

## MULTIVARIATE MATRIX REFINABLE FUNCTIONS WITH ARBITRARY MATRIX DILATION

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ABSTRACT. Characterizations of the stability and orthonormality of a multivariate matrix refinable function  $\Phi$  with arbitrary matrix dilation  $M$  are provided in terms of the eigenvalue and 1-eigenvector properties of the restricted transition operator. Under mild conditions, it is shown that the approximation order of  $\Phi$  is equivalent to the order of the vanishing moment conditions of the matrix refinement mask  $\{\mathbf{P}_\alpha\}$ . The restricted transition operator associated with the matrix refinement mask  $\{\mathbf{P}_\alpha\}$  is represented by a finite matrix  $(\mathcal{A}_{Mi-j})_{i,j}$ , with  $\mathcal{A}_j = |\det(M)|^{-1} \sum_{\kappa} \mathbf{P}_{\kappa-j} \otimes \mathbf{P}_\kappa$  and  $\mathbf{P}_{\kappa-j} \otimes \mathbf{P}_\kappa$  being the Kronecker product of matrices  $\mathbf{P}_{\kappa-j}$  and  $\mathbf{P}_\kappa$ . The spectral properties of the transition operator are studied. The Sobolev regularity estimate of a matrix refinable function  $\Phi$  is given in terms of the spectral radius of the restricted transition operator to an invariant subspace. This estimate is analyzed in an example.

### 1. INTRODUCTION

Let  $\{\mathbf{P}_\alpha\}$  be a finitely supported  $r \times r$  matrix sequence. The vectors  $\Phi$ ,  $r$ -dimensional column functions on  $\mathbb{R}^d$ , considered in this paper are solutions to functional equations of the type

$$(1.1) \quad \Phi = \sum_{\alpha \in \mathbb{Z}^d} \mathbf{P}_\alpha \Phi(M \cdot -\alpha),$$

where  $M$  is a  $d \times d$  integer matrix with  $m = |\det(M)| \geq 2$  and all eigenvalues of modulus  $> 1$ . Define

$$\mathbf{P}(\omega) := \frac{1}{m} \sum_{\alpha \in \mathbb{Z}^d} \mathbf{P}_\alpha \exp(-i\alpha\omega).$$

Then  $\mathbf{P}$  is an  $r \times r$  matrix with trigonometric polynomial entries. In the Fourier domain, functional equations (1.1) can be written as

$$(1.2) \quad \widehat{\Phi}(\omega) = \mathbf{P}({}^t M^{-1}\omega) \widehat{\Phi}({}^t M^{-1}\omega).$$

Throughout this paper,  ${}^t A$  and  $A^*$  denote the transpose and the Hermitian adjoint of a matrix  $A$  respectively.

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Equations of type (1.1) or (1.2) are called **matrix (vector) refinement equations**; the matrix  $M$  is called the **dilation matrix**;  $\mathbf{P} (\{\mathbf{P}_\alpha\})$  is called the **(matrix) refinement mask** and any solution  $\Phi$  of (1.1) is called an  $(M, \mathbf{P})$  **matrix refinable function** (or an  $(M, \mathbf{P})$  **refinable vector**).

For  $M = 2\mathbf{I}_r$ ,  $r \geq 1$ , where  $\mathbf{I}_r$  is the  $r \times r$  identity matrix, the characterizations of the stability and orthonormality of a matrix refinable function  $\Phi$  were provided in terms of the mask in [26]; the regularity estimates of  $\Phi$  were studied in [26], [19], and in [3], [24] for the case  $d = 1$ ; the existence of the distribution solution of (1.1) and the characterization of the weak stability of solutions of (1.1) were discussed in [21]. In the construction of multivariate wavelets, the dilation matrix  $M$  is involved. For  $r = 1$ , the characterizations of the stability and orthonormality of  $\Phi$ , a refinable function with matrix dilation, were proved in terms of the mask in [22]; the optimal Sobolev regularity estimate of  $\Phi$  was obtained in [15]. Our goal in this paper is to provide characterizations of the stability, orthonormality and the approximation order of an  $(M, \mathbf{P})$  refinable vector  $\Phi$  in terms of the mask, and give the regularity estimate of  $\Phi$  in terms of the spectral radius of the restricted transition operator.

Before going further, we introduce some notations used in this paper. Let  $\mathbb{Z}_+$  denote the set of all nonnegative integers, and let  $\mathbb{Z}_+^d$  denote the set of all  $d$ -tuples of nonnegative integers. We shall adopt the multi-index notations

$$\omega^\beta := \omega_1^{\beta_1} \cdots \omega_d^{\beta_d}, \quad \beta! := \beta_1! \cdots \beta_d!, \quad |\beta| := \beta_1 + \cdots + \beta_d$$

for  $\omega = {}^t(\omega_1, \dots, \omega_d) \in \mathbb{R}^d$ ,  $\beta = {}^t(\beta_1, \dots, \beta_d) \in \mathbb{Z}_+^d$ . If  $\alpha, \beta \in \mathbb{Z}_+^d$  satisfy  $\beta - \alpha \in \mathbb{Z}_+^d$ , we shall write  $\alpha \leq \beta$  and denote

$$\binom{\beta}{\alpha} := \frac{\beta!}{\alpha!(\beta - \alpha)!}.$$

For  $\beta = {}^t(\beta_1, \dots, \beta_d) \in \mathbb{Z}_+^d$ , denote

$$D^\beta := \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}},$$

where  $\partial_j = \frac{\partial}{\partial x_j}$  is the partial derivative operator with respect to the  $j$ th coordinate,  $1 \leq j \leq d$ . Except in some special cases, for  $\omega, \zeta \in \mathbb{R}^d$  we use  $\zeta\omega$  (not  ${}^t\zeta\omega$ ) to denote their scalar product.

For a finitely supported complex sequence  $c$  on  $\mathbb{Z}^d$ , its support is defined by  $\text{supp } c := \{\beta \in \mathbb{Z}^d : c(\beta) \neq 0\}$ , and for a finitely supported  $r \times r$  matrix sequence  $C$  on  $\mathbb{Z}^d$ , its support is defined by  $\text{supp } C := \bigcup \text{supp } c_{ij}$ , where  $c_{ij}$  is the  $(i, j)$ -entry of  $C$ . Throughout this paper, we assume that the matrix refinement mask  $\mathbf{P}$  satisfies  $\text{supp}\{\mathbf{P}_\alpha\} \subset [0, N]^d$  for some positive integer  $N$ .

Let  $\|x\|$  denote the Euclidean norm in  $\mathbb{R}^d$ , and let  $\text{dist}(x, y) := \|x - y\|$  be the distance between two points  $x, y \in \mathbb{R}^d$ . For two subsets  $S_1, S_2$  of  $\mathbb{R}^d$ , denote

$$\text{dist}(S_1, S_2) := \inf\{\text{dist}(x, y) : x \in S_1, y \in S_2\}.$$

For any subset  $S$  of  $\mathbb{R}^d$ , denote  $[S] := S \cap \mathbb{Z}^d$ ; and if  $S$  is a finite set of  $\mathbb{Z}^d$ , let  $|S|$  denote the number of elements in  $S$ .

For  $j = 1, \dots, r$ , let  $\mathbf{e}_j := (\delta_j(k))_{k=1}^r$  denote the standard unit vectors in  $\mathbb{R}^r$ . In this paper, for an  $r \times 1$  vector-valued function or sequence  $f = {}^t(f_1, \dots, f_r)$ , when we say that  $f$  is in a space on  $\mathbb{R}^d$  or  $\mathbb{Z}^d$ , we mean that every component  $f_i$  of  $f$  is in

this space. In particular,  $f = {}^t(f_1, \dots, f_r) \in L^2(\mathbb{R}^d)$  (or  $\mathbf{c} = (c_1, \dots, c_r) \in l^2(\mathbb{Z}^d)$ ) means that  $f_i \in L^2(\mathbb{R}^d)$  (or  $c_i \in l^2(\mathbb{Z}^d)$ ),  $i = 1, \dots, r$ , and we will use the norms

$$\|f\|_2 = \left(\sum_{i=1}^r \|f_i\|_{L^2(\mathbb{R}^d)}^2\right)^{\frac{1}{2}}, \quad \|\mathbf{c}\|_2 = \left(\sum_{i=1}^r \|c_i\|_{l^2(\mathbb{Z}^d)}^2\right)^{\frac{1}{2}}.$$

For a matrix  $A$  (or an operator  $A$  defined on a finite dimensional linear space), we say  $A$  satisfies **Condition E** if  $\rho(A) \leq 1$ , 1 is the unique eigenvalue on the unit circle and 1 is simple (the spectral radius of  $A$  is denoted by  $\rho(A)$ ).

Let  $M$  be a fixed dilation matrix with  $m = |\det(M)|$ . Then the coset spaces  $\mathbb{Z}^d/(M\mathbb{Z}^d)$  and  $\mathbb{Z}^d/({}^tM\mathbb{Z}^d)$  consist of  $m$  elements. Let  $\gamma_k + M\mathbb{Z}^d, 1 \leq k \leq m - 1$ , and  $\eta_j + {}^tM\mathbb{Z}^d, j = 0, \dots, m - 1$ , be the  $m$  distinct elements of  $\mathbb{Z}^d/(M\mathbb{Z}^d)$  and  $\mathbb{Z}^d/({}^tM\mathbb{Z}^d)$  respectively, with  $\gamma_0 = 0, \eta_0 = 0$ . Let  $C_0(\mathbb{T}^d)$  denote the space of all  $r \times r$  matrix functions with trigonometric polynomial entries. For a given matrix refinement mask  $\mathbf{P}$ , the **transition operator**  $\mathbf{T}$  associated with  $\mathbf{P}$  is defined on  $C_0(\mathbb{T}^d)$  by

(1.3)

$$\mathbf{TC}(\omega) := \sum_{j=0}^{m-1} \mathbf{P}({}^tM^{-1}(\omega + 2\pi\eta_j))C({}^tM^{-1}(\omega + 2\pi\eta_j))\mathbf{P}^*({}^tM^{-1}(\omega + 2\pi\eta_j)).$$

Assume that the support of the mask  $\{\mathbf{P}_\alpha\}$  is in  $[0, N]^d$ , and denote

$$(1.4) \quad \Omega := \left\{ \sum_{j=0}^{\infty} M^{-(j+1)}x_j : x_j \in [-N, N]^d, \forall j \in \mathbb{Z}_+ \right\}.$$

Let  $\mathbb{H}$  denote the subspace of  $C_0(\mathbb{T}^d)$  defined by

$$(1.5) \quad \mathbb{H} := \{H(\omega) \in C_0(\mathbb{T}^d) : H(\omega) = \sum_{\alpha} H_{\alpha}e^{-i\alpha\omega}, \text{supp}\{H_{\alpha}\} \subset [\Omega]\}.$$

Recall that a vector-valued function  $\Psi = {}^t(\psi_1, \dots, \psi_r)$  is called stable (orthogonal) if the integer translates of  $\psi_1, \dots, \psi_r$  form a Riesz basis (an orthonormal basis) of their closed linear span in  $L^2(\mathbb{R})$ . It has been shown that an  $(M, \mathbf{P})$  refinable vector  $\Phi$  is stable if and only if for all  $\omega \in \mathbb{T}^d$ ,  $G_{\Phi}(\omega) \geq c\mathbf{I}_r$  for some positive constant  $c$ , and that  $\Phi$  is orthogonal if and only if  $G_{\Phi}(\omega) = \mathbf{I}_r, \omega \in \mathbb{T}^d$ ; see e.g. [6], [10], [16] and [23]. Here  $G_{\Phi}(\omega)$  is the Gram matrix of  $\Phi$ , defined by

$$(1.6) \quad G_{\Phi}(\omega) := \sum_{\alpha \in \mathbb{Z}^d} \widehat{\Phi}(\omega + 2\pi\alpha)\widehat{\Phi}^*(\omega + 2\pi\alpha).$$

In the first part of Section 2, we will show that if the refinement equation (1.1) has a compactly supported solution  $\Phi$  such that  $G_{\Phi}(\omega) < \infty$  and  $\det(G_{\Phi}(0)) \neq 0$ , then  $\mathbf{P}(0)$  satisfies Condition E. Then we will provide a characterization of the existence of  $L^2$ -solutions of (1.1) under the assumption that  $\mathbf{P}(0)$  satisfies Condition E. In the last part of Section 2, we will show that the  $(M, \mathbf{P})$  refinable vector  $\Phi$  is stable if and only if the restriction  $\mathbf{T}|_{\mathbb{H}}$  of the transition operator  $\mathbf{T}$  to  $\mathbb{H}$  satisfies Condition E and the corresponding 1-eigenvector of  $\mathbf{T}|_{\mathbb{H}}$  is positive (or negative) definite on  $\mathbb{T}^d$ , and show that the  $(M, \mathbf{P})$  refinable vector  $\Phi$  is orthogonal if and only if  $\mathbf{T}|_{\mathbb{H}}$  satisfies Condition E and  $\mathbf{P}$  is a **Conjugate Quadrature Filter (CQF)**, i.e.

$$(1.7) \quad \sum_{j=0}^{m-1} \mathbf{P}({}^tM^{-1}(\omega + 2\pi\eta_j))\mathbf{P}^*({}^tM^{-1}(\omega + 2\pi\eta_j)) = \mathbf{I}_r, \quad \omega \in \mathbb{T}^d.$$

The accuracy order of the  $(M, \mathbf{P})$  refinable vector  $\Phi = {}^t(\phi_1, \dots, \phi_r)$  was considered in [11], [25] and [17] for the case  $d = 1$  and  $M = (2)$ , in [7] for  $M = 2\mathbf{I}_r$  and in [1] for the multivariate case with arbitrary dilation matrix. In Section 3, we will show that, under mild conditions,  $\Phi$  provides approximation of order  $k$ ,  $k \in \mathbb{Z}_+ \setminus \{0\}$ , if and only if the matrix refinement mask  $\mathbf{P}$  satisfies the vanishing moment conditions of order  $k$ . We will also determine explicitly the coefficients for the polynomial reproducing under the assumption that the integer shifts of  $\Phi$  ( $\phi_l(\cdot - \kappa)$ ,  $\kappa \in \mathbb{Z}^d, l = 1, \dots, r$ ) are linearly independent.

Since the spectra (eigenvalues) of a matrix can be computed directly, it is useful in practice to transfer equivalently the restricted operator  $\mathbf{T}|_{\mathbb{H}}$  to be a finite matrix, and therefore transfer the spectral problems of  $\mathbf{T}|_{\mathbb{H}}$  into those of a matrix. We will show in Section 4 that the restricted transition operator  $\mathbf{T}|_{\mathbb{H}}$  is equivalent to the matrix  $(\mathcal{A}_{M_i-j})_{i,j \in [\Omega]}$ , where  $\mathcal{A}_j$  is the  $r^2 \times r^2$  matrix given by

$$\mathcal{A}_j = \frac{1}{|\det(M)|} \sum_{\kappa \in [0, N]^d} \mathbf{P}_{\kappa-j} \otimes \mathbf{P}_{\kappa},$$

and  $\mathbf{P}_{\kappa-j} \otimes \mathbf{P}_{\kappa}$  is the Kronecker product of  $\mathbf{P}_{\kappa-j}$  and  $\mathbf{P}_{\kappa}$ . We will also consider the spectral property of  $\mathbf{T}$  in Section 4.

In the last part of this paper, Section 5, we will consider the regularity of the  $(M, \mathbf{P})$  refinable vector  $\Phi$ . An invariant subspace  $\mathbb{H}^0$  of  $\mathbb{H}$  under  $\mathbf{T}$  is found, and it is shown that  $\Phi$  is in the Sobolev space  $W^{s_0-\epsilon}(\mathbb{R}^d)$  for any  $\epsilon > 0$ , where  $s_0 := -\log \rho(\mathbf{T}|_{\mathbb{H}^0}) / (2 \log \lambda_{\max})$ ,  $\rho(\mathbf{T}|_{\mathbb{H}^0})$  is the spectral radius of the restriction  $\mathbf{T}|_{\mathbb{H}^0}$  of  $\mathbf{T}$  to  $\mathbb{H}^0$  and  $\lambda_{\max}$  is the spectral radius of the dilation matrix  $M$ . This estimate is analyzed in an example.

## 2. STABILITY AND ORTHONORMALITY

In this section, we will provide characterizations of the stability and orthonormality of the refinable vector  $\Phi$ . We first prove some lemmas.

**Lemma 2.1.** *Let  $\gamma_k + M\mathbb{Z}^d, 1 \leq k \leq m - 1$ , and  $\eta_j + {}^tM\mathbb{Z}^d, j = 0, \dots, m - 1$ , be the  $m$  distinct elements of the coset spaces  $\mathbb{Z}^d / (M\mathbb{Z}^d)$  and  $\mathbb{Z}^d / ({}^tM\mathbb{Z}^d)$  respectively, with  $\gamma_0 = 0, \eta_0 = 0$ . Then*

$$(2.1) \quad \sum_{k=0}^{m-1} e^{i2\pi {}^t\eta_j M^{-1}\gamma_k} = m\delta(j), \quad 0 \leq j \leq m - 1;$$

$$(2.2) \quad \sum_{j=0}^{m-1} e^{i2\pi \eta_j M^{-1}\gamma_k} = m\delta(k), \quad 0 \leq k \leq m - 1.$$

*Proof.* Let  $G$  be the finite abelian group consisting of  $\gamma_k + M\mathbb{Z}^d, 1 \leq k \leq m - 1$ . For any  $j, 0 \leq j \leq m - 1$ , define on  $G$  the functions  $\chi_j(g) := e^{i2\pi {}^t\eta_j M^{-1}g}, g \in G$ . Then  $\chi_j(g), j = 0, \dots, m - 1$ , form the group  $\widehat{G}$ , the character group of  $G$ . By the orthonormality relation of characters (see [4]), we have

$$(2.3) \quad \sum_{k=0}^{m-1} \chi_j(g) \overline{\chi_{j'}(g)} = m\delta_j(j'), \quad 0 \leq j, j' \leq m - 1.$$

Let  $j' = 0$ ; then (2.3) leads to (2.1). Since the transpose of  ${}^tM$  is  $M$ , (2.2) follows from (2.1). □

Let  $\Omega$  denote the domain defined by (1.4) and denote

$$\Omega_+ := \left\{ \sum_{j=0}^{\infty} M^{-(j+1)}x_j : x_j \in [0, N]^d, \forall j \in \mathbb{Z}_+ \right\}.$$

The proof of the following lemma can be carried out by modifying that of Lemma 3.1 in [15] for the case  $r = 1$ .

**Lemma 2.2.** *Assume that  $\text{supp}\{\mathbf{P}_\alpha\} \subset [0, N]^d$  and  $\Phi$  is a compactly supported  $(M, \mathbf{P})$  matrix refinable function. Let  $\mathbf{T}$  be the transition operator defined by (1.3) and  $\mathbb{H}$  the space defined by (1.5). Then*

- (i)  $\text{supp } \Phi \subset \Omega_+$ ,
- (ii)  $\mathbb{H}$  is invariant under  $\mathbf{T}$ ,
- (iii) for any  $C(\omega) \in C_0(\mathbb{T}^d)$ , there exists some  $n \in \mathbb{Z}_+$  such that  $\mathbf{T}^n C \in \mathbb{H}$ ,
- (iv) the eigenvectors of  $\mathbf{T}$  corresponding to nonzero eigenvalues belong to  $\mathbb{H}$ .

*Proof.* (i) can be obtained similarly to Lemma 3.1 in [15]. Here we verify (ii), (iii) and (iv).

For any  $H = \sum_{\ell \in \mathbb{Z}^d} H_\ell e^{-i\ell\omega} \in C_0(\mathbb{T}^d)$ , one has

$$\mathbf{P}(\omega)H(\omega)\mathbf{P}^*(\omega) = m^{-2} \sum_{\ell \in \mathbb{Z}^d} \sum_{\kappa \in [0, N]^d} \sum_{n \in \mathbb{Z}^d} \mathbf{P}_\kappa H_\ell {}^t\mathbf{P}_{\kappa-n} e^{-i\omega(n+\ell)}.$$

Thus

$$\mathbf{TH}(\omega) = m^{-2} \sum_{j=0}^{m-1} \sum_{\ell \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \sum_{\kappa \in [0, N]^d} \mathbf{P}_\kappa H_\ell {}^t\mathbf{P}_{\kappa-n} e^{-i({}^tM^{-1}(\omega+2\pi\eta_j))(n+\ell)}.$$

For any  $n \in \mathbb{Z}^d, \ell \in \mathbb{Z}^d$ , write  $n + \ell = M\tau + \gamma_k$  for some  $\tau \in \mathbb{Z}^d$  and  $k \in \mathbb{Z}_+, 0 \leq k \leq m - 1$ . By Lemma 2.1,

$$(2.4) \quad \mathbf{TH}(\omega) = m^{-1} \sum_{\tau \in \mathbb{Z}^d} \left( \sum_{\ell \in \mathbb{Z}^d} \sum_{\kappa \in [0, N]^d} \mathbf{P}_\kappa H_\ell {}^t\mathbf{P}_{\kappa-(M\tau-\ell)} \right) e^{-i\omega\tau}.$$

If  $H \in \mathbb{H}$ , then  $H = \sum_{\ell \in [\Omega]} H_\ell e^{-i\ell\omega}$  and

$$\mathbf{TH}(\omega) = m^{-1} \sum_{\tau \in \mathbb{Z}^d} \sum_{\ell \in [\Omega]} \sum_{\kappa \in [0, N]^d} \mathbf{P}_\kappa H_\ell {}^t\mathbf{P}_{\kappa-(M\tau-\ell)} e^{-i\omega\tau}.$$

If  $\mathbf{TH}(\omega) \neq 0$ , then  $M\tau - \ell \in [-N, N]^d$  for some  $\ell \in [\Omega]$ , i.e.  $M\tau \in [-N, N]^d + \Omega$ . Thus  $\tau \in M^{-1}[-N, N]^d + M^{-1}\Omega = \Omega$ , and we have

$$(2.5) \quad \mathbf{TH}(\omega) = m^{-1} \sum_{\tau \in [\Omega]} \left( \sum_{\ell \in [\Omega]} \sum_{\kappa \in [0, N]^d} \mathbf{P}_\kappa H_\ell {}^t\mathbf{P}_{\kappa-(M\tau-\ell)} \right) e^{-i\omega\tau}.$$

Hence  $\mathbb{H}$  is invariant under  $\mathbf{T}$ .

For  $C \in C_0(\mathbb{T}^d)$  and  $j \in \mathbb{Z}_+$ , denote  $\mathbf{T}^j C =: \sum_{\tau \in \mathbb{Z}^d} C^{(j)}(\tau) e^{-i\omega\tau}$ . By (2.4),

$$\text{supp}\{C^{(1)}(\tau)\} \subset M^{-1}[-N, N]^d + M^{-1} \text{supp } C.$$

Thus

$$\begin{aligned} \text{supp}\{C^{(j)}(\tau)\} &\subset M^{-1}[-N, N]^d + M^{-1} \text{supp}\{C^{(j-1)}(\tau)\} \subset \dots \\ &\subset M^{-1}[-N, N]^d + \dots + M^{-j}[-N, N]^d + M^{-j} \text{supp } C \subset \Omega + M^{-j} \text{supp } C. \end{aligned}$$

Since  $\text{dist}(\Omega, \mathbb{Z}^d \setminus [\Omega]) > 0$  and  $\lim_{j \rightarrow \infty} M^{-j} = 0$ , there exists  $n \in \mathbb{Z}_+$  such that

$$\text{dist}(\{0\}, M^{-n} \text{supp } C) < \text{dist}(\Omega, \mathbb{Z}^d \setminus [\Omega]).$$

Thus  $\text{supp}\{C^{(n)}(\tau)\} \in [\Omega]$  and  $\mathbf{T}^n C \in \mathbb{H}$ .

Finally, if  $C \in C_0(\mathbb{T}^d)$  is an eigenvector of  $\mathbf{T}$  with corresponding eigenvalue  $\lambda_0 \neq 0$ , then by (iii),  $C = \lambda_0^{-1} \mathbf{T}C = \dots = \lambda_0^{-n} \mathbf{T}^n C \in \mathbb{H}$ . □

**Lemma 2.3.** *Let  $\Phi$  be a compactly supported  $(M, \mathbf{P})$  matrix refinable function and  $G_\Phi$  be its Gram matrix defined by (1.6). If  $G_\Phi(\omega) < \infty$  for all  $\omega \in \mathbb{T}^d$ , then*

$$(2.6) \quad \mathbf{T}G_\Phi = G_\Phi,$$

and if  $\Phi \in L^2(\mathbb{R}^d)$ , then  $G_\Phi \in \mathbb{H}$ .

*Proof.* By (1.2) and the definitions of  $\mathbf{T}$ ,  $G_\Phi$ , we have

$$\begin{aligned} \mathbf{T}G_\Phi(\omega) &= \sum_{j=0}^{m-1} \sum_{\ell \in \mathbb{Z}^d} \mathbf{P}({}^t M^{-1}(\omega + 2\pi\eta_j)) \widehat{\Phi}({}^t M^{-1}(\omega + 2\pi\eta_j) + 2\pi\ell) \\ &\quad \cdot \widehat{\Phi}({}^t M^{-1}(\omega + 2\pi\eta_j) + 2\pi\ell) \mathbf{P}^*({}^t M^{-1}(\omega + 2\pi\eta_j)) \\ &= \sum_{j=0}^{m-1} \sum_{\ell \in \mathbb{Z}^d} \widehat{\Phi}(\omega + 2\pi\eta_j + 2\pi{}^t M\ell) \widehat{\Phi}^*(\omega + 2\pi\eta_j + 2\pi{}^t M\ell) \\ &= \sum_{\ell' \in \mathbb{Z}^d} \widehat{\Phi}(\omega + 2\pi\ell') \widehat{\Phi}^*(\omega + 2\pi\ell') = G_\Phi(\omega). \end{aligned}$$

By Lemma 2.2 and the Poisson summation formula,  $G_\Phi \in \mathbb{H}$  if  $\Phi \in L^2(\mathbb{R}^d)$ . □

In (2.6), the transition operator  $\mathbf{T}$  is defined by (1.3) on the function space consisting of  $r \times r$  matrix functions with every entry a  $2\pi$ -periodic function.

We will show that if there is a compactly supported solution  $\Phi$  of (1.1) satisfying  $G_\Phi(\omega) < \infty$  and  $\det G_\Phi(0) \neq 0$ , then  $\mathbf{P}(0)$  satisfies Condition E. For this, we first have

**Proposition 2.4.** *Let  $\Phi$  be a compactly supported matrix refinable function of (1.1) and let  $\mathbf{l}$  be a left (row) eigenvector of an eigenvalue  $\lambda_0$  of  $\mathbf{P}(0)$  with  $|\lambda_0| \geq 1$ . If  $G_\Phi(\omega) < \infty$ , for  $\omega \in \mathbb{T}^d$ , then*

$$(2.7) \quad \mathbf{l} \widehat{\Phi}(2\pi\beta) = 0, \quad \beta \in \mathbb{Z}^d \setminus \{0\}.$$

*Proof.* By (2.6),

$$\begin{aligned} \mathbf{l}G_\Phi(0)\mathbf{l}^* &= \mathbf{l}\mathbf{T}G_\Phi(0)\mathbf{l}^* \\ &= |\lambda_0|^2 \mathbf{l}G_\Phi(0)\mathbf{l}^* + \sum_{j=1}^{m-1} \mathbf{l}\mathbf{P}(2\pi{}^t M^{-1}\eta_j)G_\Phi(2\pi{}^t M^{-1}\eta_j)\mathbf{P}^*(2\pi{}^t M^{-1}\eta_j)\mathbf{l}^* \\ &\geq \mathbf{l}G_\Phi(0)\mathbf{l}^* + \sum_{j=1}^{m-1} \mathbf{l}\mathbf{P}(2\pi{}^t M^{-1}\eta_j)G_\Phi(2\pi{}^t M^{-1}\eta_j)\mathbf{P}^*(2\pi{}^t M^{-1}\eta_j)\mathbf{l}^*. \end{aligned}$$

Thus

$$\sum_{j=1}^{m-1} \mathbf{l}\mathbf{P}(2\pi{}^t M^{-1}\eta_j)G_\Phi(2\pi{}^t M^{-1}\eta_j)\mathbf{P}^*(2\pi{}^t M^{-1}\eta_j)\mathbf{l}^* = 0.$$

By (1.2), we have

$$\begin{aligned} & \sum_{j=1}^{m-1} \sum_{\alpha \in \mathbb{Z}^d} |\mathbf{l}\widehat{\Phi}(2\pi\eta_j + 2\pi^t M\alpha)|^2 \\ &= \sum_{j=1}^{m-1} \sum_{\alpha \in \mathbb{Z}^d} \mathbf{IP}(2\pi^t M^{-1}\eta_j)\widehat{\Phi}(2\pi^t M^{-1}\eta_j + 2\pi\alpha) \\ & \quad \cdot \widehat{\Phi}(2\pi^t M^{-1}\eta_j + 2\pi\alpha)\mathbf{P}(2\pi^t M^{-1}\eta_j)\mathbf{l}^* \\ &= \sum_{j=1}^{m-1} \mathbf{IP}(2\pi^t M^{-1}\eta_j)G_{\Phi}(2\pi^t M^{-1}\eta_j)\mathbf{P}^*(2\pi^t M^{-1}\eta_j)\mathbf{l}^* = 0. \end{aligned}$$

Therefore,

$$\mathbf{l}\widehat{\Phi}(2\pi\eta_j + 2\pi^t M\alpha) = 0, \quad 1 \leq j \leq m - 1, \alpha \in \mathbb{Z}^d.$$

For any  $\beta \in \mathbb{Z}^d \setminus \{0\}$ , there exist  $j \in \mathbb{Z}_+, 1 \leq j \leq m - 1, n \in \mathbb{Z}_+, \alpha \in \mathbb{Z}^d$  such that  $\beta = ({}^t M)^n(\eta_j + {}^t M\alpha)$ . Thus

$$\begin{aligned} \mathbf{l}\widehat{\Phi}(2\pi\beta) &= \mathbf{IP}(2\pi^t M^{-1}\beta) \cdots \mathbf{P}(2\pi^t M^{-n}\beta)\widehat{\Phi}(2\pi^t M^{-n}\beta) \\ &= \mathbf{IP}(0)^n \widehat{\Phi}(2\pi\eta_j + 2\pi^t M\alpha) = \lambda_0^n \mathbf{l}\widehat{\Phi}(2\pi\eta_j + 2\pi^t M\alpha) = 0. \end{aligned}$$

This shows (2.7). □

We note that if  $\lambda_0$  is an eigenvalue of  $\mathbf{P}(0)$  with  $|\lambda_0| \geq 1$  and  $\lambda_0 \neq 1$ , then for any left  $\lambda_0$ -eigenvector  $\mathbf{l}$  of  $\mathbf{P}(0)$ ,  $\mathbf{l}\widehat{\Phi}(2\pi\beta) = 0$  for all  $\beta \in \mathbb{Z}^d$ .

By Proposition 2.4, the following proposition can be obtained as in [21]. Its proof is presented here for the sake of completeness.

**Proposition 2.5.** *Let  $\Phi$  be a compactly supported  $(M, \mathbf{P})$  refinable vector with  $G_{\Phi}(\omega) < \infty$ . If  $\det(G_{\Phi}(0)) \neq 0$ , then  $\mathbf{P}(0)$  satisfies Condition E.*

*Proof.* Let  $\lambda_0$  be an eigenvalue of  $\mathbf{P}(0)$  with  $|\lambda_0| \geq 1$ , and  $\mathbf{l}$  be a corresponding left (row) eigenvector. If  $\lambda_0 \neq 1$ , by Proposition 2.4,  $\mathbf{l}G_{\Phi}(0)\mathbf{l}^* = \mathbf{l}\widehat{\Phi}(0)\widehat{\Phi}^*(0)\mathbf{l}^* = 0$ . On the other hand, since  $\Phi \neq 0$ , the spectral radius of  $\mathbf{P}(0) \geq 1$ . These two facts imply that if  $\det(G_{\Phi}(0)) \neq 0$ , then 1 is the only eigenvalue of  $\mathbf{P}(0)$  on the unit circle with  $\widehat{\Phi}(0)$  being a corresponding right eigenvector, and all other eigenvalues are in the unit circle. If 1 is not simple, since  $\widehat{\Phi}(0)$  is a right 1-eigenvector of  $\mathbf{P}(0)$ , then one can find a left (row) 1-eigenvector  $\mathbf{l}$  of  $\mathbf{P}(0)$  such that  $\mathbf{l}\widehat{\Phi}(0) = 0$ , which again leads to  $\mathbf{l}G_{\Phi}(0)\mathbf{l}^* = 0$ . Therefore, 1 has to be a simple eigenvalue of  $\mathbf{P}(0)$ , and hence  $\mathbf{P}(0)$  satisfies Condition E. □

**Proposition 2.6.** *Assume that (1.1) has a compactly supported solution  $\Phi$  with  $G_{\Phi}(\omega) < \infty$ . If  $\det(G_{\Phi}(2\pi^t M^{-1}\eta_j)) \neq 0, j = 0, \dots, m - 1$ , then  $\mathbf{P}(0)$  satisfies Condition E and satisfies the vanishing moment conditions of order at least one, i.e.*

$$(2.8) \quad \mathbf{lP}(2\pi^t M^{-1}\eta_j) = 0, \quad 1 \leq j \leq m - 1,$$

where  $\mathbf{l}$  is the left 1-eigenvector of  $\mathbf{P}(0)$ .

*Proof.* By Proposition 2.5,  $\mathbf{P}(0)$  satisfies Condition E; and by (2.6),

$$\begin{aligned} \mathbf{I}G_\Phi(0)\mathbf{I}^* &= \mathbf{I}TG_\Phi(0)\mathbf{I}^* \\ &= \mathbf{I}G_\Phi(0)\mathbf{I}^* + \sum_{j=1}^{m-1} \mathbf{I}\mathbf{P}(2\pi^t M^{-1}\eta_j)G_\Phi(2\pi^t M^{-1}\eta_j)\mathbf{P}^*(2\pi^t M^{-1}\eta_j)\mathbf{I}^*. \end{aligned}$$

Hence,

$$\mathbf{I}\mathbf{P}(2\pi^t M^{-1}\eta_j)G_\Phi(2\pi^t M^{-1}\eta_j)(\mathbf{I}\mathbf{P}(2\pi^t M^{-1}\eta_j))^* = 0, \quad 1 \leq j \leq m - 1.$$

Since  $G_\Phi(2\pi^t M^{-1}\eta_j) > 0$ , we have  $\mathbf{I}\mathbf{P}(2\pi^t M^{-1}\eta_j) = 0, 1 \leq j \leq m - 1.$  □

By Proposition 2.6, we have the following corollary.

**Corollary 2.7.** *If (1.1) has a compactly supported solution  $\Phi$  which is stable, then  $\mathbf{P}(0)$  satisfies Condition E and  $\mathbf{P}$  satisfies the vanishing moment conditions of order one (2.8).*

Here we note that the vanishing moment condition (2.8) is equivalent to

$$(2.9) \quad \mathbf{1} \sum_{\alpha \in \mathbb{Z}^d} \mathbf{P}_{M\alpha + \gamma_k} = 1, \quad 1 \leq k \leq m - 1.$$

In fact if (2.9) holds, then for any  $j \in \mathbb{Z}_+, 0 \leq j \leq m - 1$ , by (2.1)

$$\begin{aligned} \mathbf{I}\mathbf{P}(2\pi^t M^{-1}\eta_j) &= \frac{1}{m} \mathbf{1} \sum_{\alpha \in \mathbb{Z}^d} \mathbf{P}_\alpha e^{-i2\pi^t \eta_j M^{-1} \alpha} \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \mathbf{1} \sum_{\beta \in \mathbb{Z}^d} \mathbf{P}_{M\beta + \gamma_k} e^{-i2\pi^t \eta_j M^{-1} (M\beta + \gamma_k)} \\ &= \frac{1}{m} \sum_{k=0}^{m-1} (\mathbf{1} \sum_{\beta \in \mathbb{Z}^d} \mathbf{P}_{M\beta + \gamma_k}) e^{-i2\pi^t \eta_j M^{-1} \gamma_k} \\ &= \frac{1}{m} \sum_{k=0}^{m-1} e^{-i2\pi^t \eta_j M^{-1} \gamma_k} = \delta(j). \end{aligned}$$

Conversely, if (2.8) holds, then for any  $k \in \mathbb{Z}_+, 0 \leq k \leq m - 1$ , by (2.2)

$$\begin{aligned} 1 &= \sum_{j=0}^{m-1} \mathbf{I}\mathbf{P}(2\pi^t M^{-1}\eta_j) e^{i2\pi^t \eta_j M^{-1} \gamma_k} \\ &= \frac{1}{m} \sum_{j=0}^{m-1} \mathbf{1} \sum_{\beta \in \mathbb{Z}^d} \sum_{s=0}^{m-1} \mathbf{P}_{M\beta + \gamma_s} e^{-i2\pi^t \eta_j M^{-1} \gamma_s} e^{i2\pi^t \eta_j M^{-1} \gamma_k} \\ &= \frac{1}{m} \sum_{\beta \in \mathbb{Z}^d} \mathbf{1} \sum_{s=0}^{m-1} \mathbf{P}_{M\beta + \gamma_s} \sum_{j=0}^{m-1} e^{-i2\pi^t \eta_j M^{-1} (\gamma_s - \gamma_k)} \\ &= \frac{1}{m} \sum_{\beta \in \mathbb{Z}^d} \mathbf{1} \sum_{s=0}^{m-1} \mathbf{P}_{M\beta + \gamma_s} m \delta_k(s) = \mathbf{1} \sum_{\beta \in \mathbb{Z}^d} \mathbf{P}_{M\beta + \gamma_k}, \end{aligned}$$

and therefore (2.9) holds.

**Corollary 2.8.** *If (1.1) has a compactly supported solution  $\Phi$  which is stable, then  $\mathbf{P}(0)$  satisfies Condition E and  $\mathbf{P}$  satisfies*

$$\mathbf{l} \sum_{\alpha \in \mathbb{Z}^d} \mathbf{P}_{M\alpha + \gamma_k} = 1, \quad 1 \leq k \leq m - 1.$$

where  $\mathbf{l}$  is the left 1-eigenvector of  $\mathbf{P}(0)$ .

In the following we will assume that  $\mathbf{P}(0)$  satisfies Condition E and let  $\mathbf{r}$  be the unit right (column) 1-eigenvector of  $\mathbf{P}(0)$ . Let  $\mathbf{l}$  be the left (row) 1-eigenvector of  $\mathbf{P}(0)$  with  $\mathbf{l}\mathbf{r} = 1$ . Let  $U$  be an  $r \times r$  inverse matrix such that the first column of  $U$  is  $\mathbf{r}$  and  $U^{-1}\mathbf{P}(0)U$  is the Jordan canonical form of  $\mathbf{P}(0)$  with its  $(1, 1)$ -entry 1. Then  ${}^t\mathbf{e}_1U^{-1}$  is a left (row) 1-eigenvector of  $\mathbf{P}(0)$  with  ${}^t\mathbf{e}_1U^{-1}\mathbf{r} = {}^t\mathbf{e}_1U^{-1}U\mathbf{e}_1 = 1$ . Thus  ${}^t\mathbf{e}_1U^{-1} = \mathbf{l}$ .

Denote

$$\Pi_n(\omega) := \chi_{[-\pi, \pi]^d}({}^tM^{-n}\omega) \prod_{j=1}^n \mathbf{P}({}^tM^{-j}\omega), \quad \Pi(\omega) := \prod_{j=1}^{\infty} \mathbf{P}({}^tM^{-j}\omega).$$

Then, if  $\mathbf{P}(0)$  satisfies Condition E,  $\Pi_n$  converges to  $\Pi$  pointwise with

$$(2.10) \quad \Pi(\omega)U = (\widehat{\Phi}(\omega), \mathbf{0}, \dots, \mathbf{0}),$$

where

$$(2.11) \quad \widehat{\Phi}(\omega) := \prod_{j=1}^{\infty} \mathbf{P}({}^tM^{-j}\omega)\mathbf{r},$$

and any other compactly supported solution  $\Psi$  of (1.1) with  $\widehat{\Psi}(0) \neq 0$  is given by (2.11). About the convergence of the infinite product  $\prod_{j=1}^{\infty} \mathbf{P}({}^tM^{-j}\omega)$ , see [3], [23] for  $M = 2\mathbf{I}_r$ , and [20] for general dilation matrices  $M$ .

By (2.10), we have, for any  $r \times r$  matrix  $A$ ,

$$\begin{aligned} \Pi(\omega)A\Pi(\omega)^* &= \Pi(\omega)UU^{-1}A(U^{-1})^*U^*\Pi^*(\omega) \\ &= \widehat{\Phi}(\omega)\mathbf{e}_1^T U^{-1}A(U^{-1})^*\mathbf{e}_1\widehat{\Phi}^*(\omega) = (|\mathbf{l}A|)^*\widehat{\Phi}(\omega)\widehat{\Phi}(\omega)^*. \end{aligned}$$

We will provide in the next proposition a characterization of the existence of  $L^2$ -solutions of (1.1) under the assumption that  $\mathbf{P}(0)$  satisfies Condition E. For this, we have the following lemma.

**Lemma 2.9.** *For any  $H_1(\omega), H_2(\omega) \in C_0(\mathbb{T}^d)$ , and any positive integer  $n$ ,*

$$(2.12) \quad \int_{\mathbb{T}^d} H_1(\omega)(\mathbf{T}^n H_2)(\omega)d\omega = \int_{\mathbb{R}^d} H_1(\omega)\Pi_n(\omega)H_2({}^tM^{-n}\omega)\Pi_n^*(\omega)d\omega.$$

*Proof.* The proof of (2.12) can be carried out by induction. In fact for  $n = 1$ ,

$$\begin{aligned}
 \int_{\mathbb{T}^d} H_1(\omega) \mathbf{T}H_2(\omega) d\omega &= m \int_{\mathbb{R}^d} H_1({}^tM\omega) \sum_{j=0}^{m-1} \mathbf{P}(\omega + 2\pi^t M^{-1}\eta_j) \\
 &\quad \cdot H_2(\omega + 2\pi^t M^{-1}\eta_j) \mathbf{P}^*(\omega + 2\pi^t M^{-1}\eta_j) \chi_{\mathbb{T}^d}({}^tM\omega) d\omega \\
 &= m \int_{\mathbb{R}^d} H_1({}^tM\omega) \mathbf{P}(\omega) H_2(\omega) \mathbf{P}^*(\omega) \sum_{j=0}^{m-1} \chi_{\mathbb{T}^d}({}^tM\omega - 2\pi\eta_j) d\omega \\
 &= m \int_{\mathbb{T}^d} H_1({}^tM\omega) \mathbf{P}(\omega) H_2(\omega) \mathbf{P}^*(\omega) \sum_{\beta \in \mathbb{Z}^d} \sum_{j=0}^{m-1} \chi_{\mathbb{T}^d}({}^tM\omega - 2\pi^t M\beta - 2\pi\eta_j) d\omega \\
 &= m \int_{\mathbb{T}^d} H_1({}^tM\omega) \mathbf{P}(\omega) H_2(\omega) \mathbf{P}^*(\omega) d\omega \\
 &= \int_{\mathbb{R}^d} H_1(\omega) \mathbf{P}({}^tM^{-1}\omega) H_2({}^tM^{-1}\omega) \mathbf{P}^*({}^tM^{-1}\omega) \chi_{\mathbb{T}^d}({}^tM^{-1}\omega) d\omega \\
 &= \int_{\mathbb{R}^d} H_1(\omega) \Pi_1(\omega) H_2({}^tM^{-1}\omega) \Pi_1^*(\omega) d\omega.
 \end{aligned}$$

For  $n \in \mathbb{Z}_+ \setminus \{0\}$ , assume that (2.12) holds for any positive integers smaller than  $n$ ; then

$$\begin{aligned}
 \int_{\mathbb{T}^d} H_1(\omega) (\mathbf{T}^n H_2)(\omega) d\omega &= \int_{\mathbb{R}^d} H_1(\omega) \Pi_{n-1}(\omega) (\mathbf{T}H_2)({}^tM^{1-n}\omega) \Pi_{n-1}^*(\omega) d\omega \\
 &= m^n \int_{\mathbb{R}^d} H_1({}^tM^n\omega) \Pi_{n-1}({}^tM^n\omega) (\mathbf{T}H_2)({}^tM\omega) \Pi_{n-1}^*({}^tM^n\omega) d\omega \\
 &= m^n \int_{\mathbb{R}^d} H_1({}^tM^n\omega) \Pi_{n-1}({}^tM^n\omega) \sum_{j=0}^{m-1} \mathbf{P}(\omega + 2\pi^t M^{-1}\eta_j) H_2(\omega + 2\pi^t M^{-1}\eta_j) \\
 &\quad \cdot \mathbf{P}^*(\omega + 2\pi^t M^{-1}\eta_j) \Pi_{n-1}^*({}^tM^n\omega) \chi_{\mathbb{T}^d}({}^tM\omega) d\omega \\
 &= m^n \sum_{\beta \in \mathbb{Z}^d} \int_{\mathbb{T}^d} H_1({}^tM^n\omega) \mathbf{P}({}^tM^{n-1}\omega) \cdots \mathbf{P}({}^tM\omega) \mathbf{P}(\omega) H_2(\omega) \\
 &\quad \cdot \mathbf{P}^*(\omega) \cdots \mathbf{P}^*({}^tM^{n-1}\omega) \sum_{j=0}^{m-1} \chi_{\mathbb{T}^d}({}^tM\omega - 2\pi^t M\beta - 2\pi\eta_j) d\omega \\
 &= m^n \int_{\mathbb{T}^d} H_1({}^tM^n\omega) \mathbf{P}({}^tM^{n-1}\omega) \cdots \mathbf{P}(\omega) H_2(\omega) (\mathbf{P}({}^tM^{n-1}\omega) \cdots \mathbf{P}(\omega))^* d\omega \\
 &= \int_{\mathbb{R}^d} H_1(\omega) \Pi_n(\omega) H_2({}^tM^{-n}\omega) \Pi_n^*(\omega) d\omega.
 \end{aligned}$$

Thus the proof by induction is completed. □

**Proposition 2.10.** *Suppose that  $\mathbf{P}(0)$  satisfies Condition E. Then  $\Phi$  defined by (2.11) is in  $L^2(\mathbb{R}^d)$  if and only if there exists a positive semidefinite  $H \in \mathbb{H}$  such that  $\mathbf{T}H = H$  and  $\mathbf{1}H(0)\mathbf{1}^* > 0$ .*

*Proof.* Suppose  $\Phi \in L^2(\mathbb{R}^d)$ . Then the matrix  $H(\omega) := G_\Phi(\omega) \in \mathbb{H}$ , and  $H(\omega) \geq \mathbf{0}$ ,  $\mathbf{T}H = H$ . By Proposition 2.4,  $\mathbf{1}H(0)\mathbf{1}^* = \mathbf{1}\widehat{\Phi}(0)\widehat{\Phi}^*(0)\mathbf{1}^* = |\mathbf{1r}|^2 = 1$ .

Conversely, since the matrix  $\Pi_n(\omega)H(tM^{-n}\omega)\Pi_n^*(\omega)$  converges pointwise to the matrix

$$\Pi(\omega)H(0)\Pi(\omega)^* = (\mathbf{1}H(0)\mathbf{1}^*)\widehat{\Phi}(\omega)\widehat{\Phi}(\omega)^*,$$

we have

$$\begin{aligned} (\mathbf{1}H(0)\mathbf{1}^*) \int_{\mathbb{R}^d} |\widehat{\Phi}(\omega)|^2 d\omega &= \sum_{i=1}^r \int_{\mathbb{R}^d} \liminf_{n \rightarrow \infty} {}^t\mathbf{e}_i \Pi_n(\omega) H(tM^{-n}\omega) \Pi_n(\omega)^* \mathbf{e}_i d\omega \\ &\leq \sum_{i=1}^r \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} {}^t\mathbf{e}_i \Pi_n(\omega) H(tM^{-n}\omega) \Pi_n(\omega)^* \mathbf{e}_i d\omega < \infty. \end{aligned}$$

The last inequality follows from the fact that

$$\int_{\mathbb{R}^d} \Pi_n(\omega) H(tM^{-n}\omega) \Pi_n^*(\omega) d\omega = \int_{\mathbb{T}^d} (\mathbf{T}^n H)(\omega) d\omega = \int_{\mathbb{T}^d} H(\omega) d\omega.$$

□

About the existence of  $L^2$ -solutions of (1.1) for  $M = 2\mathbf{I}_r$ , a similar result was obtained in [21]. For the special case  $r = 1$  and  $d = 1$ , this result was given in [28].

We will use the fact that if (1.1) has a compactly supported solution which is stable, then for any  $H_1, H_2 \in \mathbb{H}$ ,

(2.13)

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \Pi_n(\omega) H_1(tM^{-n}\omega) \Pi_n(\omega)^* H_2(\omega) d\omega = \int_{\mathbb{R}^d} \Pi(\omega) H_1(0) \Pi(\omega)^* H_2(\omega) d\omega.$$

Equation (2.13) can be obtained as in [21] for the case  $M = 2\mathbf{I}_r$ , and we omit the details here.

The next theorem provides a characterization of the stability of the compactly supported  $(M, \mathbf{P})$  refinable vector  $\Phi$ .

**Theorem 2.11.** *The refinement equation (1.1) has a compactly supported solution which is stable if and only if the following conditions hold:*

- (i) *the matrix  $\mathbf{P}(0)$  satisfies Condition E,*
- (ii) *for the left (row) 1-eigenvector  $\mathbf{l}$  of  $\mathbf{P}(0)$ ,  $\mathbf{lP}(2\pi^t M^{-1} \eta_j) = 0$ ,  $1 \leq j \leq m-1$ ,*
- (iii) *the restriction transition operator  $\mathbf{T}$  to  $\mathbb{H}$  satisfies Condition E, and the corresponding 1-eigenvector is positive (or negative) definite on  $\mathbb{T}^d$ .*

*Proof.* “ $\Leftarrow$ ” Let  $H_0 \in \mathbb{H}$  be the positive definite 1-eigenvector of  $\mathbf{T}$ . By Proposition 2.10, the solution  $\Phi$  given by (2.11) is in  $L^2(\mathbb{R}^d)$ . Let  $H(\omega) = G_\Phi(\omega)$ ; then  $H(\omega) \in \mathbb{H}$  and  $\mathbf{T}H = H$ . Since the restriction  $\mathbf{T}|_{\mathbb{H}}$  of  $\mathbf{T}$  to  $\mathbb{H}$  satisfies Condition E,  $H = cH_0$  for some positive constant  $c$ . Thus  $G_\Phi(\omega) = cH_0(\omega) > 0$ , and hence  $\Phi$  is stable.

“ $\Rightarrow$ ” Let  $\Phi$  be a compactly supported solution which is stable; then  $\widehat{\Phi}(0) = c\mathbf{r}$  for some nonzero constant  $c$ . (i), (ii) follow from Proposition 2.6. To complete the proof of Theorem 2.11, it is enough to show that the restricted operator  $\mathbf{T}|_{\mathbb{H}}$  satisfies Condition E, since  $G_\Phi$  is a positive definite 1-eigenvector of  $\mathbf{T}|_{\mathbb{H}}$ .

Let  $\lambda_0$  be an eigenvalue of  $\mathbf{T}|_{\mathbb{H}}$  and  $H$  be a corresponding eigenvector. Since

$$\begin{aligned} \lambda_0^n \int_{\mathbb{T}^d} H(\omega) H(\omega)^* d\omega &= \int_{\mathbb{T}^d} \mathbf{T}^n H(\omega) H(\omega)^* d\omega \\ &= \int_{\mathbb{R}^d} \Pi_n(\omega) H(tM^{-n}\omega) \Pi_n(\omega)^* H(\omega)^* d\omega, \end{aligned}$$

the limit  $\lim_{n \rightarrow \infty} \lambda_0^n$  exists. Thus  $|\lambda_0| \leq 1$ , and 1 is the only eigenvalue of  $\mathbf{T}|_{\mathbb{H}}$  on the unit circle.

For an eigenvector  $H$  of eigenvalue 1 of  $\mathbf{T}|_{\mathbb{H}}$ , denote  $c_0 = \mathbf{1}H(0)\mathbf{1}^*$ . Then

$$\begin{aligned} & \int_{\mathbb{T}^d} (H - c_0 G_{\Phi})(H - c_0 G_{\Phi})^* d\omega \\ &= \int_{\mathbb{R}^d} \Pi_n(\omega)(H({}^t M^{-n}\omega) - c_0 G_{\Phi}({}^t M^{-n}\omega))\Pi_n(\omega)^*(H(\omega) - c_0 G_{\Phi}(\omega))^* d\omega \\ &\rightarrow \int_{\mathbb{R}^d} \Pi(\omega)(H(0) - c_0 G_{\Phi}(0))\Pi(\omega)^*(H(\omega) - c_0 G_{\Phi}(\omega))^* d\omega \\ &= \mathbf{1}(H(0) - c_0 G_{\Phi}(0))\mathbf{1}^* \int_{\mathbb{R}^d} \widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)(H(\omega) - c_0 G_{\Phi}(\omega))^* d\omega = 0. \end{aligned}$$

Thus  $H(\omega) = c_0 G_{\Phi}(\omega)$ . This implies that the geometric multiplicity of the eigenvalue 1 of  $\mathbf{T}|_{\mathbb{H}}$  is 1.

Finally we show that 1 is nondegenerate. Otherwise, there exists  $H \in \mathbb{H}$  such that  $\mathbf{T}H = G_{\Phi} + H$ . Let  $H_1 = H - c_1 G_{\Phi}$ , where  $c_1 = \mathbf{1}H(0)\mathbf{1}^*$ . Then

$$\begin{aligned} & \int_{\mathbb{T}^d} \mathbf{T}^n H_1(\omega)G_{\Phi}(\omega)^* d\omega = \int_{\mathbb{R}^d} \Pi_n(\omega)H_1({}^t M^{-n}\omega)\Pi_n(\omega)^*G_{\Phi}(\omega)^* d\omega \\ &\rightarrow \int_{\mathbb{R}^d} \Pi(\omega)(H(0) - c_1 G_{\Phi}(0))\Pi(\omega)^*G_{\Phi}(\omega)^* d\omega = 0. \end{aligned}$$

On the other hand,

$$\mathbf{T}^n H_1 = \mathbf{T}^n H - c_1 G_{\Phi} = nG_{\Phi} + H - c_1 G_{\Phi};$$

thus  $\|\int_{\mathbb{T}^d} \mathbf{T}^n H_1(\omega)G_{\Phi}(\omega)^* d\omega\| \rightarrow \infty$  as  $n \rightarrow \infty$ . This leads to a contradiction  $\square$

The next theorem provides a characterization of the orthonormality of the compactly supported  $(M, \mathbf{P})$  refinable vector  $\Phi$ .

**Theorem 2.12.** *The refinement equation (1.1) has a compactly supported solution which is orthogonal if and only if the following conditions hold:*

- (i) the mask  $\mathbf{P}$  is a CQF,
- (ii) the matrix  $\mathbf{P}(0)$  satisfies Condition E,
- (iii) for the left (row) 1-eigenvector  $\mathbf{l}$  of  $\mathbf{P}(0)$ ,  $\mathbf{lP}(2\pi^t M^{-1}\eta_j) = 0, 1 \leq j \leq m-1$ ,
- (iv) the restriction of the transition operator  $\mathbf{T}$  to  $\mathbb{H}$  satisfies Condition E.

*Proof.* “ $\Leftarrow$ ” Since  $\mathbf{P}$  is a CQF,  $\mathbf{T}\mathbf{l}_r = \mathbf{l}_r$ . Therefore by Proposition 2.10, the compactly supported solution  $\Phi$  given by (2.11) is in  $L^2(\mathbb{R}^d)$ . By (iv),  $G_{\Phi} = c\mathbf{l}_r$  for some positive constant  $c$ , and hence (1.1) has a compactly supported solution which is orthogonal.

“ $\Rightarrow$ ” (ii), (iii) and (iv) follow from the orthonormality of  $\Phi$  and Theorem 2.11. By the orthonormality of  $\Phi$ ,  $G_{\Phi}(\omega) = \mathbf{l}_r$ . Thus  $\mathbf{T}\mathbf{l}_r = \mathbf{l}_r$ , i.e.

$$\sum_{j=0}^{m-1} \mathbf{P}({}^t M^{-1}(\omega + 2\pi\eta_j))\mathbf{P}^*({}^t M^{-1}(\omega + 2\pi\eta_j)) = \mathbf{l}_r,$$

and hence  $\mathbf{P}$  is a CQF.  $\square$

3. APPROXIMATION ORDER

In this section we will consider the approximation order of the matrix refinable function  $\Phi$ . Throughout this section, we will assume the eigenvalues of the dilation matrix  $M$  are nondegenerate.

Let  ${}^tM$  be the transpose of  $M$  and  $\lambda_j, j = 1, \dots, r$ , be the eigenvalues of  $M$ . By our assumptions,  $|\lambda_i| > 1$  and every  $\lambda_i$  is nondegenerate. Thus, there exist  $d$  linearly independent vectors  $\mathbf{v}^1, \dots, \mathbf{v}^d$  such that  ${}^tM\mathbf{v}^j = \lambda_j\mathbf{v}^j, j = 1, \dots, d$ . Let

$$(3.1) \quad V := (\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^d)$$

be the  $d \times d$  matrix with column vectors  $\mathbf{v}^1, \dots, \mathbf{v}^d$ . Then

$${}^tMV = (\lambda_1\mathbf{v}^1, \dots, \lambda_d\mathbf{v}^d) = V\Lambda,$$

where  $\Lambda := \text{diag}(\lambda_1, \dots, \lambda_d)$ . Denote

$$\lambda := {}^t(\lambda_1, \dots, \lambda_d).$$

Then for any  $x \in \mathbb{R}^d, \beta \in \mathbb{Z}_+^d$ ,

$$(\Lambda x)^\beta = \lambda^\beta x^\beta.$$

For  $1 \leq j \leq d$ , let  $D_{\mathbf{v}^j}$  denote the derivative operator in the direction  $\mathbf{v}^j$ , i.e.

$$D_{\mathbf{v}^j} := (\partial_1, \dots, \partial_d)\mathbf{v}^j.$$

Then

$$D_{\mathbf{v}^j} f({}^tM\omega) = \lambda_j(D_{\mathbf{v}^j} f)({}^tM\omega).$$

For  $\beta = {}^t(\beta_1, \dots, \beta_d) \in \mathbb{Z}_+^d$ , denote

$$D_V^\beta := D_{\mathbf{v}^1}^{\beta_1} \dots D_{\mathbf{v}^d}^{\beta_d}.$$

Then we have

$$(3.2) \quad D_V^\beta f({}^tM\omega) = \lambda^\beta (D_V^\beta f)({}^tM\omega), \quad \beta \in \mathbb{Z}_+^d.$$

For a compactly supported vector-valued function  $\Psi = {}^t(\psi_1, \dots, \psi_r)$ , we denote by  $\mathcal{S}(\Psi)$  the linear space of all functions of the form  $\sum_{i=1}^r \sum_{\ell \in \mathbb{Z}^d} c_i(\ell) \psi_i(\cdot - \ell)$ , where  $\{c_i(\ell)\}_{\ell \in \mathbb{Z}^d}$  are arbitrary sequences on  $\mathbb{Z}^d$ .

We say  $\Psi$  has **accuracy** of order  $k$  if all polynomials of total degree smaller than  $k$  are contained in  $\mathcal{S}(\Psi)$ , i.e. for any  $\beta \in \mathbb{Z}_+^d, |\beta| < k$ , there exist  $y_{\beta,i}(\ell)$  such that

$$x^\beta = \sum_{i=1}^r \sum_{\ell \in \mathbb{Z}^d} y_{\beta,i}(\ell) \psi_i(x + \ell).$$

For  $\Psi \in L^2(\mathbb{R}^d)$  and  $h > 0$ , let

$$S_h(\Psi) := \{f(\frac{\cdot}{h}) : f \in \mathcal{S}(\Psi) \cap L^2(\mathbb{R}^d)\}$$

be the  $h$ -dilation of  $\mathcal{S}(\Psi) \cap L^2(\mathbb{R}^d)$ . For  $k > 0$ , we say  $\Psi$  (or  $\mathcal{S}(\Psi)$ ) provides  **$L^2$ -approximation** of order  $k$  if for every sufficiently smooth function  $f \in L^2(\mathbb{R}^d)$  and any  $h > 0$

$$\text{dist}(f, S_h(\Psi)) = O(h^k),$$

where  $\text{dist}$  here is the  $L^2$ -distance between a function and a subset of  $L^2(\mathbb{R}^d)$ .

An  $r \times 1$  vector-valued function  $\Psi$  is said to satisfy the **Strang-Fix conditions** of order  $k$  if there is a finitely supported  $1 \times r$  vector-valued sequence  $\{q_\ell\}_{\ell \in \mathbb{Z}^d}$  such that  $f := \sum_{\ell \in \mathbb{Z}^d} q_\ell \Psi(\cdot - \ell)$  satisfies

$$(3.3) \quad D^\beta \widehat{f}(2\pi\ell) = \delta(\beta)\delta(\ell), \quad \text{for } \ell \in \mathbb{Z}^d, \beta \in \mathbb{Z}_+^d, |\beta| < k.$$

About the relations among the orders of accuracy,  $L^2$ -approximation and Strang-Fix conditions of  $\Psi$ , see [13] and the references therein. The next theorem was obtained by Jia (see [13], [14]).

**Theorem 3.1.** (Jia). *Let  $\Psi = {}^t(\psi_1, \dots, \psi_r) \in L^2(\mathbb{R}^d)$  be a compactly supported vector-valued function. Assume that the sequences  $(\widehat{\psi}_j(2\pi\beta))_{\beta \in \mathbb{Z}^d}$ ,  $j = 1, \dots, r$ , are linearly independent. Then the following statements are equivalent:*

- (a)  $\Psi$  provides  $L_2$ -approximation of order  $k$ ;
- (b)  $\Psi$  has accuracy of order  $k$ ;
- (c)  $\Psi$  satisfies the Strang-Fix conditions of order  $k$ .

For a compactly supported  $(M, \mathbf{P})$  refinable vector  $\Phi$ , we will find the  $L^2$ -approximation order of  $\Phi$  in terms of the mask  $\mathbf{P}$ . For a given mask  $\mathbf{P}$ , if there exist a positive integer  $k$  and  $1 \times r$  complex vectors  $\mathbf{l}_0^\beta$ ,  $|\beta| < k$ , with  $\mathbf{l}_0^0 \neq 0$ , such that

$$(3.4) \quad \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} (i\lambda)^{\alpha-\beta} \mathbf{l}_0^\alpha D_V^{\beta-\alpha} \mathbf{P}(2\pi^t M^{-1} \eta_j) = \delta(j) \lambda^{-\beta} \mathbf{l}_0^\beta, \quad 0 \leq j \leq m-1,$$

we say that the refinement mask  $\mathbf{P}$  satisfies the **vanishing moment conditions** of order  $k$ .

We show in the next theorem that if  $\mathbf{P}$  satisfies the vanishing moment conditions of order  $k$  and  $\Phi \in L^2(\mathbb{R}^d)$  is a compactly supported  $(M, \mathbf{P})$  refinable vector with  $\mathbf{l}_0^0 \widehat{\Phi}(0) \neq 0$ , then  $\Phi$  satisfies the Strang-Fix conditions of order  $k$ .

**Theorem 3.2.** *If  $\mathbf{P}$  satisfies the vanishing moment conditions of order  $k$ , i.e. there exist  $1 \times r$  complex vectors  $\mathbf{l}_0^\beta$ ,  $|\beta| < k$ , with  $\mathbf{l}_0^0 \neq 0$  such that (3.4) holds, then any compactly supported  $(\mathbf{P}, M)$  refinable vector  $\Phi \in L^2(\mathbb{R})$  with  $\mathbf{l}_0^0 \widehat{\Phi}(0) \neq 0$  satisfies the Strang-Fix conditions of order  $k$ .*

*Proof.* Let  $f$  be the vector-valued function in  $L^2(\mathbb{R}^d)$  defined by

$$(3.5) \quad \widehat{f}(\omega) := b(\omega) \widehat{\Phi}(\omega)$$

where  $b(\omega)$  is the vector-valued function given by  $b(\omega) = \sum_{|\ell| < k} b_\ell e^{i\ell\omega}$  with

$$(3.6) \quad (-i)^{|\beta|} D_V^\beta b(0) = \sum_{|\ell| < k} ({}^t V \ell)^\beta b_\ell = \mathbf{l}_0^\beta, \quad |\beta| < k.$$

We will show that  $f$  satisfies the Strang-Fix conditions of order  $k$ .

Since  $(\partial_1, \dots, \partial_d) = (D_{v^1}, \dots, D_{v^d}) V^{-1}$ , it is enough to show that

$$(3.7) \quad D_V^\beta \widehat{f}(2\pi\ell) = c\delta(\beta)\delta(\ell), \quad \text{for } \ell \in \mathbb{Z}^d \text{ and } \beta \in \mathbb{Z}_+^d, |\beta| < k,$$

where  $c$  is a nonzero constant.

One can check that (3.4) is equivalent to

$$D_V^\beta (b(\omega) \mathbf{P}({}^t M^{-1} \omega)) |_{\omega=2\pi\eta_j} = \delta(j) \lambda^{-\beta} D_V^\beta b(0), \quad 0 \leq j \leq m-1, \beta \in \mathbb{Z}_+^d, |\beta| < k.$$

For any  $\ell \in \mathbb{Z}^d$ , there exists  $j, 0 \leq j \leq m - 1$ , such that  $\ell \in \eta_j + {}^t M \mathbb{Z}^d$ . By (3.2), one has

$$\begin{aligned} D_V^\beta \widehat{f}(2\pi\ell) &= D_V^\beta (b(\omega) \mathbf{P}({}^t M^{-1}\omega) \widehat{\Phi}({}^t M^{-1}\omega))|_{\omega=2\pi\ell} \\ &= \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} D_V^\alpha (b(\omega) \mathbf{P}({}^t M^{-1}\omega))|_{\omega=2\pi\ell} D_V^{\beta-\alpha} (\widehat{\Phi}({}^t M^{-1}\omega))|_{\omega=2\pi\ell} \\ &= \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} D_V^\alpha (b(\omega) \mathbf{P}({}^t M^{-1}\omega))|_{\omega=2\pi\eta_j} \lambda^{\alpha-\beta} D_V^{\beta-\alpha} \widehat{\Phi}(2\pi {}^t M^{-1}\ell) \\ &= \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} \lambda^{-\alpha} D_V^\alpha b(0) \delta(j) \lambda^{\alpha-\beta} D_V^{\beta-\alpha} \widehat{\Phi}(2\pi {}^t M^{-1}\ell) \\ &= \delta(j) \lambda^{-\beta} \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} D_V^\alpha b(2\pi {}^t M^{-1}\ell) D_V^{\beta-\alpha} \widehat{\Phi}(2\pi {}^t M^{-1}\ell) \\ &= \delta(j) \lambda^{-\beta} D_V^\beta \widehat{f}(2\pi {}^t M^{-1}\ell); \end{aligned}$$

the next to last equality is because if  $j = 0$ , then  $D_V^\alpha b(2\pi {}^t M^{-1}\ell) = D_V^\alpha b(0)$  by  $2\pi$ -periodicity of  $b(\omega)$ , and if  $j \neq 0$ , both sides are zero. So we have

$$(3.8) \quad D_V^\beta \widehat{f}(2\pi\ell) = \delta(j) \lambda^{-\beta} D_V^\beta \widehat{f}(2\pi {}^t M^{-1}\ell), \quad \ell \in \eta_j + {}^t M \mathbb{Z}^d.$$

If  $\ell \neq 0$ , by repeating this procedure, we have  $D_V^\beta \widehat{f}(2\pi\ell) = 0$ . And if  $\ell = 0, \beta \neq 0$ , then by (3.8),  $D_V^\beta \widehat{f}(0) = \lambda^{-\beta} D_V^\beta \widehat{f}(0)$ . Thus  $D_V^\beta \widehat{f}(0) = 0$  since  $\lambda^{-\beta} \neq 1$ . Finally, if  $\ell = 0, \beta = 0$ , then

$$\widehat{f}(0) = b(0) \widehat{\Phi}(0) = \mathbf{I}_0^0 \widehat{\Phi}(0) \neq 0.$$

Therefore we have (3.7) with  $c = \mathbf{I}_0^0 \widehat{\Phi}(0)$ , and proved Theorem 3.2. □

*Remark 3.3.* We note that  $\mathbf{I}_0^0$  in (3.4) is a left 1-eigenvector of  $\mathbf{P}(0)$ . Thus if  $\mathbf{P}(0)$  satisfies Condition E, then the solution  $\Phi \in L^2(\mathbb{R}^d)$  of (1.1) with  $\mathbf{I}_0^0 \widehat{\Phi}(0) \neq 0$  is given by (2.11), and  $\Phi$  given by (2.11) satisfies  $\mathbf{I}_0^0 \widehat{\Phi}(0) \neq 0$ .

*Remark 3.4.* Note that for a compactly supported vector-valued function  $\Psi \in L^2(\mathbb{R}^2)$ , the condition that  $(\widehat{\psi}_j(2\pi\beta))_{\beta \in \mathbb{Z}^d, j = 1, \dots, r}$ , are linearly independent in Theorem 3.1 (Jia) is equivalent to  $\det(G_\Phi(0)) \neq 0$ . Theorem 4.2 in [7] says that under the mild condition  $\det(G_\Phi(0)) \neq 0$ ,  $\Phi$  providing  $L^2$ -approximation of order  $k$  implies that the finitely supported  $1 \times r$  vector-valued sequence  $\{q_\ell\}_{\ell \in \mathbb{Z}^d}$  with  $f := \sum_{\ell \in \mathbb{Z}^d} q_\ell \Phi(\cdot - \ell)$  satisfying (3.3) is **unique**.

The above two remarks lead to the following proposition about the uniqueness of the vectors  $\mathbf{I}_0^\beta$  satisfying (3.4).

**Proposition 3.5.** *Assume that  $\mathbf{P}$  satisfies the vanishing moment conditions of order  $k$  with vectors  $\mathbf{I}_0^\beta, \beta \in \mathbb{Z}_+^d, |\beta| < k, \mathbf{I}_0^0 \neq 0$  satisfying (3.4). If (1.1) has a compactly supported solution  $\Phi \in L^2(\mathbb{R}^d)$  satisfying  $\det(G_\Phi(0)) \neq 0$ , then, up to a constant, the vectors  $\mathbf{I}_0^\beta, \beta \in \mathbb{Z}_+^d, |\beta| < k$ , are unique.*

*Proof.* Assume that  $\mathbf{I}_0^\beta, \beta \in \mathbb{Z}_+^d, |\beta| < k, \mathbf{I}_0^0 \neq 0$  are vectors satisfying (3.4). Since  $\det(G_\Phi(0)) \neq 0, \mathbf{P}(0)$  satisfies Condition E with  $\widehat{\Phi}(0)$  being a right 1-eigenvector of  $\mathbf{P}(0)$ . Hence  $\mathbf{I}_0^0 \widehat{\Phi}(0) \neq 0$ . Let  $f$  be the function defined by (3.5) with  $\{b_\ell\}$  defined by (3.6). As shown in the proof of Theorem 3.2,  $f$  satisfies (3.3). Since  $\det(G_\Phi(0)) \neq 0$ ,

by Theorem 4.2 in [7], the sequence  $\{b_\ell\}$  is unique (up to a constant). Hence the vectors  $\mathbf{l}_0^\beta$  are also unique.  $\square$

The next theorem will show that, under mild conditions,  $\mathbf{P}$  satisfying the vanishing moment conditions of order  $k$  is also necessary for  $\Phi$  to provide  $L^2$ -approximation of order  $k$ .

**Theorem 3.6.** *Assume that  $\Phi \in L^2(\mathbb{R}^d)$  is a compactly supported  $(M, \mathbf{P})$  refinable vector and  $\det(G_\Phi(2\pi^t M^{-1}\eta_j)) \neq 0$ ,  $j = 0, \dots, m-1$ . Then the following conditions are equivalent:*

- (i)  $\Phi$  provides approximation of order  $k$ ;
- (ii)  $\Phi$  has accuracy of order  $k$ ;
- (iii)  $\Phi$  satisfies the Strang-Fix conditions of order  $k$ ;
- (iv) the matrix refinement mask  $\mathbf{P}$  satisfies the vanishing moment conditions of order  $k$ .

*Proof.* The equivalence of (i), (ii) and (iii) is proved in Theorem 3.1 (Jia). Since  $\det(G_\Phi(0)) \neq 0$ , by Proposition 2.5,  $\mathbf{P}(0)$  satisfies Condition E. Thus by Remark 3.3 and Theorem 3.2, we know that (iv)  $\Rightarrow$  (iii), and we need only to show that (iii)  $\Rightarrow$  (iv).

Let  $\{q_\ell\}$  be the finitely supported  $1 \times r$  vector-valued sequence such that  $f = \sum_{\ell \in \mathbb{Z}^d} q_\ell \Phi(\cdot - \ell)$  satisfies (3.7) with  $c = 1$ . Let  $\hat{q}(\omega)$  denote the Fourier series of  $\{q_\ell\}$ ; then  $\hat{f}(\omega) = \hat{q}(\omega)\hat{\Phi}(\omega)$ . We will prove by induction that

$$(3.9) \quad D_V^\beta (\hat{q}(\omega)\mathbf{P}(^t M^{-1}\omega))|_{\omega=2\pi\eta_j} = \delta(j)\lambda^{-\beta} D_V^\beta \hat{q}(0), \quad 0 \leq j \leq m-1, \beta \in \mathbb{Z}_+^d, |\beta| < k,$$

which is equivalent to (3.4) with  $\mathbf{l}_0^\beta = (-i)^{|\beta|} D_V^\beta \hat{q}(0)$ .

First we have  $\hat{f}(0) = \hat{q}(0)\hat{\Phi}(0) \neq 0$ ; thus  $\mathbf{l}_0^0 = \hat{q}(0) \neq 0$ . Since  $\hat{f}(2\pi\kappa) = \delta(\kappa)$ ,  $\kappa \in \mathbb{Z}^d$ ,

$$\hat{q}(0)\mathbf{P}(2\pi^t M^{-1}\kappa)\hat{\Phi}(2\pi^t M^{-1}\kappa) = \delta(\kappa).$$

Hence for any  $j \in \mathbb{Z}_+$ ,  $0 \leq j \leq m-1$ , and  $\ell \in \mathbb{Z}^d$ ,

$$(3.10) \quad \hat{q}(0)\mathbf{P}(2\pi^t M^{-1}\eta_j)\hat{\Phi}(2\pi\ell + 2\pi^t M^{-1}\eta_j) = \delta(\ell)\delta(j).$$

Multiplying both sides of (3.10) by  $\hat{\Phi}^*(2\pi\ell + 2\pi^t M^{-1}\eta_j)$  and summing over  $\ell \in \mathbb{Z}^d$ ,

$$\hat{q}(0)\mathbf{P}(2\pi^t M^{-1}\eta_j)G_\Phi(2\pi^t M^{-1}\eta_j) = \delta(j)\hat{\Phi}^*(0).$$

If  $j \neq 0$ , then by the invertibility of  $G_\Phi(2\pi^t M^{-1}\eta_j)$ , we have  $\hat{q}(0)\mathbf{P}(2\pi^t M^{-1}\eta_j) = 0$ , and if  $j = 0$ , then we have

$$\hat{q}(0)\mathbf{P}(0) = \hat{\Phi}^*(0)G_\Phi(0)^{-1}.$$

On the other hand, since  $\hat{f}(2\pi\kappa) = \delta(\kappa)$ ,  $\kappa \in \mathbb{Z}^d$ , we have  $\hat{q}(0)\hat{\Phi}(2\pi\kappa) = \delta(\kappa)$ . This again leads to  $\hat{q}(0)G_\Phi(0) = \hat{\Phi}^*(0)$ , i.e.  $\hat{q}(0) = \hat{\Phi}^*(0)G_\Phi(0)^{-1}$ . Therefore we have  $\hat{q}(0)\mathbf{P}(0) = \hat{q}(0)$ , and (3.9) is true for  $\beta = 0$ .

For  $\beta \in \mathbb{Z}_+^d \setminus \{0\}$ ,  $|\beta| < k$ , assume that (3.9) is true any  $\alpha < \beta$ ,  $\alpha \in \mathbb{Z}_+^d$ . We want to prove that (3.9) holds for  $\beta$ .

Since  $D_V^\beta \hat{f}(2\pi\kappa) = 0$ , for all  $\kappa \in \mathbb{Z}^d$

$$\sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} D_V^\alpha (\hat{q}(\omega)\mathbf{P}(^t M^{-1}\omega))|_{\omega=2\pi\kappa} D_V^{\beta-\alpha} (\hat{\Phi}(^t M^{-1}\omega))|_{\omega=2\pi\kappa} = 0,$$

and hence for any  $j \in \mathbb{Z}_+, 0 \leq j \leq m - 1$ , and  $\ell \in \mathbb{Z}^d$

$$\sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} D_V^\alpha (\widehat{q}(\omega) \mathbf{P}({}^t M^{-1} \omega)) |_{\omega=2\pi\eta_j} D_V^{\beta-\alpha} (\widehat{\Phi}({}^t M^{-1} \omega)) |_{\omega=2\pi^t M\ell+2\pi\eta_j} = 0.$$

By (3.9) for  $\alpha < \beta$ ,

$$\begin{aligned} & D_V^\beta (\widehat{q}(\omega) \mathbf{P}({}^t M^{-1} \omega)) |_{\omega=2\pi\eta_j} \widehat{\Phi}(2\pi\ell + 2\pi^t M^{-1} \eta_j) \\ &= - \sum_{0 \leq \alpha < \beta} \binom{\beta}{\alpha} \lambda^{-\alpha} \delta(j) D_V^\alpha \widehat{q}(0) \lambda^{\alpha-\beta} D_V^{\beta-\alpha} \widehat{\Phi}(2\pi\ell + 2\pi^t M^{-1} \eta_j). \end{aligned}$$

If  $j \neq 0$ , then as above we have

$$D_V^\beta (\widehat{q}(\omega) \mathbf{P}({}^t M^{-1} \omega)) |_{\omega=2\pi\eta_j} G_\Phi(2\pi^t M^{-1} \eta_j) = 0$$

and therefore  $D_V^\beta (\widehat{q}(\omega) \mathbf{P}({}^t M^{-1} \omega)) |_{\omega=2\pi\eta_j} = 0$ . If  $j = 0$ , then

$$D_V^\beta (\widehat{q}(\omega) \mathbf{P}({}^t M^{-1} \omega)) |_{\omega=0} \widehat{\Phi}(2\pi\ell) + \lambda^{-\beta} \sum_{0 \leq \alpha < \beta} \binom{\beta}{\alpha} D_V^\alpha \widehat{q}(0) D_V^{\beta-\alpha} \widehat{\Phi}(2\pi\ell) = 0.$$

Since  $\widehat{f}(\omega) = \widehat{q}(\omega) \widehat{\Phi}(\omega)$  and  $D_V^\beta \widehat{f}(2\pi\ell) = 0, \ell \in \mathbb{Z}^d$ ,

$$\sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} D_V^\alpha \widehat{q}(0) D_V^{\beta-\alpha} \widehat{\Phi}(2\pi\ell) = 0.$$

Thus

$$D_V^\beta (\widehat{q}(\omega) \mathbf{P}({}^t M^{-1} \omega)) |_{\omega=0} \widehat{\Phi}(2\pi\ell) = \lambda^{-\beta} D_V^\beta \widehat{q}(0) \widehat{\Phi}(2\pi\ell).$$

This leads to

$$D_V^\beta (\widehat{q}(\omega) \mathbf{P}({}^t M^{-1} \omega)) |_{\omega=0} G_\Phi(0) = \lambda^{-\beta} D_V^\beta \widehat{q}(0) G_\Phi(0)$$

and therefore

$$D_V^\beta (\widehat{q}(\omega) \mathbf{P}({}^t M^{-1} \omega)) |_{\omega=0} = \lambda^{-\beta} D_V^\beta \widehat{q}(0).$$

It follows that (3.9) holds for  $\beta$ , so that the proof by induction is completed.  $\square$

Denote by  $\widetilde{\Phi}(x)$  the bi-infinite column from the integer shifts of  $\Phi$ :

$$\widetilde{\Phi}(x) := {}^t (\dots, {}^t \Phi(x + \ell), \dots)_{\ell \in \mathbb{Z}^d},$$

and by  $L$  the bi-infinite matrix

$$L := (\mathbf{P}_{M\alpha-\beta})_{\alpha, \beta \in \mathbb{Z}^d}.$$

Then the refinement equation (1.1) can be written as

$$L \widetilde{\Phi}(Mx) = \widetilde{\Phi}(x).$$

The characterization of the accuracy order of  $\Phi$  in terms of the eigenvalues and eigenvector structures of the infinite matrix  $L$  were studied in [11], [25] and [17] for the case  $d = 1$ . In [1], a similar characterization of the accuracy order of  $\Phi$  was obtained based on the ergodic theorem for the multivariate case with arbitrary matrix dilations  $M$  (no restriction on the diagonalization on  $M$ ), and the coefficients  $y_{\beta,i}(\kappa)$  for the polynomial reproducing  $x^\beta = \sum_{i=1}^r \sum_{\kappa \in \mathbb{Z}^d} y_{\beta,i}(\kappa) \phi_i(x + \kappa)$  were determined explicitly. In the rest of this section, under the assumption that the

integer shifts  $(\phi_i(x - \ell), 1 \leq i \leq r, \ell \in \mathbb{Z}^d)$  of  $\Phi$  are linearly independent, we will determine explicitly the coefficients  $\mathbf{y}_\ell^\beta$  for the polynomial reproducing

$$(3.11) \quad \sum_{\ell \in \mathbb{Z}^d} \mathbf{y}_\ell^\beta \Phi(x + \ell) = ({}^tVx)^\beta, \quad x \in \mathbb{R}^d, |\beta| < k,$$

where  $V$  is the matrix defined by (3.1).

**Theorem 3.7.** *Assume that  $\Phi \in L^2(\mathbb{R}^d)$  is a compactly supported  $(M, \mathbf{P})$  refinable vector and the integer shifts of  $\Phi$  are linearly independent. If  $\Phi$  has accuracy of order  $k$  with  $\mathbf{y}_\ell^\beta, \ell \in \mathbb{Z}^d, \beta \in \mathbb{Z}_+^d, |\beta| < k$ , being the  $1 \times r$  complex vectors such that (3.11) holds, then  $\mathbf{y}_\ell^\beta$  satisfy*

- (i)  $\mathbf{y}_\ell^\beta = \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} (-{}^tV\ell)^{\beta-\alpha} \mathbf{y}_0^\alpha$ ,
- (ii)  $\mathbf{y}^\beta L = \lambda^{-\beta} \mathbf{y}^\beta$ , where  $\mathbf{y}^\beta := (\dots, \mathbf{y}_\ell^\beta, \dots)_{\ell \in \mathbb{Z}^d}$ ,
- (iii) the vectors  $\mathbf{y}_0^\beta, \beta \in \mathbb{Z}_+^d, |\beta| < k$ , satisfy the vanishing moment conditions (3.4).

*Proof.* Let  $\mathbf{y}_\ell^\beta, \ell \in \mathbb{Z}^d, \beta \in \mathbb{Z}_+^d, |\beta| < k$ , be the complex vectors such that (3.11) holds. For any  $\tau \in \mathbb{Z}^d$ ,

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}^d} \mathbf{y}_{\ell+\tau}^\beta \Phi(x + \ell) &= \sum_{\ell \in \mathbb{Z}^d} \mathbf{y}_\ell^\beta \Phi(x - \tau + \ell) = ({}^tV(x - \tau))^\beta \\ &= \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} (-{}^tV\tau)^{\beta-\alpha} ({}^tVx)^\alpha \\ &= \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} (-{}^tV\tau)^{\beta-\alpha} \sum_{\ell \in \mathbb{Z}^d} \mathbf{y}_\ell^\alpha \Phi(x + \ell) \\ &= \sum_{\ell \in \mathbb{Z}^d} \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} (-{}^tV\tau)^{\beta-\alpha} \mathbf{y}_\ell^\alpha \Phi(x + \ell). \end{aligned}$$

By the linear independence of the integer shifts of  $\Phi$ ,

$$(3.12) \quad \mathbf{y}_{\ell+\tau}^\beta = \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} (-{}^tV\tau)^{\beta-\alpha} \mathbf{y}_\ell^\alpha.$$

Let  $\ell = 0$ ; then (3.12) leads to (i).

For  $\beta \in \mathbb{Z}_+^d, |\beta| < k$ , we have by (3.11)

$$({}^tVx)^\beta = \mathbf{y}^\beta \tilde{\Phi}(x) = \mathbf{y}^\beta L \tilde{\Phi}(Mx)$$

and

$$({}^tVx)^\beta = \lambda^{-\beta} (\Lambda {}^tVx)^\beta = \lambda^{-\beta} ({}^tVMx)^\beta = \lambda^{-\beta} \mathbf{y}^\beta \tilde{\Phi}(Mx).$$

By the linear independence of the integer shifts of  $\Phi$  again,

$$(3.13) \quad \mathbf{y}^\beta L = \lambda^{-\beta} \mathbf{y}^\beta, \quad \text{for } \beta \in \mathbb{Z}_+^d, |\beta| < k.$$

Finally, we verify (iii). Note that (3.13) can be written equivalently as

$$\sum_{\ell \in \mathbb{Z}^d} \mathbf{y}_\ell^\beta \mathbf{P}_{M\ell-\ell'} = \lambda^{-\beta} \mathbf{y}_{\ell'}^\beta, \quad \text{for any } \ell' \in \mathbb{Z}^d, \beta \in \mathbb{Z}_+^d, |\beta| < k,$$

and, in particular, for any  $j, 0 \leq j \leq m - 1$ ,

$$(3.14) \quad \lambda^{-\beta} \mathbf{y}_{-\gamma_j}^\beta = \sum_{\ell \in \mathbb{Z}^d} \mathbf{y}_\ell^\beta \mathbf{P}_{M\ell + \gamma_j} = \sum_{\ell \in \mathbb{Z}^d} \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} (-{}^tV\ell)^{\beta-\alpha} \mathbf{y}_0^\alpha \mathbf{P}_{M\ell + \gamma_j}.$$

For any  $\kappa \in \mathbb{Z}_+^d, |\kappa| < k$ , multiplying both side of (3.14) by

$$\lambda^{\beta-\kappa} (-{}^tV\gamma_j)^{\kappa-\beta} \binom{\kappa}{\beta}$$

and summing over  $\beta \leq \kappa$ , one has by (3.12) and  $\Lambda^tV = {}^tVM$ ,

$$\begin{aligned} \lambda^{-\kappa} \mathbf{y}_0^\kappa &= \lambda^{-\kappa} \sum_{0 \leq \beta \leq \kappa} \binom{\kappa}{\beta} (-{}^tV\gamma_j)^{\kappa-\beta} \mathbf{y}_{-\gamma_j}^\beta \\ &= \sum_{\ell \in \mathbb{Z}^d} \sum_{0 \leq \beta \leq \kappa} \sum_{0 \leq \alpha \leq \beta} \binom{\kappa}{\beta} \binom{\beta}{\alpha} \lambda^{\beta-\kappa} (-{}^tV\gamma_j)^{\kappa-\beta} (-{}^tV\ell)^{\beta-\alpha} \mathbf{y}_0^\alpha \mathbf{P}_{M\ell + \gamma_j} \\ &= \sum_{\ell \in \mathbb{Z}^d} \sum_{0 \leq \alpha \leq \kappa} \sum_{\alpha \leq \beta \leq \kappa} \binom{\kappa}{\alpha} \binom{\kappa-\alpha}{\beta-\alpha} \lambda^{\alpha-\kappa} (-{}^tV\gamma_j)^{\kappa-\beta} (-{}^tVM\ell)^{\beta-\alpha} \mathbf{y}_0^\alpha \mathbf{P}_{M\ell + \gamma_j} \\ &= \sum_{\ell \in \mathbb{Z}^d} \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} \lambda^{\alpha-\kappa} \\ &\quad \cdot \sum_{0 \leq \tau \leq \kappa-\alpha} \binom{\kappa-\alpha}{\tau} (-{}^tV\gamma_j)^{\kappa-\alpha-\tau} (-{}^tVM\ell)^\tau \mathbf{y}_0^\alpha \mathbf{P}_{M\ell + \gamma_j} \\ &= \sum_{\ell \in \mathbb{Z}^d} \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} \lambda^{\alpha-\kappa} (-{}^tV(M\ell + \gamma_j))^{\kappa-\alpha} \mathbf{y}_0^\alpha \mathbf{P}_{M\ell + \gamma_j}. \end{aligned}$$

Thus for any  $\kappa \in \mathbb{Z}_+^d, |\kappa| < k$ ,

$$(3.15) \quad \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} (-\lambda)^{\alpha-\kappa} \mathbf{y}_0^\alpha \sum_{\ell \in \mathbb{Z}^d} ({}^tV(M\ell + \gamma_j))^{\kappa-\alpha} \mathbf{P}_{M\ell + \gamma_j} = \lambda^{-\kappa} \mathbf{y}_0^\kappa.$$

For any  $s \in \mathbb{Z}_+, 0 \leq s \leq m - 1$ , multiplying both side of (3.15) by  $e^{-2\pi^t\eta_s M^{-1}\gamma_j}$  and summing over  $j = 0, \dots, m - 1$ , one has by Lemma 2.1,

$$\begin{aligned} &\sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} (-\lambda)^{\alpha-\kappa} \mathbf{y}_0^\alpha \sum_{j=0}^{m-1} \sum_{\ell \in \mathbb{Z}^d} ({}^tV(M\ell + \gamma_j))^{\kappa-\alpha} \mathbf{P}_{M\ell + \gamma_j} e^{-2\pi^t\eta_s M^{-1}\gamma_j} \\ &= \lambda^{-\kappa} \mathbf{y}_0^\kappa \sum_{j=0}^{m-1} e^{-2\pi^t\eta_s M^{-1}\gamma_j} = m \lambda^{-\kappa} \mathbf{y}_0^\kappa \delta(s). \end{aligned}$$

Thus

$$\frac{1}{m} \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} (-\lambda)^{\alpha-\kappa} \mathbf{y}_0^\alpha \sum_{\ell' \in \mathbb{Z}^d} ({}^tV\ell')^{\kappa-\alpha} \mathbf{P}_{\ell'} e^{-2\pi^t\eta_s M^{-1}\ell'} = \lambda^{-\kappa} \mathbf{y}_0^\kappa \delta(s).$$

On the other hand, one has

$$\begin{aligned} & \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} (i\lambda)^{\alpha-\kappa} \mathbf{y}_0^\alpha D_V^{\kappa-\alpha} \mathbf{P}(2\pi^t M^{-1} \eta_s) \\ &= \frac{1}{m} \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} (i\lambda)^{\alpha-\kappa} \mathbf{y}_0^\alpha \sum_{\ell \in \mathbb{Z}^d} (-i^t V \ell)^{\kappa-\alpha} \mathbf{P}_\ell e^{-i^t \eta_s M^{-1} \ell} \\ &= \frac{1}{m} \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} (-\lambda)^{\alpha-\kappa} \mathbf{y}_0^\alpha \sum_{\ell \in \mathbb{Z}^d} (i^t V \ell)^{\kappa-\alpha} \mathbf{P}_\ell e^{-i^t \eta_s M^{-1} \ell}. \end{aligned}$$

Therefore for any  $s \in \mathbb{Z}_+, 0 \leq s \leq m - 1, \kappa \in \mathbb{Z}_+^d, |\kappa| < k,$

$$\sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} (i\lambda)^{\alpha-\kappa} \mathbf{y}_0^\alpha D_V^{\kappa-\alpha} \mathbf{P}(2\pi^t M^{-1} \eta_s) = \delta(s) \lambda^{-\kappa} \mathbf{y}_0^\kappa,$$

and the proof of (iii) is completed. □

*Remark 3.8.* By Proposition 3.5,  $\mathbf{y}_0^\beta, \beta \in \mathbb{Z}_+^d, |\beta| < k,$  are the unique vectors satisfying (3.4). Thus the unique coefficients  $\mathbf{y}_\ell^\beta$  for the reproducing polynomial are given by (i) of Theorem 3.7, and they satisfy (ii) of Theorem 3.7.

#### 4. THE RESTRICTED TRANSITION OPERATOR

Assume that  $\mathbf{P}$  is a matrix refinement mask with  $\text{supp}\{\mathbf{P}_\alpha\} \subset [0, N]^d$  for some positive integer  $N,$  and  $\Phi$  is a compactly supported  $(M, \mathbf{P})$  refinable vector. It was shown in Section 2 that to decide whether  $\Phi$  is stable (orthogonal) or not, we need only to check the properties of the spectra (eigenvalues) and the 1-eigenvector of the restriction  $\mathbf{T}|_{\mathbb{H}}$  of the transition operator  $\mathbf{T}$  to  $\mathbb{H},$  where  $\mathbb{H}$  is the finite dimensional space defined by (1.5) and  $\mathbf{T}$  is the transition operator defined by (1.3). It is useful in practice to transfer equivalently the restricted operator  $\mathbf{T}|_{\mathbb{H}}$  to a finite matrix, since eigenvalues and eigenvectors of a finite matrix can be computed directly. In this section, we give the representing matrix  $\mathcal{T}$  of  $\mathbf{T}|_{\mathbb{H}},$  and then study the spectral properties of  $\mathbf{T}.$

For  $H(\omega) = \sum_{\ell \in [\Omega]} H_\ell e^{-i\ell\omega} \in \mathbb{H},$  by (2.5), under the basis  $\{e^{-i\ell\omega}\}_{\ell \in [\Omega]}$  of  $\mathbb{H}, \mathbf{T}$  transfers the sequence  $\{H_\ell\}_{\ell \in [\Omega]}$  into another sequence:

$$\left\{ m^{-1} \sum_{\ell \in [\Omega]} \sum_{\kappa \in [0, N]^d} \mathbf{P}_\kappa H_\ell {}^t \mathbf{P}_{\kappa - (M\tau - \ell)} \right\}_{\tau \in [\Omega]}.$$

Now let us look at the matrices of the form  $\mathbf{P}_\kappa H_\ell {}^t \mathbf{P}_\tau.$  Let  $Q = (Q(1), \dots, Q(r))$  be an  $r \times r$  matrix with  $Q(j)$  the  $j$ th column, and define an  $r^2 \times 1$  vector  $\text{vec}(Q)$  by

$$\text{vec}(Q) := {}^t ({}^t Q(1), \dots, {}^t Q(r)).$$

Then we have the following lemma.

**Lemma 4.1.** *Let  $P, Q, H$  be  $r \times r$  matrices, then*

$$(4.1) \quad \text{vec}(PH {}^t Q) = (Q \otimes P) \text{vec}(H),$$

where  $Q \otimes P = (q_{ij} p)_{1 \leq i, j \leq r},$  the Kronecker product of matrices  $Q$  and  $P.$

*Proof.* Let  $P(i), H(i)$  denote the  $i$ th column of  $P$  and  $H$ , respectively, and let  $q_{ij}$  be the  $(i, j)$ -entry of  $Q$ . Then the  $j$ th column of  $PH^tQ$  is

$$PH (q_{ji})_{i=1}^r = \sum_{i=1}^r q_{ji}PH(i) = (q_{j1}P, \dots, q_{jr}P)^t ({}^tH(1), \dots, {}^tH(r)).$$

Thus

$$\begin{aligned} \text{vec}(PH^tQ) &= {}^t({}^t(PH(q_{1i})_{i=1}^r), \dots, {}^t(PH(q_{ri})_{i=1}^r)) \\ &= (q_{ji}P)_{1 \leq j \leq r, 1 \leq i \leq r} {}^t({}^tH(1), \dots, {}^tH(r)) = (Q \otimes P)\text{vec}(H). \end{aligned}$$

□

About formula (4.1) for more general matrices, one can refer to [12], and in particular, one has that, for any  $1 \times r$  vectors  $\mathbf{v}, \mathbf{u}$  and  $r \times r$  matrix  $Q$ ,

$$(4.2) \quad (\mathbf{v} \otimes \mathbf{u})\text{vec}(Q) = \mathbf{u}Q^t\mathbf{v},$$

where  $\mathbf{v} \otimes \mathbf{u}$  denotes the Kronecker product of  $\mathbf{v}, \mathbf{u}$ .

For  $j \in \mathbb{Z}^d$ , define  $r^2 \times r^2$  matrices

$$\mathcal{A}_j := m^{-1} \sum_{\ell \in [0, N]^d} \mathbf{P}_{\ell-j} \otimes \mathbf{P}_\ell,$$

and define an  $(r^2|[[\Omega]]|) \times (r^2|[[\Omega]]|)$  matrix

$$(4.3) \quad \mathcal{T} := (\mathcal{A}_{Mi-j})_{i, j \in [\Omega]}.$$

For  $f = \sum_{j \in [\Omega]} f_j e^{-i\omega j} \in \mathbb{H}$ , let  $\text{vec}(f)$  be the  $(r^2|[[\Omega]]|) \times 1$  vector defined by

$$\text{vec}(f) := {}^t(\dots, {}^t(\text{vec}(f_j)), \dots)_{j \in [\Omega]}.$$

Then from (2.5) and (4.1), for any  $\tau \in [\Omega]$ ,

$$\begin{aligned} \text{vec}((\mathbf{T}H)_\tau) &= m^{-1} \sum_{\ell \in [\Omega]} \sum_{\kappa \in [0, N]^d} \text{vec}(\mathbf{P}_\kappa H_\ell {}^t\mathbf{P}_{\kappa-(M\tau-\ell)}) \\ &= m^{-1} \sum_{\ell \in [\Omega]} \sum_{\kappa \in [0, N]^d} (\mathbf{P}_{\kappa-(M\tau-\ell)} \otimes \mathbf{P}_\kappa)\text{vec}(H_\ell) \\ &= \sum_{\ell \in [\Omega]} \mathcal{A}_{M\tau-\ell} \text{vec}(H_\ell) = (\mathcal{T} \text{vec}(H))(\tau). \end{aligned}$$

Hence we have

**Theorem 4.2.** *The restriction of the transition operator  $\mathbf{T}$  to  $\mathbb{H}$  is equivalent to the matrix  $\mathcal{T}$  defined by (4.3) under the basis  $\{e^{-i\omega \ell}\}_{\ell \in [\Omega]}$  of  $\mathbb{H}$ , and for  $H \in \mathbb{H}$*

$$(4.4) \quad \text{vec}(\mathbf{T}H) = \mathcal{T} \text{vec}(H).$$

Lemma 2.2, Theorem 2.11, Theorem 2.12 and Theorem 4.2 lead to the following two corollaries.

**Corollary 4.3.** *The refinement equation (1.1) has a compactly supported solution which is stable if and only if the following conditions hold:*

- (i) the matrix  $\mathbf{P}(0)$  satisfies Condition E,
- (ii) for the left (row) 1-eigenvector  $\mathbf{l}$  of  $\mathbf{P}(0)$ ,  $\mathbf{l}\mathbf{P}(2\pi^t M^{-1}\eta_j) = 0$ ,  $1 \leq j \leq m-1$ ,
- (iii) the finite matrix  $\mathcal{T}$  satisfies Condition E and the corresponding right 1-eigenvector  $\mathbf{v}$  is such that  $H_0(\omega)$  is positive (or negative) definite on  $\mathbb{T}^d$ , where  $H_0(\omega)$  is the unique matrix function in  $\mathbb{H}$  satisfying  $\text{vec}(H_0) = \mathbf{v}$ .

**Corollary 4.4.** *The refinement equation (1.1) has a compactly supported solution which is orthogonal if and only if the following conditions hold:*

- (i) *the mask  $\mathbf{P}$  is a CQF,*
- (ii) *the matrix  $\mathbf{P}(0)$  satisfies Condition E,*
- (iii) *for the left (row) 1-eigenvector  $\mathbf{1}$  of  $\mathbf{P}(0)$ ,  $\mathbf{1P}(2\pi^t M^{-1}\eta_j) = 0$ ,  $1 \leq j \leq m-1$ ,*
- (iv) *the finite matrix  $\mathcal{T}$  satisfies Condition E.*

By (4.4),  $\mathbf{v}$  is an eigenvector of  $\mathcal{T}$  if and only if the matrix-valued function  $H(\omega)$  in  $\mathbb{H}$  with  $\text{vec}(H) = \mathbf{v}$  is an eigenvector of  $\mathbf{T}$ , and furthermore  $\mathbf{v}, H(\omega)$  correspond to the same eigenvalue. Therefore to study the spectral properties of  $\mathbf{T}$ , we need only to consider those of the matrix  $\mathcal{T}$ . In the rest of this section, we will discuss the spectral properties of  $\mathcal{T}$ . In the following, we will assume that the eigenvalues of the dilation matrix  $M$  are nondegenerate, and let  $\lambda_j, 1 \leq j \leq d$ , be the eigenvalues of  $M$ . Let  $V$  denote the matrix defined by (3.1). We also assume that  $\mathbf{P}$  satisfies the vanishing moment condition of order  $k$  for some positive integer  $k$ , i.e.  $\mathbf{P}$  satisfies (3.4) for some vectors  $\mathbf{l}_0^\beta, \beta \in \mathbb{Z}_+^d, |\beta| < k$ , with  $\mathbf{l}_0^0 \neq 0$ .

Let  $k_0 \in \mathbb{Z}_+, k_0 \leq k$ , be the largest integer such that there exist  $1 \times r$  complex vectors  $\mathbf{l}_0^\beta, \beta \in \mathbb{Z}_+^d, k \leq |\beta| \leq k + k_0 - 1$ , satisfying

$$(4.5) \quad \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} (i\lambda)^{\alpha-\beta} \mathbf{l}_0^\alpha D_V^{\beta-\alpha} \mathbf{P}(0) = \lambda^{-\beta} \mathbf{l}_0^\beta.$$

If all the numbers  $\lambda^{-\beta}, k \leq |\beta| \leq k + k_0 - 1$ , are not eigenvalues of  $\mathbf{P}(0)$  for some  $k_0 \in \mathbb{Z}_+$ , then the vectors  $\mathbf{l}_0^\beta, \beta \in \mathbb{Z}_+^d, k \leq |\beta| \leq k + k_0 - 1$ , can be chosen iteratively by

$$\mathbf{l}_0^\beta (\lambda^{-\beta} \mathbf{I}_r - \mathbf{P}(0)) = \sum_{0 \leq \alpha < \beta} \binom{\beta}{\alpha} (i\lambda)^{\alpha-\beta} \mathbf{l}_0^\alpha (D_V^{\beta-\alpha} \mathbf{P})(0).$$

For the case  $r = 1$ , since  $\mathbf{P}(0) = 1, k_0 = k$ .

Let  $B(\omega) = \sum_{\ell \in \mathbb{Z}_+^d, |\ell| < k+k_0} B_\ell e^{i\ell\omega}$  be the vector trigonometric polynomial satisfying

$$(4.6) \quad D_V^\beta B(0) = i^{|\beta|} \mathbf{l}_0^\beta, \quad \beta \in \mathbb{Z}_+^d, |\beta| < k + k_0.$$

The coefficients  $B_\kappa, 1 \times r$  vectors, can be gotten by the following equations:

$$\sum_{|\ell| < k+k_0} ({}^t V \ell)^\beta B_\ell = \mathbf{l}_0^\beta, \quad \beta \in \mathbb{Z}_+^d, |\beta| < k + k_0.$$

By (3.2), for any  $j \in \mathbb{Z}_+, 0 \leq j \leq m - 1$ ,

$$\begin{aligned} & D_V^\beta (B({}^t M \omega) \mathbf{P}(\omega)) |_{\omega=2\pi^t M^{-1}\eta_j} \\ &= \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} \lambda^\alpha \left( (D_V^\alpha B)({}^t M \omega) D_V^{\beta-\alpha} \mathbf{P}(\omega) \right) |_{\omega=2\pi^t M^{-1}\eta_j} \\ &= \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} \lambda^\alpha (D_V^\alpha B)(0) D_V^{\beta-\alpha} \mathbf{P}(\omega) |_{\omega=2\pi^t M^{-1}\eta_j} \\ &= \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} (i\lambda)^\alpha \mathbf{l}_0^\alpha D_V^{\beta-\alpha} \mathbf{P}(2\pi^t M^{-1}\eta_j). \end{aligned}$$

Thus the vanishing moment conditions (3.4) and (4.5) can be written equivalently in the forms

$$(4.7) \quad \begin{aligned} D_V^\beta (B({}^t M \omega) \mathbf{P}(\omega)) |_{\omega=2\pi {}^t M^{-1} \eta_j} &= \delta(j) D_V^\beta B(0), \\ \beta \in \mathbb{Z}_+^d, |\beta| < k, 0 \leq j < m, \end{aligned}$$

and

$$(4.8) \quad D_V^\beta (B({}^t M \omega) \mathbf{P}(\omega)) |_{\omega=0} = D_V^\beta B(0), \quad \beta \in \mathbb{Z}_+^d, k \leq |\beta| < k + k_0.$$

Let  $\mathbf{l}_0^\beta$ ,  $\beta \in \mathbb{Z}_+^d, |\beta| < k + k_0$ , be the row vectors satisfying (3.4) and (4.5). For  $\kappa \in \mathbb{Z}^d$ , define row vectors  $\mathbf{l}_\kappa^\beta$  by

$$(4.9) \quad \mathbf{l}_\kappa^\beta := \sum_{0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} (-{}^t V \kappa)^{\beta-\alpha} \mathbf{l}_0^\alpha, \quad \text{for } \beta \in \mathbb{Z}_+^d, |\beta| < k + k_0,$$

and then define  $1 \times (r^2|\Omega|)$  vectors  $\mathbf{L}_\Omega^\beta$  by

$$(4.10) \quad \mathbf{L}_\Omega^\beta := (\dots, \mathbf{l}^\beta(\kappa), \dots)_{\kappa \in [\Omega]}$$

with

$$\mathbf{l}^\beta(\kappa) := \sum_{0 \leq \alpha \leq \beta} (-1)^\alpha \binom{\beta}{\alpha} \bar{\mathbf{l}}_{-\kappa}^\alpha \otimes \mathbf{l}_0^{\beta-\alpha}, \quad \kappa \in \mathbb{Z}^d.$$

**Lemma 4.5.** *For any  $\beta \in \mathbb{Z}_+^d, |\beta| < k + k_0$ , let  $\mathbf{L}_\Omega^\beta$  be the vectors defined by (4.10). Then for any  $H \in \mathbb{H}$*

$$\mathbf{L}_\Omega^\beta \text{vec}(H) = (-i)^{|\beta|} D_V^\beta (B(\omega) H(\omega) B^*(\omega)) |_{\omega=0}.$$

*Proof.* By (4.2), for any  $\beta \in \mathbb{Z}_+^d, |\beta| < k + k_0$ , and any  $H \in \mathbb{H}$

$$\begin{aligned} \mathbf{L}_\Omega^\beta \text{vec}(H) &= \sum_{\kappa} \mathbf{l}^\beta(\kappa) \text{vec}(H_\kappa) = \sum_{\kappa} \sum_{0 \leq \alpha \leq \beta} (-1)^\alpha \binom{\beta}{\alpha} \mathbf{l}_0^{\beta-\alpha} H(\kappa) (\mathbf{l}_{-\kappa}^\alpha)^* \\ &= \sum_{\kappa} \sum_{0 \leq \alpha \leq \beta} (-1)^\alpha \binom{\beta}{\alpha} \mathbf{l}_0^{\beta-\alpha} H(\kappa) \sum_{0 \leq \gamma \leq \alpha} ({}^t V \kappa)^\gamma \binom{\alpha}{\gamma} (\mathbf{l}_0^{\alpha-\gamma})^* \\ &= \sum_{\kappa} \sum_{0 \leq \alpha \leq \beta} \sum_{0 \leq \gamma \leq \alpha} (-1)^\alpha \binom{\beta}{\alpha} \\ &\quad \cdot ({}^t V \kappa)^\gamma \binom{\alpha}{\gamma} (-i)^{|\beta-\alpha|} D_V^{\beta-\alpha} B(0) H(\kappa) i^{|\alpha-\gamma|} D_V^{\alpha-\gamma} B^*(0) \\ &= (-i)^{|\beta|} \sum_{0 \leq \alpha \leq \beta} \sum_{0 \leq \gamma \leq \alpha} \binom{\beta}{\alpha} \binom{\alpha}{\gamma} D_V^{\beta-\alpha} B(0) \sum_{\kappa} (-i {}^t V \kappa)^\gamma H(\kappa) D_V^{\alpha-\gamma} B^*(0) \\ &= (-i)^{|\beta|} \sum_{0 \leq \alpha \leq \beta} \sum_{0 \leq \gamma \leq \alpha} \binom{\beta}{\alpha} \binom{\alpha}{\gamma} D_V^{\beta-\alpha} B(0) D_V^\gamma H(0) D_V^{\alpha-\gamma} B^*(0) \\ &= (-i)^{|\beta|} D_V^\beta (B(\omega) H(\omega) B^*(\omega)) |_{\omega=0}. \end{aligned}$$

□

For  $\beta \in \mathbb{Z}_+^d, |\beta| < k + k_0$ , denote

$$E_\beta := \{\beta' : \lambda^{\beta'} = \lambda^\beta, \beta' \in \mathbb{Z}_+^d, |\beta'| < k + k_0\}.$$

**Theorem 4.6.** For any  $\beta \in \mathbb{Z}_+^d, |\beta| < k + k_0$ , let  $\mathbf{L}_\Omega^\beta$  be the vectors defined by (4.10). Then

$$(4.11) \quad \mathbf{L}_\Omega^\beta \mathcal{T} = \lambda^{-\beta} \mathbf{L}_\Omega^\beta.$$

If there exists a  $\beta' \in E_\beta$  such that  $\mathbf{L}_\Omega^{\beta'} \neq \mathbf{0}$ , then  $\lambda^{-\beta}$  is an eigenvalue of  $\mathcal{T}$  with a corresponding left eigenvector  $\mathbf{L}_\Omega^{\beta'}$ .

*Proof.* We need only to show that for any  $H \in \mathbb{H}$ ,

$$\mathbf{L}_\Omega^\beta \mathcal{T} \text{vec}(H) = \lambda^{-\beta} \mathbf{L}_\Omega^\beta \text{vec}(H).$$

In fact, by (4.4) and Lemma 4.5,

$$\begin{aligned} (i\lambda)^\beta \mathbf{L}_\Omega^\beta \mathcal{T} \text{vec}(H) &= (i\lambda)^\beta \mathbf{L}_\Omega^\beta \text{vec}(\mathbf{T}H) \\ &= D_V^\beta (B({}^t M\omega)(\mathbf{T}H)({}^t M\omega)B^*({}^t M\omega))|_{\omega=0} \\ &= \sum_{j=0}^{m-1} D_V^\beta (B({}^t M\omega)\mathbf{P}(2\pi\omega + 2\pi^t M^{-1}\eta_j) \\ &\quad \cdot H(2\pi\omega + 2\pi^t M^{-1}\eta_j)\mathbf{P}(2\pi\omega + 2\pi^t M^{-1}\eta_j)^* B^*({}^t M\omega))|_{\omega=0} \\ &= \sum_{j=0}^{m-1} \sum_{0 \leq \alpha \leq \beta} \sum_{0 \leq \gamma \leq \alpha} \binom{\beta}{\alpha} \binom{\alpha}{\gamma} D_V^\alpha (B({}^t M\omega)\mathbf{P}(\omega))|_{\omega=2\pi^t M^{-1}\eta_j} \\ &\quad \cdot D_V^\gamma H(\omega)|_{\omega=2\pi^t M^{-1}\eta_j} D_V^{\beta-\alpha-\gamma} (B({}^t M\omega)\mathbf{P}(\omega))^*|_{\omega=2\pi^t M^{-1}\eta_j}. \end{aligned}$$

Since for any  $\beta, \alpha, \gamma \in \mathbb{Z}_+^d$  with  $|\beta| < k + k_0$  and  $\gamma \leq \alpha \leq \beta$ , we have the inequality  $\min(|\alpha|, |\beta - \alpha - \gamma|) < k$ , it follows, from (4.7) and (4.8), that

$$\begin{aligned} (i\lambda)^\beta \mathbf{L}_\Omega^\beta \mathcal{T} \text{vec}(H) &= \sum_{0 \leq \alpha \leq \beta} \sum_{0 \leq \gamma \leq \alpha} \binom{\beta}{\alpha} \binom{\alpha}{\gamma} D_V^\alpha (B({}^t M\omega)\mathbf{P}(\omega))|_{\omega=0} \\ &\quad \cdot D_V^\gamma H(\omega)|_{\omega=0} D_V^{\beta-\alpha-\gamma} (B({}^t M\omega)\mathbf{P}(\omega))^*|_{\omega=0} \\ &= \sum_{0 \leq \alpha \leq \beta} \sum_{0 \leq \gamma \leq \alpha} \binom{\beta}{\alpha} \binom{\alpha}{\gamma} D_V^\alpha B(0) D_V^\gamma H(0) D_V^{\beta-\alpha-\gamma} B^*(0) \\ &= D_V^\beta (B(\omega)H(\omega)B^*(\omega))|_{\omega=0} = i^{|\beta|} \mathbf{L}_\Omega^\beta \text{vec}(H). \end{aligned}$$

Therefore  $\mathbf{L}_\Omega^\beta \mathcal{T} \text{vec}(H) = \lambda^{-\beta} \mathbf{L}_\Omega^\beta \text{vec}(H)$ . The second statement of Theorem 4.2 follows from (4.11), and the proof of Theorem 4.6 is completed.  $\square$

Since  $\mathbf{L}_\Omega^0 = (\mathbf{l}_0^0, \dots, \mathbf{l}_0^0) \neq 0$ , 1 is an eigenvalue of  $\mathbf{T}$ . In the case  $r = 1, d = 1, M = (2)$ , then  $\Omega = [-N, N]$  and  $k_0 = k$ . For any  $n \in \mathbb{Z}_+, n \leq 2k - 1$ , the vector  $((-N)^n, \dots, (-1)^n, 0^n, 1^n, \dots, N^n)$  (with  $0^n := \delta(n)$ ) is the generalized left eigenvector of the eigenvalue  $2^{-n}$  of  $\mathcal{T}$ , and hence  $2^{-n}, 0 \leq n \leq 2k - 1$ , are eigenvalues of  $\mathbf{T}$  (see [5]). Theorem 4.6 says that for  $\beta \in \mathbb{Z}_+^d, |\beta| < k + k_0$ , if there exists  $\beta' \in E_\beta$  such that  $\mathbf{L}_\Omega^{\beta'} \neq \mathbf{0}$ , then  $\lambda^{-\beta}$  is an eigenvalue of  $\mathbf{T}$ . If the refinement equation (1.1) has a compactly supported solution  $\Phi$  with  $\Phi \in W^s(\mathbb{R}^d)$  for some  $s \geq 0$ , then one can show similarly as in [19] that  $\mathbf{L}_\Omega^\beta \neq \mathbf{0}$  for  $\beta \in \mathbb{Z}_+^d, |\beta| \leq \min(k + k_0 - 1, 2s)$ , and hence  $\lambda^{-\beta}$  are eigenvalues of  $\mathbf{T}$ . In this paper, for  $s \geq 0$ , we say a vector-valued function  $f = (f_1, \dots, f_r)$  is in the Sobolev space  $W^s(\mathbb{R}^d)$  if every component  $f_j$  of  $f$  satisfies  $(1 + |\omega|^2)^{\frac{s}{2}} \widehat{f}_j(\omega) \in L^2(\mathbb{R}^d), 1 \leq j \leq r$ .

The vectors  $\mathbf{L}_\Omega^\beta$  play an important role in estimating the Sobolev regularity of the refinable vector  $\Phi$ , which will be done in the next section.

5. SOBOLEV REGULARITY ESTIMATES

Assume that  $\mathbf{P}$  ( $\{\mathbf{P}_\alpha\}$ ) is a matrix refinement mask satisfying (3.4) and (4.5) for some positive integers  $k, k_0$  with  $k_0 \leq k$ , and  $\Phi$  is a compactly supported  $(M, \mathbf{P})$  refinable vector. Suppose  $\text{supp}\{\mathbf{P}_\alpha\} \subset [0, N]^d$ , and let  $\mathbb{H}$  be the space defined by (1.5). In this section, we will estimate the Sobolev regularity of  $\Phi$  in terms of the spectral radius of the restriction of the transition operator  $\mathbf{T}$  to an invariant subspace  $\mathbb{H}^0$  of  $\mathbb{H}$ .

For  $j \in \mathbb{Z}_+, 1 \leq j \leq r$ , and  $\alpha \in \mathbb{Z}_+^d, |\alpha| < k$ , let  ${}_j\mathbf{l}_\Omega^\alpha, {}_j\mathbf{r}_\Omega^\alpha$  be the  $1 \times (r^2|\Omega|)$  vectors defined by

$$(5.1) \quad {}_j\mathbf{l}_\Omega^\alpha := (\cdots, {}_j\mathbf{l}^\alpha(\kappa), \cdots)_{\kappa \in [\Omega]}, \quad {}_j\mathbf{r}_\Omega^\alpha := (\cdots, {}_j\mathbf{r}^\alpha(\kappa), \cdots)_{\kappa \in [\Omega]}$$

with

$${}_j\mathbf{l}^\alpha(\kappa) := {}^t\mathbf{e}_j \otimes \mathbf{l}_\kappa^\alpha, \quad {}_j\mathbf{r}^\alpha(\kappa) := \bar{\mathbf{l}}_{-\kappa}^\alpha \otimes {}^t\mathbf{e}_j, \quad \kappa \in \mathbb{Z}^d.$$

**Lemma 5.1.** For  $j, 1 \leq j \leq r$ , and  $\alpha \in \mathbb{Z}_+^d, |\alpha| \leq k - 1$ , let  ${}_j\mathbf{l}_\Omega^\alpha, {}_j\mathbf{r}_\Omega^\alpha$  be the row vectors defined by (5.1). Then for any  $H \in \mathbb{H}$ ,

$$\begin{aligned} {}_j\mathbf{l}_\Omega^\alpha \text{vec}(H) &= i^\alpha D_V^\alpha (B(\omega)H(\omega)\mathbf{e}_j)|_{\omega=0}, \\ {}_j\mathbf{r}_\Omega^\alpha \text{vec}(H) &= (-i)^\alpha D_V^\alpha ({}^t\mathbf{e}_j H(\omega)B^*(\omega))|_{\omega=0}. \end{aligned}$$

*Proof.* For any  $H \in \mathbb{H}$  with  $H(\omega) = \sum_{\kappa \in [\Omega]} H_\kappa e^{-i\kappa\omega}$ ,

$$\begin{aligned} D_V^\alpha (B(\omega)H(\omega)\mathbf{e}_j)|_{\omega=0} &= \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\gamma} D_V^\gamma B(0) D_V^{\alpha-\gamma} H(0)\mathbf{e}_j \\ &= i^\alpha \sum_{\kappa} \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\gamma} (-{}^tV\kappa)^{\alpha-\gamma} \mathbf{l}_0^\gamma H_\kappa \mathbf{e}_j = i^\alpha \sum_{\kappa} \mathbf{l}_\kappa^\alpha H_\kappa \mathbf{e}_j \\ &= i^\alpha \sum_{\kappa} ({}^t\mathbf{e}_j \otimes \mathbf{l}_\kappa^\alpha) \text{vec}(H_\kappa) = i^\alpha {}_j\mathbf{l}_\Omega^\alpha \text{vec}(H). \end{aligned}$$

The proof of the second formula is similar, and it is omitted here. □

Let  $\mathbb{H}^0$  be the subspace of  $\mathbb{H}$  defined by

$$(5.2) \quad \mathbb{H}^0 := \{H \in \mathbb{H} : \mathbf{L}_\Omega^\beta \text{vec}(H) = 0, {}_j\mathbf{l}_\Omega^\alpha \text{vec}(H) = 0 \text{ and } {}_j\mathbf{r}_\Omega^\alpha \text{vec}(H) = 0, \forall \beta, \alpha \in \mathbb{Z}_+^d, |\beta| < k + k_0, |\alpha| < k, 1 \leq j \leq r\}.$$

**Proposition 5.2.** The subspace  $\mathbb{H}^0$  of  $\mathbb{H}$  defined by (5.2) is invariant under  $\mathbf{T}$ .

*Proof.* By Theorem 4.6, for any  $H \in \mathbb{H}^0$  and  $\beta \in \mathbb{Z}_+^d, |\beta| < k + k_0$ ,

$$\mathbf{L}_\Omega^\beta \text{vec}(\mathbf{T}H) = \mathbf{L}_\Omega^\beta \mathbf{T} \text{vec}(H) = \lambda^{-\beta} \mathbf{L}_\Omega^\beta \text{vec}(H) = 0.$$

By Lemma 5.1, for any  $\alpha \in \mathbb{Z}_+^d, |\alpha| < k$ , the equalities  ${}_j\mathbf{l}_\Omega^\alpha \text{vec}(H) = 0$  and  ${}_j\mathbf{r}_\Omega^\alpha \text{vec}(H) = 0$  for all  $j, 1 \leq j \leq r$ , are equivalent to  $D_V^\alpha (B(\omega)H(\omega))|_{\omega=0} = 0$  and  $D_V^\alpha (H(\omega)B^*(\omega))|_{\omega=0} = 0$ , respectively. One can check by (4.7) and (4.8) that  $D_V^\alpha (B(\omega)\mathbf{T}H(\omega))|_{\omega=0} = 0$  ( $D_V^\alpha (\mathbf{T}H(\omega)B^*(\omega))|_{\omega=0} = 0$  resp.) for all  $\alpha \in \mathbb{Z}_+^d, |\alpha| < k$ , if  $D_V^\alpha (B(\omega)H(\omega))|_{\omega=0} = 0$  ( $D_V^\alpha (H(\omega)B^*(\omega))|_{\omega=0} = 0$  resp.) for  $\alpha \in \mathbb{Z}_+^d, |\alpha| < k$ . Thus  $\mathbb{H}^0$  is invariant under  $\mathbf{T}$ . □

Let  $\mathbf{T}|_{\mathbb{H}^0}$  denote the restriction of  $\mathbf{T}$  to  $\mathbb{H}^0$ . We will want to find the Sobolev regularity estimate of  $\Phi$  in terms of the the spectral radius  $\rho(\mathbf{T}|_{\mathbb{H}^0})$  of  $\mathbf{T}|_{\mathbb{H}^0}$ , and therefore we need to find the maximum of the moduli of the eigenvalues of  $\mathbf{T}|_{\mathbb{H}^0}$ . Since the product of the left and right eigenvectors of a simple eigenvalue of a matrix is not zero, Theorem 4.6 leads to the following corollary,

**Corollary 5.3.** *If  $\lambda^{-\beta}$  with  $\beta \in \mathbb{Z}_+^d, |\beta| < k + k_0$ , is a simple eigenvalue of  $\mathcal{T}$  and there exists  $\beta' \in E_\beta$  such that  $\mathbf{L}_\Omega^{\beta'} \neq 0$ , then  $\lambda^{-\beta}$  is not an eigenvalue of  $\mathbf{T}|_{\mathbb{H}^0}$ .*

The next proposition provides a way to find the eigenvalues of  $\mathbf{T}|_{\mathbb{H}^0}$ . Let  $\mathcal{L}_\Omega$  be the  $r^2|[\Omega]|$  by  $\binom{d + k + k_0 - 1}{d}$  matrix defined by

$$\mathcal{L}_\Omega := (\cdots, {}^t(\mathbf{L}_\Omega^\beta), \cdots)_{\beta \in \mathbb{Z}_+^d, |\beta| \leq k + k_0 - 1},$$

and for  $j, 1 \leq j \leq r$ , let  $L_j$  and  $R_j$  be the  $r^2|[\Omega]|$  by  $\binom{d + k - 1}{d}$  matrices defined by

$$L_j := (\cdots, {}^t(j\mathbf{l}_\Omega^\alpha), \cdots)_{\alpha \in \mathbb{Z}_+^d, |\alpha| \leq k - 1}, \quad R_j := (\cdots, {}^t(j\mathbf{r}_\Omega^\alpha), \cdots)_{\alpha \in \mathbb{Z}_+^d, |\alpha| \leq k - 1}.$$

Then define the  $r^2|[\Omega]|$  by  $\binom{d + k + k_0 - 1}{d} + 2r \binom{d + k - 1}{d}$  matrix  $M_\Omega$  by

$$M_\Omega := (\mathcal{L}_\Omega, L_1, \cdots, L_r, R_1, \cdots, R_r).$$

**Proposition 5.4.** *Assume that  $\lambda_0$  is a nonzero eigenvalue of  $\mathbf{T}$ . Then  $\lambda_0$  is an eigenvalue of  $\mathbf{T}|_{\mathbb{H}^0}$  if and only if  $\text{rank}({}^tM_\Omega(\mathbf{u}_1, \cdots, \mathbf{u}_l)) < l$ , where  $\mathbf{u}_1, \cdots, \mathbf{u}_l$  constitute a basis of the  $\lambda_0$ -eigenspace of  $\mathcal{T}$ .*

*Proof.* Note that  $\lambda_0$  is a nonzero eigenvalue of  $\mathbf{T}|_{\mathbb{H}^0}$  if and only if  $\lambda_0$  is a nonzero eigenvalue of  $\mathcal{T}$  with a corresponding right eigenvector  $\mathbf{u}$  satisfying

$$(5.3) \quad {}^tM_\Omega \mathbf{u} = 0.$$

By the fact that for any matrices  $M_1, M_2$  (with the product  $M_1M_2$  meaningful),  $\text{rank}(M_1M_2) \leq \min(\text{rank}M_1, \text{rank}M_2)$ , we know that if  $\text{rank}({}^tM_\Omega(\mathbf{u}_1, \cdots, \mathbf{u}_l)) \geq l$ , then  $\text{rank}({}^tM_\Omega(\mathbf{u}_1, \cdots, \mathbf{u}_l)) = l$ , and therefore any linear combinations of  $\mathbf{u}_1, \cdots, \mathbf{u}_l$  does not satisfies (5.3). Thus  $\lambda_0$  is not an eigenvalue of  $\mathbf{T}|_{\mathbb{H}^0}$ .

If  $\text{rank}({}^tM_\Omega(\mathbf{u}_1, \cdots, \mathbf{u}_l)) = l_0 < l$ , we assume without loss of generality that the rank of  ${}^tM_\Omega(\mathbf{u}_1, \cdots, \mathbf{u}_{l_0})$  is  $l_0$ . Thus  ${}^tM_\Omega \mathbf{u}_j, j = 1, \cdots, l_0$ , are linearly independent, while  ${}^tM_\Omega \mathbf{u}_j, j = 1, \cdots, l_0 + 1$ , are linearly dependent. Hence we can find constants  $c_1, \cdots, c_{l_0}$  such that

$$\mathbf{v} := c_1 \mathbf{u}_1 + \cdots + c_{l_0} \mathbf{u}_{l_0} + \mathbf{u}_{l_0+1}$$

satisfies (5.3), i.e.  $\lambda_0$  is an eigenvalue of  $\mathbf{T}|_{\mathbb{H}^0}$  with  $H_0 \in \mathbb{H}$  given by  $\text{vec}(H_0) = \mathbf{v}$ , with  $\mathbf{v}$  being a corresponding eigenvector.  $\square$

Proposition 5.4 provides an easy way to find eigenvalues of  $\mathbf{T}|_{\mathbb{H}^0}$ , and its proof shows how to find the corresponding eigenvector. By Proposition 5.4, we have the following corollary.

**Corollary 5.5.** *The spectral radius  $\rho(\mathbf{T}|_{\mathbb{H}^0})$  of  $\mathbf{T}|_{\mathbb{H}^0}$  is the maximum of the moduli of all eigenvalues  $\lambda_0$  of  $\mathcal{T}$  satisfying  $\text{rank}({}^tM_\Omega(\mathbf{u}_1, \cdots, \mathbf{u}_l)) < l$ , where  $\mathbf{u}_1, \cdots, \mathbf{u}_l$  are a basis of the  $\lambda_0$ -eigenspace of  $\mathcal{T}$ .*

For the next proposition, we need to consider the transition operators on other spaces. Denote  $\mathcal{N} := \max(N, k + k_0)$  and

$$\Omega_1 := \left\{ \sum_{j=0}^{\infty} M^{-(j+1)} x_j : x_j \in [-\mathcal{N}, \mathcal{N}]^d, \forall j \in \mathbb{Z}_+ \right\}.$$

Let  $\mathbb{H}_{\Omega_1}$  denote the space of all  $r \times r$  matrices with each entry a trigonometric polynomial whose Fourier coefficients are supported in  $[\Omega_1]$ , and let  $\mathbf{T}_{\Omega_1}$  denote the operator  $\mathbf{T}$  restricted to  $\mathbb{H}_{\Omega_1}$ . Then  $\mathbf{T}_{\Omega_1}$  is a linear operator on  $\mathbb{H}_{\Omega_1}$  leaving  $\mathbb{H}_{\Omega_1}$  and  $\mathbb{H}$  invariant, and the representing matrix of  $\mathbf{T}_{\Omega_1}$  is

$$\mathcal{T}_{\Omega_1} := (\mathcal{A}_{2i-j})_{i,j \in [\Omega_1]}.$$

Let  $\mathbb{H}_{\Omega_1}^0$  be the subspace of  $\mathbb{H}_{\Omega_1}$  defined as follows:  $H \in \mathbb{H}_{\Omega_1}^0$  if and only if  $\mathbf{L}_{\Omega_1}^\beta \text{vec}(H) = 0, {}_j \mathbf{L}_{\Omega_1}^\alpha \text{vec}(H) = 0$  and  ${}_j \mathbf{r}_{\Omega_1}^\alpha \text{vec}(H) = 0$  for all  $\beta, \alpha \in \mathbb{Z}_+^d, |\beta| < k + k_0, |\alpha| < k, 1 \leq j \leq r$ . In this case  $\mathbf{L}_{\Omega_1}^\beta, {}_j \mathbf{L}_{\Omega_1}^\alpha$  and  ${}_j \mathbf{r}_{\Omega_1}^\alpha$  are  $1 \times (r^2 |[\Omega_1]|)$  vectors defined by (4.9) and (5.1), respectively, with  $\Omega_1$  instead of  $\Omega$ . It can be shown similarly that  $\mathbb{H}_{\Omega_1}^0$  is invariant under  $\mathbf{T}_{\Omega_1}$ , and we let  $\mathbf{T}|_{\mathbb{H}_{\Omega_1}^0}$  denote the restriction of  $\mathbf{T}_{\Omega_1}$  ( $\mathbf{T}$ ) to  $\mathbb{H}_{\Omega_1}^0$ . Let  $H_0 \in \mathbb{H}_{\Omega_1}$  be defined by

$$(5.4) \quad H_0(\omega) = \sum_{j=1}^d (1 - \cos(\omega_j))^{k+k_0} \mathbf{I}_r, \quad \omega = {}^t(\omega_1, \dots, \omega_d) \in \mathbb{R}^d.$$

Then  $H_0(\omega) \in \mathbb{H}_{\Omega_1}^0$ , and thus  $\mathbb{H}_{\Omega_1}^0$  is nontrivial. By Lemma 2.2, the eigenvectors of  $\mathbf{T}_{\Omega_1}$  corresponding to nonzero eigenvalues are in  $\mathbb{H}$ . Therefore  $\mathbf{T}_{\Omega_1}$  ( $\mathbf{T}|_{\mathbb{H}_{\Omega_1}^0}$  resp.) and the restriction  $\mathbf{T}|_{\mathbb{H}}$  of  $\mathbf{T}$  to  $\mathbb{H}$  ( $\mathbf{T}|_{\mathbb{H}^0}$  resp.) have the same nonzero eigenvalues. Hence  $\rho(\mathbf{T}|_{\mathbb{H}^0}) = \rho(\mathbf{T}|_{\mathbb{H}_{\Omega_1}^0})$ , where  $\rho(\mathbf{T}|_{\mathbb{H}^0})$  and  $\rho(\mathbf{T}|_{\mathbb{H}_{\Omega_1}^0})$  denote the spectral radii of  $\mathbf{T}|_{\mathbb{H}^0}$  and  $\mathbf{T}|_{\mathbb{H}_{\Omega_1}^0}$ , respectively.

The following proposition is obtained by modifying the proof of Proposition 4.4 in [26] or Proposition 3.3 in [19].

Choose a vector norm on the space  $\mathbb{H}_{\Omega_1}^0$  and define the operator (matrix) norm  $\|\mathbf{T}|_{\mathbb{H}_{\Omega_1}^0}\|$  with respect to this vector norm. Then

$$\lim_{n \rightarrow \infty} \|(\mathbf{T}|_{\mathbb{H}_{\Omega_1}^0})^n\|^{1/n} = \rho(\mathbf{T}|_{\mathbb{H}_{\Omega_1}^0}) = \rho(\mathbf{T}|_{\mathbb{H}^0}).$$

**Proposition 5.6.** *Assume that  $\mathbf{P}$  satisfies conditions (3.4) and (4.5), and  $\rho(\mathbf{T}|_{\mathbb{H}^0})$  is the spectral radius of  $\mathbf{T}|_{\mathbb{H}^0}$ . Then for any  $\epsilon > 0$ , for the corresponding  $(M, \mathbf{P})$  matrix refinable function  $\widehat{\Phi}$ , there exists a constant  $c$  independent of  $n$  such that*

$$\int_{\mathbb{D}_n} \left| \widehat{\Phi}(w) \right|^2 dw \leq c(\rho(\mathbf{T}|_{\mathbb{H}^0}) + \epsilon)^n,$$

where  $\mathbb{D}_n := {}^t M^n \mathbb{T}^d \setminus ({}^t M^{n-1} \mathbb{T}^d)$ ,  $n \in \mathbb{Z}_+$ .

*Proof.* Let  $H_0(\omega) \in \mathbb{H}_{\Omega_1}^0$  be defined by (5.4). Since  ${}^t M^{-1} \mathbb{T}^d$  is a neighborhood of the origin, there exists a positive integer  $q$  such that  $\frac{1}{q} \mathbb{T}^d \subset {}^t M^{-1} \mathbb{T}^d$ . Note that for  $\omega \in \mathbb{D}_n$ ,  $\widehat{\Phi}(\omega) = \Pi_n(\omega) \widehat{\Phi}({}^t M^{-n} \omega)$ , and for  $\omega \in \mathbb{T}^d \setminus (\frac{1}{q} \mathbb{T}^d)$ ,  $H_0(\omega) \geq c_0 \mathbf{I}_r$  with  $c_0 = d(1 - \cos(\frac{\pi}{q}))^{k+k_0} > 0$ . Thus by the continuity of  $\widehat{\Phi}(\omega)$  on  $\mathbb{T}^d$ , we have for any

positive integer  $n$ ,

$$\begin{aligned} \int_{\mathbb{D}_n} \widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)d\omega &= \int_{\mathbb{D}_n} \Pi_n(\omega)\widehat{\Phi}({}^tM^{-n}\omega)\widehat{\Phi}^*({}^tM^{-n}\omega)\Pi_n^*(\omega)d\omega \\ &\leq c \int_{\mathbb{D}_n} \Pi_n(\omega)\Pi_n^*(\omega)d\omega \leq c \int_{{}^tM^n\mathbb{T}^d \setminus (\frac{1}{q}{}^tM^n\mathbb{T}^d)} \Pi_n(\omega)\Pi_n^*(\omega)d\omega \\ &\leq c \int_{{}^tM^n\mathbb{T}^d \setminus (\frac{1}{q}{}^tM^n\mathbb{T}^d)} \Pi_n(\omega)H_0({}^tM^{-n}\omega)\Pi_n^*(\omega)d\omega \\ &\leq c \int_{\mathbb{R}^d} \Pi_n(\omega)H_0({}^tM^{-n}\omega)\Pi_n^*(\omega)d\omega = c \int_{\mathbb{T}^d} (\mathbf{T}_{\Omega_1}^n H_0)(\omega)d\omega, \end{aligned}$$

where the last equality follows from Lemma 2.9. Since the Hilbert-Schmidt norm  $\|Q\|_2 = \sqrt{\text{Tr}(QQ^*)}$  is an equivalent norm for finite matrices, by applying the trace operation, we obtain

$$\int_{\mathbb{D}_n} |\widehat{\Phi}(\omega)|^2 d\omega = \int_{\mathbb{D}_n} \text{Tr}(\widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)) d\omega \leq c_\epsilon \left(\rho(\mathbf{T}|_{\mathbb{H}_{\Omega_1}^0}) + \epsilon\right)^n = c_\epsilon (\rho(\mathbf{T}|_{\mathbb{H}^0}) + \epsilon)^n$$

with  $c_\epsilon$  independent of  $n$ . □

Proposition 5.6 together with the usual Littlewood-Paley technique leads to the following Sobolev estimate of the refinable vector  $\Phi$ .

**Theorem 5.7.** *Assume that  $\mathbf{P}$  satisfies (3.4) and (4.5). Then the  $(M, \mathbf{P})$  matrix refinable function  $\Phi$  is in  $W^s(\mathbb{R}^d)$  for any  $s < s_0 := -\log \rho(\mathbf{T}|_{\mathbb{H}^0}) / (2 \log \lambda_{\max})$ , where  $\rho(\mathbf{T}|_{\mathbb{H}^0})$  is the spectral radius of  $\mathbf{T}|_{\mathbb{H}^0}$  and  $\lambda_{\max} := \max\{|\lambda_1|, \dots, |\lambda_d|\}$ .*

*Proof.* For the dilation matrix  $M$ , there exists some  $n_0 \in \mathbb{Z}_+$  such that  $\mathbb{T}^d \subset ({}^tM)^{n_0+1}\mathbb{T}^d$ . For  $s < s_0$ , let  $\epsilon > 0$  be a constant satisfying

$$s < -\log(\epsilon + \rho(\mathbf{T}|_{\mathbb{H}^0})) / (2 \log \lambda_{\max}).$$

Since

$$\int_{\mathbb{D}_n} |\widehat{\Phi}(w)|^2 d\omega \leq c(\epsilon + \rho(\mathbf{T}|_{\mathbb{H}^0}))^n,$$

for some constant  $c$  independent of  $n$ , and  $\widehat{\Phi}$  is continuous on  $\mathbb{T}^d$ , it follows that

$$\begin{aligned} &\int_{\mathbb{R}^d} (1 + |\omega|^2)^s |\widehat{\Phi}(\omega)|^2 d\omega \\ &\leq \int_{\mathbb{T}^d} (1 + |\omega|^2)^s |\widehat{\Phi}(\omega)|^2 d\omega + \sum_{n=1}^{\infty} \int_{{}^tM^{n_0+n}\mathbb{T}^d \setminus {}^tM^{n-1}\mathbb{T}^d} (1 + |\omega|^2)^s |\widehat{\Phi}(\omega)|^2 d\omega \\ &= \int_{\mathbb{T}^d} (1 + |\omega|^2)^s |\widehat{\Phi}(\omega)|^2 d\omega + \sum_{n=1}^{\infty} \sum_{j=0}^{n_0} \int_{\mathbb{D}_{n+j}} (1 + |\omega|^2)^s |\widehat{\Phi}(\omega)|^2 d\omega \\ &\leq c + c \sum_{n=1}^{\infty} \sum_{j=0}^{n_0} (\lambda_{\max})^{2(n+j)s} (\epsilon + \rho(\mathbf{T}|_{\mathbb{H}^0}))^n < \infty. \end{aligned}$$

Therefore  $\Phi \in W^s(\mathbb{R}^d)$ . □

Let  $C^\gamma(\mathbb{R}^d)$  denote the space defined as the following way: if  $\gamma = n + \gamma'$  with  $n \in \mathbb{Z}_+$  and  $0 \leq \gamma' < 1$ , then  $f \in C^\gamma(\mathbb{R}^d)$  if and only if  $f \in C^{(n)}(\mathbb{R}^d)$  and  $f^{(n)}$  is

uniformly Hölder continuous with exponent  $\gamma'$ , i.e.

$$|D^\beta f(x + y) - D^\beta f(x)| \leq c|y|^{\gamma'}, \quad \text{for any } \beta \in \mathbb{Z}_+^d, |\beta| = n,$$

for some constant  $c$  independent of  $x, y \in \mathbb{R}^d$ . With the well-known inclusion

$$W^s(\mathbb{R}^d) \subset C^\gamma(\mathbb{R}^d), \quad \text{for } s > \gamma + \frac{d}{2},$$

Theorem 5.7 leads to the following corollary.

**Corollary 5.8.** *Suppose  $\mathbf{P}$  satisfies conditions (3.4) and (4.5). Then the  $(M, \mathbf{P})$  matrix refinable function  $\Phi \in C^\gamma(\mathbb{R}^d)$  for any  $\gamma < -\frac{d}{2} - \log \rho(\mathbf{T}|_{\mathbb{H}^0}) / (2 \log \lambda_{\max})$ , where  $\rho(\mathbf{T}|_{\mathbb{H}^0})$  is the spectral radius of  $\mathbf{T}|_{\mathbb{H}^0}$  and  $\lambda_{\max} := \max\{|\lambda_1|, \dots, |\lambda_d|\}$ .*

Assume that the refinement mask  $\{\mathbf{P}_\alpha\}$  is a finitely supported real  $r \times r$  matrix sequence and  $\mathbf{P}$  satisfies the vanishing moment conditions of order  $k$  (3.4) and (4.5) for some  $k_0$  with real vectors  $\mathbf{l}_0^\beta, |\beta| < k + k_0$ . Let  $\mathbb{H}_r$  denote the space of all  $r \times r$  matrices with each entry a trigonometric polynomial whose Fourier coefficients are real and supported in  $[\Omega]$ . Then  $\mathbb{H}_r$  is invariant under  $\mathbf{T}$ . Define the subspace  $\mathbb{H}_{\text{sym}}$  of  $\mathbb{H}_r$  by

$$\begin{aligned} \mathbb{H}_{\text{sym}} := \{H \in \mathbb{H}_r : & \quad H^* = H, \quad \mathbf{L}_\Omega^\beta \text{vec}(H) = 0 \text{ and} \\ & \quad \mathbf{j} \mathbf{l}_\Omega^\alpha \text{vec}(H) = 0, \forall \beta, \alpha \in \mathbb{Z}_+^d, |\beta| < k + k_0, |\alpha| < k, 1 \leq j \leq r\}. \end{aligned}$$

Then  $\mathbb{H}_{\text{sym}}$  is a linear space over the field  $\mathbb{R}$  and is invariant under  $\mathbf{T}$ . Let  $\mathbf{T}|_{\mathbb{H}_{\text{sym}}}$  denote the restriction of  $\mathbf{T}$  to  $\mathbb{H}_{\text{sym}}$ . Then, as above, we can obtain the Sobolev regularity estimate of the compactly supported  $(M, \mathbf{P})$  refinable vector  $\Phi$  in terms of the spectral radius of  $\mathbf{T}|_{\mathbb{H}_{\text{sym}}}$ .

**Theorem 5.9.** *Assume that the refinement mask  $\{\mathbf{P}_\alpha\}$  is a finitely supported real  $r \times r$  matrix sequence and  $\mathbf{P}$  satisfies (3.4) and (4.5) with real vectors  $\mathbf{l}_0^\beta, |\beta| < k + k_0$ . Then the  $(M, \mathbf{P})$  matrix refinable function  $\Phi$  is in  $W^s(\mathbb{R}^d)$  for any  $s < s_0 := -\log \rho(\mathbf{T}|_{\mathbb{H}_{\text{sym}}}) / (2 \log \lambda_{\max})$ , where  $\rho(\mathbf{T}|_{\mathbb{H}_{\text{sym}}})$  is the spectral radius of  $\mathbf{T}|_{\mathbb{H}_{\text{sym}}}$  and  $\lambda_{\max} := \max\{|\lambda_1|, \dots, |\lambda_d|\}$ .*

In [19], the Sobolev regularity estimates of the B-splines defined by knots  $0, 0, 1, 1$  and  $0, 1, 1, 2$ , the GHM-orthogonal scaling functions in [8] and two refinable vectors from [2] are analyzed. To finish this paper, we analyze an example from [9] about refinable bivariate splines.

**Example 5.10.** Let  $\phi_1$  denote the “pyramid function” with support on the square with vertices  $(2, 1), (1, 2), (0, 1)$  and  $(1, 0)$  which is continuous, satisfies  $\phi_1(1, 1) = 1$  and is linear on each of the four triangles formed by the boundary and the two diagonals of its support. Let  $\phi_2$  be the “pyramid function” with support on  $[1, 2]^2$ , i.e.

$$\phi_2(x_1, x_2) = \phi_1(x_1 + x_2 - 1, x_1 - x_2).$$

Let  $\Phi := {}^t(\phi_1, \phi_2)$ . Then  $\Phi$  satisfies the matrix refinement equations (1.1) with  $M = 2\mathbf{I}_2$  and the matrix refinement mask given by (refer to [9])

$$\mathbf{P}(\omega) := \frac{1}{8} \begin{pmatrix} z_1 + z_2 + 2z_1z_2 + z_1^2z_2 + z_1z_2^2 & (1 + z_1)(1 + z_2) \\ 2(z_1z_2)^2 & z_1z_2(1 + z_1)(1 + z_2) \end{pmatrix},$$

where  $z_1 = e^{-i\omega_1}, z_2 = e^{-i\omega_2}$ . In this case  $\eta_j = \gamma_j, j = 0, \dots, 3$ , and they are the vertices of  $[0, 1]^2$ , and  $1, \frac{1}{4}$  are eigenvalues of  $\mathbf{P}(0), N = 2, \Omega = [-2, 2]^2$ . One has

$$\mathbf{P}(0) = \frac{1}{8} \begin{pmatrix} 6 & 4 \\ 2 & 4 \end{pmatrix}, \quad \mathbf{P}(\pi\eta_j) = \frac{1}{8} \begin{pmatrix} -2 & 0 \\ 2 & 0 \end{pmatrix}, \quad j = 1, 2, 3.$$

Thus  $\mathbf{l}_0^{(00)} = {}^t(1, 1)$  is the unique (up to a nonzero constant) vector satisfying (3.4) for  $\beta = (00)$ , and we have

$$\begin{aligned} D^{(10)}\mathbf{P}(0) &= D^{(10)}\mathbf{P}(0) = \frac{-i}{8} \begin{pmatrix} 6 & 2 \\ 4 & 6 \end{pmatrix}, \\ D^{(10)}\mathbf{P}(\pi, 0) &= D^{(01)}\mathbf{P}(0, \pi) = \frac{-i}{8} \begin{pmatrix} -2 & -2 \\ 4 & 2 \end{pmatrix}, \\ D^{(10)}\mathbf{P}(0, \pi) &= D^{(01)}\mathbf{P}(\pi, 0) = D^{(10)}\mathbf{P}(\pi, \pi) = D^{(01)}\mathbf{P}(\pi, \pi) = \frac{-i}{8} \begin{pmatrix} -2 & 0 \\ 4 & 0 \end{pmatrix}. \end{aligned}$$

One can obtain that  $\mathbf{l}_0^{(10)} = \mathbf{l}_0^{(01)} = {}^t(1, \frac{3}{2})$  satisfy (3.4) for  $\beta = (10)$  and  $\beta = (01)$ , respectively, and there are no such vectors  $\mathbf{l}_0^\beta$  that satisfy (3.4) for all  $\beta \in \mathbb{Z}_+^2$  with  $|\beta| = 2$ . Though  $\frac{1}{4}$  is an eigenvalue of  $\mathbf{P}(0)$ , there are vectors  $\mathbf{l}_0^{(20)} = \mathbf{l}_0^{(02)} = {}^t(1, 2), \mathbf{l}_0^{(11)} = {}^t(1, \frac{9}{4})$  and  $\mathbf{l}_0^{(30)} = \mathbf{l}_0^{(03)} = {}^t(1, \frac{9}{4}), \mathbf{l}_0^{(21)} = \mathbf{l}_0^{(12)} = {}^t(1, 3)$  satisfying (4.5) for  $\beta = (20), (02), (30), (03), (21)$  and  $(12)$ , respectively. To check the stability of  $\Phi$ , we need to compute the eigenvalues of the  $100 \times 100$  matrix

$$\mathcal{T}_{[-2, 2]^2} = (\mathcal{A}_{2i-j})_{i, j \in [-2, 2]^2}.$$

We find for  $\beta \in \mathbb{Z}_+^d, |\beta| \leq 3$ , that  $\mathbf{L}_{[-2, 2]^2}^\beta \neq 0$ . Thus by Theorem 4.2,  $1, \frac{1}{2}, \frac{1}{4}$  and  $\frac{1}{8}$  are eigenvalues of  $\mathcal{T}$ . In fact the eigenvalues of  $\mathcal{T}$  are  $1, \frac{1}{2}(2), \frac{1}{4}(5), \frac{1}{8}(12), \frac{1}{16}(24)$  and  $0(56)$ . Here for an eigenvalue  $\lambda_0$ , the notation  $\lambda_0(l)$  means that the algebraic multiplicity of  $\lambda_0$  is  $l$ . Thus  $\mathcal{T}_{[-2, 2]^2}$  and the transition operator  $\mathbf{T}$  restricted to  $\mathbb{H}_{[-2, 2]^2}$ , denoted by  $\mathbf{T}_{[-2, 2]^2}$ , satisfy Condition E. We find that the 1-eigenvector of  $\mathbf{T}_{[-2, 2]^2}$  is

$$H(\omega) = \begin{pmatrix} 8 + e^{i\omega_1} + e^{i\omega_2} + e^{-i\omega_1} + e^{-i\omega_2} & 1 + e^{i\omega_1} + e^{i\omega_2} + e^{i(\omega_1 + \omega_2)} \\ 1 + e^{-i\omega_1} + e^{-i\omega_2} + e^{-i(\omega_1 + \omega_2)} & 4 \end{pmatrix}.$$

Checking directly,  $H(\omega) > 0$  for all  $\omega \in \mathbb{T}^2$ ; hence  $\Phi$  is stable. By Theorem 3.6,  $\mathcal{S}(\Phi)$  provides approximation of order 2.

To estimate the regularity by our method, we need only to find the maximum of the moduli of the eigenvalues of  $\mathbf{T}_{[-2, 2]^2}|_{\mathbb{H}^0}$ , the restriction of  $\mathbf{T}_{[-2, 2]^2}$  to the invariant subspace  $\mathbb{H}^0$  of  $\mathbb{H}_{[-2, 2]^2}$  defined by (5.2). By Corollary 5.3 and Proposition 5.4, we find that  $1, \frac{1}{2}$  and  $\frac{1}{4}$  are not eigenvalues of  $\mathbf{T}_{[-2, 2]^2}|_{\mathbb{H}^0}$ , and  $\frac{1}{8}$  is an eigenvalue of  $\mathbf{T}_{[-2, 2]^2}|_{\mathbb{H}^0}$  with a corresponding eigenvector  $H^0(\omega) = \sum_{\ell \in [-1, 1]^2} H_\ell e^{-i\ell\omega}$  given by

$$\begin{aligned} H_{-1-1} &= {}^t H_{11} = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}, \quad H_{-10} = {}^t H_{10} = \begin{pmatrix} -6 & 6 \\ 0 & 0 \end{pmatrix}, \\ H_{0-1} &= {}^t H_{01} = \begin{pmatrix} 0 & 6 \\ 0 & -6 \end{pmatrix}, \quad H_{00} = \begin{pmatrix} -10 & 4 \\ 4 & -8 \end{pmatrix}, \end{aligned}$$

and  $H_{-11} = {}^t H_{1-1} = \mathbf{0}$ . Thus  $\rho(\mathbf{T}_{[-2, 2]^2}|_{\mathbb{H}^0}) = \frac{1}{8}$ , and it follows from Theorem 5.7 or Theorem 5.9 that  $\Phi \in W^{\frac{3}{2}-\epsilon}(\mathbb{R}^2)$  for any  $\epsilon > 0$ . On the other hand, the Fourier

transform of  $\Phi$  is (see [9])

$$\begin{aligned}\widehat{\phi}_1(\omega_1, \omega_2) &= 4e^{-i(\omega_1+\omega_2)} \frac{\omega_1 \sin \omega_2 - \omega_2 \sin \omega_1}{\omega_1 \omega_2 (\omega_1^2 - \omega_2^2)}, \\ \widehat{\phi}_2(\omega_1, \omega_2) &= \frac{1}{2} e^{-\frac{3}{2}i(\omega_1+\omega_2)} \widehat{\phi}_1\left(\frac{\omega_1 + \omega_2}{2}, \frac{\omega_1 - \omega_2}{2}\right).\end{aligned}$$

Thus  $\Phi \in W^s(\mathbb{R}^2)$  if and only if  $s < \frac{3}{2}$ , and our estimate on the Sobolev regularity of  $\Phi$  is optimal.

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