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A COMBINATORIAL PROOF OF BASS'S EVALUATIONS OF THE IHARA-SELBERG ZETA FUNCTION FOR GRAPHS

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This paper is dedicated to Gian-Carlo Rota, on his millionth$_2$'s birthday.

Abstract. We derive combinatorial proofs of the main two evaluations of the Ihara-Selberg zeta function associated with a graph. We give three proofs of the first evaluation all based on the algebra of Lyndon words. In the third proof it is shown that the first evaluation is an immediate consequence of Amitsur's identity on the characteristic polynomial of a sum of matrices. The second evaluation of the Ihara-Selberg zeta function is first derived by means of a sign-changing involution technique. Our second approach makes use of a short matrix-algebra argument.

1. Introduction

We are pleased to dedicate the present paper to our *bon maître* Rota who has been a great promoter of combinatorial methods. Convinced that combinatorics was hidden in many branches of mathematics (see, e.g., [14]), he has successfully persuaded his followers to unearth its treasures, study them for their own sake and propose a fruitful symbiosis with the mainstream of mathematics.

Rota's pioneering paper [13] made the *Möbius function*, and hence its associated *zeta function*, a central unifying concept in combinatorics and elsewhere. The present paper is devoted to the calculation of a zeta function, not of a partially ordered set, as it has been so successfully done in the past by Rota and his disciples (see, e.g., [17]), but of a tree lattice.

Digging out those combinatorial treasures is not always an easy task, since very often a language barrier has to be overcome. One such example, that we were fortunate to discover, is Bass's [3] evaluation of the Ihara-Selberg zeta function for a graph. Thanks to his superb and very lucid talk (Temple Mathematics Colloquium, May 1995) we were introduced to the algebraic set-up of his derivation and led to the core of his paper. Of great help also have been his transparencies, copies of which he was kind enough to send us.

In calculating the zeta function of a tree lattice Bass [4] was led to determine the following invariant for a finite connected unoriented graph G. To express his result he first transformed G into an *oriented* graph by letting each edge whose

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ends are vertices i and j give rise to two *oriented edges* going from i to j and from j to i. Let c_0 (resp. $2\,c_1$) be the number of vertices (resp. of oriented edges). He then introduced the class $\mathcal{R}$ of *prime*, *reduced* cycles of G, a class that in general is infinite, and formed the product

$$\eta(u) = \prod_{\gamma \in \mathcal{R}} (1 - u^{|\gamma|}), \tag{1.1}$$

where $|\gamma|$ denotes the length of the cycle γ. The product $\eta(u)$ is usually called the *Ihara-Selberg function* associated with the graph G.

His main result was to show that the expansion of $\eta(u)$ as an infinite series is actually a *polynomial* in u giving two explicit formulas for it, first as the determinant of a matrix of order $2\,c_1$ that depends on the *successiveness* of the *edges* (a notion that will be defined below), namely

$$\eta(u) = \det(I - u\,T), \tag{1.2}$$

second, as a product

$$\eta(u) = (1 - u^2)^{c_1 - c_0} \det \Delta(u), \tag{1.3}$$

where $\Delta(u)$ is a matrix of order c_0 that depends on the *connectedness* of the *vertices*. The definitions of T and $\Delta(u)$ will be given in full detail later on.

To prove (1.2) and (1.3) Bass makes use of keen algebraic techniques. In particular, the Jacobi formula $\det \exp A = \exp \operatorname{tr} A$ plays a key role in the derivation of (1.2). As this classical formula has been derived by combinatorial methods ([6], [20]), it was challenging to use those methods to find combinatorial proofs of both formulas (1.2) and (1.3). This is the purpose of the paper.

With the present combinatorial approach we can show that (1.2) can be derived in a more general context. Instead of counting the cycles by the *counter* $u^{|\gamma|}$, we can keep track of the successiveness property within each cycle γ by mapping γ onto a monomial $\beta(\gamma)$ in the so-called successiveness variables $b(e, e')$. As we shall see, the determinantal expression (1.2) can be derived in three different manners, all based on the the algebra of *Lyndon words*.

The concept of Lyndon words has been crucial in the foundations of Free Differential Calculus, initiated by Chen, Fox and Lyndon [5] and pursued by Schützenberger [15], [16] and Viennot [19]. The standard material on the subject can be found in the book by Lothaire ([8], chap. 5). Hereafter we just recall a few basic properties

Start with a finite nonempty set X supposed to be totally ordered and consider the free monoid X^* generated by X. Let $<$ be the lexicographic order on X^* derived from the total order on X. A *Lyndon word* is defined to be a nonempty word in X^* which is prime, i.e., not the power l'^r of any other word l' for any $r \geq 2$, and which is also minimal in the class of its cyclic rearrangements. Let L denote the set of all Lyndon words. The following property, due to Lyndon, can be found in [8], p. 67 (Theorem 5.1.5):

(1.4) *Each nonempty word $w \in X^*$ can be uniquely written as a nonincreasing juxtaposition product of Lyndon words:*

$$w = l_1 l_2 \ldots l_n, \qquad l_k \in L, \qquad l_1 \geq l_2 \geq \cdots \geq l_n.$$

With each Lyndon word l let us associate a variable denoted by $[l]$. Assume that all those variables $[l]$ are distinct and commute with each other. Furthermore, let $\mathcal{B}$ be a square matrix whose entries $b(x,x')$ $(x,x'\in X)$ form another set of commuting variables.

If $w=x_1x_2\dots x_m$ is a nonempty word in X^*, define

$$\beta_{\mathrm{circ}}(w):=b(x_1,x_2)b(x_2,x_3)\dots b(x_{m-1},x_m)b(x_m,x_1)$$

and $\beta_{\mathrm{circ}}(w)=1$ if w is the empty word. Notice that all the words in the same cyclic class have the same β_{circ}-image. Also define

$$\beta([l]):=\beta_{\mathrm{circ}}(l) \tag{1.5}$$

for each Lyndon word l. Now form the **Z**-algebras of formal power series in the variables $[l]$ and in the variables $b(x,x')$, and by linearity make β to be a continuous homomorphism. It makes sense to consider the product

$$\Lambda:=\prod_{l\in L}\bigl(1-[l]\bigr) \tag{1.6}$$

as well as its inverse Λ^{-1}. We can also consider the images of Λ and Λ^{-1} under β. We have

$$\beta(\Lambda)=\prod_{l\in L}\bigl(1-\beta([l])\bigr);$$

and

$$\beta(\Lambda^{-1})=\bigl(\beta(\Lambda)\bigr)^{-1}.$$

We further define two maps β_{dec} and β_{vert} ("dec" for "decreasing" and "vert" for "vertical") as follows. If $(l_1,l_2,\dots,l_n)$ is the nonincreasing factorization of a word w in Lyndon words, as defined in (1.4), let

$$\beta_{\mathrm{dec}}(w):=\beta_{\mathrm{circ}}(l_1)\,\beta_{\mathrm{circ}}(l_2)\dots\beta_{\mathrm{circ}}(l_n).$$

Now when the m letters of a word $w=x_1x_2\dots x_m$ are rearranged in nondecreasing order, we obtain a word $\widetilde{w}=\tilde{x}_1\tilde{x}_2\dots\tilde{x}_m$ called the *nondecreasing rearrangement* of w. Then define

$$\beta_{\mathrm{vert}}(w):=b(\tilde{x}_1,x_1)b(\tilde{x}_2,x_2)\dots b(\tilde{x}_m,x_m).$$

Also define $\beta_{\mathrm{dec}}(w)=\beta_{\mathrm{vert}}(w):=1$ when w is the empty word.

By convention let X^* denote the *sum* of all the words w $(w\in X^*)$ and use the notation

$$\beta_{\mathrm{dec}}(X^*):=\sum_{w\in X^*}\beta_{\mathrm{dec}}(w)$$

with an analogous notation for $\beta_{\mathrm{vert}}(X^*)$.

Theorem 1.1. *We have the identities*

$$\beta(\Lambda^{-1})=\beta_{dec}(X^*); \tag{1.7}$$

$$\beta_{dec}(X^*)=\beta_{vert}(X^*); \tag{1.8}$$

$$\beta_{vert}(X^*)=\bigl(\det(I-\mathcal{B})\bigr)^{-1}; \tag{1.9}$$

$$\beta(\Lambda)=\det(I-\mathcal{B}). \tag{1.10}$$

Notice that the conjunction of (1.7), (1.8) and (1.9) implies the identity

$$\beta(\Lambda^{-1}) = \big(\det(I-\mathcal{B})\big)^{-1} \tag{1.11}$$

and therefore (1.10). The proofs of (1.7), (1.8) and (1.9) are given in section 2. As we shall see, they are all classical, or preexist in other contexts. The proof of (1.10) itself is given in section 4. Thus we already have two independent proofs of (1.10).

The direct proof of (1.10) heavily relies on the techniques developed (or not yet developed) in the algebra of Lyndon words. Section 3 is then devoted to recalling classical results on Lyndon words and proving new ones. Section 4 contains the construction of an involution of X^* that shows that $\beta(\Lambda)$ reduces to a *finite* sum $\beta(\mathbf{G})$ that is easily expressible as $\det(I-\mathcal{B})$.

Our third proof of (1.10) was suggested to us by Jouanolou [7] after the first author had discussed the contents of a first version of the paper during the October 1996 session of the *Séminaire Lotharingien.* It is based on a specialization of Amitsur's identity [2] on the characteristic polynomial of a finite sum of matrices $A_1+\cdots+A_k$. For each Lyndon word $l=i_1i_2\dots i_p$ whose letters belong to the set $[k]=\{1,2,\dots,k\}$ let A_l be the matrix product $A_l := A_{i_1}A_{i_2}\dots A_{i_p}$.

Then Amitsur's identity can be stated as

$$\det(I-(A_1+\cdots+A_k)) = \prod_{l\in L}\det(I-A_l), \tag{1.12}$$

where the product is extended over all Lyndon words in the alphabet $[k]$.

In section 5 we reproduce the (short) proof of Amitsur's identity (1.12) due to Reutenauer and Schützenberger [12]. As will be seen, (1.10) is a mere consequence of (1.12). Thus the shortest proof of identity (1.10) has to be borrowed from Classical Matrix Algebra.

In section 6 we show how identity (1.2) fits into the present context. It is shown that when X is taken as the set E of all oriented edges of the graph G and each variable $b(x,x')$ is equal to 0 when the edge x' is the reverse of x or is not the successor of x, and equal to u otherwise, identity (1.10) reduces to (1.2).

As named by Bass [3], the inverse of $\eta(u)$, as given in (1.2), is the *zeta function* of the underlying tree lattice, so that $\eta(u)$ itself may be called the *Möbius function* of the tree lattice. Accordingly, when proving (1.9) (resp. (1.10)) we calculate the zeta function (resp. the Möbius function) of the tree lattice.

There is *a priori* no extension of (1.3) in which the information on the edge successiveness can be kept other than a simple counting of the *reduced* prime cycles. We are then left to prove (1.3) itself, but we present two new proofs, one purely combinatorial derived in section 7, which is based on the constructions of several involutions on words. The second one is of matrix-algebra nature.

After submitting the present paper for publication in the fall of 1996 our attention was drawn by Ahumada (Mulhouse), (who himself published an early paper on the subject [1]), to the paper by Stark and Terras that had just appeared [18]. The latter authors also have a proof of identity (1.10) when L is restricted to the set of *reduced* prime cycles. Finally, Stanton (Minneapolis) was kind enough to send us a preprint by Northshield [10] who also has elementary proofs of both identities (1.2) and (1.3).

2. The zeta function approach

When the infinite product Λ is developed as an infinite series in the variables $[l]$, we get the sum of all the *commuting* monomials $[l_1]\,[l_2]\,\ldots\,[l_n]$, or, equivalently, the sum of the nonincreasing words $[l_{i_1}]\,[l_{i_2}]\,\ldots\,[l_{i_n}]$ $(l_{i_1} \geq l_{i_2} \geq \cdots \geq l_{i_n})$. Hence, as

$$\beta(\Lambda^{-1}) = \sum \beta(l_{i_1})\,\beta(l_{i_2})\,\ldots\beta(l_{i_n}) = \sum_{w\in X^*} \beta_{\mathrm{dec}}(w) = \beta_{\mathrm{dec}}(X^*),$$

because of Lyndon's theorem (1.4) and by definition of β_{dec}, we obtain (1.7).

Let $|w|$ denote the length of each word $w \in X^*$. As both mappings β_{dec} and β_{vert} transform a word *of length* m into a monomial in the variables $b(x, x')$ *of degree* m, identity (1.7) is equivalent to

$$\sum_{|w|=m} \beta_{\mathrm{dec}}(w) = \sum_{|w|=m} \beta_{\mathrm{vert}}(x)$$

for all $m \geq 0$. Therefore (1.7) is proved if and only if the following proposition holds.

(2.1) *There exists a bijection Φ of X^* onto itself having the following property: if $w = x_1x_2\ldots x_m$ belongs to X^*, then $\Phi(w) = w' = x'_1x'_2\ldots x'_m$ is a rearrangement of w and $\beta_{\mathrm{vert}}(w') = \beta_{\mathrm{dec}}(w)$.*

The construction of such a bijection has been given in [4] (theorem 4.11) and also in [8], pp. 198-199. However the construction must be slightly modified to fit in the present derivation. We illustrate the construction of the bijection with an example. Let $X = \{1, 2, \ldots, 5\}$ and $w = 3, 4, 5, 1, 2, 4, 2, 1, 2, 3, 1, 2, 4, 2$. The factorization $(l_1, l_2, \ldots, l_n)$ of w as a nonincreasing sequence of Lyndon words (as defined in (1.5)) is $(3, 4, 5;\ 1, 2, 4, 2;\ 1, 2, 3, 1, 2, 4, 2)$. For the construction of the bijection another factorization is used, the *decreasing factorization* $(d_1, d_2, \ldots, d_r)$ of w simply defined by cutting w before every letter x of w which is *smaller than or equal to* each letter to its left. With the working example $(d_1, d_2, \ldots, d_r) = (3, 4, 5;\ 1, 2, 4, 2;\ 1, 2, 3;\ 1, 2, 4, 2)$. Notice that each Lyndon word l_i is the juxtaposition product of contiguous factors d_j. Moreover

$$\begin{aligned}\beta_{\mathrm{dec}}(w) &= \beta_{\mathrm{circ}}(l_1)\,\beta_{\mathrm{circ}}(l_2)\ldots\beta_{\mathrm{circ}}(l_n)\\ &= \beta_{\mathrm{circ}}(d_1)\,\beta_{\mathrm{circ}}(d_2)\ldots\beta_{\mathrm{circ}}(d_r).\end{aligned} \tag{2.2}$$

To obtain w' we form the product of the so-called *dominated circuits* (see [8], chap. 10)

$$\Delta(w) = \left(\begin{array}{c|c|c|c} 4\,5\,3 & 2\,4\,2\,1 & 2\,3\,1 & 2\,4\,2\,1 \\ 3\,4\,5 & 1\,2\,4\,2 & 1\,2\,3 & 1\,2\,4\,2 \end{array}\right).$$

In $\Delta(w)$ the top word in each factor is obtained from the bottom factor d_j by making a right to left cyclic shift of d_j.

Next we reshuffle the columns of $\Delta(w)$ in such a way that the mutual order of two columns with the same top entry is not modified but the top row becomes nonincreasing:

$$\begin{pmatrix} 5\;4\;4\;4\;3\;3\;2\;2\;2\;2\;2\;1\;1\;1 \\ 4\;3\;2\;2\;5\;2\;1\;4\;1\;1\;4\;2\;3\;2 \end{pmatrix}.$$

The resulting bottom word is the word $\Gamma^{-1}(\Delta(w))$ as described in [8], p. 199, except the construction has been given with the *reverse* order of X.

Now exchange top and bottom words and rewrite the resulting biword from right to left:

$$\begin{pmatrix} 2\,3\,2\,4\,1\,1\,4\,1\,2\,5\,2\,2\,3\,4 \\ 1\,1\,1\,2\,2\,2\,2\,2\,3\,3\,4\,4\,4\,5 \end{pmatrix}.$$

Finally, reshuffle the columns of the last biword so that the top word becomes *nondecreasing*, still keeping the mutual order of any two columns having the same top entry invariant:

$$w' = \begin{pmatrix} 1\,1\,1\,2\,2\,2\,2\,2\,3\,3\,4\,4\,4\,5 \\ 2\,2\,2\,1\,1\,3\,4\,4\,1\,4\,2\,2\,5\,3 \end{pmatrix}.$$

Then w' is defined to be the bottom word of the above biword. Moreover

$$\beta_{\mathrm{dec}}(w) = \beta_{\mathrm{vert}}(w').$$

With the working example the latter monomial is equal to

$$b(1,2)^3\, b(2,1)^2\, b(2,3)\, b(2,4)^2\, b(3,1)\, b(3,4)\, b(4,2)^2\, b(4,5)\, b(5,3).$$

Identity (1.9) is essentially the MacMahon Master Theorem identity (see [9], pp. 93-96, or [4], chap. 5). This achieves the proof of (1.11). Notice that the combination of (1.7), (1.8) and (1.10) provides a new proof of the Master Theorem identity.

3. Lyndon and Donlyn words

As already defined in the introduction a *Lyndon word* is a nonempty word in X^* which is prime and also minimal in its class of cyclic rearrangements. Let L denote the set of all Lyndon words. The following properties (3.1)–(3.3) can be found in [8], pp. 65 and 66 (Propositions 5.1.2 and 5.1.3):

(3.1) *A nonempty word in X^* is a Lyndon word if and only if it is strictly smaller that any of its proper right factors.*

(3.2) *A nonempty word in X^* is a Lyndon word if and only if it is of length one or the juxtaposition product lm of two Lyndon words l, m such that $l < m$.*

Let l be a Lyndon word; if $|l| \geq 2$ let m_0 be the proper right factor of maximal length such that $m_0 \in L$. Write $l = l_0 m_0$. The factorization (l_0, m_0) of l is called the *standard factorization* of l.

(3.3) *If (l_0, m_0) is the standard factorization of a Lyndon word l of length $|l| \geq 2$, then l_0 is also a Lyndon word and $l_0 < l_0 m_0 < m_0$.*

We will also need the following two properties, apparently not stated in the standard texts, but essential in our derivation.

(3.4) *A factorization (l_0, m_0) of a Lyndon word l into two nonempty factors is the standard factorization of l if and only if $m_0 l_0$ is the second smallest cyclic rearrangement of l (the smallest one being l itself).*

For obvious reasons we shall call the word $m_0 l_0$ a *Donlyn* word. We reproduce the short proof kindly provided by Perrin [11].

Notice that if (l_0, m_0) is the standard factorization of l, then m_0 is necessarily the smallest proper right factor of l. Let (l_1, m_1) be another factorization of l. Either m_1 does not start with m_0 and then $m_0 l_0 < m_1 l_1$, or $m_1 = m_0 m_2$ for some word m_2. In the latter case, as m_2 is a proper right factor of l, we have $l < m_2$ and then $m_0 l_0 < m_0 l < m_0 m_2 l_1 = m_1 l_1$. The converse is immediate.

(3.5) *Let l, m be two Lyndon words such that $l < m$. Then (l, m) is the standard factorization of lm if and only if m is less than each of the cyclic rearrangements of l other than l.*

Proof. Assume that (l, m) is the standard factorization of lm and let $l = l'l''$ with both l' and l'' nonempty. If $l'' = mm''$, then $m < l''l'$. If l'' does not start with m, then $m < l''$; otherwise, we would have $l'' < m$ and then $l''m < m$ which contradicts the fact that m is the smallest proper right factor of lm. Now if $m = l''m'$, then $l''m' = m < l''m = l''l''m'$ implies $m' < l''m' = m$ and this contradicts the fact that m is a Lyndon word. Accordingly, m cannot start with l'' and the inequality $m < l''$ implies $m < l''l'$.

Conversely, suppose that m is less than each of the cyclic rearrangements of l other than l. If (l, m) is not the standard factorization of lm, then $l = l'l''$, with l', l'' nonempty, $l''m \in L$ and $l''m < m$. By assumption, we also have $m < l''l'$. Therefore, $l''m < m < l''l'$. This implies that $m = l''m'$ with $m < m' < l'$, so that $m < m' < l' < l$. But the inequality $m < l$ cannot hold as lm is a Lyndon word. □

4. The Möbius function approach

The *content* of a word w is defined as the set $\mathrm{Cont}(w)$ of all distinct letters occurring in w. A nonempty word of X^* is said to be *multilinear*, if all its letters are distinct. Two words w and w' are said to be *disjoint*, if they have no letter in common.

Denote by $[L]$ the set of all commuting variables $[l]$ associated with each Lyndon word l. If w is a prime word, it is the cyclic rearrangement of a unique Lyndon word l. We will also write $[w] = [l]$, regarding each variable $[l]$ as being associated with the *class of cyclic rearrangements* of the word l. We next form the free Abelian monoid $\mathrm{Ab}[L]$ generated by $[L]$ and consider the following sequence $\mathrm{Ab}[L] \supset \mathcal{D} \supset \mathcal{G}$ defined as follows. Each monomial $\pi = [l_1]\,[l_2]\,\ldots\,[l_r]$ belongs to $\mathcal{D}$, if and only if the Lyndon words $l_1, l_2, \ldots, l_r$ are all *distinct*. It belongs to $\mathcal{G}$ if furthermore every element $x \in X$ occurs *at most once* in the set $\mathrm{Cont}(\pi) = \mathrm{Cont}(l_1 l_2 \ldots l_r)$. In such a case all the Lyndon words l_k are necessarily multilinear. As X is finite, the set $\mathcal{G}$ is necessarily *finite*. Moreover each element $\pi \in \mathcal{G}$ may be regarded as a *permutation* of the set $\mathrm{Cont}(\pi) \subset X$. The number r of factors in π is called the *degree* of π and denoted by $\deg \pi$.

The expansion of Λ (defined in (1.6)) is the infinite series

$$\Lambda = \sum_{\pi \in \mathcal{D}} (-1)^{\deg \pi} \pi. \tag{4.1}$$

We can also form the *polynomial*

$$\mathbf{G} := \sum_{\pi \in \mathcal{G}} (-1)^{\deg \pi} \pi. \tag{4.2}$$

The definition of the homomorphism β was given in (1.5).

Theorem 4.1. *We have the identity*

$$\beta(\Lambda) = \beta(\mathbf{G}), \tag{4.3}$$

so that $\beta(\Lambda)$ is a polynomial.

The proof of Theorem 4.1 is based on an *involution* $\pi \mapsto \pi'$ of $\mathcal{D} \setminus \mathcal{G}$ such that $\deg \pi + \deg \pi' = 0 \bmod 2$ that is defined as follows.

Construction of the involution. Say that $\pi = [l_1]\,[l_2]\,\ldots\,[l_r]$ is a *good companion* if it belongs to $\mathcal{G}$. If $\pi = [l_1]\,[l_2]\,\ldots\,[l_n]$ is a bad companion (an element of $\mathcal{D} \setminus \mathcal{G}$), let x be the *smallest* letter that occurs more than once in $l_1 l_2 \ldots l_r$. If l_i contains x, let $xu_1, xu_2, \ldots, xu_s$ be the list of all cyclic rearrangements of l_i that start with x. Write such a list for each of the words $l_1, l_2, \ldots, l_r$ and combine all those lists. It is essential to notice that all the elements in the list are *distinct*, because it is so for all the cyclic rearrangements of a Lyndon word and by assumption all the Lyndon words $l_1, l_2, \ldots, l_r$ are themselves distinct.

Now choose a total order on X such that $x = \min X$ and consider the lexicographic order on X^* with respect to that total order. Furthermore, write the previous list in *increasing order*

$$\mathrm{List}(\pi) = (xu_1, xu_2, xu_3, \ldots) \tag{4.4}$$

and consider the *smallest two elements* xu_1, xu_2. Either they come from the same factor l_i (case (i)), or from two different factors (case (ii)).

In case (i) write $xu_1 = xvxw$, $xu_2 = xwxv$ $(v, w \in X^*)$ so that $[l_i] = [xvxw] = [xwxv]$. Then define

$$\pi = [l_1]\,[l_2]\,\ldots [l_r] \;\mapsto\; \pi' = [l_1]\,\ldots\,[l_{i-1}]\,[xv]\,[xw]\,[l_{i+1}]\,\ldots [l_r].$$

In case (ii) suppose $[xu_1] = [l_1]$, $[xu_2] = [l_2]$. Then define

$$\pi = [l_1]\,[l_2]\,\ldots\,[l_r] \;\mapsto\; \pi' = [xu_1 xu_2]\,[l_3]\,\ldots\,[l_r].$$

In case (i) the word $xu_1 = xvxw$ that is first in $\mathrm{List}(\pi)$ is necessarily a Lyndon word (with respect to the latter total order on X). Furthermore, the pair (xv, xw) is the standard factorization of $xvxw$ by Property (3.4). Therefore, both xv, xw are Lyndon words and accordingly prime by Property (3.3).

On the other hand, as $xvxw$ and $xwxv$ are the smallest two elements in $\mathrm{List}(\pi)$ and since $xv < xvxw < xw < xwxv < xu_3 < \cdots$, both xv and xw are smaller than all the other words xu_k for each $k \geq 3$. It also follows from Property (3.4) that xw is less than all the cyclic rearrangements of xv other than xv. Accordingly, the smallest two elements in $\mathrm{List}(\pi')$ are xv and xw. Consequently, $(\pi')' = \pi$.

In case (ii) the two words xu_1 and xu_2 coming from two different factors are necessarily Lyndon words. As $xu_1 < xu_2$, Property (3.2) implies that $xu_1 xu_2$ is also a Lyndon word and therefore is prime. On the other hand, as $xu_1 < xu_1 xu_2 < xu_2$, the word $xu_1 xu_2$ is less than all cyclic rearrangements of xu_1 that may occur in $\mathrm{List}(\pi)$ other than all the words xu_k for each $k \geq 3$, in particular it is less than all cyclic rearrangements of xu_1 that may occur in $\mathrm{List}(\pi)$ other than xu_1. It follows from Property (3.5) that (xu_1, xu_2) is the standard factorization of $xu_1 xu_2$. Accordingly, $xu_1 xu_2$ and $xu_2 xu_1$ are the smallest two elements in $\mathrm{List}(\pi')$. Hence $(\pi')' = \pi$.

This shows that $\pi \mapsto \pi'$ is a well defined involution of $\mathcal{D} \setminus \mathcal{G}$. Moreover it satisfies

$$\beta(\pi) = \beta(\pi').$$

Therefore (4.3) holds.

Let $\pi = [l_1]\,[l_2]\,\ldots\,[l_r]$ be a monomial belonging to $\mathcal{G}$. As noted before, π may be regarded as a *permutation* of $\mathrm{Cont}(\pi)$. The set $\mathcal{G}$ is then the set of all permutations of *subsets* of X. The summand $(-1)^{\deg \pi}\beta(\pi)$ in $\beta(\mathbf{G})$ is then the term in the expansion of $\det(I-\mathcal{B})$ associated with the permutation π (see, e.g., [20], § 1). Thus

$$\beta(\mathbf{G}) = \sum_{\pi\in\mathcal{G}} (-1)^{\deg \pi}\,\beta(\pi) = \det(I-\mathcal{B}). \tag{4.5}$$

This yields identity (1.10) in view of Theorem 4.1.

5. Amitsur's identity

Reutenauer and Schützenberger [12] gave the following short proof of the Amitsur identity (1.12): Lyndon's factorization theorem (1.4) may be expressed as $\prod_l (1-l)^{-1} = X^* = (\sum_w w)$ $(w\in X^*)$ where the product is taken over all Lyndon words in nonincreasing order. As $X^* = (1-X)^{-1} = (\sum_w w)$ $(w\in X^*)$, we can deduce $(1-X)^{-1} = \prod_l (1-l)^{-1}$. Now form the inverse of the latter identity and replace X by a set of matrices $\{A_1,\ldots,A_k\}$. Taking the determinant of both sides yields identity (1.12).

Next Amitsur's identity (1.12) specializes into (1.10) in the following manner. Let $N = 2c_1$, $k = N\times N$ and consider the lexicographic order on the pairs (i,j) $(1\le i,j\le N)$. If (i,j) is the m-th pair, let A_m be the matrix whose entries are all null except the (i,j)-entry which is equal to $b(i,j)$. Then $A_1+\cdots+A_k = \mathcal{B}$.

Consider a word $l = (i_1,j_1)(i_2,j_2)\ldots(i_p,j_p)$ in the alphabet $\{(1,1),\ldots,(N,N)\}$. If $j_1 = i_2$, $j_2 = i_3$, $\ldots$, $j_{p-1} = i_p$, then A_l is the matrix whose all entries are null except the (i_1,j_p)-entry which is equal to $b(i_1,i_2)b(i_2,i_3)\cdots b(i_{p-1},i_p)b(i_p,j_p)$. If the above contiguity relations for the entries $b(i,j)$ do not hold, A_l is zero.

Now remember that $\det(I-A_l)$ is the alternating sum of the *diagonal* minors of the matrix A_l. Accordingly, when the word l satisfies the above contiguity relations *and* $j_p = i_1$, we have

$$\det(I-A_l) = 1 - b(i_1,i_2)b(i_2,i_3)\cdots b(i_{p-1},i_p)b(i_p,i_1).$$

In the other cases, $\det(I-A_l) = 1$.

The infinite product in (1.12) can then be restricted to the Lyndon words $l = (i_1,j_1)(i_2,j_2)\ldots(i_p,j_p)$ in the alphabet $\{(1,1),\ldots,(N,N)\}$ satisfying $j_1 = i_2$, $j_2 = i_3$, $\ldots$, $j_{p-1} = i_p$ and $j_p = i_1$. But those words are in bijection with the Lyndon words $i_1 i_2 \ldots i_p$ in the alphabet $[N]$. This proves identity (1.10).

6. Bass's results

As said in the introduction Bass's calculations deal with an oriented graph having c_0 vertices labelled $1,2,\ldots,c_0$ and $2\,c_1$ oriented edges. Notice that each loop around vertex i in the original unoriented graph gives rise to *two oriented* loops around i in the oriented graph. Each oriented edge e going from vertex i, called the *origin* of e, to vertex j, called the *end* of e, has a unique *reverse* edge going from j to i that will be denoted by $J(e)$ or by $\overline{e}$. Let V be the set of vertices and E be the set of oriented edges so that $\#V = c_0$ and $\#E = 2c_1$.

Say that an oriented edge e' is a *successor* of an oriented edge e, if the end of e and the origin of e' coincide. An *oriented path* from vertex i to vertex j is a linear sequence of oriented edges $e_1e_2\ldots e_m$ $(m\ge 1)$ such that for every $k = 1,2,\ldots,m-1$

the edge e_{k+1} is a sucessor of e_k; moreover the origin of e_1 is i while the end of e_m is j. The integer m is the *length* of the oriented path. It will be convenient to consider the free monoid E^* generated by the edge set E and see the oriented paths as particular elements of E^*.

When $j = i$ the oriented path is called a *pointed cycle*. The oriented path $e_1e_2\dots e_m$ is said to be *reduced*, if $J(e_1) \neq e_2$, $J(e_2) \neq e_3$, ... , $J(e_{m-1}) \neq e_m$, $J(e_m) \neq e_1$. A pointed cycle is said to be *prime*, if it cannot be expressed as the product δ^r of a given pointed cycle δ for any $r \geq 2$.

Two pointed cycles δ and δ' are said to be (cyclically) *equivalent*, if they are cyclic rearrangements of each other, i.e., if they can be expressed as words $\delta = e_1e_2\dots e_m$ and $\delta' = e_ke_{k+1}\dots e_me_1\dots e_{k-1}$ in E^* for some k ($1 \leq k \leq m$). Each equivalence class is called a *cycle*. The cycle containing the pointed cycle δ will be denoted by $[\delta]$. This notation will not conflict with our previous notation for the variables $[l]$ as we shall see.

If a pointed cycle is prime (resp. reduced, resp. of length m), all the elements in its equivalence class are prime (resp. reduced, resp. of length m). We can then speak of *prime, reduced cycles*. The *length* of a cycle γ will be denoted by $|\gamma|$. Let $\mathcal{P}$ (resp. $\mathcal{R}$) denote the set of all *prime* (resp. *prime* and *reduced*) cycles. The ingredients of (1.1) are then fully defined.

The further notions introduced by Bass are the following.

(i) For each $i = 1, 2, \dots, c_0$ let E_i (resp. $\mathcal{L}(E_i)$) be the set of the oriented edges going out of vertex i (resp. the vector space spanned by the basis E_i). The *outer degree* of vertex i is the number of oriented edges going out of i. Let $Q(i)$ be equal to that outer degree minus one, so that, as the graph is assumed to be connected, $Q(i) \geq 0$. Let $\mathcal{Q}$ be the diagonal matrix $\operatorname{diag}(Q(1), \dots, Q(c_0))$. With those notations we have: $\dim \mathcal{L}(E_i) = Q(i) + 1$. The direct sum of all the $\mathcal{L}(E_i)$'s will be denoted by $\mathcal{L}(E)$, so that $\dim \mathcal{L}(E) = \sum\limits_{i=1}^{n} (Q(i) + 1) = 2\, c_1$.

(ii) The *successiveness map* "Succ" is defined as follows: let e be an oriented edge going from vertex i to vertex j. Then

$$\operatorname{Succ}(e) := \sum_{e' \in E_j} e'. \tag{6.1}$$

In other words, $\operatorname{Succ}(e)$ is the sum of all the successors of e. The mappings Succ and the reverse map J may be regarded as endomorphisms of $\mathcal{L}(E)$. Then $T = \operatorname{Succ} - J$ is the endomorphism occurring in formula (1.2).

(iii) The *connectedness matrix* $\mathcal{K} = (K(i, j))$ ($1 \leq i, j \leq c_0$). Let $E_{i,j}$ be the set of all oriented edges going from vertex i to vertex j. Then $K(i, j) := |E_{i,j}|$. Notice that $K(j, i) = K(i, j)$ and $K(i, i) \geq 2$ if there is a loop around i. The matrix $\Delta(u)$ occurring in (1.3) is the matrix

$$\Delta(u) = I - u\, \mathcal{K} + u^2\, \mathcal{Q}. \tag{6.2}$$

To recover the first evaluation (1.2) we have to take the following ingredients:

(i) $X = E$, the set of oriented edges;

(ii) ignore each variable $b(e, e')$ when e' is *not* a successor of e or when $e' = J(e)$ (mapping it to 0) and make all the other variables equal to u. Call β_u the corresponding homomorphism β.

If a cycle is prime, it contains a unique pointed cycle which is also a Lyndon word l. We then denote the cycle by $[l]$. We have

$$\beta_u([l]) = \begin{cases} u^{|l|}, & \text{if } [l] \text{ is reduced;} \\ 0, & \text{otherwise;} \end{cases}$$

and then

$$\beta_u(\Lambda) = \prod_{\gamma \in \mathcal{R}} (1 - u^{|\gamma|}).$$

Also if $\pi = [l_1]\,[l_2]\,\ldots\,[l_r]$ is a monomial whose components are prime reduced cycles, we have $\beta_u(\pi) = u^{|\operatorname{Cont}\pi|}$. Let $\mathcal{H}$ be the set of the monomials $\pi = [l_1]\,[l_2]\,\ldots\,[l_r]$ such that each $[l_k]$ is a prime reduced cycle and every edge occurs at most once in $l_1 l_2 \ldots l_r$ and let

$$\mathbf{H} := \sum_{\pi \in \mathcal{H}} (-1)^{\deg \pi} \pi.$$

Then

$$\beta_u(\mathbf{G}) = \beta_u(\mathbf{H}),$$

so that (4.3) becomes

$$\beta_u(\Lambda) = \beta_u(\mathbf{H}). \tag{6.3}$$

As $\det(I - \mathcal{B})$ reduces to $\det(I - u\,T)$, formula (4.5) becomes

$$\beta_u(\mathbf{H}) = \sum_{\pi \in \mathcal{H}} (-1)^{\deg \pi} u^{|\operatorname{Cont}\pi|} = \det(I - u\,T). \tag{6.4}$$

7. A purely combinatorial proof of formula (1.3)

Our purpose is to give a combinatorial proof of the identity

$$(1-u^2)^{\frac{1}{2}|E|+|V|} \beta_u(\mathbf{H}) = (1-u^2)^{|E|} \det \Delta(u), \tag{7.1}$$

which is obviously equivalent to (1.3) because of (6.4). The determinant $\Delta(u)$ was defined in (6.2).

Our strategy will be to introduce a class of permutation graphs with colored edges, called *chaps* and consider the sum of the weights of all those chaps. That sum will be computed in two different ways. We will soon define *polite chaps* and later *good chaps*. It will be shown that the weighted sum of the impolite chaps is zero, as well as the weighted sum of the bad chaps. This is achieved by defining appropriate involutions that will partition all the impolite chaps into pairs each of whose members' weight is the negative of the other, and similarly for the bad chaps. It will then follow that the sum of the weights of the polite chaps equals the sum of the weights of the good chaps. The former will turn out to be the right side of (7.1) while the latter will turn out to be the left side of (7.1).

7.1. **Introducing chaps.** Suppose that the set E of all oriented edges of G is totally ordered. A *chap* may be seen as a *permutation graph* Ch (i.e., a collection of disjoint cycles) whose vertices — call them *supervertices* — are the vertices and the edges of the original graph, i.e., the elements of $V \cup E$, and whose edges — call them *superedges* — are *colored* in the following sense. Let e, e' be two oriented edges (not necessarily distinct) going out of the same vertex i, let j be the end of e and let e'' be a successor of e (its origin is then vertex j). By definition the only possible *colored superedges* of a chap are the following

$$i \xrightarrow{1} i; \quad i \xrightarrow{2} e; \quad e \xrightarrow{3a} i; \quad e \xrightarrow{3b} i;$$
$$e \xrightarrow{4a} j; \quad e \xrightarrow{4b} j; \quad e \xrightarrow{5} e; \quad e \xrightarrow{6} e'; \quad e \xrightarrow{7} e''.$$

Color	1	2	$3a$	$3b$	$4a$	$4b$	5	6	7
Weight	$1-u^2$	$u(1-u^2)$	u	$-u$	1	-1	$1-u^2$	u^2	$-u$

With each of the nine colors is associated a weight as shown in the table above.

The *weight* of a chap is defined to be the product of the weights of the superedges times the *sign* of the graph permutation.

A chap is *polite* if its superedges are of the form:

$$i \xrightarrow{1} i; \quad i \xrightarrow{2} e; \quad e \xrightarrow{3b} i; \quad e \xrightarrow{4a} j; \quad e \xrightarrow{5} e;$$

where e is an edge of origin i and end j. A chap that is not polite will be called *impolite*. If a chap is impolite, there exists a vertex i such that one of the following conditions holds from some edges e, e', e'': (A) $e \xrightarrow{3a} i$, $e \in E_i$; (B) $e \xrightarrow{4b} j$, $e \in E_i$; (C) $e \xrightarrow{6} e'$, $e \in E_i$; (D) $e \xrightarrow{7} e''$, $e \in E_i$. Denote by i the smallest such vertex and let e be the *smallest* oriented edge in E_i which is the origin of a superedge colored $3a$, $4b$, 6 or 7. Accordingly, one the following six conditions holds:

(1) $e \xrightarrow{3a} i \xrightarrow{2} e'$;

(2) $e \xrightarrow{4b} j \xrightarrow{2} e'$;

(3) $e \xrightarrow{6} e'$, $i \xrightarrow{1} i$;

(3′) $e \xrightarrow{6} e'$, $e'' \xrightarrow{x} i \xrightarrow{2} e'''$ with $x = 3a$, $3b$, $4a$ or $4b$.

(4) $e \xrightarrow{7} e'$, $j \xrightarrow{1} j$, $e' \in E_j$;

(4′) $e \xrightarrow{7} e'$, $e'' \xrightarrow{x} j \xrightarrow{2} e'''$, $e' \in E_j$ with $x = 3a$, $3b$, $4a$ or $4b$.

If (1) (resp. (2)) occurs within an impolite chap Ch, transform Ch into another (impolite) chap Ch′ by replacing occurrence (1) (resp. (2)) by occurrence (3) (resp. (4)) and conversely. Finally, if (3′) occurs, perform the change: $e \xrightarrow{6} e'''$, $e'' \xrightarrow{x} i \xrightarrow{2} e'$ and if (4′) occurs, perform the change $e \xrightarrow{7} e'''$ and $e'' \xrightarrow{x} j \xrightarrow{2} e'$. Those changes preserve the absolute value of the weight and reverse its sign.

It follows that the sum of the weights of all impolite chaps is zero. Hence the sum of the weights of all chaps equals the sum of the weights of the polite chaps. We will now proceed to compute it.

7.2. **The sum of the weights of the polite chaps.** Each polite chap consists of cycles where superedges 2 and 4a intertwine

$$i_1 \xrightarrow{2} e_1 \xrightarrow{4a} i_2 \xrightarrow{2} e_2 \xrightarrow{4a} i_3 \ \cdots \ i_k \xrightarrow{2} e_k \xrightarrow{4a} i_1$$

as well as 2-cycles of the form $i \xrightarrow{2} e \xrightarrow{3b} i$, the other vertices and edges being fixed points: $i \xrightarrow{1} i$, $e \xrightarrow{5} e$.

A cycle of the first kind has weight $u^k(1-u^2)^k$, while a cycle of the second kind has weight $-u^2(1-u^2)$. To the product of all these cycles we must multiply by $(1-u^2)$ raised to the power of the number of remaining edges and vertices. Let V_1 (resp. V_2, resp. V_3) be the set of vertices belonging to the cycles of the first kind (resp. of the second kind, resp. of the form $e \xrightarrow{5} e$). Since each cycle of the first kind has the same number of vertices and edges, and each cycle of the second kind has one vertex and one edge, the total weight is

$$(1-u^2)^{|E|} \times u^{|V_1|} \times (-u^2)^{|V_2|} \times (1-u^2)^{|V_3|}.$$

This is the same as $(1-u^2)^{|E|+|V|} \times (u/(1-u^2))^{|V_1|} \times (-u^2/(1-u^2))^{|V_2|}$.

Remember that $E_{i,j}$ denote the set of all oriented edges in the graph G going from vertex i to vertex j and $|E_{i,j}| = K(i,j)$. A polite chap Ch is then characterized by a sequence $(V_1, V_2, V_3, \sigma, f, g)$, where

(i) (V_1, V_2, V_3) is a partition of the vertex set V in disjoint subsets;
(ii) σ is a permutation of V_1;
(iii) $f : V_1 \to E$ is a mapping such that $[\sigma(i) = j] \Rightarrow [f(i) \in E_{i,j}]$;
(iv) $g : V_2 \to E$ is a mapping such that $g(i) \in E_i$.

Write $\alpha = u/(1-u^2)$ and $\beta = -u^2/(1-u^2)$. As the sign of π is given by $\varepsilon(\sigma)\,(-1)^{|V_1|+|V_2|}$, the sum of the weights of the polite chaps is equal to

$$(1-u^2)^{|E|+|V|} \sum \varepsilon(\sigma)(-\alpha)^{|V_1|}(-\beta)^{|V_2|},$$

extended over all sequences $(V_1, V_2, V_3, \sigma, f, g)$. Now the last summation, say, S is equal to

$$\begin{aligned} S &= \sum_{((V_1,V_2,V_3),\sigma)} \varepsilon(\sigma)\,(-\alpha)^{|V_1|}(-\beta)^{|V_2|} \prod_{i \in V_1} K(i,\sigma(i)) \times \prod_{j \in V_2} \deg j \\ &= \sum_{(V_1,V_2,V_3)} \det(-\alpha\, K(i,j))_{(i,j \in V_1)} \times (-\beta)^{|V_2|} \prod_{j \in V_2} \deg j \\ &= \det(I - \beta(I + \mathcal{Q}) - \alpha\, \mathcal{K}), \end{aligned}$$

where $\mathcal{Q}$ and $\mathcal{K}$ are the two matrix ingredients of the matrix $\Delta(u)$ defined in section 6. Hence the sum of all the weights of the polite chaps (and hence the sum of the weights of *all* chaps) equals

$$\begin{aligned} (1-u^2)^{|E|+|V|} \det\Big(I + \frac{u^2}{1-u^2}\,(I+\mathcal{Q}) - \frac{u}{1-u^2}\,\mathcal{K}\Big) \\ = (1-u^2)^{|E|} \det(I - u\,\mathcal{K} + u^2\,\mathcal{Q}) \\ = (1-u^2)^{|E|} \det \Delta(u), \end{aligned}$$

the right side of (7.1).

7.3. **Good and bad chaps.** A chap is *hopelessly bad* if it contains superedges colored $3a$, $3b$, $4a$, or $4b$. It is immediate that the sum of all the weights of the hopelessly bad chaps is zero since superedges colored $3a$ and $3b$ annihilate each other, as do those colored $4a$ and $4b$. It is also clear that if a superedge 2 is present, then the chap must be a hopelessly bad chap, since whenever a vertex goes to an edge, some edge must go to a vertex through a superedge necessarily colored $3a$,

$3b$, $4a$, $4b$. For the remaining chaps, the only way a vertex can be mapped onto is onto itself (superedge 1), that explains the factor of $(1-u^2)^{|V|}$ on the left side of (7.2). We can now forget about the vertices and focus on the interaction of the edges.

Having purged the hopelessly bad chaps, we can only have chaps with superedges colored 5, 6, 7, that we shall further split into:

$e \xrightarrow{5a} e$ with weight 1;
$e \xrightarrow{5b} e$ with weight $-u^2$;
$e \xrightarrow{6a} e'$ (e' having the same origin as e) with weight u^2;
$e \xrightarrow{6b} e'$ (e' having the same origin as e but $e' \neq e$) with weight u^2;
$e \xrightarrow{7a} \overline{e}$ ($\overline{e} = J(e)$) with weight $-u$;
$e \xrightarrow{7b} e'$ (e' a successor of e but different from $\overline{e}$) with weight $-u$.

A not hopelessly bad chap is nevertheless *very bad* if it contains superedges colored $5b$ or $6a$. These two cases annihilate each other so we can easily execute all the very bad chaps. It follows that a chap is *not very bad* if it contains superedges colored $5a$, $6b$, $7a$, $7b$.

Finally, a chap Ch is a *bad chap* if one of the three properties takes place:
(i) there is an edge e such that $e \xrightarrow{5a} e$ and $\overline{e} \xrightarrow{6b} e'$ occur for some e';
(ii) there is an edge e such that $e \xrightarrow{6b} e'$ and $\overline{e} \xrightarrow{7b} e''$ occur for some e', e'' ;
(iii) there is an edge e such that the sequence $e \xrightarrow{7a} \overline{e} \xrightarrow{7b} e''$ occurs for some e''.

If Ch is a bad chap let e be the smallest offending edge. We define Ch' by making the obvious transposition, i.e., by replacing the occurrence in case (i) by the occurrence in case (iii) and conversely, and replacing (ii) by $e \xrightarrow{6b} e''$ and $\overline{e} \xrightarrow{7b} e'$. It is clear that $\mathrm{Ch} \mapsto \mathrm{Ch}'$ is an involution of the set of the not very bad chaps that reverses the sign and preserves the absolute value of the weight.

A non-bad chap will be called a *good chap*. It is then a chap containing superedges colored $5a$, $6b$, $7a$, $7b$ and having the following properties:
(i) whenever $e \xrightarrow{5a} e$ occurs, then either $\overline{e} \xrightarrow{5a} \overline{e}$, or $\overline{e} \xrightarrow{7b} e'$ occurs;
(ii) whenever $e \xrightarrow{6b} e'$ occurs, then either $\overline{e} \xrightarrow{7a} e$ or $\overline{e} \xrightarrow{6b} e''$ occurs;
(iii) whenever $e \xrightarrow{7a} \overline{e}$ occurs, then either $\overline{e} \xrightarrow{6b} e'$ or $\overline{e} \xrightarrow{7a} e$ occurs.

7.4. **Enumerating the good chaps.** Referring to the left side of (7.1) we are left to prove that the weighted sum of all the good chaps is equal to

$$\beta_u(\mathbf{H}) \times (1-u^2)^{|E|/2} = \sum_{\pi \in \mathcal{H}} (-1)^{\deg \pi} u^{|\,\mathrm{Cont}(\pi)|} \times (1-u^2)^{|E|/2}.$$

Say that τ is a *back-track involution*, if there exists a subset $F(\tau)$ of the edge set E such that $J(F(\tau)) = F(\tau)$ and τ is the restriction of J to $F(\tau)$. Let $\mathcal{T}$ denote the set of all back-track involutions. (Notice that $\mathcal{T} \subset \mathcal{G}$.) With those notations we have

$$(1-u^2)^{|E|/2} = \sum_{\tau \in \mathcal{T}} (-1)^{\deg \tau} u^{|F(\tau)|}.$$

Denote by $w(\mathrm{Ch})$ and $\varepsilon(\mathrm{Ch})$ the weight and the sign of a (good) chap Ch, respectively. We are left to prove the identity

$$\sum_{\text{Ch good chap}} \varepsilon(\mathrm{Ch})\, w(\mathrm{Ch}) = \sum_{\pi\in\mathcal{H}} (-1)^{\deg\pi} u^{|\operatorname{Cont}(\pi)|} \times \sum_{\tau\in\mathcal{T}} (-1)^{\deg\tau} u^{|F(\tau)|}.$$

Construction of a bijection $Ch \mapsto (\pi,\tau)$ of the set of good chaps onto $\mathcal{H}\times\mathcal{T}$. The definition of a good chap shows that there are six cases to consider depending on the colors of superedges going out of each pair $e,\overline{e}$. The bijection is shown in the next table. For instance, in case (2) we define $\pi(\overline{e}) = e'$; furthermore $e \notin \operatorname{Cont}(\pi)$ and $e,\overline{e} \notin F(\tau)$. The definition of τ is straightforward. To obtain π we start with the cycles of Ch and make the local modifications indicated.

	Ch	π	τ
(1)	$e \xrightarrow{5a} e$ $\overline{e} \xrightarrow{5a} \overline{e}$		
(2)	$e \xrightarrow{5a} e$ $\overline{e} \xrightarrow{7b} e'$	$\overline{e} \longrightarrow e'$	
(3)	$e \xrightarrow{6b} e'$ $\overline{e} \xrightarrow{7a} e$	$\overline{e} \longrightarrow e'$	$e \longrightarrow \overline{e}$ $\overline{e} \longrightarrow e$

	Ch	π	τ
(4)	$e \xrightarrow{6b} e'$ $\overline{e} \xrightarrow{6b} e''$	$e \longrightarrow e''$ $\overline{e} \longrightarrow e'$	$e \longrightarrow \overline{e}$ $\overline{e} \longrightarrow e$
(5)	$e \xrightarrow{7a} \overline{e}$ $\overline{e} \xrightarrow{7a} e$		$e \longrightarrow \overline{e}$ $\overline{e} \longrightarrow e$
(6)	$e \xrightarrow{7b} e'$ $\overline{e} \xrightarrow{7b} e''$	$e \longrightarrow e'$ $\overline{e} \longrightarrow e''$	

In the construction of π no edge e is mapped onto its reverse $\overline{e}$, so that $\pi \in \mathcal{H}$. The inverse bijection is described by means of the same table.

What remains to be proved is the identity

$$\varepsilon(\mathrm{Ch})w(\mathrm{Ch}) = (-1)^{\deg\pi}\, u^{|\operatorname{Cont}(\pi)|}\, (-1)^{\deg\tau}\, u^{|F(\tau)|}. \tag{7.2}$$

In cases (1), (2) and (6) there is no modification in the composition of the cycles when we go from Ch to π. In case (3) the supervertex e is deleted from the cycle containing $\overline{e}$, but the transposition $e \leftrightarrow \overline{e}$ occurs in τ. In case (4) two cycles of π are made out of a single one, or a single cycle is made out of two existing ones. Therefore the sign changes, but again $e \leftrightarrow \overline{e}$ occurs in τ. Finally, in case (5) the transposition $e \xleftrightarrow{7b} \overline{e}$ is transformed into the transposition $e \leftrightarrow \overline{e}$ in τ. Hence

$$\varepsilon(h) = \varepsilon(\pi)\,\varepsilon(\tau)$$

For each $i = 1,\dots,6$ let n_i be the number of pairs $(e,\overline{e})$ falling into case (i). The weight of Ch (not counting the contribution due to the vertices) is equal to

$$\begin{aligned} w(\mathrm{Ch}) &= (-u)^{n_2} u^{2n_3} (-u)^{n_3} u^{4n_4} (-u)^{2n_5} (-u)^{2n_6} \\ &= (-u)^{n_2+n_3+2n_4+2n_6}\, (-u)^{2n_3+2n_4+2n_5} \\ &= (-u)^{|\operatorname{Cont}(\pi)|}\, (-u)^{|F(\tau)|}. \end{aligned}$$

Altogether

$$\begin{aligned} \varepsilon(\mathrm{Ch})w(\mathrm{Ch}) &= \varepsilon(\pi)\, (-u)^{|\operatorname{Cont}(\pi)|}\, \varepsilon(\tau)\, (-u)^{|F(\tau)|} \\ &= (-1)^{\deg\pi}\, u^{|\operatorname{Cont}(\pi)|}\, (-1)^{\deg\tau}\, u^{|F(\tau)|}. \end{aligned}$$

8. A matrix-algebraic proof of (1.3)

Let $(u(i,j))$ $(1 \le i,j \le c_0)$ and $(v(i))$ $(1 \le i \le c_0)$ be two sets of commuting variables. Introduce the *common origin map* "Com" as follows: if e is an oriented edge that goes from vertex i to vertex j, define:

$$\mathrm{Com}(e) := \sum_{e' \in E_i,\, e' \neq e} e'; \tag{8.1}$$

$$\mathrm{Com}(\mathbf{v})(e) := v(i)\,\mathrm{Com}(e). \tag{8.2}$$

Thus $\mathrm{Com}(e)$ is the sum of all edges, other than e, that have the same origin as e. Keeping the same notations we further define

$$\mathrm{Succ}(\mathbf{u})(e) := u(i,j)\,\mathrm{Succ}(e), \tag{8.3}$$

so that

$$A := I + \mathrm{Succ}(\mathbf{u}) + \mathrm{Com}(\mathbf{v}) \tag{8.4}$$

is an endomorphism of $\mathcal{L}(E)$. Finally, for each $i = 1, 2, \ldots, c_0$ let $\Delta(i,i) := 1 + K(i,i)u(i,i) + Q(i)v(i)$ and form the matrix

$$\Delta = \begin{pmatrix} \Delta(1,1) & K(1,2)u(1,2) & \ldots & K(1,c_0)u(1,c_0) \\ K(1,2)u(1,2) & \Delta(2,2) & \ldots & K(2,c_0)u(2,c_0) \\ \vdots & \vdots & \ddots & \vdots \\ K(c_0,1)u(c_0,1) & K(c_0,2)u(c_0,2) & \ldots & \Delta(c_0,c_0) \end{pmatrix}.$$

Proposition 8.1. *The determinant of A factorizes as*

$$\det A = \det \Delta \times \prod_{i=1}^{c_0} (1 - v(i))^{Q(i)}. \tag{8.5}$$

Proof. There is no confusion in denoting both the endomorphism and its corresponding matrix by the same symbol. For each $i,j = 1, 2, \ldots, c_0$ let $A(i,j)$ be the linear map, induced by A, that maps the space $\mathcal{L}(E_j)$ into $\mathcal{L}(E_i)$. Its corresponding matrix is of dimension $(Q(i)+1) \times (Q(j)+1)$. The matrix A itself is fully described by the contents of all the blocks $A(i,j)$ $(i,j = 1, 2, \ldots, c_0)$.

If B is a matrix of order $n \times m$, denote by $B_{1,\bullet}, B_{2,\bullet}, \ldots, B_{n,\bullet}$ its n rows (from top to bottom) and by $B_{\bullet,1}, B_{\bullet,2}, \ldots, B_{\bullet,m}$ its m columns (from left to right). Next define σB to be the matrix whose rows are $B_{1,\bullet}, B_{2,\bullet} - B_{1,\bullet}, \ldots, B_{n,\bullet} - B_{1,\bullet}$. Also let αB be the matrix whose rows are $B_{\bullet,1} + B_{\bullet,1} + \cdots + B_{\bullet,m}, B_{\bullet,2}, \ldots, B_{\bullet,n}$.

First apply $\sigma\,\alpha$ to the blocks $A(i,j)$ $(i > j)$ *below* the diagonal of the matrix A and $\alpha\,\sigma$ to the other blocks $A(i,j)$ $(i \le j)$. It is easily seen that those transformations keep invariant the value of the determinant. Its value does not change either if we further make the following shift of rows *and* columns in the resulting matrix: $1 \to 1$, $Q(1) + 2 \to 2$, $Q(1) + Q(2) + 3 \to 3, \ldots, Q(1) + \cdots + Q(c_0 - 1) + c_0 \to c_0$. We obtain the matrix

$$D = \begin{pmatrix} \Delta & \star & \ldots & \star \\ 0 & (1 - v(1))I_{Q(1)} & \ldots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (1 - v(c_0))I_{Q(c_0)} \end{pmatrix}$$

where Δ is the matrix defined above. Hence

$$\det A = \det D = \det \Delta \times \prod_{i=1}^{c_0} (1 - v(i))^{Q(i)}.$$

$\square$

Using the endomorphism $T = \mathrm{Succ} - J$ defined in section 6, we have $\mathrm{Com} = TJ$. Accordingly, if we let $v(i) = u^2$ for all i and replace all the $u(i,j)$ by $-u$ in the definition of A, we get $A = I - u(T+J) + u^2 TJ = (I - uT)(I - uJ)$. But $\det(I - uJ)$ is clearly equal to $(1-u^2)^{c_1}$. Hence $\det \Delta \prod_{i=1}^{n} (1-u^2)^{Q(i)} = \det \Delta (1-u^2)^{2c_1 - c_0} = \det(1 - uT)\det(I - uJ)$, so that $\det(I - uT) = \det \Delta (1-u^2)^{c_1 - c_0}$, which is Bass's identity (1.3).

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