $L' = T(-4)L$.

As above, we divide the set of reduced geodesics into subsets of type I and II. The following statements are analogous to the results obtained above.

Theorem 20. *Let $L = (\eta, \theta)$ where $\eta, \theta \in V - \mathcal{P}$. If L_i is a reduced geodesic of type II, then $2ht(L_i) \geq k_F$.*

Moreover, if L_i cuts a vertical face of a v -component in D , then $2ht(L_i) \geq k_G$.

Theorem 21. *Let $L = (\eta, \theta)$ where $\eta, \theta \in V - \mathcal{P}$. Then*

$$k(L) = 2 \sup ht(L_i),$$

the supremum being taken over all geodesics L_i of type II.

Let $L = (\eta, \theta)$ be the axis of a loxodromic element $h \in G$. Let L^o be a fundamental domain of the cyclic group generated by h on L chosen so that it consists of whole arcs $\lambda_1, \dots, \lambda_p$. Note that $\lambda_{i+p} = h(\lambda_i)$ and $L_{i+p} = L_i$ for all i . Thus the sequence T_i , as generated by Algorithm I, is also periodic, $T_{i+p} = T_i$ for all i , and $h = T_1 \cdots T_p$. Note that if $\text{Stab}(L, G) = \langle S, S' | S^2 = S'^2 = id \rangle$, then, with the proper choice of λ_1 , there is an additional symmetry $\lambda_{p-i} = S(\lambda_i)$, $i = 1, \dots, p-1$. We have the following.

Theorem 22. *The sequence of arcs (7) of a geodesic L is periodic if and only if L is the axis of a loxodromic element. (If $\lambda_{i+p} = \lambda_i$ and $h = T_1 \cdots T_p$, then $h(L) = L$.)*

Let $L = (\eta, \theta)$ be the axis of a loxodromic element $h \in G$. Suppose that $L = L_0$ is reduced. There are only finitely many reduced geodesics $L_1, \dots, L_p = L_0$ in the G -orbit of L and Algorithm I can be used to find all of them. Also,

$$k(\eta) = k(\theta) = k(L) = 2 \sup ht(L_i), \quad 1 \leq i \leq p,$$

where $L = T_0 \cdots T_i L_{i+1}$ and the sequence T_i is generated by Algorithm I.

In particular, if the fundamental domain of $\text{Stab}(L, G)$ on L belongs to $K(\infty)$ (in which case $p = 1$ or 2), then $k(L) = 2ht(L)$.

Remark. Let \mathbf{B} be the ball model of the $(n + 1)$ -dimensional hyperbolic space. Let G be a geometrically finite discrete subgroup of the group of orientation-preserving isometries of \mathbf{B} . For some fixed $w \in \mathbf{B}$, let $D(w)$ denote the Dirichlet polytope with center at w and let $K(w) = G_w D(w)$. Upon replacing the region $K(\infty)$ by $K(w)$ in the definition of Algorithm I we can define a similar algorithm for the ball model.

4. EXTREMAL GEODESICS

In the rest of this paper, the group G , which is originally defined to be a subgroup of orientation-preserving isometries of H^{n+1} , will be sometimes extended by reflections, which, in this paper, are the isometries of order two. The set of all points of H^{n+1} fixed by a reflection R is its *axis* a_R and we usually say that R is the reflection with respect to a_R . The axis a_R of a reflection R is a hyperbolic subspace of H^{n+1} . If $\text{codim}(a_R)$ is odd, then R is an orientation-reversing isometry of H^{n+1} ; otherwise R preserves orientation. When $n = 2$, an element $R \in B_d \subset G$ such that $\text{tr}(R) = 0$ will be simply called a reflection. In that case, the axis a_R of R is a geodesic in H^3 and R reverses the orientation in any hemisphere in H^3 through a_R .

A geodesic L in H^{n+1} with the endpoints at $\eta, \eta' \in V$ is said to be *extremal* if

$$k(L) = 2 \sup_{g \in G} \text{ht}(gL) = |\eta - \eta'|, \quad g \in G.$$

Algorithm I can be used to find an extremal geodesic in the G -orbit of L since the set of reduced geodesics for L contains an extremal geodesic if it exists. By Theorem 22, if L is the axis of a loxodromic element in G , then the number of reduced geodesics is finite and an extremal geodesic in the G -orbit exists. Let L be extremal. Then $h_L = 2\text{ht}(L) = k(L) \in \mathcal{M}(G)$. Since the Hurwitz constant $C(G) \geq 1/k(L)$ for any geodesic L , we have obtained a lower bound for $C(G)$. Thus, by Theorem 1,

$$1/k(L) \leq C(G) \leq 1/k_G,$$

and if a critical edge of D lies on the extremal geodesic L , then $k_G = k(L)$ and $C(G) = 1/k_G$ in Theorem 1.

Here simple criteria of extremality of a geodesic (see Corollary 24 and Lemma 25) are applied to some Bianchi groups. For each extremal geodesic found, Algorithm I is used to find the set of reduced geodesics. In conclusion, it is shown that if the endpoints v and v' of a critical edge σ of D are cusps and there is a reflection $R \in G$ such that $v' = R(v)$, then the Hurwitz constant $C(G) = k(\sigma) = |v' - v|$ and it is an accumulation point in the Lagrange and Markov spectra (Corollary 30).

It is clear that an extremal geodesic L does not cut any horosphere $Q(g, h_L)$. (Otherwise, $\text{ht}(g^{-1}L) > \text{ht}(L)$.) As above, let L° be the fundamental domain of $\text{Stab}(L, G)$ on L . Assume that L° passes through the sets $K(u_i)$, $u_i = g_i(\infty)$, $i = 1, \dots, r$. We have the following.

Lemma 23. *The following statements are equivalent:*

1. A geodesic L is extremal.
2. L° belongs to the closure of $\mathcal{N}(h_L)$.
3. L° cuts none of the horospheres $Q(g_i, h_L)$, $1, \dots, r$.

In particular, we have the following.

Corollary 24. *If $L^\circ \subset K(\infty)$ then L is extremal.*

In the examples below, besides Corollary 24, the following simple criterion of extremality of a geodesic will be used.

Lemma 25. *Let B be a vertical face of a v -cell $N(v)$ and let $s \in B$ be the foot of the perpendicular from v into B . Assume that there are reflections $R \in G_s$ and $W \in G_v$ such that the axis a_R of R lies in B and a_R and the axis of W do not meet.*

If the axis L of the loxodromic element $\Phi = RW \in G$ cuts the vertical face of $A(v)$ which lies in B , then L is an extremal geodesic.

Proof. Since the fundamental domain of $\text{Stab}(L, G)$ on L lies in $N(v)$, L is extremal if and only if it cuts none of the horospheres $Q(g_i, h_L)$, $g_i(\infty)$ being a vertex of $N(v)$. Since $W \in \text{Stab}(L, G)$, we can confine ourself to those horospheres whose bases belong to B . (For others, $2ht(g_i^{-1}L) < k_F$.) Assume that $g_i(\infty) \in B$. Since $Q = Q(g_i, k)$ and L both are orthogonal to a_R , L cuts Q if and only if the point $b = L \cap a_R$ is inside Q . If b belongs to the vertical face of $A(v)$, then $h_L \geq h_B$. By the definition of $N(v, h_L)$, since $b \in A(v, h_L) \subset N(v, h_L)$, none of the horoballs $\mathcal{R}(g_i, h_L)$ contains b . Thus, L is extremal. \square

Theorem 5.5 implies the following.

Lemma 26. *Suppose that a geodesic L meets a face gB , $g \in G$, where B is a vertical face of $N(v)$. If $ht(L) < h_B/2$, then L is not extremal.*

Let $n = 2$ and G be a Bianchi group. For the vertical faces B of v -cells shown in Figures 1–3 we have $h_B = \sqrt{3}, \sqrt{2}, 1$ respectively. By Lemma 26, if geodesic L cuts a triangular face B of a v -cell, then $k(L) \leq \sqrt{3}$. Thus, if $k(L) > \sqrt{3}$, L does not cut a hyperbolic Farey triangle. When $d = 2$ or 7 there is up to G -equivalence only one extremal geodesic which does not cut the triangles whereas when $d = 11$ there are infinitely many such geodesics (see [22], [24], [32]).

Suppose that σ , a critical edge of D with the endpoints $v, v' \in H^{n+1}$ and summit s , lies on a geodesic L . The groups G_v and $G_{v'}$ are spherical. If each of them contains reflections which fix L , then σ contains a fundamental domain of $\text{Stab}(L, G)$ on L . Hence, by Corollary 24, L is extremal, and the Hurwitz constant $C(G) = 1/k_G$. It is known (see [10], p. 226) that the group of symmetries of a regular polytope contains a reflection with respect to its center unless it is a dihedral group of order $2q$, q odd, or the tetrahedral group.

On the other hand, for $n = 2$, if G_v is a dihedral group of order $2q$, q odd, and L is an axis of symmetry of order two or the tetrahedral group, then L cuts a horospherical face of $N(v, k_G)$. Hence, L is not extremal and $C(G) < 1/k_G$. It explains why, when G is a Bianchi group $PGL_2(O_d)$, the equality $C(G) = 1/k_G$ holds for $d = 1$ or 2 but does not hold when $d \equiv 3 \pmod{4}$ (see [14], [7], [35]). The case when G is a Hecke group G_q is considered in [34], [36]. In that case, G_v is a dihedral group of order $2q$ and the equality $C(G) = 1/k_G$ holds if and only if q is even.

When applying the results obtained to Diophantine approximation in V we can use the well known results on the structure of the discrete spherical groups (see e.g. [4]).

When $n = 1$ and G_v is a finite or infinite dihedral group, all extremal geodesics L for which $L \cap N(v)$ contains a fundamental domain of $\text{Stab}(L, G)$ on L are enumerated in [34]. In the Examples 27, 28, and 31 below we apply Corollary 24 and

Lemma 25 to find such extremal geodesics for some Bianchi groups $PGL_2(O_d)$ with the generator $x \rightarrow \bar{x}$, $x \in V = \mathbf{C}$, adjoined.

Denote by \mathcal{H}_n the dihedral group, and by A_4 and S_4 the tetrahedron and cube symmetry groups respectively. The isometric fundamental domains for these groups for square-free $d \leq 19$ are described in [5], [28]. In [7], for $d = 1, 2, 3, 6$ and 7 , the vertex and edge subgroups of G are given as well as the presentation of G as a graph amalgamation product. In [14], for $d = 1, 2, 3, 7, 11$ and 19 , the groups G_v are described and the orbits $G_v(\infty)$ are found. We shall use these results in the examples below. In Examples 27, 28 and §5, the following notation will be used: J, J'' are the reflections in the vertical planes in H^3 through the imaginary and real axes in $V = \mathbf{C}$ respectively; S is the reflection in the unit sphere in H^3 with center at the origin in \mathbf{C} ; $R(b)$, $b \in O_d$, is the reflection in the vertical line through $b/2$; $W = SJ$. We choose D so that $J(D) = D$ and enumerate the extremal geodesics up to this symmetry.

Example 27. When $d = 3, 7$ or 11 , the fundamental domain D has only one vertex $v \neq \infty$. The stabilizer of v is A_4 for $d = 3$ and 11 , and for $d = 7$, $G_v = \mathcal{H}_3$ (see [5], [28], [14], [7]). We choose D so that v lies above the imaginary axis. We have

$$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, R(b) = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix},$$

$$R' = \begin{pmatrix} -1 & 0 \\ -\bar{\omega} & 1 \end{pmatrix}, R'' = \begin{pmatrix} a & \omega(1-a) \\ \bar{\omega} & -a \end{pmatrix},$$

where $R' = SRS$, $R'' = RR'R$, $a = |\omega|_{-1}^2$, and we abbreviate $R = R(\omega)$. Let $R_1 = JRJ$, $R'_1 = WRW$, $R''_1 = R_1R'_1R_1$. Then $R, R', R'' \in G_s$ and $R_1, R'_1, R''_1 \in G_{s'}$. Here, points s and $s' = Js$ in H^3 are the summits of the critical edges of D . They lie above $\omega/2$ and $-\bar{\omega}/2$ respectively. Let

$$\Phi_1 = \begin{pmatrix} 0 & 1 \\ 1 & -\omega \end{pmatrix}, \Phi_2 = \begin{pmatrix} \bar{\omega} & -2 \\ -2 & \omega \end{pmatrix},$$

$$\Phi_3 = \begin{pmatrix} 0 & 1 \\ 1 & -2\omega \end{pmatrix}, \Phi_4 = \begin{pmatrix} 1 & -\omega \\ -\omega & \omega^2 - 1 \end{pmatrix}.$$

Since the axis of the reflection W does not meet the axes of R and R_1 , the elements $\Phi_1 = WR$ and $\Phi'_1 = WR' = R_1W$ are loxodromic, and their axes L_1 and L'_1 are extremal by Corollary 24. When $d = 3$, the axes of W, R'' , and R''_1 meet. For $d = 7$, $R' = R''$ and $R'_1 = R''_1$. When $d = 11$, the axes of W and R'' do not meet and the axis L_2 of $\Phi_2 = WR''$ is extremal by Corollary 24. The endpoints $\eta, \eta' \in V$ of L_1 are the roots of $f_1(x) = x^2 - \omega x - 1 = 0$ and $k(L_1) = k(\eta) = |\eta - \eta'| = |\omega^2 + 4|$. Thus, $k(L_1) = k(L'_1) = 13^{1/4}, 8^{1/4}$, and $5^{1/4}$ for $d = 3, 7$ and 11 respectively. Similarly, $k(L_2) = \sqrt{5}/2$. Notice that $1/k(L_1)$ is the Hurwitz constant for $d = 3$ and 7 , and $1/k(L_2)$ is when $d = 11$ (see e.g. [35]).

Denote $u = (\omega, 1) \in H^3$. The reflections $R_2 = R(2\omega)$ and $R'_2 = RR(0)WR$ belong to G_u . The axis of W does not meet the axes of R_2 and R'_2 . The axes L_3, L_4 of the loxodromic elements $\Phi_3 = WR_2$ and $\Phi_4 = WR'_2$ respectively are extremal by Lemma 25.

We have $k(L_3) = 2|\omega^2 + 1|^{1/2}$ and $k(L_4) = |\omega^4 + 4|^{1/2}/|\omega|$. When $d = 3$, $1/k(L_4) = 13^{-1/4}$ and $1/k(L_3) = 1/2$ are the first and second minima in the Lagrange

spectrum. For $d = 7$, $1/k(L_4) = 3^{-1/2}$ is the second minimum in the Lagrange spectrum (see e.g. [21]).

In all the cases above, when applying Algorithm I, we can choose D as in Example 15 so that its floor lies in the unit sphere with center at the origin and $S_i = W$. The sequence of reduced geodesics obtained is periodic. The length of the period is two and the number of reduced geodesics is two if the reflection W fixes a reduced geodesic and it is equal to four if W does not.

Let $L = L_1$. Then there are exactly two reduced geodesics in the G -orbit of L : $L = T_0T_1L$ and $L' = T_1L$ where $T_0 = U(\omega)W$, $T_1 = U(-\omega)W$.

Let $d = 11$ and $L = L_2$. Since an arc of L lies in the floor of D , L is not reduced. But $L' = R(L)$ is. Let $T_0 = U(\omega)W$, $T_1 = U(\bar{\omega})W$. There are four reduced geodesics: $L' = T_0T_1L'$, $L'' = T_1L'$, and $W(L')$, $W(L'')$ with their orientation changed for the opposite.

The geodesic $L = L_3$ is reduced. Let $T_0 = U(2\omega)W$, $T_1 = U(-2\omega)W$. There are exactly two reduced geodesics $L = T_0T_1L$ and $L' = T_1L$.

Let $d = 7$ and $L = L_4$. L is reduced. Denote $T_0 = U(\sqrt{-7})W$, $T_1 = U(-\sqrt{-7})W$. There are exactly two reduced geodesics $L = T_0T_1L$ and $L' = T_1L$.

Example 28. Let G be the extended Bianchi group B_{15} or $B_{19} = PGL_2(O_{19})$ with the generator $x \rightarrow \bar{x}$, $x \in V = \mathbf{C}$, adjoined (see [33] or [35]). The fundamental domain D is bounded by the unit spheres in H^3 with centers at 0 , ω , $-\bar{\omega} \in V$, by the spheres of radius r with centers at $\omega/2$ and $-\bar{\omega}/2$ where $r = 1/\sqrt{2}$ if $d = 15$ and $r = 1/2$ if $d = 19$, and by vertical planes through the sides of the triangle with vertices at 0 , ω , and $-\bar{\omega}$. D is divided into two v -components by the vertical plane through the points $\omega/2$ and $-\bar{\omega}/2$ in V .

First let $d = 19$. There are two vertices $v = (4/\sqrt{-19}, \sqrt{3/19})$, $w = (6/\sqrt{-19}, \sqrt{2/19})$ and $\text{Stab}(v, G) = \mathcal{H}_3$ and $\text{Stab}(w, G) = A_4$ (see e.g. [14]). Let

$$W' = \begin{pmatrix} 2 & -\omega \\ \bar{\omega} & -2 \end{pmatrix}, \Phi_2 = \begin{pmatrix} -2 & \omega \\ -\bar{\omega} & 3 \end{pmatrix}, \Phi_3 = \begin{pmatrix} 2 & \omega \\ \bar{\omega} & 2 \end{pmatrix}, \Phi_5 = \begin{pmatrix} -3 & \omega + 2 \\ \omega & 2 - \omega \end{pmatrix},$$

$W'' = RW'R$, $W'_1 = JW'J$ and $W''_1 = JW''J$. Let the reflections W , R , R_1 , R_2 be defined as in Example 27. Then W , W' , $W'_1 \in G_v$ and W'' , $W''_1 \in G_w$. The axes of reflections R and $R(0)$ pass through the vertical face of $\mathcal{A}(v)$ which is orthogonal to the edge σ of D (σ lies on the axis of W'). By Corollary 24, the axes L_1 , L_2 , L_3 of the loxodromic elements $\Phi_1 = WR$, $\Phi_2 = W'R$ and $\Phi_3 = W'R(0)$ respectively are extremal. (On the other hand, the axis L_4 of W'_1R with $ht(L_4) = 5^{-1/4}/2$ is not extremal since it does not cut $K(\infty)$.) Thus, we have $k(L_1) = 5^{1/4}$, $k(L_2) = 1$, and $k(L_3) = 2$.

Consider now the v -cell $N(w)$. The reflections $W'' = RW'R$, $W''_1 = JW''_1J$ belong to G_w . The axes of R and R_2 pass through the vertical face of $\mathcal{A}(w)$ which is orthogonal to the edge σ' of D (σ' lies on the axis of W''). By Lemma 25, the axes L_2 , L_5 , L'_3 of the loxodromic elements $W''R = RW'$, $\Phi_5 = W''_1R$ and $W''R_2 = RW'R(0)R$ respectively are extremal. Thus, we have $k(L_2) = 1$, $k(L_5) = (9/5)^{1/4} = 1.15829$, and $k(L'_3) = 2$. Note that the Hurwitz constant $C(B_{19}) = 1/k(L_2) = 1$.

Now let $d = 15$. The extended Bianchi group B_{15} is not generated by reflections (see [27]). The fundamental domain D consists of two v -components $\mathcal{A}(v)$ and

$\mathcal{A}(w)$ where $v = (-3/\sqrt{-15}, \sqrt{2/5})$ and $w = (\sqrt{-15}/3, 1/\sqrt{3})$. Denote

$$W' = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 - \bar{\omega} & 1 + \omega \\ 1 + \omega & 1 + \bar{\omega} \end{pmatrix}, \quad \Phi_7 = \begin{pmatrix} 1 & -1 - \omega \\ 1 + \bar{\omega} & -5 \end{pmatrix},$$

$W'_1 = RW'$, $W'' = JW'J$, $W''_1 = JW'_1J$, $S' = RSR$, and $S'' = R_1SR_1$. Then $S, W, W', W'' \in G_v = \mathcal{H}_2$ and $S', S'', W'_1, W''_1 \in G_w = \mathcal{H}_3$. Note that $R, W', W'_1 \in \text{Stab}(u, G)$, the Klein 4-group, where $u = (\omega/2, 1/\sqrt{2})$. The reflection W' and loxodromic elements WR and $W''R$ have the common axis L_1 which is an extremal geodesic with $k(L_1) = \sqrt{2}$. The reflection W and loxodromic elements $W'R(0)$ and $W''R(0)$ also have the common axis L_3 which is an extremal geodesic with $k(L_3) = 2$. Let $J' = RJR$ be the reflection in the vertical plane in H^3 through the line $\text{Re}z = 1/2$. The reflection W'_1 and loxodromic elements $J'\Phi_7J'$, $S''R$ and $S''W'$ have the common axis L'_7 which is an extremal geodesic with $k(L_7) = k(L'_7) = k(J(L'_7)) = \sqrt{2}$ where L_7 is the axis of Φ_7 . In $N(w)$, the axes of W'_1 and R , W'_1 and R_2 , and W'_1 and R'_2 do not meet. Hence the axes L_6, L_3 and L_4 of W''_1R , W'_1R_2 and $W'_1R'_2$ respectively are extremal and $k(L_6) = 6^{1/4}$, $k(L_3) = 2$, and $k(L_4) = 3^{1/2}$. We find that $k_G = \sqrt{7}/2$ and, since the critical edges of D are not extremal, $1/\sqrt{2} \leq C(B_{15}) < 2/\sqrt{7}$. (It will be shown in §5 that $C(B_{15}) = 1/\sqrt{2}$.)

Finally, let $L = L_8$ be the geodesic through the vertices $w_1 = Rw$ and $w_2 = R_1w$. The endpoints of L_8 are the roots of $f_8(x) = 3x^2 - i\sqrt{15}x - 3 = 0$. The stabilizer of L_8 is generated by W and $W_2 = J'W'_1J'$. Let s be the point of intersection of the axis of W with L_8 . Then the arc of L_8 with endpoints s and w_1 is a fundamental domain of $\text{Stab}(L_8, G)$. It cuts the common face of $N(v)$ and $N(w_1)$ with vertices $0, \omega/2$ and ∞ . Since L_8 does not cut the horospheres with bases at these vertices when $k = h_L = \sqrt{21}/3$, by Lemma 23(2), L_8 is extremal and $k(L_8) = \sqrt{21}/3$.

When applying Algorithm I we choose D so that the floor of D consists of three faces which lie on the spheres with centers at the points $0, \omega/2$, and $\bar{\omega}/2$ in V , and we define Algorithm I so that $S_i \in \{W, W', \bar{W}'\}$ for any i . Figure 4 shows the upper one-half of the projection of D from ∞ into \mathbf{C} .

If $L = L_1$, then, as in Example 27, there are two reduced geodesics in the G -orbit of L : $L = T_0T_1L$ and $L' = T_1L$ where $T_0 = U(\omega)W$, $T_1 = U(-\omega)W$.

Let $d = 19$. The geodesic $L = L_2$ lies in the floor of D and it is reduced. Let $T_0 = U(\omega)W$, $T_1 = U(\bar{\omega})W$. There are four reduced geodesics: $L = T_0T_1L$, $L' = T_1L$, and $W(L)$, $W(L')$ with their orientation changed for the opposite (cf. Example 27, $d = 11$, $L = L_2$).

Let $d = 15$ and $L = L_7 = RJ(L'_7)$. The endpoints of L_7 lie in $(1 + \omega)\mathbf{R}$ and it is the axis of Φ_7 . Let $T_0 = U(1 + \omega)W$, $T_1 = U(1 + \bar{\omega})W$. There are four reduced geodesics: $L = T_0T_1L$, $L' = T_1L$, and $W(L)$, $W(L')$ with their orientation changed for the opposite.

Let $L = L_3$. Then, for $d = 19$, there are four reduced geodesics in the G -orbit of L : $L = W\bar{W}'WW'(L)$, $\bar{W}'WW'(L)$, $WW'(L)$, and $W'(L)$; and there are only two $L = \bar{W}'W'(L)$, and $W'(L)$ when $d = 15$.

Let σ be an edge of D whose endpoints v and v' are cusps of D . For $n = 1$, it was shown in [34] that the geodesic $L \supset \sigma$ is extremal and $k(L)$ is not isolated in the Lagrange spectrum provided there is a reflection $R \in G_s$ such that $v' = Rv$. (Here s is the summit of σ .) We shall generalize this result to higher dimensional spaces.

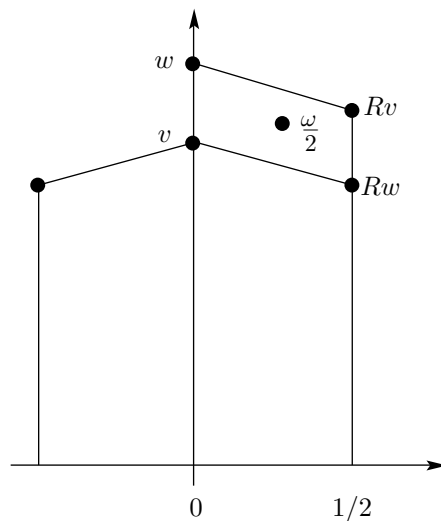


FIGURE 4

Theorem 29. *Suppose that v is a cusp of D . Let B be a vertical face of a v -cell $N(v)$ and let $s \in B$ be the foot of the perpendicular from v into B . Assume that there are reflections $R \in G_s$ and $W \in G_v$ such that the axis a_R of R lies in B and a_R and the axis of W do not meet. Then the axis of the loxodromic element RW as well as the geodesic $L = (v, R(v))$ is extremal. Moreover, $k(L)$ is an accumulation point in the Lagrange and Markov spectra.*

Proof. Let P_n run through the set of all parabolic elements in G_v . The axes of the reflections $W_n = P_n^{-1}WP_n \in G_v$ converge to v and the axes L_n of the loxodromic elements W_nR converge to L . The sequence of geodesics L_n contains a subsequence of geodesics which cut the vertical face of $\mathcal{A}(v)$ that lies in B and, by Lemma 25, they are extremal. \square

Corollary 30. *Assume that the geodesic L in Theorem 29 is a critical edge of D . Then the Hurwitz constant $C(G) = 1/k(L)$ and it is an accumulation point in the Lagrange and Markov spectra.*

Example 31. Let $G = GPL_2(O_{15})$. The fundamental domain D of G has the cusp $v = (\omega/2, 0)$ and vertex at $w = (2\sqrt{-15}/7, \sqrt{3}/7) \in H^3$. It consists of two v -components $\mathcal{A}(v)$ and $\mathcal{A}(w)$, where $\mathcal{A}(v)$ is bounded by vertical planes through the sides of the rhombus with vertices $(0, 0)$, $c = (-4/\sqrt{-15}, 0)$, $(\omega, 0)$, $c' = ((-7 + \sqrt{-15})/2\sqrt{-15}, 0)$, unit spheres with centers at $(0, 0$ and $(\omega, 0)$, and spheres with radius $1/\sqrt{15}$ and centers at c and c' ; and the component $\mathcal{A}(w)$ is bounded by vertical planes through the sides of the triangle with vertices $(\omega, 0)$, $(-\bar{\omega}, 0)$, c , unit spheres with centers at $(\omega, 0)$ and $(\bar{\omega}, 0)$ and the sphere with center at c and radius $1/\sqrt{15}$ (see [5], [28]). Since c is the center of the circumscribed circle of the triangle with vertices 0 , ω , $-\bar{\omega}$, there are three critical edges σ , σ' , and σ'' with the endpoints at v and $v' = (-\bar{\omega}/2, 0)$, v and w , and v' and w respectively in the fundamental domain D' bounded by the vertical planes through the sides of this triangle. Thus we have $k_G = 1/2$. It can be easily shown that the edges

σ' and σ'' are not extremal. Let B be the face of $\mathcal{A}(v)$ whose projection into V is the interval with endpoints at $(0, 0)$ and c . The reflections W and R defined as in Examples 27, 28, and 31 with their roles interchanged satisfy the hypothesis of Theorem 29. Hence the edge σ is extremal. By Corollary 30, the Hurwitz constant $C(G) = 1/k_G = 2$ and it is an accumulation point in $\mathcal{L}(G)$ and $\mathcal{M}(G)$.

Example 32. Let $n = 4$ and G be the same as in Example 12. It is shown in [36] that the endpoints of a critical edge σ are cusps and that $k_G = 1$. It can be shown that the conditions of Theorem 29 and Corollary 30 are satisfied and, therefore, $C(G) = 1/k_G = 1$ and it is a limit point in $\mathcal{L}(G)$ and $\mathcal{M}(G)$.

5. DIOPHANTINE APPROXIMATION IN $\mathbf{Q}\sqrt{-15}$

In this section, Theorem 2 will be proved. Here G is an extended Bianchi group B_{15} . Assume that any two v -cells $N(v_i)$ and $N(v_{i+1})$ in the sequence $N(v_i)$, $i = 1, \dots, n$, have at least one common edge. Generalizing the definition of $k(v)$ introduced in [34], define $k(v_1, \dots, v_n)$ to be the largest k such that any geodesic passing through $N(v_i, k)$, $i = 1, \dots, n$, cuts a horospherical face of at least one of $N(v_i, k)$, $i = 1, \dots, n$. Assume that a geodesic L passes through v -cells $N(v_i)$, $i = 1, \dots, n$. We shall say that L is *feasible* for this sequence of v -cells if L does not cut horospherical faces of $N(v_i, h_L)$, $i = 1, \dots, n$. Thus, by definition, $k_G < h_L = 2ht(L)$ for a feasible L .

When $d = 15$, there are only two types of v -cells: tetrahedra and octahedra (see Example 28). Tetrahedra are congruent to $N(v)$ with vertices $0, \omega/2, -\bar{\omega}/2, \infty$ and octahedra to $N(w)$ with vertices $\omega, -\bar{\omega}, \omega/2, -\bar{\omega}/2, (-1 + 2\omega)/3, \infty$. Each tetrahedron has common faces with octahedra only. Each octahedron has two common faces with octahedra and six with tetrahedra. Octahedra with common faces have a common axis which is the axis of an elliptic element of order three (see Figure 5).

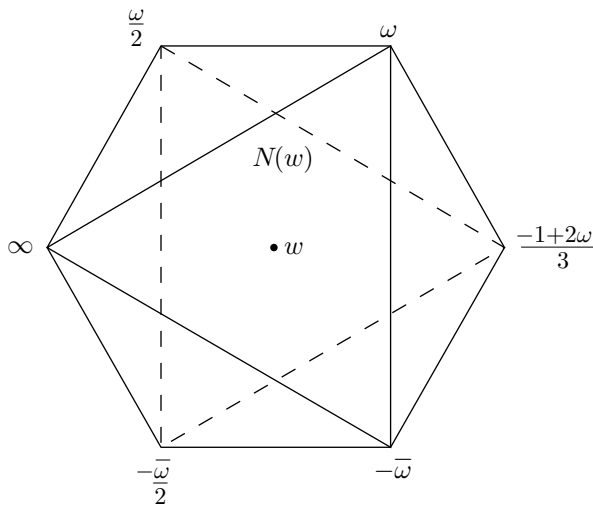


FIGURE 5

If an extremal geodesic L cuts such a common face (which is congruent to the triangle with vertices $0, 1, \infty$), then $k(L) \geq \sqrt{3}$. Thus,

$$(9) \quad k(w_1, J''w_1) = \sqrt{3}.$$

If a geodesic L meets the axis a_R of the reflection R and it does not cut the horospheres with bases at $\omega/2$ and ∞ (which are tangent when $k = \sqrt{2}$), then $h_L \geq \sqrt{2}$. Since the extremal geodesic L_1 and L'_7 meet a_R and $k(L_1) = k(L'_7) = \sqrt{2}$, we have

$$(10) \quad k(v, Rv) = k(w, w_1) = \sqrt{2}$$

where $w_1 = Rv$. Similarly, if L meets the vertical line through the origin, which is the axis of the reflection $R(0)$, then $h_L \geq 2$. Since the axis of W is such a geodesic and it is extremal (see Example 28),

$$(11) \quad k(v, R(0)v) = k(w, R(0)w_1) = 2.$$

Assume that $k(L) < \sqrt{2}$. It follows from (9), (10) and (11) that L cannot pass through two tetrahedra or two octahedra in succession. Assume that L passes successively through a tetrahedron T_1 , octahedron O_1 , tetrahedron T_2 , and octahedron O_2 . Let B be the common face of T_1 and O_1 and B' the common face of O_1 and T_2 . Let τ be the reflection in H^3 with respect to the center of O_1 . It follows that the tetrahedra T_1 and T_2 can have (a) a common edge; (b) only one common vertex; or (c) they do not intersect, in which case $\tau(B) = B'$. Replacing, if necessary, L by $g(L)$ for some $g \in G$, we can assume that L passes through one of the following sequences of v -cells: (a) $N(v), N(w), N(Rv)$, or $N(v), N(w_1), N(Rv)$; (b) $N(v), N(w_1), N(J'v)$; or (c) $N(v), N(w_1), N(\tau(v))$; and that it is feasible for the corresponding sequence. Here τ is the reflection with respect to w_1 .

On the other hand, octahedra O_1 and O_2 always have a common edge M . By applying some $g \in G$, if necessary, we can assume that M is a vertical edge of $N(w_1)$, and that L passes either through $N(w_1), N(v), N(w)$ (when M passes through $\omega/2$) or through $N(w_1), N(v), N(Jw_1)$ (when M passes through 0).

Lemma 33. $k(v, w, Rv) \geq \sqrt{2}, k(v, w_1, Rv) \geq \sqrt{2}$.

Proof. Assume that L passes through $N(v)$ and $N(Rv)$ and that it is feasible for these v -cells. Let $f(x, y) = (x - \theta y)(x - \theta' y)$ where θ and θ' are the endpoints of L . We say that f is *extremal* if $|f(x, y)| \geq n(x, y)$ for all $x/y = g(\infty), g \in G$. Here $n(x, y)$ is the norm of the ideal generated by x and y . It is clear that f is extremal if and only if L is. Let $\Delta = \Delta(f) = (\theta - \theta')^2/4$. Then $h_L = 2|\Delta|^{1/2} < \sqrt{2}$. Since $k_G = \sqrt{7}/2$ (see Example 28), we can assume that $7/16 < |\Delta| < 1/2$. It is clear from geometry (see Figure 6) that $0 \leq \arg(\Delta) \leq 2 \arg(\omega) = 4\alpha$.

Thus, we can assume that

$$(12) \quad \Delta \in E_1 := \{z \in \mathbf{C} : 7/16 < |z| < 1/2; 0 \leq \arg(z) \leq 2 \arg(\omega)\}.$$

To show that there are no feasible geodesics with $\Delta \in E_1$ we shall utilize the approach which was used in [30] and [31] to find the discrete part of the Markov spectrum for the Gaussian field. Let $b = (\theta + \theta')/2$. We can assume that

$$(13) \quad b \in P := \{z \in \mathbf{C} : 0 \leq \operatorname{Re}(z) \leq 1/2, 0 \leq \operatorname{Im}(z) \leq \sqrt{15}/4\}.$$

It is clear that f is extremal if and only if $|f(z, 1)| = |(z - b)^2 - \Delta| \geq n(x, y)|y|^{-2}$ for any $z = x/y = g(\infty), g \in G$. Thus, if for a fixed b the open disks $K(z_i)$ with

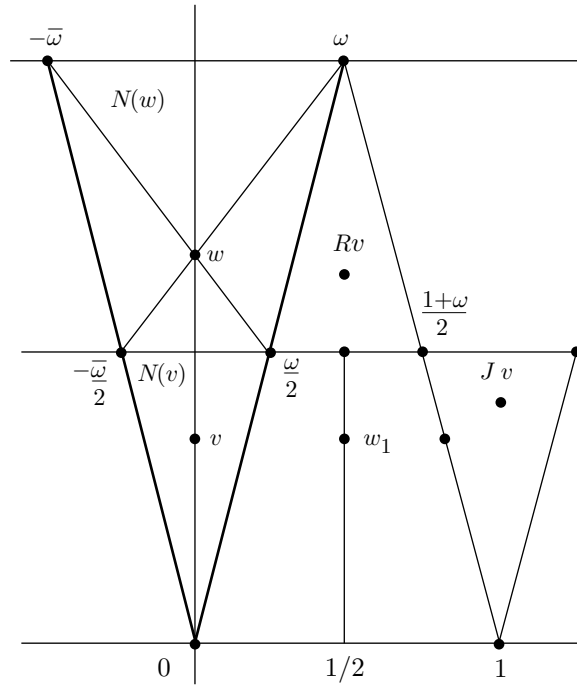


FIGURE 6

centers at $C(z_i) = (z_i - b)^2$ and radii $r(z_i) = n(x_i, y_i)|y_i|^{-2}$, $i = 1, \dots, j$, cover the region E_1 , there are no feasible geodesics with center at $b \in \mathbf{C}$ and $h_L < \sqrt{2}$ which cut $N(v)$ and $N(Rv)$. We shall show that such a covering exists for any $b \in P$. Only the disks $K(z)$ of radii $r(z) = 1$ and $1/2$ will be used.

Denote $B_m = \{z \in \mathbf{C} : |z^2 - 1/2| < 1/m\}$, $m = 1, 2$. Thus, B_1 is a convex region bounding a Cassinian oval, and B_2 is the interior of a lemniscate (these two curves have the same foci $\pm 1/\sqrt{2}$). A part of B_2 bounded by one leaf of the lemniscate is also convex (see Figure 7).

Let $B_m(z, \eta) = z + e^{i\eta}B_m$. It is clear that if $b \in B_m(z, \eta)$, then $e^{2i\eta}/2 \in K(z)$, and, since $r(z) \geq 1/2$, if $b \in B_m(z, \eta) \cap B_m(z, \eta')$, $0 < \eta < \eta' < \pi/2$, then the arc of the circle $|z| = 1/2$, $\eta < \arg(z) < \eta'$, is covered by $K(z)$.

Let $a_1 = 1/2 + i0.6890$, $a_2 = 0.4024 + i0.6900$, $a_3 = i0.7070$, $a_4 = 1/2 + i0.7450$, $a_5 = i0.7450$, $a_6 = i0.945$. Divide the rectangle P in (13) into five polygons (see Figure 8): P_1 is the pentagon with vertices $0, 1/2, a_1, a_2, a_3$; P_2 is the hexagon with vertices $\omega/2, 1/4 + \omega/2, a_4, a_2, a_3, a_5$; P_3 is the triangle with vertices a_1, a_2, a_4 , P_4 is the triangle with vertices $a_5, a_6, \omega/2$; and P_5 is the triangle with vertices $a_6, i\sqrt{15}/4, \omega/2$.

1. Since $P_1 \subset B_1(0, \eta)$ for any $\eta \in [0, \pi/2]$, and $C(0) = b^2/4$, $E_1 \subset K(0)$ if $b \in P_1$.

2. Let $b \in P_2 \subset B_2((1 + \omega)/2, \eta) \cap B_1(0, \eta')$, $0 \leq \eta < \alpha < \eta' \leq \pi$. Then $1/2, \omega/4 \in \bar{K}((1 + \omega)/2)$ and $\omega/4, -1/2 \in \bar{K}(0)$. Hence $E_1 \subset K(0) \cup K((1 + \omega)/2)$.

3. Let $\delta_1 = e^{i0.2618}/2$. Let $b \in P_3 \subset B_2((2 + \omega)/2, \eta) \cap B_2((1 + \omega)/2, \eta') \cap B_1(0, \eta'')$, $0 \leq \eta \leq 0.1309 \leq \eta' \leq \alpha \leq \eta'' \leq 2\alpha$. Then $|C((1 + \omega)/2)| < 1/2$ and

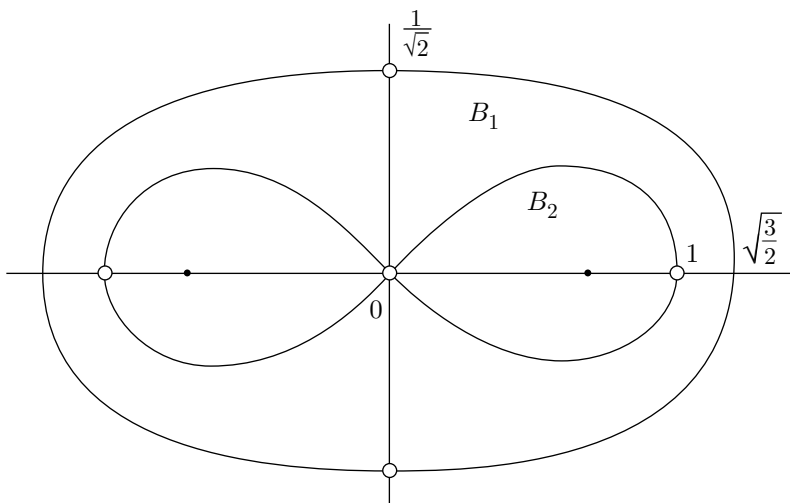


FIGURE 7

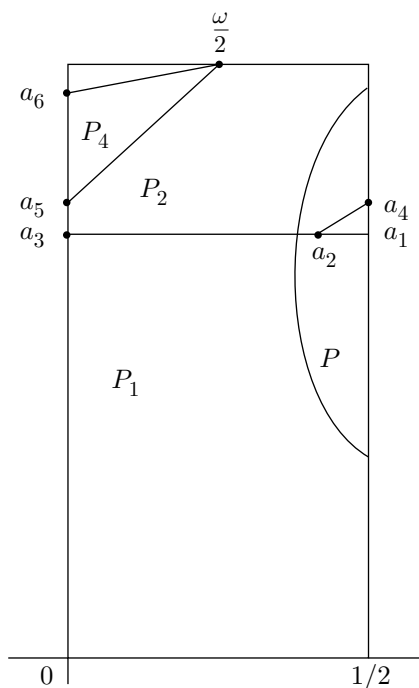


FIGURE 8

$\omega^2/8, \omega/4 \in K(0)$, $\omega/4, \delta_1 \in K((1+\omega)/2)$ and $\delta_1, 1/2, 7/16 \in K((2+\omega)/2)$. Thus, $E_1 \subset K(0) \cup K((1+\omega)/2) \cup K((2+\omega)/2)$.

4. Let $\delta_2 = e^{i1.658}/2$. Let $b \in P_4 \subset B_2(\omega/2, \eta) \cap B_1(0, \eta')$, $0 \leq \eta \leq 0.827 \leq \eta' \leq \pi$. Then $1/2, \delta_2 \in K(\omega/2)$; $\delta_2, -1/2 \in K(0)$, and since $|C(\omega/2)| < 1/2$, $E_1 \subset K(0) \cup K(\omega/2)$.

5. Let $b \in P_5 \subset B_2(\omega/2, \eta) \cap B_1(\omega, \eta') \cap B_1(0, \eta'')$, $0 \leq \eta \leq 0.827 \leq \eta' \leq \pi/2 - \alpha \leq \eta'' \leq \pi$. Then $1/2, \delta_2 \in K(\omega/2)$, $i/2, e^{3\pi/4}/2 \in K(\omega)$, and $-\bar{\omega}/4, -1/2 \in K(0)$. Hence $E_1 \subset K(0) \cup K(\omega) \cup K(\omega/2)$.

Thus, for any $b \in P$

$$E_1 \subset K(0) \cup K(\omega) \cup K(\omega/2) \cup K((1 + \omega)/2) \cup K((2 + \omega)/2)$$

which completes the proof. □

Lemma 34. $k(v, w_1, J'v) \geq \sqrt{2}$.

Proof. Let L be a feasible geodesic with $h_L < \sqrt{2}$ which passes through $N(v)$ and $N(J'v)$. Then L cuts the geodesic faces of $N(v, \sqrt{2})$ and $N(J'v, \sqrt{2})$ that lie in the vertical faces of $N(v)$ and $N(J'v)$ with vertices $0, \omega/2, \infty$ and $1, (1 + \omega)/2, \infty$ respectively (see Figures 6 and 9). It follows that

$$\Delta \in E_2 := \{z \in \mathbf{C} : 7/16 < |z| < 1/2; |\arg(z)| < \pi/3\}$$

and

$$b \in P' := \{z \in P : |z - (2 + \omega)/3| < 0.45\}$$

(see Figure 8). By Lemma 33, $E_2 \cap E_1$ is covered by the disks $K(z)$ for any $b \in P' \subset P$. Hence it is enough to show that such a covering of $E'_2 = \{z \in E_2 : -\pi/3 < \arg(z) \leq 0\}$ exists for any $b \in P'$. Let $P_6 = P' \cap B_1(1, 0)$ and $P_7 = P' - P_6$. If $b \in P_6 \subset B_1(1, \eta)$, $-\pi/6 < \eta \leq 0$, then $E'_2 \subset K(1)$. If $b \in P_7 \subset B_2(-\bar{\omega}/2, \eta)$, $-\pi/6 < \eta \leq 0$, then $E'_2 \subset K(-\bar{\omega}/2)$. The lemma is proved. □

Remark. Note that $k(v, w_1) \leq k(L_8) = \sqrt{21}/3$ (see Example 28).

Proof of Theorem 2. It follows from (9), (10), (11), Lemmas 33 and 34 that the inequality $k(L) < \sqrt{2}$ may hold only if L passes through an alternating sequence of polyhedra $\dots, T_1, O_1, T_2, \dots$, such that $\tau(T_1) = T_2$ for any two consecutive tetrahedra T_1 and T_2 in the sequence. Here τ is the reflection in H^3 with respect to the center of the octahedron O_1 . As mentioned above, we have to consider two cases.

1. Assume that L passes through $N(w), N(v), N(w_1)$. Then L cuts also $N(\tau'(v))$ and $N(\tau(v))$ where τ' and τ are the reflections in H^3 with respect to w and w_1 respectively. The projections of $N(\tau'(v))$ and $N(\tau(v))$ from ∞ into \mathbf{C} are the triangles with vertices $\omega, -\bar{\omega}, (-1 + \omega)/3$ and $1, (1 + \omega)/2, (1 + \omega)/3$ respectively. But if L passes through $N(\tau'(v))$ and $N(\tau(v))$, it does not meet $N(v)$ (see Figure 6). The contradiction obtained shows that no such geodesic L exists.

2. Now assume that L passes through $N(w_1), N(v), N(Jw_1)$. Then L cuts also $N(\tau(v))$ and $N(\tau''(v))$ where $\tau'' = J\tau J$ is the reflection in H^3 with respect to Jw_1 . Let B be the common geodesic face of $N(w_1, \sqrt{2})$ and $N(\tau(v), \sqrt{2})$ and let $B' = J(B)$. (B is congruent to the geodesic face shown in Figure 9.) Then L cuts both B and B' . It is clear from geometry that the geodesic L' through the vertices $u = ((2 + \omega)/4, 1/\sqrt{8})$ and $u' = Ju = ((2 - \bar{\omega})/4, 1/\sqrt{8})$ of B and B' respectively has the smallest height among all the geodesics which cut both B and B' . But $ht(L') = \sqrt{33}/8 > 1/\sqrt{2} > ht(L)$ which contradicts the assumption.

Thus, no geodesic with $k(L) < \sqrt{2}$ exists and the Hurwitz constant $C(B_{15}) = 1/\sqrt{2}$. It is attained at the extremal geodesics L_1 and L'_7 (see Example 28) whose endpoints are $(\omega \pm \sqrt{\omega})/2$ and $(\omega \pm \sqrt{-\omega})/2$ respectively. □

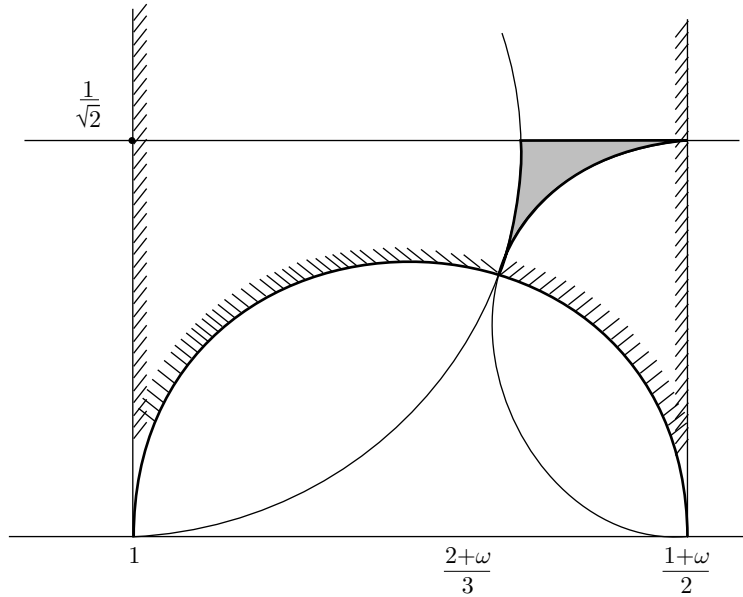


FIGURE 9

6. ISOLATION THEOREM

In this section, we generalize to n -dimensional euclidean spaces, $n \geq 3$, the isolation theorems proved in [8], p. 25, and [31] for $n = 1$ and 2.

Let L be a geodesic in H^{n+1} with endpoints η and θ in V . Let $g \in G$ and let $r = r(g)$ be the radius of the isometric sphere $I(g)$ with center at $x = g^{-1}(\infty) \in V$. Denote

$$l(g, \eta) = |x - \eta|/r.$$

Then (see [1], p. 129)

$$l(g, \eta) = |g'(\eta)|^{-1/2}$$

where $g'(x)$ is the Jacobian of g . Denote

$$f_L(g) = l(g, \eta)l(g, \theta).$$

Then (see [19], p. 8)

$$f_L(g) = \frac{|\eta - \theta|}{|g(\eta) - g(\theta)|}$$

which implies the following.

Lemma 35. *Let L be a geodesic in H^{n+1} . Then L cuts a horosphere $Q = Q(g, h_L)$ if and only if $f_L(g^{-1}) < 1$, and L is tangent to Q if and only if $f_L(g^{-1}) = 1$. Thus L is extremal if and only if $f_L(g) \geq 1$ for any $g \in G$.*

Proof. A geodesic L cuts the horosphere Q if and only if $ht(g^{-1}(L)) > h_L/2$ which is equivalent to $f_L(g^{-1}) < 1$. □

Remark. If G is a Vahlen group and $g(\infty) = ac^{-1}$, $g \in G$, then we also have the following representation: $f_L(g) = |\eta c - a||\theta c - a|$.

Denote by $C(x, \alpha)$ the spherical cone in V with vertex at η , axis through x , and vertex angle α . Let x_1, \dots, x_{n+1} be the vertices of a regular tetrahedron in V with center at η . Since η divides a height of the tetrahedron in ratio $n : 1$, the cones $C(x_1, \alpha), \dots, C(x_{n+1}, \alpha)$ cover V provided $\alpha \geq \alpha_n = \pi/4 + (\arcsin(1/n))/2$.

Suppose now that L is the axis of a loxodromic element $h \in G$. Then $h(\eta) = \eta$ and $h(\theta) = \theta$. Suppose, as we may, that $h^k(x) \rightarrow \eta$ as $k \rightarrow \infty$ for any $x \in V, x \neq \theta$. By the chain rule, $(gh)'(\eta) = g'(\eta)h'(\eta)$. Denote $\lambda^{-2} = |h'(\eta)|$. Since $f_L(gh^n) = f_L(g)$, we have

$$l(gh^n, \eta) = \lambda^{-n}l(g, \eta), \quad l(gh^n, \theta) = \lambda^n l(g, \theta).$$

Suppose that the geodesic L is extremal. For every $c = g(\infty) \in V, g \in G$, let $u(c)$ be the unit tangent vector at η to the circle through c, η , and θ in the direction of c . Let points $c_i = g_i(\infty) \in V, g_i \in G, i = 1, \dots, r$, be such that $f_L(g_i^{-1}) = 1$ and every cone $C(x, \alpha), \alpha > \alpha_n$, contains at least one of the vectors $u(c_i), i = 1, \dots, r$. Since the angle between $u(h^k c_i)$ and $u(h^k c_j)$ does not depend on k , every cone $C(x, \alpha), \alpha > \alpha_n$, contains an infinite sequence of points $h^k c_{i_k}, 1 \leq i_k \leq r$, for all integer k greater than some fixed positive constant.

Let H be a hemisphere in H^{n+1} orthogonal to L and let $S = H \cap V$. The region R in V bounded by spheres S and hS is a fundamental domain in V of the infinite cyclic subgroup of $\text{Stab}(L, G)$ generated by h . Assume that the points $c_1, \dots, c_r \in K = R \cup hR \cup \dots \cup h^s R$.

Denote $K_m = h^{sm} K$. Let L' be a geodesic with endpoints at η' and θ' such that $\eta' \in K_m$ for some m and $|\eta' - \eta| < \epsilon, |\theta' - \theta| < \epsilon$ for some $\epsilon > 0$. There is $g \in G$ such that $c = g(\infty) \in C(\eta', \alpha_n) \cap K_{m-1}$ and $f_L(g^{-1}) = 1$. Then $h^{2s} c \in K_{m+1}$. Since the distance between η and the center of a sphere $S \in V$ with radius ρ is less than ρ^2/h_L , there is a constant C such that

$$\lambda^{2s}(1 - C\epsilon^2)|\eta - c| < |\eta' - \eta| < (1 + C\epsilon^2)|\eta - c|, \quad c = g(\infty),$$

where fixed $\epsilon > |\eta' - \eta|$. Suppose that $n \geq 2$. Then $\alpha_n \leq \pi/3$. It follows that there is a constant C' such that

$$|\eta' - c| < (1 - \lambda^{2s} + \lambda^{4s})^{1/2} |\eta - c| (1 + C'\epsilon^2).$$

Let $\mu < (1 - \lambda^{2s} + \lambda^{4s})^{1/2}$. There is a constant C'' such that

$$|\theta' - c| \leq |\theta - c| + |\theta' - \theta| < (1 + C''\epsilon)|\theta - c|.$$

Thus, for a sufficiently small $\epsilon, f_{L'}(g^{-1}) = r^2|\eta' - c||\theta' - c| < \mu$.

We have proved the following isolation theorem.

Theorem 36. *Suppose that an extremal geodesic L in H^{n+1} with endpoints η and θ in V is the axis of a loxodromic element h in G . Let $u(c)$ be the unit tangent vector at η to the circle through c, η , and θ in the direction of c . Suppose that there are points $c_i = g_i(\infty) \in V, g_i \in G$ such that $f_L(g_i^{-1}) = 1, i = 1, \dots, r$, and every cone $C(x, \alpha), \alpha > \pi/4 + (\arcsin(1/n))/2$, contains at least one of the vectors $u(c_i), i = 1, \dots, r$. Then there are $k' > k(L)$ and an $\epsilon > 0$ depending only on L such that $k(L') > k'$ for any geodesic L' with endpoints at η' and θ' for which*

$$(14) \quad |\eta' - \eta| < \epsilon, \quad |\theta' - \theta| < \epsilon.$$

Suppose that an extremal geodesic L is the axis of a loxodromic $h \in G$ and that L contains a critical edge σ of D which is orthogonal to the vertical face B of v -cell $N(v)$. Then the Hurwitz constant $C(G) = 1/k(L)$. Assume that the group

of exterior automorphisms of L in G acts transitively on the set of vertices of B . (We say $g \in G$ is an *exterior* automorphism of L if g fixes L pointwise.) Then the hypothesis of Theorem 36 is satisfied since, as it is easily seen, for any vertices c_i and c_j of B , the angle between the geodesic M_i through $s = \sigma \cap B$ and c_i and M_j through s and c_j is equal to the angle between the circular arcs through η , c_i , θ and η , c_j , θ . Thus, we have the following.

Corollary 37. *Suppose that each of the critical edges σ of the fundamental domain D of the discrete group G acting in H^{n+1} lies on the axis L of an loxodromic element in G which is an extremal geodesic. Let σ be orthogonal to B , a vertical face of a v -cell. If the group of exterior automorphisms of L acts transitively on the set of the vertices of B , then the Hurwitz constant of G is isolated in its Lagrange and Markov spectra.*

Proof. By assumption, the Hurwitz constant $C(G) = 1/k(L)$. Since L satisfies Theorem 36, we can confine ourselves to consideration of only extremal geodesics. Assume that for any $\delta > 0$ there is an extremal geodesic L' such that $k(L) < k(L') < k(L) + \delta$. Then, as follows from Theorem 5, for sufficiently small δ , L' cuts the geodesic faces $B_o \subset B$ and $B'_o \subset B'$ of $N(v, k(L'))$ where B and B' are faces of the v -cell $N(v)$ cut by some L containing a critical edge of D . Thus, the inequalities (14) are satisfied and by Theorem 36 L' is not extremal which contradicts the assumption. (Note that the diameter of a geodesic face of $N(v, k(L'))$ is small and approaches zero as $\delta \rightarrow 0$.) \square

In all the examples considered in [35] and [36], the group G is generated by reflections. Hence, Corollary 37 is applicable to G , and the Hurwitz constants are isolated in the Lagrange and Markov spectra of G when $n = 1$ and $G = G_q$, the Hecke group with even $q > 2$; or $n = 2$ and $G = B_d$, the extended Bianchi group, $d = 1, 2, 5$, or 6 ; or $n = 3$ and G is the discrete subgroup $SV(\mathbf{Z}^4)$, of the Vahlen's group of Clifford matrices (see Example 12); or $n = 4$ and $G = PSL(2, \mathcal{H})$ where \mathcal{H} is the ring of Hurwitz integral quaternions. Some of these or even stronger results are obtained in [15], [23], [21], [30], [31], and [22]. But the isolation of the approximation constants for the imaginary quadratic fields $\mathbf{Q}(\sqrt{-5})$ and $\mathbf{Q}(\sqrt{-6})$, and for the three-dimensional euclidean space is a new result. This isolation phenomenon should be compared with the case when the endpoints of the critical edge σ of D are cusps when the Hurwitz constant is a limit point in the spectra $\mathcal{L}(G)$ and $\mathcal{M}(G)$ (see Example 32).

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