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EMBEDDINGS OF OPEN MANIFOLDS

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Abstract. Let $TOP(M)$ be the simplicial group of homeomorphisms of M. The following theorems are proved.

Theorem A. *Let M be a topological manifold of dim ≥ 5 with a finite number of tame ends ε_i, $1 \leq i \leq k$. Let $TOP^{ep}(M)$ be the simplicial group of end preserving homeomorphisms of M. Let W_i be a periodic neighborhood of each end in M, and let $p_i : W_i \to \mathbb{R}$ be manifold approximate fibrations. Then there exists a map $f : TOP^{ep}(M) \to \prod_i TOP^{ep}(W_i)$ such that the homotopy fiber of f is equivalent to $TOP_{cs}(M)$, the simplicial group of homeomorphisms of M which have compact support.*

Theorem B. *Let M be a compact topological manifold of dim ≥ 5, with connected boundary ∂M, and denote the interior of M by $Int\ M$. Let $f : TOP(M) \to TOP(Int\ M)$ be the restriction map and let $\mathcal{G}$ be the homotopy fiber of f over $id_{Int\ M}$. Then $\pi_i\ \mathcal{G}$ is isomorphic to $\pi_i\ \mathcal{C}(\partial M)$ for $i > 0$, where $\mathcal{C}(\partial M)$ is the concordance space of ∂M.*

Theorem C. *Let $q_0 : W \to \mathbb{R}$ be a manifold approximate fibration with dim $W \geq 5$. Then there exist maps $\alpha : \pi_i\ TOP^{ep}(W) \to \pi_i\ TOP(\hat{W})$ and $\beta : \pi_i\ TOP(\hat{W}) \to \pi_i\ TOP^{ep}(W)$ for $i > 1$, such that $\beta \circ \alpha \simeq id$, where $\hat{W}$ is a compact and connected manifold and W is the infinite cyclic cover of $\hat{W}$.*

0. Introduction

In this paper we study the homotopy type of the simplicial group of homeomorphisms of an open manifold of dimension ≥ 5 into itself. There has been extensive research about the homotopy type of $TOP(M)$, for a compact topological manifold M. For example, see [4], [9], [38] and the survey papers [10], [11] and [19]. But, if M is a noncompact manifold, very little about this simplicial group is known.

Let M be a topological manifold of dim ≥ 5 with a finite number of tame ends ε_i, $1 \leq i \leq k$. Each end ε_i of M has a neighborhood W_i which is a finitely dominated infinite cyclic cover of a compact and connected manifold. Hughes and Ranicki showed in [13] that for each W_i, there exists a manifold approximate fibration over $\mathbb{R}$, $p_i : W_i \to \mathbb{R}$. The neighborhood W_i is called a periodic neighborhood of M.

Denote by $TOP^{ep}(M)$ the simplicial group of end preserving homeomorphisms of M. Let $TOP_{cs}(M)$ be the simplicial group of homeomorphisms of M which have compact support. Then $TOP^{ep}(M) \subset TOP(M)$ and $TOP_{cs}(M) \subset TOP^{ep}(M)$.

With those notations, the main result of Section 2 is

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Theorem A. *There exists a map* $f : TOP^{ep}(M) \to \prod_i TOP^{ep}(W_i)$ *such that the homotopy fiber of* f *is equivalent to* $TOP_{cs}(M)$.

Hughes' Approximate Isotopy Covering Theorem – Relative Version, and Siebenmann's Recognition Criterion for I–regular neighborhoods have an important role in the proof of this result.

Let $\mathcal{GE}_{\varepsilon_i}(\mathcal{N}(\varepsilon_i), M)$, $1 \leq i \leq k$, be the simplicial set of equivalence classes of germs of embeddings of a neighborhood of ε_i into M which send ε_i into itself.

The proof of Theorem A is given in two steps. In the first step we show that the map $TOP^{ep}(M) \to \prod_i \mathcal{GE}_{\varepsilon_i}(\mathcal{N}(\varepsilon_i), M)$ is a fibration with fiber $TOP_{cs}(M)$, using Siebenmann's Isotopy Extension Theorem.

In the second step we show that $\prod_i TOP^{ep}(W_i) \to \prod_i \mathcal{GE}_{\varepsilon_i}(\mathcal{N}(\varepsilon_i), M)$ is a homotopy equivalence. This homotopy equivalence is a generalization of the Kister–Mazur Theorem: $TOP(\mathbb{R}^n\ ;0) \simeq \mathcal{GE}_0(\mathcal{N}(0), \mathbb{R}^n)$. A new proof of this theorem is given in Section 2, Corollary 2.6.

As an application of Theorem A, a new proof of a theorem of Anderson, Hsiang and Hatcher [3] is given in Section 2, Theorem 2.9.

Kuiper and Lashof in [23] proved a theorem where they express $TOP(\mathbb{R}^n)$ in terms of $TOP(D^n)$ and the concordance space for S^{n-1}, $\mathcal{C}(S^{n-1})$, i.e.

Kuiper–Lashof Theorem. $\mathcal{C}(S^{n-1}) \to TOP(D^n) \to TOP(\mathbb{R}^n)$ *is a homotopy fibration sequence.*

In this work, the Kuiper–Lashof Theorem is generalized: D^n is replaced by any compact manifold M and $\mathbb{R}^n$ by the interior of M. That is the main result of Section 3.

Theorem B. *Let* M *be a compact topological manifold of dim* ≥ 5, *with connected boundary* ∂M, *and denote the interior of* M *by* $Int\ M$. *Let* $f : TOP(M) \to TOP(Int\ M)$ *be the restriction map and let* $\mathcal{G}$ *be the homotopy fiber of* f *over* $id_{Int\ M}$. *Then,* $\pi_i\ \mathcal{G}$ *is isomorphic to* $\pi_i\ \mathcal{C}(\partial M)$ *for* $i > 0$, *where* $\mathcal{C}(\partial M)$ *is the concordance space of* ∂M.

Siebenmann's Isotopy Extension Theorem for CS sets [34] has an important role in the proof of this result.

The map f in Theorem B is not necessarily a fibration, and an example is given of a self–homeomorphism ρ of $Int\ M$ which is not the restriction of a self–homeomorphism of M but ρ is isotopic to the identity map.

Finally, in Section 4 we prove

Theorem C. *Let* $q_0 : W \to \mathbb{R}$ *be a manifold approximate fibration with dim* $W \geq 5$. *Then*

1. *there exists a manifold approximate fibration* $q : \hat{W} \to S^1$ *such that the following diagram commutes :*

$$\begin{array}{ccc} W & \xrightarrow{q_0} & \mathbb{R} \\ \downarrow & & \downarrow{\scriptstyle exp} \\ \hat{W} & \xrightarrow[q]{} & S^1 \end{array}$$

2. $\pi_n\ TOP^{ep}(W)$ *is a direct summand of* $\pi_n\ TOP(\hat{W})$ *for* $n > 1$, *where* $\hat{W}$ *is a compact and connected manifold and* W *is the infinite cyclic cover of* $\hat{W}$.

The proof of this theorem uses results of Sections 2 and 3.

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1. Preliminaires

In this section, we establish definitions, results and some properties of the objects that will be used below.

The following definition and examples may be found in Siebenmann [34].

Definition 1.1. A *stratified set* X, in Siebenmann's sense, is a metrizable space X with a filtration $\emptyset = X^{-1} \subset X^0 \subset \cdots \subset X^{k-1} \subset X^k \subset \cdots \subset X$ by closed subsets X^k, $k \geq -1$, such that for each $k \geq 0$, the components of $X^k - X^{k-1}$ are open in $X^k - X^{k-1}$.

It is a *top stratified set* if $X^k - X^{k-1}$ is a topological k-manifold without boundary, called the *k-stratum* of X.

A stratified set X is *locally cone–like* if for each $x \in X$, say $x \in X^k - X^{k-1}$, there is an open neighborhood U of x in $X^k - X^{k-1}$, a compact stratified set of finite dimension L (called a *link* of x in X) and a stratum-preserving homeomorphism of $U \times cL$ onto an open neighborhood of x in X. (cL is the open cone in L. Regard U as a stratified set with $U = U^k - U^{k-1}$.)

A *CS set* is a locally cone–like top stratified set.

Example 1. A topological m-manifold X is a CS set. Here $X^k = X$ for $k \geq m$, $X^{m-1} = \partial X$, and $X^i = \emptyset$ for $i \leq m-2$.

Example 2. Let M be a compact topological manifold with connected boundary ∂M. The topological space $X = Int\ M \cup \{\infty\}$, the one-point compactification of $Int\ M$, is a CS set. The space $Y = \partial M * S^0$, where $\partial M * S^0$ denotes the join of ∂M and S^0, is a CS set.

A *mock open cone* is a locally compact metric space C with a homotopy $\gamma_t : C \to C$, with $0 \leq t \leq 1$, such that

1. γ_t, $0 \leq t < 1$, is an isotopy of id_C, through homeomorphisms,
2. $\gamma_0 = id_{|C}$, $\quad \gamma_1(C) = v \in C \quad$ and $\quad \gamma_t(v) = v$, $\forall t$.

A topological stratified set X is a *locally weakly cone–like set* (WCS) if for each $x \in X^k - X^{k-1}$, there is a mock open cone C with vertex v and a homeomorphism $\theta : \mathbb{R}^k \times C \to U$, where U is an open neighborhood of x in X, such that $\theta^{-1}(X^k) = \mathbb{R}^k \times v$.

Example 3. Open cones on compact sets are trivial examples of mock open cones.

Example 4. Let W be a connected topological manifold of dim ≥ 5. Assume W is proper homotopy equivalent to (or even properly dominated by) $F \times \mathbb{R}$, with F a finite connected CW complex. Assume e_+ is one of the two end points of W. Then $C = W \cup e_+$ is a non-trivial example of a mock open cone. A homotopy γ_t of $W \cup e_+$ to e_+ can be constructed by an engulfing argument such that (1) and (2) hold and, for each t, γ_t fixes points outside some compact set in W (depending this time on t). See [34, §5].

Let M be a manifold and U be an open subset of M. If K is a subset of M with $K \subset U$, let $\underline{Emb}(U, M; K)$ denote the space of proper embeddings of U into M which are the identity on K, and let $\underline{Emb}(U, M)$ denote $\underline{Emb}(U, M; \emptyset)$. A neighborhood of $h \in \underline{Emb}(U, M; K)$ is of the form

$$N(h) = \{g \in \underline{Emb}(U, M; K)/d(g(x), h(x)) < \epsilon, \forall x \in C\},$$

where C is a compact subset of U, $\epsilon > 0$ and d is the metric on M.

Theorem 1.2 (Deformation Theorem). *Let X be a Hausdorff, locally compact, locally connected topological space (CS set or WCS set), $K \subset X$ be a compact set and $V \subset X$ be an open neighborhood of K. If $h : V \to X$ is an open embedding sufficiently near to the inclusion $i : V \hookrightarrow X$ in $\underline{Emb}(V, X)$, then there exists an isotopy h_t, $0 \le t \le 1$, of h through open embeddings $h_t : V \to X$ such that $h_1 = i$ on K and $h_t = h$ outside some compact set in V (independent of t and even of h). Furthermore, the isotopy is standard in the sense that it is constructed to be a continuous function on h as h varies sufficiently near i.* See [34], and for sufficiently near see [7].

Note. Let A be a subset of a topological space X and $x \in X$. A is a *neighborhood* of x if A contains an open set containing x.

Lemma 1.3. *Let X be a Hausdorff, locally compact, locally connected topological space; let K and U be subsets of X such that U is an open neighborhood of the compact set K. Then K has a compact neighborhood C in X such that $C \subset U$.*

Proof. Since X is locally compact, $x \in K$ contains a compact neighborhood C_x such that $C_x \subset U$. Thus, for each $x \in K$ the collection $\mathcal{A} = \{\mathring{C}_x\}_{x \in K}$ is an open cover of K. And since K is compact, this implies that there exists a finite subcollection $\{\mathring{C}_{x_1}, \mathring{C}_{x_2}, \ldots, \mathring{C}_{x_n}\}$ that also covers K. Thus, let $C = \bigcup_{i=1}^{n} C_{x_i}$ be the compact neighborhood of K in X and $C \subset U$ (since each $C_{x_i} \subset U$). $\square$

Theorem 1.4 (Siebenmann's Isotopy Extension Theorem to (X, K)). *Let X be a Hausdorff, locally compact, locally connected topological space (CS set or WCS set); let K and V be subsets of X such that V is an open neighborhood of the compact set K, and such that K has a compact frontier in V. Let $f_t : V \to X$, $t \in I^n$, be a continuous family of embeddings, and let $f_t(K)$ be closed. Assume that f_t respects strata. Then there exists a continuous family of homeomorphisms $F_t : X \to X$, $t \in I^n$, fixed outside some compact set, such that $F_{t_0} = id$, $F_t|_K = f_t$, $\forall t \in I^n$ and F_t respects strata.* See [34, Theorem 6.5].

Remark 1. By Lemma 1.3, the isotopy F_t in Siebenmann's Isotopy Extension Theorem above can be chosen such that $F_t = f_t$ in some compact neighborhood C of K in X.

This theorem implies

Theorem 1.5. *With the same notation as in Theorem 1.4, the restriction map $TOP(X) \to \mathcal{GE}(\mathcal{N}(K), X)$ is a (Kan) fibration.*

Here $\mathcal{GE}(\mathcal{N}(K), X)$ denotes the simplicial set of embeddings $f : U \times \Delta^k \to X \times \Delta^k$ commuting with the projection on Δ^k, where U is an open neighborhood of K and two such embeddings f and $f' : U' \times \Delta^k \to X \times \Delta^k$ are identified if they agree in a smaller neighborhood of K. Let $TOP(X)$ denote the simplicial set of homeomorphisms of X. See [5], [24].

Proof. In fact,

$$\begin{tikzcd} \Delta^k \times \{0\} \arrow[r, "\beta"] \arrow[d] & TOP(X) \arrow[d, "r"] \\ \Delta^k \times I \arrow[r, "\alpha"'] & \mathcal{GE}(\mathcal{N}(K), X) \end{tikzcd}$$

Let α be a $(k+1)$–simplex of $\mathcal{GE}(\mathcal{N}(K), X)$ given by an embedding $q : U \times \Delta^k \times I \to X \times \Delta^k \times I$, where Δ^{k+1} is identified with $\Delta^k \times I$ and U is a neighborhood of K in X. Let $C \subset U$ be a compact neighborhood of K given by Lemma 1.3. Suppose we are given a lift of the 0–level of α to a k–simplex β of $TOP(X)$. Thus β is given by the homeomorphism $p : X \times \Delta^k \to X \times \Delta^k$ such that $p = q$ on $C \times \Delta^k$. Let $i : U \times \Delta^k \hookrightarrow X \times \Delta^k$ be the inclusion map. Consider the composition $U \times \Delta^k \times I \xrightarrow{q} U \times \Delta^k \times I \xrightarrow{i \times id_I} X \times \Delta^k \times I$, which is a family of embeddings.

From Theorem 1.4 applied to (X, C), there exists an isotopy of homeomorphisms $f : X \times \Delta^k \times I \to X \times \Delta^k \times I$ such that $f = q$ on $C \times \Delta^k \times I$ and $f|_{X \times \Delta^k} = p$. Thus this describes a $(k+1)$–simplex of $TOP(X)$ which is the required lift of α. □

Lemma 1.6. *Let X and Y be connected Kan simplicial sets with base points x and y respectively. Let $f : X \to Y$ be a base point preserving map. If $E(f)$, the homotopy fiber of f over y, is contractible then f is a homotopy equivalence.* For the definition of $E(f)$ see [25].

Lemma 1.7. *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & C \arrow[d, "g'"] \\ B \arrow[r, "f'"'] & D \end{tikzcd}$$

be a commutative diagram of connected based Kan simplicial sets. Then the data determines simplicial maps α and β between the homotopy fibers, $\alpha : E(f) \to E(f')$ and $\beta : E(g) \to E(g')$. Thus, $E(\alpha)$, the homotopy fiber of α, is weak homotopy equivalent to the homotopy fiber $E(\beta)$ of β.

Remark 2. See Adams [1] for the proof of the analogous result for topological spaces.

We refer to Siebenmann's thesis [30] for definition and basic results on ends and tame ends. An end of a manifold is tame if it has a sequence of connected neighborhoods satisfying certain properties. The ends of the interior of a compact manifold are examples of tame ends.

Manifolds with tame ends arise in Siebenmann [31], [33] as finitely dominated infinite cyclic covers of compact manifolds.

Let X be a compact space and $f : X \to S^1$ be a continuous map. Let Y be an infinite cyclic cover of X induced by f from $exp : \mathbb{R} \to S^1$. Then, there exist a proper map $p : Y \to \mathbb{R}$ and a generating covering translation $T : Y \to Y$ such that $pT(y) = p(y) + 1$, $\forall y \in Y$. See [14] and [33].

Definition 1.8. A neighborhood V of an end ε of M is a *periodic neighborhood* if V is homeomorphic to a finitely dominated infinite cyclic cover of a connected and compact manifold.

Remark 3. Tame ends of an open manifold of dimension ≥ 5 have periodic neighborhoods. See Siebenmann [31]. This is a special case of the Main Theorem in [18] (see page 1 and let $B =$ point). See also [8].

We now recall some definitions on manifold approximate fibrations. See [14].

Definition 1.9. Let X and B be topological spaces. Given $\epsilon > 0$, a map $p : X \to B$ is an *ϵ–fibration* if for any space Z and maps $f : Z \to X$, $F : Z \times I \to B$ such that $F(z, 0) = pf(z)$ for $z \in Z$, there exists a map $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(z, 0) = f(z)$ and $p\tilde{F}$ is ϵ-close to F.

An *approximate fibration* is a map $p : X \to B$ which is an ϵ–fibration for every $\epsilon > 0$.

A *manifold approximate fibration* is a proper map $p : X \to B$ which is an approximate fibration and such that X is a finite dimensional manifold without boundary.

The map p in the definition of an approximate fibration is not necessarily onto. But if $p : X \to B$ is an approximate fibration then the image of p in any path component of B is either empty or dense. In particular, the standard inclusion $(0, 1) \hookrightarrow [0, 1]$ is an approximate fibration. If p is a closed map then the image of p is closed and hence is either empty or all of a particular path component.

Let $p : X \to \mathbb{R}$ be a manifold approximate fibration.

Recall from [14] that a k–simplex of the simplicial group $TOP^c\big(X \xrightarrow{p} \mathbb{R}\big)$ of controlled homeomorphisms of X is a homeomorphism $h : X \times \Delta^k \times [0, 1) \to X \times \Delta^k \times [0, 1)$ such that h commutes with the projection on $\Delta^k \times [0, 1)$ and the compositions

$$X \times \Delta^k \times [0, 1) \xrightarrow{h} X \times \Delta^k \times [0, 1) \xrightarrow{p \times id} \mathbb{R} \times \Delta^k \times [0, 1)$$

and

$$X \times \Delta^k \times [0, 1) \xrightarrow{h^{-1}} X \times \Delta^k \times [0, 1) \xrightarrow{p \times id} \mathbb{R} \times \Delta^k \times [0, 1)$$

extend continuously to maps

$$X \times \Delta^k \times [0, 1] \to \mathbb{R} \times \Delta^k \times [0, 1]$$

via $p \times id : X \times \Delta^k \times [0, 1] \to \mathbb{R} \times \Delta^k \times [0, 1]$.

Recall from [16] that a k–simplex of the simplicial group $TOP^b\big(X \xrightarrow{p} \mathbb{R}\big)$ of bounded homeomorphisms of X consists of a homeomorphism $h : X \times \Delta^k \to X \times \Delta^k$ commuting with the projection on $\Delta^k \times [0, 1)$, and such that h is bounded in the $\mathbb{R}$–direction. Note that a map $f : X \to Y$ between two topological spaces is called *bounded* if there exists a number $c > 0$, which depends on f, such that for each $x \in X$, $\| p_2 f(x) - p_1(x) \| < c$, where $p_1 : X \to \mathbb{R}$ and $p_2 : Y \to \mathbb{R}$.

Remark 4. Hughes and Ranicki in [13, Lemma 7.7] showed that a topological manifold M of dimension ≥ 5 admits an approximate fibration to $\mathbb{R}$ if and only if M is a finitely dominated infinite cyclic cover of a compact space.

Theorem 1.10. *Let W be a connected manifold of dim ≥ 5 and let $p : W \to \mathbb{R}$ be a manifold approximate fibration. Then the following simplicial groups are homotopy equivalent:*

1. $TOP^{ep}(W)$,
2. $TOP^{b}(W \xrightarrow{p} \mathbb{R})$,
3. $TOP^{c}(W \xrightarrow{p} \mathbb{R})$,

where $TOP^{ep}(W)$ denotes the simplicial group of end preserving homeomorphisms of W. See [16].

Theorem 1.11. (Hughes'Approximate Isotopy Covering Theorem − Relative Version). *Let $p : M \to B$ be a manifold approximate fibration with dim $M \geq 5$, and let B be a metric space. Let C and $\tilde{C}$ be closed subsets of B such that $C \subset int\ \tilde{C}$, let α be an open cover of B and let $h_t : B \to B$ be an isotopy which is supported on C. Then there exists an isotopy $H_t : M \to M$, $0 \leq t \leq 1$, such that pH_t is α–close to $h_t p$, for each t, and H_t is supported on $p^{-1}(\tilde{C})$.*

Proof. In [16, Theorem 6.1] the case where $C = \emptyset$ is deduced from the Approximation Theorem in [12]. The proof of the relative version is the same except one uses a Relative Approximation Theorem. □

We refer to Siebenmann [35] for definitions on I–regular neighborhoods. We summarize the basic results of I–regular neighborhoods. The proofs are essentially in [35], [36], [37].

Let Y be a topological space and X be any subset of Y.

I–Compression Axiom. (Y, X) *satisfies I–compression axiom if for any neighborhood U of X in Y there exists a neighborhood $V \subset U$ so that V is I–compressible towards X in U.*

Remark 5. Under the hypothesis of the I–compression axiom, every regular neighborhood of X in Y is an I–regular neighborhood. See [36, Remark 1.7].

Theorem 1.12 (Uniqueness of I–regular neighborhoods). *If E and E' are two I–regular neighborhoods of X in Y, then there exists an isotopy of embeddings $g_t : E \to Y$, $0 \leq t \leq 1$, fixing a neighborhood of X in Y (independent of t) and such that $g_0 = \imath$, where $\imath : E \hookrightarrow Y$ is the inclusion and $g_1(E) = E'$.* See [35, Theorem 1.4] or [36, Theorem 2.2].

Theorem 1.13 (Recognition Criterion). *Suppose Y is locally compact and $X \subset Y$ is compact. Then an open neighborhood U of X in Y is I–regular if and only if U is σ–compact, and for each compact set $K \subset U$ there exists a compact set $L \subset U$ such that K is I–compressible towards X in L.* See [35, Theorem 3.1] or [36, Theorem 4.1].

Remark 6. Siebenmann in [34, page 254] says that if (Y, X) is compact and metrizable, an open neighborhood U of X is regular if and only if K is compressible towards X in U for each compact set $K \subset U$.

Theorem 1.14. *Let W be a topological manifold of dim ≥ 5, let ε be an isolated end of W and let $W \cup \varepsilon$ be the one-point compactification of W. Suppose that ε admits I–regular neighborhoods in $(\partial W) \cup \varepsilon$. If ε is tame then ε admits I–regular neighborhoods in $W \cup \varepsilon$.* See [37, §2].

Remark 7. In the theorem above, if some neighborhood U of ε is such that $(\partial W) \cap U = \emptyset$, in particular if $\partial W = \emptyset$, then trivially ε admits I–regular neighborhoods in $(\partial W) \cup \varepsilon$.

Remark 8. In Theorem 2.7 we will prove that $W \cup e_+$ is an I–regular neighborhood, for W a total space of a manifold approximate fibration over $\mathbb{R}$.

2. Manifolds with tame ends

Let M be a non-compact, separable topological manifold of dimension ≥ 5, with compact (possibly empty) boundary ∂M, and let M have a finite number of ends $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$, each one tame. By Remark 3 and Remark 4, for each end ε_i of M, choose a periodic neighborhood W_i and a manifold approximate fibration $p_i : W_i \to \mathbb{R}$.

Let $TOP(M)$ denote the simplicial group of homeomorphisms of M, where a k-simplex is a homeomorphism $h : M \times \Delta^k \to M \times \Delta^k$ commuting with the projection on Δ^k. Let $TOP^{ep}(M)$ denote the simplicial subgroup of $TOP(M)$ of homeomorphisms of M which preserve all the ends of M. Notice that $TOP^{ep}(M)$ is the union of certain components of $TOP(M)$.

Let $TOP_{cs}(M)$ be the simplicial subgroup of $TOP^{ep}(M)$ of homeomorphisms of M with compact support.

Let X be a topological space, $K \subset X$ a compact set. Let $\mathcal{GE}_K(\mathcal{N}(K), X)$ be the simplicial set of equivalence classes of germs of embeddings whose k-simplices are represented by embeddings $h : U \times \Delta^k \to X \times \Delta^k$ commuting with the projection on Δ^k, for some open neighborhood U of K in X and such that $h(K) = K$. Two such embeddings $h_i : U_i \times \Delta^k \to X \times \Delta^k$, $i = 1, 2$, are equivalent if they agree on $U_3 \times \Delta^k$, where $U_3 \subset U_1 \cap U_2$.

Let $A \subset X$. Let $TOP(X\ rel\ A)$ denote the simplicial group whose k-simplices are homeomorphisms $h : X \times \Delta^k \to X \times \Delta^k$ commuting with the projection on Δ^k and which restrict to the identity on A.

Theorem A. *There exists a map $f : TOP^{ep}(M) \to \prod_i TOP^{ep}(W_i)$ such that the homotopy fiber of f is equivalent to $TOP_{cs}(M) \subset TOP^{ep}(M)$.*

Henceforth we shall assume that M has just one tame end ε, with a periodic neighborhood W and a manifold approximate fibration $p : W \to \mathbb{R}$. Denote by e_+ and e_- the two ends of W. The general case follows easily.

The main result follows from the analysis of the diagram

$$\begin{array}{ccccc}
G & \longrightarrow & TOP^{ep}(M) & \xrightarrow{\ f\ } & TOP^{ep}(W) \\
\big\uparrow & & \big\| & & \big\uparrow{\scriptstyle g} \\
TOP_{cs}(M) & \longrightarrow & TOP^{ep}(M) & \xrightarrow{\ \varsigma\ } & \mathcal{GE}_{\varepsilon}(\mathcal{N}(\varepsilon), M)
\end{array}$$

where the following will be proved:

1. The restriction map ς is a fibration with fiber $TOP_{cs}(M)$.
2. The map g is a homotopy equivalence.

In this diagram G denotes the homotopy fiber of f and $\mathcal{GE}_{\varepsilon}(\mathcal{N}(\varepsilon), M)$ denotes the simplicial set of equivalence classes of germs of embeddings of a neighborhood of ε into M which send ε into itself.

The proof of (1) is given in Theorem 2.1.

In order to prove (2) we construct, in Theorem 2.5, a homotopy equivalence $\delta : TOP^{ep}(W) \to \mathcal{GE}_{\varepsilon}(\mathcal{N}(\varepsilon), M)$. Then let g be a homotopy inverse to δ.

From Theorem 1.10 we have that $TOP^{ep}(W)$ is homotopy equivalent to $TOP^{c}\big(W \xrightarrow{p} \mathbb{R}\big)$.

Proof of Theorem A. Assuming (1) and (2) above, it follows that G is homotopy equivalent to $TOP_{cs}(M)$ in the diagram above, where f is the composition map $f = g\varsigma$. □

Let $W \hookrightarrow M$ be a periodic neighborhood of ε. Let $TOP_W(M) \subset TOP(M)$ be the subsimplicial group of homeomorphisms of M which restrict to a homeomorphism of W.

Corollary A1. *(i) $BTOP_W(M) \to BTOP(M)$ is a homotopy equivalence.*
(ii) $BTOP_W(M) \to BTOP^{ep}(W)$ is a fibration.

Theorem 2.1. *The restriction map $\varsigma : TOP^{ep}(M) \to \mathcal{GE}_\varepsilon(\mathcal{N}(\varepsilon), M)$ is a fibration.*

Proof. This follows by applying Theorem 1.5 to $X = M \cup \varepsilon$ and $K = \varepsilon$. Notice that Example 4 (Section 1) of a mock open cone implies that $M \cup \varepsilon$ is a WCS set.

The fiber of ς over the standard embedding is $TOP_{cs}(M)$. □

Proposition 2.2. *Let X be a topological space, $K \subset X$ a compact set, and let V be an open neighborhood of K in X. Then the inclusion $V \subset X$ induces a map $\phi : \mathcal{GE}_K(\mathcal{N}(K), V) \to \mathcal{GE}_K(\mathcal{N}(K), X)$ which is a homotopy equivalence.*

Proof. Let $h : U \to V$ be a representative of the class $[h]$ in $\mathcal{GE}_K(\mathcal{N}(K), V)$, where U is a neighborhood of K in V. Then $i \circ h : U \to V$ is an embedding such that $i \circ h(K) = K$, where i is the inclusion map. Thus define $\phi : \mathcal{GE}_K(\mathcal{N}(K), V) \to \mathcal{GE}_K(\mathcal{N}(K), X)$ by $\phi[h] = [i \circ h]$.

Conversely, let $g : U' \to X$ be an embedding representative of the class $[g]$ in $\mathcal{GE}_K(\mathcal{N}(K), X)$, where U' is a neighborhood of K in X such that $g(K) = K$. Since $V \subset X$ and $g(K) = K$, $g^{-1}(V) \supset K$ is an open set. Let L be a neighborhood of K such that $L \subset g^{-1}(V)$. Denote $g' = g|_L$. Then $\bar{g} : L \to V$ such that $\bar{g}(y) = g'(y)$ for $y \in L$ is an embedding in $\mathcal{GE}_K(\mathcal{N}(K), V)$. Thus define $\psi : \mathcal{GE}_K(\mathcal{N}(K), X) \to \mathcal{GE}_K(\mathcal{N}(K), V)$ by $\psi[g] = [\bar{g}]$.

We have $\phi \circ \psi = id_{\mathcal{GE}_K(\mathcal{N}(K), X)}$ and $\psi \circ \phi = id_{\mathcal{GE}_K(\mathcal{N}(K), V)}$. □

Corollary 2.3. *The map $\phi : \mathcal{GE}_{e_+}(\mathcal{N}(e_+), W) \to \mathcal{GE}_\varepsilon(\mathcal{N}(\varepsilon), M)$ is a homotopy equivalence.*

Proof. This follows from Proposition 2.2, where $K = e_+$ which is also the end of M, $V = W$ and $X = M$. □

Proposition 2.4. *The restriction map $\eta : TOP^{ep}(W) \to \mathcal{GE}_{e_+}(\mathcal{N}(e_+), W)$ is a homotopy equivalence.*

Proof. This is implied by the following claims. □

Claim 1. *η is a fibration.*

Proof. This follows from Theorem 2.1 with $M = W$.

The fiber of η over the standard embedding is $TOP(W\ rel\ \mathcal{N}(e_+))$. □

Claim 2. *$TOP(W\ rel\ \mathcal{N}(e_+)) \simeq *$.*

Proof. Let $p : W \to \mathbb{R}$ be a manifold approximate fibration and $W_k = p^{-1}(k, +\infty)$ be a neighborhood of e_+ in W. Let $h : W \to W$ be a homeomorphism such that $h|_{W_k} = id$.

Consider a homeomorphism $g : \mathbb{R} \to \mathbb{R}$ such that $g|_{(k,+\infty)} = id$, where $(k, +\infty)$ is a neighborhood $+\infty$ of in $\mathbb{R}$.

An isotopy of g to the identity, fixing $(k, +\infty)$, is given by $g_s : \mathbb{R} \to \mathbb{R}$, $0 \leq s \leq 1$:

$$g_s(t) = \begin{cases} g(t + \frac{s}{1-s}) - \frac{s}{1-s} & \text{if } 0 \leq s < 1, \\ id & \text{if } s = 1. \end{cases}$$

g_s is continuous near 1: given $t \in \mathbb{R}$, choose s close enough to 1 so that $t + \frac{s}{1-s} > k$. Then $g(t + \frac{s}{1-s}) = t + \frac{s}{1-s}$. Thus, $g_s(t) = t + \frac{s}{1-s} - \frac{s}{1-s} = t$.

By Theorem 1.11 there exists a continuous family of homeomorphisms $G_s : W \to W$, $0 \leq s \leq 1$, such that $G_1 = id$ and $(p \times id_I)G_s$ is close to $g_s(p \times id_I)$. G_s is an isotopy of h and the identity, fixing a neighborhood of e_+ contained in W_k. □

Claim 3. *η is onto on π_0.*

Proof. Let N be a neighborhood of e_+ in W such that N is also a total space of a manifold approximate fibration $q : N \to \mathbb{R}$. Applying Corollary 2.8, there exists an isotopy of embeddings $h_t : N \to W$, $0 \leq t \leq 1$, such that $h_0 =$ inclusion $\imath : N \hookrightarrow W$, $h_1 =$ homeomorphism, and there exists a smaller neighborhood V of e_+ in W such that $h_t|_V = \imath|_V$ for all t. Let $f : N \to W$ such that $f(e_+) = e_+$ be an embedding in $\mathcal{GE}_{e_+}(\mathcal{N}(e_+), W)$. Applying Corollary 2.8 again to $f(N) \subset W$, we get an isotopy of embeddings $g_t : f(N) \to W$ such that $g_0 =$ inclusion $\imath : f(N) \hookrightarrow W$, $g_1 =$ homeomorphism, and there exists a smaller neighborhood V' of e_+ in $f(N)$ such that $g_t|_{V'} = g_0|_{V'}$.

Define an isotopy of embeddings $s_t : N \to W$, $0 \leq t \leq 1$, by the composition $s_t = fg_t$ so that $s_0 = f$, $s_1 =$ homeomorphism, and there exists a smaller neighborhood V'' of e_+ such that $s_t|_{V''} = f|_{V''}$.

Define $F : W \to W$ by $F = s_1(g_1)^{-1}$. Then F is a homeomorphism such that $F|_{V \cap f^{-1}(V')} = f|_{V \cap f^{-1}(V')}$, i.e., F is a homeomorphism which is germ equivalent to f at e_+. □

Claim 4. *Any two fibers of η are isomorphic.*

Proof. Let $F_0 = TOP(W\ rel\ \mathcal{N}(e_+))$ be the fiber of η over the standard embedding $i : \mathcal{N}(e_+) \to W$. In particular, $id_W \in F_0$. Let $g \in \mathcal{GE}_{e_+}(\mathcal{N}(e_+), W)$ and let F be the fiber of η over g, i.e., the simplicial group of a homeomorphism h of W into itself such that $h|_{\mathcal{N}(e_+)} = g$. Construct an isomorphism $H : F_0 \to F$ as follows. Let h be an element in F. Define $H : F_0 \to F$ by $H(f) = h \circ f$, with $f \in F_0$. Since $f|_{\mathcal{N}(e_+)} = id|_{\mathcal{N}(e_+)}$, we have that $h \circ f|_{\mathcal{N}(e_+)} = h|_{\mathcal{N}(e_+)}$. Thus, $H(f) = h \circ f$ is in F, i.e. $\eta(h) = \eta(h \circ f)$.

Define the inverse of H, $H_{-1} : F \to F_0$, by $H^{-1}(g) = h^{-1} \circ g$. It is well defined because h is a homeomorphism.

Clearly $H^{-1} \circ H = id_{F_0}$ and $H \circ H^{-1} = id_F$. □

Theorem 2.5. *The map $\delta : TOP^{ep}(W) \to \mathcal{GE}_\varepsilon(\mathcal{N}(\varepsilon), M)$ is a homotopy equivalence.*

Proof. This follows from Corollary 2.3 and Proposition 2.4, where $\delta = \phi \circ \eta$. □

As a corollary we have

Corollary 2.6 (Kister - Mazur Theorem)**.** *The restriction map $TOP(\mathbb{R}^n; 0) \to \mathcal{GE}_0(\mathcal{N}(0), \mathbb{R}^n)$ is a homotopy equivalence, where $TOP(\mathbb{R}^n; 0)$ denotes the simplicial group of homeomorphisms of $\mathbb{R}^n$ which fixes the origin.*

Proof. In this proof we will use the following claims: □

Claim 1. $TOP^{ep}(S^{n-1} \times \mathbb{R}) \cong TOP(\mathbb{R}^n; 0)$.

Proof. Let $h : (\mathbb{R}^n - 0) \to S^{n-1} \times \mathbb{R}$ be a homeomorphism.

Given a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(0) = 0$, define $\bar{f} : S^{n-1} \times \mathbb{R} \to S^{n-1} \times \mathbb{R}$ by $\bar{f} = h \circ f \circ h^{-1}$, which is an end preserving homeomorphism. Conversely, given an end preserving homeomorphism $g : S^{n-1} \times \mathbb{R} \to S^{n-1} \times \mathbb{R}$, define $\bar{g} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\bar{g}(x) = \begin{cases} h^{-1} \circ g \circ h(x) & \text{for } x \neq 0, \\ 0 & \text{or } x = 0. \end{cases}$$

Since g is end preserving, $\bar{g}$ is continuous in 0. □

Claim 2. $\mathcal{GE}_{-\infty}(\mathcal{N}(-\infty), S^{n-1} \times \mathbb{R}) \simeq \mathcal{GE}_0(\mathcal{N}(0), \mathbb{R}^n)$.

Proof. Analogous to Claim 1.

Then, applying Proposition 2.4, where $W = S^{n-1} \times \mathbb{R}$ and p is the projection map, we have that $TOP^{ep}(S^{n-1} \times \mathbb{R}) \simeq \mathcal{GE}_{-\infty}(\mathcal{N}(-\infty), S^{n-1} \times \mathbb{R})$. And by Claim 1 and Claim 2 we have the corollary. □

Theorem 2.7. *Let $p : W \to \mathbb{R}$ be a manifold approximate fibration and let dim $W \geq 5$. Then $W \cup e_+$ is an I-regular neighborhood of e_+.*

Proof. It follows from Siebenmann [33] that both of the ends e_+, e_- of W are tame ends. Then, using Theorem 1.14 and Remark 7, we have that e_+ (resp. e_-) admits I-regular neighborhoods in $W \cup e_+$ (resp. $W \cup e_-$), i.e. $(W \cup e_+, e_+)$ (resp. $(W \cup e_-, e_-)$) satisfies the I-compression axiom. Thus, since $(W \cup e_+, e_+)$ satisfies the I-compression axiom, it follows from Remark 5 that it is enough to show that $W \cup e_+$ is a regular neighborhood of e_+. And to show this we apply Remark 6 to $Y = W \cup \{e_+, e_-\}$, $U = W \cup e_+$, together with Theorem 1.11.

Let $K = p^{-1}[k, \infty) \cup e_+$ be a compact set, $K \subset W \cup e_+$, and let V be a neighborhood of e_+. Choose r such that $r > k$ and $p^{-1}[k, \infty) \subset V$.

We will apply Theorem 1.11 to $C = [k-1, r+2]$, $\tilde{C} = [l+1, r+3]$, where $l + 1 < k - 1$, and to the isotopy $h_t : \mathbb{R} \to \mathbb{R}$, $0 \leq t \leq 1$, such that $h_0 = id$, $h_1(x) > r + 1$ for $x \geq k$ and h_t is supported on C. Thus, by Theorem 1.11 there exists an isotopy $H_t : W \to W$, $0 \leq t \leq 1$, such that pH_t is α-close to $h_t p$, for each t, and H_t is supported on $p^{-1}(\tilde{C})$.

Notice the sequence of real numbers $1 < l+1 < k-1 < k < r < r+1 < r+2 < r+3$.

The isotopy h_t of $\mathbb{R}$ is defined by $h_0 = id$ and

$$h_1(x) = \begin{cases} x & \text{if } x > r+2 \text{ or } x < k-1, \\ x(r-k+2) + (k-1)(k-r-1) & \text{if } k-1 \leq x < k, \\ r + 1 + \frac{x-k}{r-k+2} & \text{if } k \leq x \leq r+2. \end{cases}$$

Since H_t is supported on $p^{-1}(\tilde{C})$, H is the identity on

$$p^{-1}(-\infty, l+1) \cup p^{-1}(r+3, \infty),$$

where $p^{-1}(-\infty, l+1) \supset W - p^{-1}[l, \infty)$ and $p^{-1}(r+3, \infty)$ is a neighborhood of e_+.

Now we verify that $H_1(K) \subset V$. Let $x \in K$. Then $pH_1(x)$ is α-close to $h_1(p(x))$. Since $p(x) \geq k$, it follows that $pH_1(x) \geq r$ (because $h_1(p(x)) \geq r + 1$ by the construction of h_1). It means that $H_1(x) \subset p^{-1}([r, \infty)) \subset V$.

Since H_t is fixed on a neighborhood of e_+, we can extend H_t to $\bar{H}_t : W \cup e_+ \to W \cup e_+$ by $\bar{H}_t|_W = H_t$ and $\bar{H}_t(e_+) = e_+$. Thus, K is compressible towards e_+ in U. □

Corollary 2.8. *Let $p : W \to \mathbb{R}$ be a manifold approximate fibration and suppose that U is an open neighborhood of e_+ in W such that U is also the total space of a manifold approximate fibration $q : U \to \mathbb{R}$. Then there exists an isotopy of embeddings $h_t : U \to W$, $0 \leq t \leq 1$, such that, $h_0 = \imath$, where $\imath$ is the inclusion map $\imath : U \hookrightarrow W$, h_1 is a homeomorphism and h_t fixes a smaller neighborhood V of e_+.*

Proof. This follows from Theorem 1.12, where $E = U \cup e_+$ and $E' = W \cup e_+$ are I-regular neighborhoods. □

We now use Theorem 2.1 to give an alternative proof of Anderson and Hsiang's Theorem [3] as given in the next theorem.

Let N be a compact, connected manifold and let $p : N \times \mathbb{R} \to \mathbb{R}$ be the projection map.

Theorem 2.9 (Anderson-Hsiang-Hatcher)**.** $\Omega(TOP^b(N \times \mathbb{R})) \simeq TOP(N \times I\ rel\ \partial)$.

Proof. Fact (*): If X is a topological space, $x \in X$ is a base point, and $\Delta : X \to X \times X$ is the diagonal map, then the homotopy fiber of Δ at (x, x) is homotopy equivalent to $\Omega(X, x)$.

This fact will be used in the proof of this theorem.

From Theorem 2.1 applied to $M = N \times \mathbb{R}$ we have that the restriction maps

$$\mu_+ : TO^{ep}(N \times \mathbb{R}) \to \mathcal{GE}_{+\infty}(\mathcal{N}(+\infty), N \times \mathbb{R})$$

and

$$\mu_- : TOP^{ep}(N \times \mathbb{R}) \to \mathcal{GE}_{-\infty}(\mathcal{N}(-\infty), N \times \mathbb{R})$$

are fibrations. The homotopy fiber of the map

$$\Phi : TOP^{ep}(N \times \mathbb{R}) \to \mathcal{GE}_{+\infty}(\mathcal{N}(+\infty), N \times \mathbb{R}) \times \mathcal{GE}_{-\infty}(\mathcal{N}(-\infty), N \times \mathbb{R})$$

is

$$TOP(N \times \mathbb{R}\ rel\ \{\mathcal{N}(+\infty), \mathcal{N}(-\infty)\})$$

which is homotopy equivalent to $TOP(N \times I\ rel\ \partial)$.

So, we construct the following diagram:

$$\begin{tikzcd}
 & TOP^b(N \times \mathbb{R}) \arrow[r, "\Psi"] \arrow[d, "i"', "\simeq"] & \mathcal{GE}_{+\infty}(\mathcal{N}(+\infty), N \times \mathbb{R}) \times \mathcal{GE}_{-\infty}(\mathcal{N}(-\infty), N \times \mathbb{R}) \arrow[d, Rightarrow, no head] \\
TOP_{cs}(N \times \mathbb{R}) \arrow[r] & TOP^{ep}(N \times \mathbb{R}) \arrow[r, "\Phi"] & \mathcal{GE}_{+\infty}(\mathcal{N}(+\infty), N \times \mathbb{R}) \times \mathcal{GE}_{-\infty}(\mathcal{N}(-\infty), N \times \mathbb{R})
\end{tikzcd}$$

where Ψ is the composition $\Psi = \Phi \circ i$ with i a homotopy equivalence. See Theorem 1.10.

Then, the homotopy fiber of the map Ψ is equivalent to the fiber of Φ , which is homotopy equivalent to $TOP(N \times I\ rel\ \partial)$. Finally, by fact (*), the homotopy fiber of Φ at (incl, incl) is equivalent to $\Omega(TOP^b(N \times \mathbb{R}))$. In other words, $TOP(N \times I\ rel\ \partial)$ is homotopy equivalent to $\Omega(TOP^b(N \times \mathbb{R}))$. □

3. Manifolds which are the interior of a compact manifold

In this section, a generalization of the Kuiper–Lashof Theorem is given for a non-compact manifold which is the interior of a compact manifold with connected boundary.

Through this section all embeddings are proper.

Let M be a compact topological manifold of dimension ≥ 5, with connected boundary ∂M, and denote the interior of M by $Int\ M$.

Let $\mathcal{C}(\partial M)$ denote the space of concordances of ∂M.

Let $f : TOP(M) \to TOP(Int\ M)$ be the restriction map, and let $\mathcal{G}$ be the homotopy fiber of f over $id_{Int\ M}$.

Theorem B. *$\pi_i\ \mathcal{G}$ is isomorphic to $\pi_i\ \mathcal{C}(\partial M)$, for $i > 0$.*

This result follows from the of the diagram

(∗∗)

$$\begin{tikzcd}
 & \mathcal{G} \arrow[r, leftrightarrow] \arrow[d] & \mathcal{H} \arrow[d] & \mathcal{C}(\partial M) \arrow[d] \\
TOP(M\ rel\ \mathcal{N}(\partial M)) \arrow[r] \arrow[d, "i"] & TOP(M) \arrow[r, "r"] \arrow[d, "f"] & \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M) \arrow[d, "u"] & TOP(\partial M) \arrow[l, "v"'] \arrow[d, "g"] \\
TOP(Int\ M\ rel\ \infty) \arrow[r] & TOP(Int\ M) \arrow[r, "s"] & \mathcal{GE}_{\infty}(\mathcal{N}(\infty), Int\ M) \arrow[r, "j"] & TOP^b(\partial M \times \mathbb{R})
\end{tikzcd}$$

where the following will be proved:

(1) the restriction maps r and s are fibrations, with fibers $TOP(M\ rel\ \mathcal{N}(\partial M))$ and $TOP(Int\ M\ rel\ \infty)$;

(2) the maps j and v are homotopy equivalences;

(3) the diagrams (I)

$$\begin{tikzcd}
TOP(M) \arrow[r, "r"] \arrow[d, "f"] & \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M) \arrow[d, "u"] \\
TOP(Int\ M) \arrow[r, "s"] & \mathcal{GE}_{\infty}(\mathcal{N}(\infty), Int\ M)
\end{tikzcd}$$

and (II)

$$\begin{tikzcd}
\mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M) \arrow[d, "u"] & TOP(\partial M) \arrow[l, "v"'] \arrow[d, "g"] \\
\mathcal{GE}_{\infty}(\mathcal{N}(\infty), Int\ M) & TOP^b(\partial M \times \mathbb{R}) \arrow[l, "j"']
\end{tikzcd}$$

are commutative.

The proof of (1) will be given in Theorems 3.1 and 3.2. The maps j and v will be constructed in Theorems 3.5 and 3.10. The construction depends on the choice of a collar for ∂M. In Remarks 3.6 and 3.11 we have (3).

It was proved by Anderson and Hsiang [3] that $\mathcal{C}(\partial M)$ is the homotopy fiber of the map $g = - \times id_{\mathbb{R}} : TOP(\partial M) \to TOP^b(\partial M \times \mathbb{R})$.

Proof of Theorem B. Let $\mathcal{H}$ denote the homotopy fiber of u. Assume (1) – (3) above. Then:

1. $i : TOP(M\ rel\ \mathcal{N}(\partial M)) \to TOP(Int\ M\ rel\ \infty)$ is an isomorphism (Remark 3.12).
2. Lemma 1.7 applied to the square (I) implies that $\pi_i\ \mathcal{G} \cong \pi_i\ \mathcal{H}$, for $i > 0$.

3. Since the square (II) commutes, and j and v are homotopy equivalences, and we get that $\pi_i\ \mathcal{H} \cong \pi_i\ \mathcal{C}(\partial M)$, for $i > 0$.

Thus, $\pi_i\ \mathcal{G} \cong \pi_i\ \mathcal{C}(\partial M)$, for $i > 0$. □

The technique used cannot be applied for the case $i = 0$, because it works only for connected sets. See Lemma 1.7 and Remark 2.

Theorem 3.1. *The restriction map* $r : TOP(M) \to \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M)$ *is a fibration.*

Proof. This follows from Theorem 1.5. The fiber of r over the inclusion map is $TOP(M\ rel\ \mathcal{N}(\partial M))$. □

Theorem 3.2. *The restriction map* $s : TOP(Int\ M) \to \mathcal{GE}_\infty(\mathcal{N}(\infty), Int\ M)$ *is a fibration.*

Proof. This is a special case of Theorem 2.1, where $M = Int\ M$. The fiber of s is $TOP(Int\ M\ rel\ \infty)$. □

The homotopy equivalence $j : TOP^b(\partial M \times \mathbb{R}) \to \mathcal{GE}_\infty(\mathcal{N}(\infty), Int\ M)$ is based on Lemmas 3.3 and 3.4, and on a choice of a collar for ∂M in M.

Choose a collar $c : \partial M \times [0, 1) \to M$ for ∂M in M. c induces an isomorphism of simplicial sets.

Lemma 3.3. $i_c : \mathcal{GE}_{+\infty}(\mathcal{N}(+\infty), \partial M \times \mathbb{R}) \to \mathcal{GE}_\infty(\mathcal{N}(\infty), Int\ M)$ *is a homotopy equivalence.*

Proof. With the above choice of a collar c, this follows from Corollary 2.3 with $W = \partial M \times \mathbb{R}$. □

Lemma 3.4. *The restriction map* $\mu : TOP^b(\partial M \times \mathbb{R}) \to \mathcal{GE}_{+\infty}(\mathcal{N}(+\infty), \partial M \times \mathbb{R})$ *is a homotopy equivalence.*

Proof. This follows from Theorem 1.10 and Proposition 2.4 for the special case where $W = \partial M \times \mathbb{R}$ and $p : \partial M \times \mathbb{R} \to \mathbb{R}$ is the projection map. □

Theorem 3.5. *The map* $j : TOP^b(\partial M \times \mathbb{R}) \to \mathcal{GE}_\infty(\mathcal{N}(\infty), Int\ M)$ *is a homotopy equivalence.*

Proof. The proof follows from Lemmas 3.3 and 3.4 as indicated in the diagram

$$TOP^b(\partial M \times \mathbb{R}) \underset{\mu}{\overset{3.4}{\simeq}} \mathcal{GE}_{+\infty}(\mathcal{N}(+\infty), \partial M \times \mathbb{R}) \underset{i_c}{\overset{3.3}{\simeq}} \mathcal{GE}_\infty(\mathcal{N}(\infty), Int\ M).$$

Thus $j = \mu \circ i_c$. □

Remark 3.6. The commutativity of square I follows by inspectiom since the maps r, s, u and f are all restriction maps.

Now, with the same choice of the collar c we will construct the homotopy equivalence v. This construction is based on Lemma 3.7 through Proposition 3.9. For this, we define maps α, β, γ, v and k such that $k = \alpha \circ \gamma^{-1}$ and $v = \gamma \circ \beta$, as follows.

The map $\gamma : \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), \partial M \times [0, 1)) \to \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M)$ is defined in terms of the collar c, and it is a homotopy equivalence.

The map $\alpha : \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), \partial M \times [0, 1)) \to TOP(\partial M)$ is defined as the restriction map and we will show that it is a homotopy equivalence.

The map $\beta : TOP(\partial M) \to \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), \partial M \times [0,1))$ is defined as $\beta = - \times id_{[0,1)}$, and it is a homotopy equivalence. Thus the map $k : \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M) \to TOP(\partial M)$, defined as the restriction map, is a homotopy equivalence, and the map $v : TOP(\partial M) \to \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M)$ defined by $v = \gamma \circ \beta$ is a homotopy equivalence.

Lemma 3.7. *The map* $\gamma : \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), \partial M \times [0,1)) \to \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M)$ *is a homotopy equivalence.*

Proof. This follows from Corollary 2.3. □

Proposition 3.8. *The restriction map* $\alpha : \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), \partial M \times [0,1)) \to TOP(\partial M)$ *is a homotopy equivalence.*

Proof. The Isotopy Extension Theorem for topological manifolds [6, Corollary 1.4] applied here implies by Theorem 1.5 that α is a (Kan) fibration, and it is surjective. The fiber of α over the identity map is $\mathcal{GE}(\mathcal{N}(\partial M), \partial M \times [0,1);\ rel\ \partial M)$, the simplicial set of equivalence classes of germs of embeddings from a neighborhood of ∂M to $\partial M \times [0,1)$ which restrict to the identity on ∂M.

We will show that $\pi_i\ \mathcal{GE}(\mathcal{N}(\partial M), \partial M \times [0,1);\ rel\ \partial M)$ is trivial for all i.

For any $0 < a < 1$, we have a map

$$\begin{aligned} r_a : Emb\ (\partial M \times [0,a),\ \partial M \times [0,1); rel\ (\partial M \times 0)) \\ \to \mathcal{GE}(\mathcal{N}(\partial M), \partial M \times [0,1);\ rel\ \partial M) \end{aligned}$$

which sends each embedding into its class of germ.

We will prove the following two facts.

(1) Given any map $\lambda : S^n \to \mathcal{GE}(\mathcal{N}(\partial M), \partial M \times [0,1);\ rel\ \partial M)$, there exist a number a_0 (which we will denote simply by a) and a map

$$\bar{\lambda} : S^n \to Emb\ (\partial M \times [0,a), \partial M \times [0,1); rel\ (\partial M \times 0))$$

such that $r_a \circ \bar{\lambda} \simeq \lambda$ and

(2) Given any $\bar{\lambda} : S^n \to Emb\ (\partial M \times [0,a), \partial M \times [0,1); rel\ (\partial M \times 0))$, there exist $h_s : S^n \to Emb\ (\partial M \times [0,a), \partial M \times [0,1); rel\ \mathcal{N}(\partial M \times 0))$, $0 \le s < \infty$, such that $h_0 = \bar{\lambda}$ and for all $s > 0$, $h_s \in Emb\ (\partial M \times [0,a), \partial M \times [0,1); rel\ (\partial M \times 0))$. □

Proof of item (1). Let $\lambda : S^n \to \mathcal{GE}(\mathcal{N}(\partial M), \partial M \times [0,1);\ rel\ \partial M)$ be a continuous map. For each $z \in S^n$, let $b_z : \partial M \times [0, a_z) \to \partial M \times [0,1)$ be a representative of the class $\lambda(z) \in \mathcal{GE}(\mathcal{N}(\partial M), \partial M \times [0,1);\ rel\ \partial M)$, where $\partial M \times [0,a_z)$ is a neighborhood of ∂M in $\partial M \times [0,1)$. By continuity of λ, b_z is such that the map $S^n \to (0,1)$; $z \mapsto a_z$ is continuous, and since S^n is compact, b_z has a minimum value, say $a > 0$. Then $b_{z_|} : \partial M \times [0,a) \to \partial M \times [0,1)$ still is the same class $\lambda(z)$. Then consider $\bar{\lambda} : S^n \to Emb\ (\partial M \times [0,a), \partial M \times [0,1);\ rel\ (\partial M \times 0))$ such that $z \mapsto b_z$ and $r_a : Emb(\partial M \times [0,a), \partial M \times [0,1); rel\ (\partial M \times 0)) \to \mathcal{GE}(\mathcal{N}(\partial M), \partial M \times [0,1);\ rel\ \partial M)$ which sends each embedding into its class of germ such that $r_a \circ \bar{\lambda} \simeq \lambda$. □

Proof of item (2). Let $f : \partial M \times [0,a) \to \partial M \times [0,1)$ be an embedding such that $f|_{\partial M \times 0} = id$. We define an isotopy $h_s : \partial M \times [0,a) \to \partial M \times [0,1)$ in the following way.

Set $I_s = [-s, a)$, for $s \in [0, \infty)$. First, define an auxiliary family of embeddings $f_s : \partial M \times I_s \to \partial M \times I_s$ by

$$f_s(x,t) = \begin{cases} (x,t) & \text{if } t \in [-s,0], \\ f(x,t) & \text{if } t \in [0,a). \end{cases}$$

Since $f|_{\partial M\times 0} = id$, f_s is well defined, it is continuous, and it is an embedding $\forall s \in [0,1]$. Also, $f_0 = f$.

Now, for each $s \in [0,\infty)$ consider the homeomorphisms $g_s : [-s,a) \to [0,a)$ defined by $g_s(t) = \frac{a(t+s)}{a+s}$. Notice that $g_0 = id_{[0,a)}$.

Finally define an isotopy $h_s : \partial M \times [0,a) \to \partial M \times [0,1)$ by

$$h_s(x,t) = (id_{\partial M} \times g_s) \circ f_s \circ (id_{\partial M} \times (g_s^{-1})(x,t) = (id_{\partial M} \times g_s) \circ f_s(x, (g_s^{-1})(t)).$$

We have $h_0 = f_0 = f$, and for $t \in [0, \frac{sa}{s+a}]$ we have $(g_s)^{-1}(t) \leq 0$, which implies $f_s(x,(g_s)^{-1}(t)) = (x,(g_s)^{-1}(t))$. Thus, for $t \in [0, \frac{sa}{s+a}]$,

$$h_s(x,t) = (id_{\partial M} \times g_s) \circ f_s(x,(g_s)^{-1}(t)) = (id_{\partial M} \times g_s)(x,(g_s)^{-1}(t)) = (x,t).$$

This shows that $\pi_0\ Emb\ (\partial M \times [0,a), \partial M \times [0,1); rel\ \partial M) = 0$.

Analogously, for $i \geq 1$, $\pi_i\ Emb\ (\partial M \times [0,a), \partial M \times [0,1); rel\ \partial M) = 0$.

Consider $f : S^n \times \partial M \times [0,a) \to \partial M \times [0,1)$ such that for each $z \in S^n$, $f|_{\partial M\times 0} = id$.

Set $I_s = [-s,a)$, for $s \in [0,\infty)$. Define an auxiliary family of embeddings $f_s : S^n \times \partial M \times I_s \to \partial M \times I_s$ by

$$f_s(z,x,t) = \begin{cases} (x,t) & \text{if } t \in [-s,0], \\ f(z,x,t) & \text{if } t \in [0,a). \end{cases}$$

And f_s has the same properties as before.

Consider the same family of homeomorphisms g_s. Then define an isotopy $h_s : S^n \times \partial M \times [0,a) \to \partial M \times [0,1)$ by

$$\begin{aligned} h_s(z,x,t) &= (id_{\partial M} \times g_s) \circ f_s \circ (id_{S^n} \times id_{\partial M} \times (g_s^{-1})(z,x,t) \\ &= (id_{\partial M} \times g_s) \circ f_s(z,x,(g_s^{-1})(t)). \end{aligned}$$

If we apply the map r_a to this homotopy, we then get a homotopy in $\mathcal{GE}(\mathcal{N}(\partial M), \partial M \times [0,1);\ rel\ \partial M)$ such that $r_a \circ\ h_0 = \lambda$ and $\forall s > 0$, $r_a \circ h_s \in \mathcal{GE}(\mathcal{N}(\partial M), \partial M \times [0,1);\ rel\ \partial M)$. □

Proposition 3.9. *The restriction map $k : \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M) \to TOP(\partial M)$ is a homotopy equivalence.*

Proof. The map $k = \alpha \circ \gamma^{-1}$ is indicated in the following diagram:

$$\mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M) \xrightarrow{\gamma^{-1}} \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), \partial M \times [0,1)) \xrightarrow{\alpha} TOP(\partial M),$$

where γ^{-1} and α are homotopy equivalences, which are proved in Lemma 3.7 and 3.8. So, k is a homotopy equivalence. □

Theorem 3.10. *The map $v : TOP(\partial M) \to \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M)$ is a homotopy equivalence.*

Proof. The map $v = \gamma \circ \beta$ is indicated in the following diagram:

$$TOP(\partial M) \xrightarrow{\beta} \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), \partial M \times [0,1)) \xrightarrow{\alpha} \mathcal{GE}_{\partial M}(\mathcal{N}(\partial M), M),$$

where γ is the homotopy equivalence in Lemma 3.7, β is defined by $\beta = - \times id_{[0,1)}$, and $\alpha \circ \beta = id_{TOP(\partial M)}$. Then β and α are homotopy equivalences. $\square$

Remark 3.11. The commutativity of square II follows by inspection, where the maps v and j are homotopy equivalences (by using the same choice of a collar), the map u is the restriction map and the map $g = - \times id_{\mathbb{R}}$.

Remark 3.12. Clearly, the map $i : TOP(M\ rel\ \mathcal{N}(\partial M)) \to TOP(Int\ M\ rel\ \infty)$ is an isomorphism.

Notice that the map $f : TOP(M) \to TOP(Int\ M)$ in the diagram (**) is not necessarily a fibration. Consider the following example.

Example. Let M be the cylinder $S^1 \times [0,1]$. There is a homeomorphism $\tau : Int\ M \to Int\ M$ that is not a restriction of a self-homeomorphism of M. However, τ is isotopic to the restriction of a self-homeomorphism of M. In other words, the map induced by the restriction $r : TOP(M) \to TOP(Int\ M)$ is not a Kan fibration.

Represent the point $x \in S^1 \times (0,\infty)$ by $x = (e^{i\theta}, t)$, where $\theta \in [0, 2\pi)$ and $t \in (0,\infty)$.

Let $\sigma : (0,\infty) \to (0,1)$ be any homeomorphism, and for any $s \in [0,1]$ let $\rho_s : S^1 \times (0,\infty) \to S^1 \times (0,\infty)$ be a family of homeomorphisms defined by $\rho_s(e^{i\theta}, t) = (e^{i(\theta + 2\pi ts)}, t)$. Then $\tau_s : S^1 \times (0,1) \to S^1 \times (0,1)$, defined by $\tau_s = (id \times \sigma) \circ \rho_s \circ (id \times \sigma^{-1})$, is an isotopy from ρ_s to id. For $s = 1$, τ_1 is not a restriction of any homeomorphism from $S^1 \times [0,1]$ into itself because the image of the sequence $a_n = (e^{i\theta_0}, 1 - 1/n)$ for any fixed $\theta_0 \in [0, 2\pi)$ by τ_1 does not converge.

4. Wrapping homeomorphisms around a circle

Let W be a manifold without boundary of dimension ≥ 5.

Theorem C (Wrapping homeomorphism around a circle)**.** *Let $q_0 : W \to \mathbb{R}$ be a manifold approximate fibration. Then*:

(1) There exists a manifold approximate fibration $q : \hat{W} \to S^1$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "q_0"] \arrow[d] & \mathbb{R} \arrow[d, "exp"] \\ \hat{W} \arrow[r, "q"'] & S^1 \end{tikzcd}$$

(2) $\pi_n\ TOP^{ep}(W)$ is a direct summand of $\pi_n\ TOP(\hat{W})$, for $n > 1$, where $\hat{W}$ is a compact and connected manifold and W is the infinite cyclic cover of $\hat{W}$.

Before proving this theorem we will give some definitions.

For any topological manifold B, let $MAF(B)$ be the simplicial set of manifold approximate fibrations over B (see [14, page 12]). If $B = S^1$, then a vertex of $MAF(S^1)$ is $q : \hat{W} \to S^1$, and if $B = \mathbb{R}$, a vertex of $MAF(\mathbb{R})$ is $q_0 : W \to \mathbb{R}$.

Let $\iota : \mathbb{R} \hookrightarrow S^1$ be an orientation preserving embedding. Then the map $q_| : q^{-1}(\iota(\mathbb{R})) \to \mathbb{R}$ is a manifold approximate fibration, called the *fiber germ* of q over ι. We say that q has fiber germ q_0 if and only if there exists a controlled homeomorphism between a manifold approximate fibration $q_0 : W \to \mathbb{R}$ and $q_|$. See [14]. Then ι induces a map $\iota^* : MAF(S^1) \to MAF(\mathbb{R})$ which sends a manifold approximate fibration $\hat{W} \to S^1$ to a manifold approximate fibration $W \to \mathbb{R}$. We shall prove Theorem C by given a homotopy left inverse for ι^*.

By [14, Theorem 1.4] we have the following commutative diagram:

$$(***) \qquad \begin{array}{ccc} MAF(S^1) & \xrightarrow[d]{\simeq} & Map(S^1, MAF(\mathbb{R})) \\ {\scriptstyle \iota^*}\downarrow & & {\scriptstyle r}\downarrow \\ MAF(\mathbb{R}) & \xrightarrow[d]{\simeq} & Map(\mathbb{R}, MAF(\mathbb{R})) \end{array}$$

The maps ι^* and r are the restriction maps induced by ι. In order to give a left inverse to ι^*, we construct a left inverse to r which determines a left inverse to ι^*.

Lemma 4.1. *The restriction map* $r : Map(S^1, MAF(\mathbb{R})) \to Map(\mathbb{R}, MAF(\mathbb{R}))$ *has a homotopy left inverse.*

Proof. Let $f \in Map(S^1, MAF(\mathbb{R}))$. Then the map r, induced by ι, is such that $r(f) = f \circ \iota \in Map(\mathbb{R}, MAF(\mathbb{R}))$. Let $* \in S^1$. Define the restriction map $r_| : Map(S^1, MAF(\mathbb{R})) \to Map(*, MAF(\mathbb{R}))$ such that $r_|(f) = f_{|_*} : * \to MAF(\mathbb{R})$ and $*$ goes to $f(*)$. $r_|$ has a homotopy left inverse $s : Map(*, MAF(\mathbb{R})) \to Map(S^1, MAF(\mathbb{R}))$ defined as follows. Let $x \in Map(*, MAF(\mathbb{R}))$. So, x is a map $x : * \to MAF(\mathbb{R})$; $* \mapsto g$. Thus the map $s : Map(*, MAF(\mathbb{R})) \to Map(S^1, MAF(\mathbb{R}))$ is such that $x \mapsto c_x$, where c_x is the constant map, $c_x(z) = g$. Then, $r_| \circ s : Map(*, MAF(\mathbb{R})) \to Map(*, MAF(\mathbb{R}))$ is the identity. Thus, applying any isomorphism $Map(\mathbb{R}, MAF(\mathbb{R})) \cong Map(*, MAF(\mathbb{R}))$ which sends $0 \in \mathbb{R}$ to $*$, we have that s is a homotopy left inverse of r. Since ι^* preserves base point, so do r and s. ☐

Proof of Theorem C (1). From Lemma 4.1 and diagram (***), s determines (up to homotopy) a homotopy left inverse to ι^*.

Thus, given any manifold approximate fibration $q_0 : W \to \mathbb{R}$, there exists a manifold approximate fibration $\hat{q} : \hat{W} \to S^1$ such that with an orientation preserving embedding ι, $q' : W' \to \mathbb{R}$ is controlled homeomorphic to $q_0 : W \to \mathbb{R}$. In fact, consider the infinite cyclic cover of $\hat{W}$ and S^1. Form the pullback

$$\begin{array}{ccc} W' & \xrightarrow{\hookrightarrow} & \hat{W} \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle \hat{q}} \\ \mathbb{R} & \xrightarrow[exp]{\hookrightarrow} & S^1 \end{array}$$

Then

$$\begin{array}{l} W' = \hat{q}^{-1}(exp(\mathbb{R})) \hookrightarrow \hat{W} \\ {\scriptstyle q'}\downarrow \\ \mathbb{R} \end{array}$$

is a manifold approximate fibration (by Corollary 12.14 in [14]), and $q' = \hat{q}_|$ is fiber germ of $\hat{q}$ over exp.

By Corollary 12.14 in [14] we have a manifold approximate fibration $q' : W' \to \mathbb{R}$. From the uniqueness of fiber germs [14], any two fiber germs of a manifold approximate fibration over a connected oriented manifold are controlled homeomorphic. So it follows that q' is controlled homeomorphic to q_0. $\square$

Let $MAF(S^1)_q$ denote the component of $MAF(S^1)$ containing q, and let $MAF(\mathbb{R})_{q_0}$ denote the component of $MAF(\mathbb{R})$ containing q_0.

By [14, Corollary 7.12] we have a commutative diagram

$$(****)\qquad \begin{array}{ccc} BTOP^c(\hat{W} \xrightarrow{q} S^1) & \xrightarrow{\simeq} & MAF(S^1)_q \\ \big\downarrow & & \big\downarrow \\ BTOP^c(W \xrightarrow{q_0} \mathbb{R}) & \xrightarrow{\simeq} & MAF(\mathbb{R})_{q_0} \end{array}$$

where the horizontal maps are homotopy equivalences.

Proof of Theorem C(2). From Lemma 4.1, diagram (***) and diagram (****) we have that the map $\iota_| : BTOP^c(\hat{W} \xrightarrow{q} S^1) \to BTOP^c(W \xrightarrow{q_0} \mathbb{R})$, induced by ι^*, has a homotopy left inverse $s_| : BTOP^c(W \xrightarrow{q_0} \mathbb{R}) \to BTOP^c(\hat{W} \xrightarrow{q} S^1)$, induced by the left inverse of ι^*. The maps $\iota_|$ and $s_|$ preserve base points. Thus $\iota_| \circ s_| \simeq id$ implies that $\pi_i\ (TOP^c(W \xrightarrow{q_0} \mathbb{R}))$ is a direct summand of $\pi_i\ (TOP^c(\hat{W} \xrightarrow{q} S^1))$.

By [17, Theorem 1.1], where $B = S^1$, the forget control map

$$\phi : TOP^c(\hat{W} \xrightarrow{q} S^1) \to TOP^h(\hat{W} \xrightarrow{q} S^1)$$

is a homotopy split injective, where $TOP^h(\hat{W} \xrightarrow{q} S^1)$ denotes the homotopy fiber of the simplicial map $\Psi : TOP(\hat{W}) \to Map(\hat{W}, S^1)$ defined by $\Psi(h) = q \circ h$, where the homeomorphism $h : \hat{W} \to \hat{W}$ is a vertex of $TOP(\hat{W})$ and $Map(\hat{W}, S^1)$ denotes the simplicial set of maps from $\hat{W}$ to S^1. Hence, a vertex of $TOP^h(\hat{W} \xrightarrow{q} S^1)$ consists of a homeomorphism $h : \hat{W} \to \hat{W}$ together with a homotopy from $q \circ h$ to q. The elements of $TOP^h(\hat{W} \xrightarrow{q} S^1)$ are called *homotopically controlled*. Thus, $\pi_i\ (TOP^c(\hat{W} \xrightarrow{q} S^1))$ is a direct summand of $\pi_i\ (TOP^h(\hat{W} \xrightarrow{q} S^1))$.

From the fibration sequence $TOP^h(\hat{W} \xrightarrow{q} S^1) \to TOP(\hat{W}) \to Map(\hat{W}, S^1)$ we have the long exact sequence in homotopy

$$\cdots \to \pi_i\ TOP^h(\hat{W} \xrightarrow{q} S^1) \to \pi_i\ TOP(\hat{W}) \to \pi_i\ Map(\hat{W}, S^1) \to \cdots .$$

With the fibration $exp : \mathbb{R} \to S^1$, when $\hat{W}$ is a CW complex, then the map $Map(\hat{W}, \mathbb{R}) \to Map(\hat{W}, S^1)$ is a fibration. Since $\mathbb{R} \simeq *$, we have $Map(\hat{W}, \mathbb{R}) \simeq Map(\hat{W}, *) \simeq *$. Thus, $* \simeq Map(\hat{W}, \mathbb{R}) \to Map(\hat{W}, S^1)_{certain\ components}$ (i.e. components of the homotopy trivial map) implies $\pi_i\ Map(\hat{W}, S^1)_* = 0$, for $i > 1$. Thus, $\pi_i\ TOP^h(\hat{W} \xrightarrow{q} S^1) \cong \pi_i\ TOP(\hat{W})$, for $i > 1$.

So, $\pi_i\ TOP^c(W \xrightarrow{q_0} \mathbb{R})$ is a direct summand of $\pi_i\ TOP^c(\hat{W} \xrightarrow{q} S^1)$; likewise $\pi_i\ TOP^c(\hat{W} \xrightarrow{q} S^1)$ is a direct summand of $\pi_i\ TOP^h(\hat{W} \xrightarrow{q_0} S^1)$, and by Theorem 1.11, $\pi_i\ TOP^c(W \xrightarrow{q_0} \mathbb{R}) \cong \pi_i\ TOP^{ep}(W)$.

Since W is the infinite cyclic cover of $\hat{W}$ induced by $q : \hat{W} \to S^1$ from $exp : \mathbb{R} \to S^1$, the map $p : W \hookrightarrow \hat{W}$ induces a map $TOP(\hat{W}) \to TOP^{ep}(W)$.

Hence, for $i > 1$, $\pi_i\ TOP^{ep}(W)$ is a direct summand of $\pi_i\ TOP(\hat{W})$. $\square$

Lemma 4.2. *$\pi_1\ Map(\hat{W}, S^1) \simeq \mathbb{Z}$ and $\pi_0\ Map(\hat{W}, S^1) \simeq H^1(\hat{W}, \mathbb{Z})$.*

Proof. Let $\hat{W}$ be a connected compact manifold and consider the fibration sequence $\mathbb{Z} \hookrightarrow \mathbb{R} \overset{exp}{\to} S^1$. Then $Map(\hat{W}, S^1)$ is a fibration and since $\mathbb{R} \simeq *$, $Map(\hat{W}, \mathbb{R}) \simeq Map(\hat{W}, *) \simeq *$, which implies $\pi_i\ Map(\hat{W}, S^1)_* = 0$, for $i > 1$. From the fibration sequence $Map(\hat{W}, \mathbb{Z}) \hookrightarrow Map(\hat{W}, \mathbb{R}) \to Map(\hat{W}, S^1)$ we have a exact sequence in homotopy

$$\cdots \to 0 \to \pi_1\ Map(\hat{W}, S^1) \to \pi_0\ Map(\hat{W}, \mathbb{Z}) \to 0 \to \cdots$$

which implies $\pi_1\ Map(\hat{W}, S^1)_* \cong Map(\hat{W}, \mathbb{Z}) \simeq \mathbb{Z}$.

Now, S^1 is a topological group, so $Map(\hat{W}, S^1)$ is an H-space, which implies any two path components of $Map(\hat{W}, S^1)$ are homotopy equivalent. Thus,

$$\pi_0\ Map(\hat{W}, S^1) = [\hat{W}, S^1] = [\hat{W}, K(\mathbb{Z}, 1)] = H^1(\hat{W}, \mathbb{Z}).$$

$\square$

Conclusion: If $\hat{W}$ is a connected, compact manifold, then $Map(\hat{W}, S^1) \overset{weak}{\simeq} H^1(\hat{W}, \mathbb{Z}) \times S^1$.

Remark 4.3. By Lemma 4.2, $\pi_1\ Map(\hat{W}, S^1) \simeq \mathbb{Z}$ and $\pi_0\ Map(\hat{W}, S^1) \simeq H^1(\hat{W}, \mathbb{Z})$. Thus,

$$\cdots \to 0 \to \pi_1\ TOP^h(\hat{W} \overset{q}{\to} S^1) \rightarrowtail \pi_1\ TOP(\hat{W}) \to \mathbb{Z} \to \cdots .$$

And hence,

$$\begin{tikzcd}
 & \pi_1\ TOP^c(\hat{W} \overset{q}{\to} S^1) \arrow[r, "c"] \arrow[d, "a", "\text{direct summand}"'] & \pi_1\ TOP(\hat{W}) \arrow[d, equal] & \\
0 \arrow[r] & \pi_1\ TOP^h(\hat{W} \overset{q}{\to} S^1) \arrow[r, tail] & \pi_1\ TOP(\hat{W}) \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

So, c is injective.

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